HYPERCOVERS AND SIMPLICIAL PRESHEAVES

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ABSTRACT. We use hypercovers to study the homotopy theory of simplicial presheaves. The main result says that model structures for simplicial presheaves involving local weak equivalences can be constructed by localizing at the hypercovers. One consequence is that the fibrant objects can be explicitly described in terms of a hypercover descent condition. These ideas are central to constructing realization functors on the homotopy theory of schemes [DI1, Is]. We give a few other applications for this new description of the homotopy theory of simplicial presheaves.

CONTENTS

1. INTRODUCTION

This paper is concerned with the subject of homotopical sheaf theory, as it has developed over time in the articles [I, B, BG, Th, Jo, J1, J2, J3, J4]. Given a fixed Grothendieck site $\mathcal{C}$, one wants to consider contravariant functors F defined on $\mathcal{C}$ whose values have a homotopy type associated to them. The most basic question is: what should it mean for F to be a sheaf? The desire is for some kind of local-to-global property—also called a *descent* property—where the value of F on an object X can be recovered by homotopical methods from the values on a cover. Perhaps the earliest instance where such a concept had to be tackled was in algebraic geometry, where people had to deal with presheaves of chain complexes defined on a space X. Because of its abelian nature this could be handled by classical homological algebra, and led to the Grothendieck definition of hypercohomology. Much later, people encountered the non-abelian example of algebraic K-theory. Here the site $\mathcal{C}$ is a category of schemes, and the functor F assigns to each scheme X its algebraic K-theory spectrum $K(X)$. Thomason's paper [Th] (building on earlier work from [B, BG]) combined homotopy theory and sheaf theory to study the descent properties of this functor.

The work of [BG, Jo, J2] brought the use of model categories into this picture. In the most modern of these [J2], Jardine defined a model category structure on presheaves of simplicial sets with the property that the weak equivalences are *local* in nature. Classical invariants such as sheaf cohomology arise in this setting as homotopy classes of maps into certain Eilenberg-MacLane objects, and the whole theory can in some sense be regarded as the study of non-additive sheaf cohomology. Jardine's model structure has recently served as the foundation from which Morel and Voevodsky built their $\mathbb{A}^1$-homotopy theory for schemes [MV].

One important ingredient missing from Jardine's work is a description of the fibrant objects. They can be characterized in terms of a certain lifting property, but this is not so enlightening and not very useful in practice. Intuitively it has always been clear that the fibrant objects should be the simplicial presheaves that satisfy some kind of descent. Our main goal in this paper is to clarify this issue and give an explicit interpretation of fibrancy for simplicial presheaves.

To explain the basic ideas, let's assume our site is the category of topological spaces equipped with the usual open covers. A presheaf of sets F is a sheaf if $F(X)$ is the equalizer of $\prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_a \cap U_b)$ whenever $\{U_a\}$ is an open cover of X. This equalizer is in fact the same as the inverse limit of the entire cosimplicial diagram

$$\prod_a F(U_a) \rightrightarrows \prod_{a,b} F(U_{ab}) \Rrightarrow \cdots$$

where we have abbreviated $U_{a_0 \ldots a_n}$ for $U_{a_0} \cap \cdots \cap U_{a_n}$ and have refrained from drawing the codegeneracies for typographical reasons. For a presheaf of *simplicial* sets (or taking values in some other homotopical objects like *spectra*), it is natural to replace the limit by a homotopy limit. So one requires that $F(X)$ be weakly equivalent to the homotopy limit of the above cosimplicial diagram. This property, when it holds for all open covers, is called **Čech descent**. It can also be expressed in a slightly more compact way, if one recalls that the Čech complex $\check{C}U$ associated to a cover $\{U_a\}$ of X is the simplicial object $[n] \mapsto \coprod_{a_0,\ldots,a_n} U_{a_0 \cdots a_n}$. Then F satisfies Čech descent if the natural map

$$F(X) \to \operatorname*{holim}_n F(\check{C}U_n)$$

is a weak equivalence.

A motivating example is given by the functor $\mathcal{T}op^{op} \to \mathcal{S}pectra$ taking X to E^X, where E is a fixed spectrum and E^X denotes the function spectrum. This functor has Čech descent, because X is weakly equivalent to the homotopy colimit of the Čech complex for any open cover—this was shown in [DI1, Thm. 1.1].

Now, it is not true that the fibrant objects in Jardine's model category are just the simplicial presheaves which satisfy Čech descent (although this erroneous claim has appeared in a couple of preprints). See the appendix, Example A.10, for an example. What we show in this paper is that one has to instead consider descent for all *hypercovers*. A hypercover is a simplicial object U, augmented by X, which is similar to a Čech complex except in level n we only need to have a *cover* of the n-fold intersections $U_{a_0 \cdots a_n}$. A precise definition requires a morass of machinery (see Section 4). A simplicial presheaf F satisfies **descent for the hypercover**

$\boldsymbol{U} \to \boldsymbol{X}$ if the natural map

$$F(X) \to \operatorname*{holim}_n F(U_n)$$

is a weak equivalence (see Definition 4.3).

What we will show is that the fibrant objects in Jardine's model category are essentially the simplicial presheaves which satisfy descent for all hypercovers:

Theorem 1.1. *The fibrant objects in Jardine's model category $sPre(\mathcal{C})_{\mathcal{L}}$ are those simplicial presheaves that*

(1) are fibrant in the injective model structure $sPre(\mathcal{C})$, and
(2) satisfy descent for all hypercovers $U \to X$.

The *injective* model structure on $sPre(\mathcal{C})$ just refers to Jardine's model structure for the discrete topology on $\mathcal{C}$ (see Section 2 for more about this). The fibrancy conditions for this model structure are awkward to describe, but they also aren't very interesting—they have no dependence on the Grothendieck topology, only on the shape of the underlying category $\mathcal{C}$. The conditions require that each $F(X)$ be a fibrant simplicial set, certain maps $F(X) \to F(Y)$ be fibrations, and more complicated conditions of a similar 'diagrammatic nature'. In practice such conditions are not very important, and in fact there's a way to get around them completely by using the *projective* version of Jardine's model structure; see Theorem 1.3 below and the discussion in Section 2.

We'd like to point out that the above theorem can be re-interpreted in terms of giving 'generators' and 'relations' for the homotopy theory of simplicial presheaves, in the manner introduced by [D]. Using the language of that paper, we prove

Theorem 1.2. *Jardine's model category $sPre(\mathcal{C})_{\mathcal{L}}$ is Quillen equivalent to the* universal homotopy theory *$U\mathcal{C}/S$ constructed by*

(1) Formally adding homotopy colimits to the category $\mathcal{C}$, to create $U\mathcal{C}$; and then
(2) Imposing relations requiring that for every hypercover $U \to X$, the map $\operatorname{hocolim}_n U_n \to X$ is a weak equivalence.

In other words, the result says that everything special about the homotopy theory of simplicial presheaves can be derived from the basic fact that one can reconstruct X as the homotopy colimit of any of its hypercovers. The above theorem is crucial to the construction of étale realization functors for $\mathbb{A}^1$-homotopy theory [Is], as well as the analogous question about topological realization functors [DI1].

One advantage of the model structure $U\mathcal{C}/S$ over the model structure $sPre(\mathcal{C})_{\mathcal{L}}$ is that the fibrant objects are much easier to describe. The inexplicit fibrancy conditions of the injective model structure are replaced by a much simpler condition. Compare the following result to Theorem 1.1.

Theorem 1.3. *The fibrant objects in the model category $U\mathcal{C}/S$ are those simplicial presheaves that*

*(1) are objectwise fibrant (*i.e.*, each $F(X)$ is a fibrant simplicial set), and*
(2) satisfy descent for all hypercovers $U \to X$.

The main ideas we use to prove these results are very simple, and worth summarizing. They exactly parallel classical facts about CW-complexes. The two key ingredients are:

(i) In the category of simplicial presheaves one can construct objects analogous to CW-complexes, the only difference being that one has different kinds of 0-simplices corresponding to the different representable presheaves rX. (And as a consequence, there are different kinds of n-simplices corresponding to the objects $\Delta^n \otimes rX$.) Every simplicial presheaf has a cellular approximation built up out of representables in this way (see [D, Section 2.6]).

(ii) Weak equivalences for simplicial presheaves are characterized by a certain 'local lifting criterion', where lifting problems can be solved by passing from a representable object to the pieces of a cover. See Proposition 3.1 and the paper [DI2].

From these two basic principles, it's inevitable that hypercovers will arise in the solution of lifting problems. One starts building a lift inductively on a CW-approximation, and the obstructions to extending the lift are made to vanish by passing to a finer cover at each stage. Thus, one finds oneself inductively constructing a hypercover. These ideas are explored in detail in Section 5.

This paper came into existence because we needed to use Theorems 6.2, 7.6, and A.6(c,d) in other work. We originally hoped for a very short paper, but to actually write down complete proofs one has to be able to manipulate hypercovers with a certain amount of ease—and the literature on this subject is not the most helpful. So in the end a large portion of the paper has been devoted to carefully setting down the machinery of hypercovers, hopefully in a way that will be usable by other people. For this reason the paper sometimes takes on an expository tone. We have tried to be clear and thorough, and for good or bad this has come at the expense of brevity. Also, one of our goals has been to adopt definitions which can be applied to any Grothendieck site, not just the classical ones which get used most often. The result is theorems which are simple enough to state and prove, but sometimes hard to apply in practice. To complement this, we have included the reductions to Verdier sites (Section 8) and internal hypercovers (Section 9) one can implement for sites like those encountered in algebraic geometry. This subject of homotopical sheaf theory is rapidly finding applications in many contexts, so we have tried to give a presentation that is clear enough, and general enough, to be useful to a variety of practitioners.

1.4. **Organization of the paper.**

In Section 2 we review the basic model categories that will be used throughout the paper. One of these is Jardine's model structure, and the other is a Quillen equivalent version which has fewer cofibrations and more fibrations. We assume throughout that the reader is familiar with the theory of model categories—the original reference for this subject is [Q], but we generally follow [H] in notation and terminology. [Ho] is also a good reference.

Section 3 reviews material from [DI2] on lifting properties for simplicial presheaves, and how these can be used to characterize local weak equivalences. Section 4 introduces the machinery needed for defining and working with hypercovers. The section is a bit long, and serves mostly as a reference section for the rest of the paper—it can comfortably be skimmed the first time through.

In Section 5 we show how hypercovers enter into the solution of lifting problems in the homotopy theory of simplicial presheaves. These are the key observations which are needed for the main results. The proofs of these main results are then given in Section 6, where they appear as Theorem 6.2 and Corollary 6.3.

One application of the results on hypercovers is to realization functors from the homotopy theory of schemes—this is treated in the papers [DI1, Is]. In Section 7 we give a few more applications. One of the most interesting, given in Section 7.1, is a much simpler approach to the change-of-site functors of [MV]. We also discuss a generalization of the Verdier hypercovering theorem in Theorem 7.6.

In applications one rarely wants to work with *all* hypercovers, because this is just too broad a class of objects. In the case of the 'geometric' sites which are most commonly used, one can adopt more restrictive definitions and have all the above results still go through. These reductions are explored in Sections 8 and 9. We axiomatize what is necessary into the notion of a *Verdier site*, which comes equipped with a special class of 'basal hypercovers'. These ideas appear sporadically in Sections 6 and 7, but hopefully the reader can just refer back to the later sections as necessary.

Finally, the paper contains an appendix which explores the difference between Čech descent and hypercover descent. Again, the principal motivation comes from the fact that Čech descent is more easily dealt with in practice. We show, among other things, that having descent for Čech complexes is equivalent to having descent for all *bounded* hypercovers (the ones where the refinement process stops at some finite level). This is an important ingredient in [DI1].

1.5. **Notation and Terminology.** If X is an object of a site $\mathcal{C}$, then the **representable** simplicial presheaf rX on $\mathcal{C}$ is given by the formula $rX(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. Note that each simplicial set $rX(Y)$ is discrete. If U is a simplicial object of $\mathcal{C}$, then rU is the simplicial presheaf given by the formula $rU(Y)_n = \mathrm{Hom}_{\mathcal{C}}(Y, U_n)$—these, of course, are usually not discrete. We frequently abuse notation and write simply X (or U) for the presheaf rX (or rU).

If S is a scheme, then $\mathcal{S}ch/S$ denotes the category of schemes of finite-type over S. The full subcategory of schemes which are smooth over S is denoted Sm/S. Finally, in a simplicial model category we write $\mathrm{Map}(A, B)$ for the simplicial mapping space.

2. Model structures on simplicial presheaves

We start by recalling that for any small category $\mathcal{C}$ there are two Quillen equivalent model structures on the category of diagrams $s\mathcal{S}et^{\mathcal{C}}$. In each case a map $D \to E$ is a weak equivalence if $D(c) \to E(c)$ is a weak equivalence of simplicial sets for each c in $\mathcal{C}$. Such a map is usually called an **objectwise weak equivalence**. In the **projective model structure** on $s\mathcal{S}et^{\mathcal{C}}$ one defines a map $D \to E$ to be

(1) a *fibration* if every $D(c) \to E(c)$ is a fibration of simplicial sets (*i.e.*, $D \to E$ is an objectwise fibration), and
(2) a *cofibration* if it has the left-lifting-property with respect to the acyclic fibrations.

Dually, in the **injective model structure** the cofibrations are objectwise and the fibrations have the right-lifting-property with respect to acyclic cofibrations. The names 'projective' and 'injective' come from the analogy between the two usual model structures on chain complexes of R-modules. For notational convenience, the projective model structure is denoted $U\mathcal{C}$ (as was done in [D], where it was pointed out that $U\mathcal{C}$ has a certain universal property) and the injective model structure is denoted $sPre(\mathcal{C})$.

When $\mathcal{C}$ comes equipped with a Grothendieck topology, then one can construct refinements of these model structures which reflect the topology on $\mathcal{C}$. A map of simplicial presheaves $F \to G$ is a **local weak equivalence** if it induces isomorphisms on all sheaves of homotopy groups [I, Jo, J3]. In this paper we will use an alternative characterization in terms of homotopy liftings, described below.

Jardine's model structure on simplicial presheaves is the left Bousfield localization of $sPre(\mathcal{C})$ at the class $\mathcal{L}$ of local weak equivalences; we denote this localization as $sPre(\mathcal{C})_{\mathcal{L}}$. Of course since $\mathcal{L}$ is a *class* of maps there is no *a priori* guarantee that the Bousfield localization exists, but Jardine was able to construct it directly—it is only after the fact that one can identify it as a localization.

Similarly, one can define a model structure $U\mathcal{C}_{\mathcal{L}}$ by localizing $U\mathcal{C}$ at the same class $\mathcal{L}$ (cf. [Bl, Thm. 1.5]). The identity maps induce a Quillen equivalence $U\mathcal{C}_{\mathcal{L}} \to sPre(\mathcal{C})_{\mathcal{L}}$, so once again these are projective and injective versions of the same underlying homotopy theory. The injective version has the advantage that every object is cofibrant, but in the projective version the fibrant objects are easier to understand and the representable presheaves are still cofibrant. Also, it is usually easier to construct functors out of the projective version [D]. We state most of our results only in terms of $sPre(\mathcal{C})_{\mathcal{L}}$, but analogous statements for $U\mathcal{C}_{\mathcal{L}}$ are also true with only minor differences between the proofs.

Both $U\mathcal{C}$ and $sPre(\mathcal{C})$ are proper, simplicial model categories: if F is a simplicial presheaf and K is a simplicial set then $K \otimes F$ and F^K are defined objectwise, by

$$(K \otimes F)(X) = K \times F(X) \qquad \text{and} \qquad (F^K)(X) = F(X)^K.$$

From general considerations [H, Thm. 4.1.1], all localizations of $U\mathcal{C}$ and $sPre(\mathcal{C})$ that we consider are also left proper, simplicial model categories.

Remark 2.1. If F is a simplicial presheaf, then one obtains a diagram $D_F\colon \Delta^{op} \to sPre(\mathcal{C})$ by sending $[n]$ to F_n. Here F_n is just a presheaf of sets, but we can regard it as a discrete simplicial presheaf in the obvious way. The realization of this simplicial diagram is precisely F. The Bousfield-Kan map $\operatorname{hocolim} D_F \to |D_F|$ is a weak equivalence in this case, by some basic model category theory. So any simplicial presheaf F is weakly equivalent to $\operatorname{hocolim} D_F$. This observation will be needed often.

3. Local weak equivalences and local lifting properties

Local weak equivalences are usually defined in terms of sheaves of homotopy groups. Here we recall a different description which is more suitable for our purposes. See [DI2] for the proof that the two definitions agree and for more details on the results in this section.

First, recall that if X is in $\mathcal{C}$ and F and G are simplicial presheaves, then a diagram such as

$$\begin{tikzcd} \Lambda^{n,k} \otimes X \arrow[r] \arrow[d] & F \arrow[d] \\ \Delta^n \otimes X \arrow[r] & G \end{tikzcd}$$

has **local liftings** if there exists a covering sieve R of X such that for every map $U \to X$ in the sieve, the diagram one obtains by restricting from X to U has a lifting $\Delta^n \otimes U \to F$. These liftings are not required to be compatible for the different U's. A map $F \to G$ is called a **local fibration** if it has local liftings with respect to the maps $\Lambda^{n,k} \otimes X \to \Delta^n \otimes X$, for all X in $\mathcal{C}$. A simplicial presheaf is called **locally fibrant** if $F \to *$ is a local fibration.

Proposition 3.1 ([DI2, Th. 6.15]). *A map $F \to G$ between locally fibrant simplicial presheaves is a local weak equivalence if and only if every square*

$$\begin{tikzcd} \partial\Delta^n \otimes X \arrow[r] \arrow[d] & F \arrow[d] \\ \Delta^n \otimes X \arrow[r] & G \end{tikzcd}$$

has local ***relative homotopy-liftings****, in the following sense: after restricting to the pieces $U \to X$ of some covering sieve, one has maps $\Delta^n \otimes U \to F$ making the upper triangle commute on the nose and the lower triangle commute up to simplicial homotopy relative to $\partial\Delta^n \otimes U$.*

The reader may consult [DI2] for a detailed discussion of this kind of relative-homotopy-lifting property.

The following two results from [DI2] will be used later. Recall that a map is a **local acyclic fibration** if it is both a local fibration and a local weak equivalence.

Proposition 3.2 ([DI2, Prop. 7.2]). *A map $F \to G$ admits local liftings in every square*

$$\begin{tikzcd} \partial\Delta^n \otimes X \arrow[r] \arrow[d] & F \arrow[d] \\ \Delta^n \otimes X \arrow[r] \arrow[ur, dotted] & G \end{tikzcd}$$

if and only if it is a local acyclic fibration.

One consequence of the above result is that local acyclic fibrations are closed under pullbacks (in [J2] this was proven only when the domain and codomain are locally fibrant).

Proposition 3.3 ([J4, Lemma 19],[DI2, Cor. 7.4]). *Let $F \to G$ be a local fibration (resp., local acyclic fibration). If $K \hookrightarrow L$ is an inclusion of finite simplicial sets, then the induced map*

$$F^L \to F^K \times_{G^K} G^L$$

is a local fibration (resp., local acyclic fibration).

Let $f\colon E \to B$ be a map between presheaves of sets. One says that f is a **generalized cover** (or *local epimorphism*) if it has the following property: given any map $rX \to B$, there is a covering sieve $R \hookrightarrow X$ such that for every element $U \to X$ in R, the composite $rU \to rX \to B$ lifts through f. The 'generalized' adjective is there to remind us that we are looking at a map between presheaves, not actual objects of the site. In the case where B is representable and E is a coproduct $\coprod E_a$ of representables, f is a generalized cover precisely when the sieve generated by the maps $\{E_a \to B\}$ is a covering sieve of B.

For a simplicial presheaf F, let $\tilde{M}_n F$ denote the 0th object of $F^{\partial\Delta^n}$ (the 'tilde' is to distinguish this from a slightly different construction used later in the paper). This is the presheaf of sets whose value $\tilde{M}_n F(X)$ is the set of all maps $\partial\Delta^n \to F(X)$.

There is a natural map $F_n \to \tilde{M}_n F$ induced by $F^{\Delta^n} \to F^{\partial\Delta^n}$. Proposition 3.2 can be rephrased as saying that $F \to G$ is a local acyclic fibration if and only if the maps

$$F_n \to \tilde{M}_n F \times_{\tilde{M}_n G} G_n \tag{3.4}$$

are generalized covers, for all $n \geq 0$. Using this observation, most properties of generalized covers can automatically be seen to hold for local acyclic fibrations.

4. Background on hypercovers

This section contains the necessary machinery for defining and working with hypercovers. Unfortunately there is quite a bit of annoying category theory, and some readers may wish to only skim this section their first time through. This should be enough to understand the basic definitions that are used throughout the paper. In Section 4.12 we recall the coskeleton and degeneration functors, which appear when passing between simplicial objects and truncated simplicial objects. These notions are used later in the paper, but only in fairly technical contexts.

4.1. The definition.

Definition 4.2. *Let X belong to $\mathcal{C}$ and suppose that U is a simplicial presheaf with an augmentation $U \to X$. This map is called a* ***hypercover*** *of X if each U_n is a coproduct of representables, and $U \to X$ is a local acyclic fibration.*

Using (3.4) one can rewrite the second condition in a more explicit way: it says that the maps $U_0 \to X$, $U_1 \to U_0 \times_X U_0$, and $U_n \to \tilde{M}_n U$ (for $n \geq 1$) are all generalized covers. This is not particularly enlightening, but it's easy to provide some intuition behind it. For convenience we assume our Grothendieck topology is given by a basis of covering families. Then the easiest examples of hypercovers are the Čech complexes, which have the form

$$\cdots \coprod U_{a_0 a_1 a_2} \rightrightarrows \coprod U_{a_0 a_1} \rightrightarrows \coprod U_{a_0} \longrightarrow X$$

for some chosen covering family $\{U_a \to X\}$. Here $U_{a_0 \cdots a_n}$ is the fibre-product $U_{a_0} \times_X \cdots \times_X U_{a_n}$. The Čech complexes are the hypercovers for which the maps $U_1 \to U_0 \times_X U_0$ and $U_n \to \tilde{M}_n U$ are all *isomorphisms.* In an arbitrary hypercover one takes the iterated fibre-products at each level but then is allowed to refine that object further, by taking a generalized cover of it. We refer the reader to [AM, Section 8] for further discussion of hypercovers.

Next is the formal definition of hypercover descent:

Definition 4.3. *An objectwise-fibrant simplicial presheaf F* ***satisfies descent*** *for a hypercover $U \to X$ if $F(X)$ is weakly equivalent to the homotopy limit of the diagram*

$$\prod_a F(U_0^a) \rightrightarrows \prod_a F(U_1^a) \rightrightarrows \cdots,$$

where the products range over the representable summands of each U_n. If F is not objectwise-fibrant, we say it satisfies descent if some objectwise-fibrant replacement for F does.

The definition has been arranged so that if $F \to G$ is an objectwise weak equivalence, then F satisfies descent for $U \to X$ if and only if G does. While the definition reflects our intuitive notion of descent, the next lemma gives a more concise reformulation in terms of simplicial mapping spaces.

Lemma 4.4.

(i) A simplicial presheaf F satisfies descent for a hypercover $U \to X$ if and only if $\mathrm{Map}(X, \hat{F}) \to \mathrm{Map}(U, \hat{F})$ *is a weak equivalence of simplicial sets, where $\hat{F}$ is an injective-fibrant replacement for F.*

(ii) Let U' be a cofibrant replacement for U in $U\mathcal{C}$. Then F satisfies descent for $U \to X$ if and only if $\mathrm{Map}(X, \hat{F}) \to \mathrm{Map}(U', \hat{F})$ *is a weak equivalence of simplicial sets, where $\hat{F}$ is an objectwise-fibrant replacement for F.*

Note that any split hypercover (see Definition 4.13) is cofibrant in $U\mathcal{C}$, in which case one can apply (ii) with $U' = U$.

Proof. This is by general nonsense. Consider the diagram $\Delta^{op} \to sPre(\mathcal{C})$ given by $[n] \to U_n$, and let $\tilde{U}$ be its homotopy colimit. This is not the same as U, but there is a map $\tilde{U} \to U$ which is an objectwise weak equivalence (see Remark 2.1). Let $\hat{F}$ be an injective-fibrant replacement for F, which *a fortiori* is an objectwise-fibrant replacement as well. Then $\mathrm{Map}(\tilde{U}, \hat{F})$ is weakly equivalent to $\mathrm{Map}(U, \hat{F})$ since $\tilde{U} \to U$ is a weak equivalence between injective-cofibrant objects. But $\mathrm{Map}(\tilde{U}, \hat{F})$ is

$$\mathrm{Map}(\operatorname*{hocolim}_n U_n, \hat{F}) \simeq \operatorname*{holim}_n \mathrm{Map}(U_n, \hat{F}) \simeq \operatorname*{holim}_n \textstyle\prod_a \hat{F}(U_n^a).$$

Since $\mathrm{Map}(X, \hat{F})$ is equal to $\hat{F}(X)$, the condition that $\mathrm{Map}(X, F) \to \mathrm{Map}(U, F)$ be a weak equivalence is a direct translation of the homotopy limit formulation in Definition 4.3. This proves (i).

For (ii), note that each U_n is cofibrant in $U\mathcal{C}$ and so $\tilde{U} = \operatorname{hocolim}_n U_n$ is also cofibrant. In other words $\tilde{U}$ is a cofibrant replacement for U, and so $\tilde{U} \simeq U'$. If $\hat{F}$ is an objectwise-replacement for F then $\mathrm{Map}(U', F) \simeq \mathrm{Map}(\tilde{U}, F)$, and as in (i) the latter is equivalent to $\operatorname{holim}_n \prod_a F(U_n^a)$. The rest of the proof is the same. $\square$

A more elegant way to phrase the above result is to say that F satisfies descent for $U \to X$ if and only if $\mathrm{hMap}(X, F) \to \mathrm{hMap}(U, F)$ is a weak equivalence of simplicial sets, where $\mathrm{hMap}(-,-)$ denotes a homotopy function complex [H, Ch. 17] in either $U\mathcal{C}$ or $sPre(\mathcal{C})$.

4.5. **Machinery.**

Definition 4.2 is very compact, but it's not always such an easy thing to work with. For the rest of this section we will set down more convenient techniques for constructing and working with hypercovers. This material is used throughout the paper, but many readers will want to skip ahead and refer back to this section only when needed.

Let $\mathcal{M}$ be a category which is complete and co-complete—in our applications $\mathcal{M}$ is $Pre(\mathcal{C})$, but for the moment let us work in the more general setting. Let Δ_+ denote the augmented cosimplicial indexing category: it is obtained by adjoining an initial object $[-1]$ to Δ. Let $s_+\mathcal{M}$ denote the category of functors $\Delta_+^{op} \to \mathcal{M}$, i.e. the category of augmented simplicial objects. We regard a simplicial set K as belonging to $s_+\mathcal{S}et$ by letting K_{-1} consist of a single point.

If S is a set and X belongs to $\mathcal{M}$, let $X^{\cdot S}$ denote a product of copies of X indexed by the elements of S. Given a simplicial set K and an object W of $s_+\mathcal{M}$, we regard these as functors $K\colon \Delta_+^{op} \to \mathcal{S}et$ and $W\colon \Delta_+^{op} \to \mathcal{M}$ and then form the resulting end, denoted $\hom_+(K,W)$:

$$\hom_+(K,W) := \mathrm{eq}\Big[\prod_n W_n^{\cdot K_n} \rightrightarrows \prod_{[n]\to[m]} W_m^{\cdot K_n}\Big].$$

The + subscript is to remind us of the augmentations.

Remark 4.6. As with any end, this construction exhibits a useful adjointness property. If Z is in $\mathcal{M}$, then the maps $Z \to \hom_+(K,W)$ in $\mathcal{M}$ correspond bijectively with the maps $Z \otimes K \to W$ in $s_+\mathcal{M}$. Here $Z \otimes K$ is the augmented simplicial object which in dimension n is a coproduct, indexed by the set K_n, of copies of Z.

In the unaugmented simplicial category $s\mathcal{M}$, we can compute unaugmented ends $\hom(K,W)$ in an analogous way. Again, this construction is right adjoint to tensoring with K.

The following lemma can be proved with the above adjointness property and the Yoneda lemma.

Lemma 4.7. *Let $W \to X$ be an augmented simplicial object (that is, X is the augmentation).*

(i) $\hom_+(K,W)$ *is isomorphic to* $\hom^X(K,W)$, *where* $\hom^X(K,W)$ *is computed in the unaugmented simplicial overcategory* $s(\mathcal{M}\downarrow X)$.
(ii) $\hom_+(K,W) \cong \hom(K,W)$ *if K is connected.*
(iii) $\hom_+(\emptyset,W) \cong X$, *and so for any simplicial set K there is a canonical map* $\hom_+(K,W) \to X$.
(iv) $\hom_+(\Delta^n,W) \cong W_n$.
(v) $\hom_+(-,W)$ *takes colimits of simplicial sets to limits in $\mathcal{M}\downarrow X$. In other words, if* $K = \mathrm{colim}_i K_i$, *then* $\hom_+(K,W) \cong \lim_i^X \hom_+(K_i,W)$.

Definition 4.8. *The object* $\hom_+(\partial\Delta^n,W)$ *is the* ***nth augmented matching space M_nW***. *The induced map* $\hom_+(\Delta^n,W) \to \hom_+(\partial\Delta^n,W)$, *which we may now write as* $W_n \to M_nW$, *is the* ***nth matching map*** *for W. Note that $W_0 \to M_0W$ is just the augmentation since $\partial\Delta^0 = \emptyset$.*

We have chosen to work with these augmented constructions only because they seem to make for the most compact and intuitive proofs. Note that the augmented matching objects and maps are the ones that arise when considering Reedy model structures of simplicial objects in $(\mathcal{M}\downarrow X)$ [H, Ch. 16]. For $n \geq 2$, M_nW is isomorphic to $\tilde{M}_nU = \hom(\partial\Delta^n,W)$ because $\partial\Delta^n$ is connected. The following lemma is a reformulation of (3.4).

Lemma 4.9. *An augmented simplicial presheaf $U \to X$ is a hypercover iff each U_n is a coproduct of repesentables and the maps $U_n \to M_nU$ are all generalized covers.*

Definition 4.10. *A hypercover $U \to X$ is* ***bounded*** *if there exists an $n \geq 0$ such that the maps $U_k \to M_kU$ are isomorphisms for all $k > n$. The smallest such n for which this is true is called the* ***height*** *of the hypercover, and denoted* $\mathrm{ht}\, U$.

We have already remarked that the hypercovers of height 0 are precisely the Čech complexes. If one thinks of the nth level of a hypercover as refining the $(n+1)$-fold 'intersections' of the objects in previous levels, then a bounded hypercover is one where the refinement process stops at some point. The following lemma is a minor ingredient in the discussion of coskeleta in Section 4.12 below, but the ideas from the proof reappear several times throughout the paper.

Lemma 4.11. *If $U \to X$ is a bounded hypercover of height at most n, then the induced maps $\hom_+(\Delta^k, U) \to \hom_+(\mathrm{sk}_n \Delta^k, U)$ are isomorphisms for all k.*

Proof. When $k \leq n$, the result is easy because Δ^k equals $\mathrm{sk}_n \Delta^k$. In general, Δ^k is obtained from $\mathrm{sk}_n \Delta^k$ by gluing on finitely many simplices of dimension at least $n+1$. It suffices to show that $\hom_+(L, U) \to \hom_+(K, U)$ is an isomorphism if L is obtained from K by attaching a simplex of dimension i, where $i > n$. Using Lemma 4.7 we obtain a pullback square

$$\begin{array}{ccc} \hom_+(L,U) & \longrightarrow & \hom_+(K,U) \\ \downarrow & & \downarrow \\ \hom_+(\Delta^i,U) & \longrightarrow & \hom_+(\partial\Delta^i,U). \end{array}$$

The bottom map is the matching map $U_i \to M_i U$, which is an isomorphism since $i > n$. Hence the top map is also an isomorphism. □

4.12. **Skeleta, coskeleta, and split objects.**
We continue to assume that $\mathcal{M}$ is complete and cocomplete. Let $s\mathcal{M}_{\leq n}$ and $s_+\mathcal{M}_{\leq n}$ denote the categories of n-truncated simplicial objects and augmented n-truncated simplicial objects over $\mathcal{M}$. There is an obvious forgetful functor $s_+\mathcal{M} \to s_+\mathcal{M}_{\leq n}$ called sk_n, and this has a right adjoint called cosk_n. These are the **skeleta** and **coskeleta** functors for augmented simplicial objects. If W belongs to $s_+\mathcal{M}$, we abbreviate $\mathrm{cosk}_n \mathrm{sk}_n W$ to just $\mathrm{cosk}_n W$.

The kth object of $\mathrm{cosk}_n U$ is

$$[\mathrm{cosk}_n U]_k \cong \hom_+(\Delta^k, \mathrm{cosk}_n U) \cong \hom_+(\mathrm{sk}_n \Delta^k, U)$$

(use Remark 4.6 for the second isomorphism). In particular, the $(n+1)$st object of $\mathrm{cosk}_n U$ is what we have been calling $M_{n+1}U$. Observe also, using Lemma 4.11, that a hypercover U has height at most n if and only if $U \cong \mathrm{cosk}_n U$.

Now, the functor sk_n also has a left adjoint $\mathrm{dgn}_n \colon s_+\mathcal{M}_{\leq n} \to s_+\mathcal{M}$, called the ***n*-degeneration** functor. The simplicial object $\mathrm{dgn}_n U$ is obtained from U by freely adding the images of the degeneracies in dimensions higher than n (and so, in particular, note that the augmentations are irrelevant). The object $[\mathrm{dgn}_n U]_{n+1}$ is called the **$(n+1)$st latching object** for U and is denoted $\boldsymbol{L_{n+1}U}$. This latching object is the one that arises when considering Reedy model structures of simplicial diagram categories [H, Ch. 16]. Note that U, $\mathrm{dgn}_n U$, and $\mathrm{cosk}_n U$ all have the same n-skeleton, so there are canonical maps $\mathrm{dgn}_n U \to U \to \mathrm{cosk}_n U$; looking in level $n+1$ gives $L_{n+1}U \to U \to M_{n+1}U$.

Definition 4.13. *An object W of $s_+\mathcal{M}$ is said to be **split**, or to have **free degeneracies**, if there exist subobjects $N_k \hookrightarrow W_k$ such that the canonical maps $N_k \amalg L_k W \to W_k$ are isomorphisms for all $k \geq 0$. This is equivalent to requiring that the canonical map*

$$\coprod_{\sigma} N_\sigma \to W_k$$

is an isomorphism, where the variable σ ranges over all surjective maps in Δ of the form $[k] \to [n]$, N_σ denotes a copy of N_n, and the map $N_\sigma \to W_k$ is the one induced by $\sigma^* \colon W_n \to W_k$ (see [AM, Def. 8.1]*).*

The idea is that the objects N_k represent the non-degenerate part of W in dimension k, and that the leftover degenerate part is as free as possible. The same definition as above can be applied to augmented simplicial objects, and the result is that such an object is split if and only if it is split when one forgets the augmentation.

We are particularly interested in **split hypercovers**. If $U \to X$ is a split hypercover then $L_k U$ is a summand of U_k, each $L_k U$ is a coproduct of representables, and each representable summand of $L_k U$ is the image under some degeneracy of a representable from U_{k-1} (but not uniquely). It follows from [D, Cor. 9.4] that split hypercovers are cofibrant in $U\mathcal{C}_{\mathcal{L}}$, which is why we care about them.

4.14. **Computing matching objects.**
Suppose that $U \to X$ is an augmented simplicial presheaf which in each level is a coproduct of representables. Note that (1) the decomposition of U_n into a coproduct of representables is unique up to permutations of the summands, and (2) to give a map $\amalg_i A_i \to \amalg_j B_j$ between coproducts of representables corresponds to giving, for each index i, an index $j(i)$ and a map $A_i \to B_{j(i)}$. Because of these remarks, one can construct a simplicial set K by taking K_n to be the set of representable summands of U_n. We'll refer to K as the **indexing simplicial set** for U.

Now suppose $a\colon L \to K$ is a map of simplicial sets. If $\Delta^{op}L$ denotes the opposite category of simplices of L [H, Def. 16.1.15], there is an obvious diagram $\Delta^{op}L \to sPre(\mathcal{C}) \downarrow X$ which sends a k-simplex σ to the representable which is the summand of U_k corresponding to $a(\sigma)$. We'll write $U(a)$ for the limit of this diagram.

The following observation is straightforward (use Remark 4.6):

Proposition 4.15. *There is an isomorphism of presheaves*

$$\hom_+(L, U) \cong \coprod_{a\colon L\to K} U(a).$$

In particular, the matching object $M_n U$ is isomorphic to $\coprod_{a\colon \partial\Delta^n \to K} U(a)$.

Note that $\Delta^{op}L$ is an infinite category. If L has the property that every nondegenerate simplex has nondegenerate faces (e.g. $L = \partial\Delta^n$), then one can use a smaller version. Let $\Delta^{op}_{nd}L$ be the subcategory whose objects are the non-degenerate simplices, and where the maps correspond to face maps. Under the above assumption on L, it is an easy exercise to check that $\Delta^{op}_{nd}L \hookrightarrow \Delta^{op}L$ is final (use the fact that in any simplicial set a degenerate simplex is an iterated degeneracy of a unique nondegenerate simplex). Hence the limit $U(a)$ can be computed over $\Delta^{op}_{nd}L$ in practice.

5. Hypercovers and lifting problems

In Proposition 3.1 we saw how local weak equivalences relate to solutions of homotopy-lifting problems after passing from a representable to the elements of a covering sieve. These liftings could not necessarily be made compatible on the different pieces of the sieve, however. In this section we show that one *can* arrange for this kind of compatibility by using hypercovers.

The proof of our main result (Theorem 6.2) is mostly formal, except for the key ingredient provided by the next proposition. Recall that, just as for ordinary covering families, a refinement of a hypercover $U \to X$ is another hypercover $V \to X$ that factors through U.

Proposition 5.1. *Let $F \to G$ be a local acyclic fibration and let $K \to L$ be a cofibration of finite simplicial sets. For any square*

$$\begin{tikzcd} K \otimes U \arrow[r] \arrow[d] & F \arrow[d] \\ L \otimes U \arrow[r] & G \end{tikzcd} \tag{5.2}$$

in which $U \to X$ is a hypercover, there exists another hypercover $V \to X$ refining U and liftings as in the following diagram:

$$\begin{tikzcd} K \otimes V \arrow[r] \arrow[d] & K \otimes U \arrow[r] & F \arrow[d] \\ L \otimes V \arrow[r] \arrow[urr, dotted] & L \otimes U \arrow[r] & G. \end{tikzcd} \tag{5.3}$$

To summarize the basic idea of the proof of Proposition 5.1, let's assume that $K \to L$ is $\emptyset \to *$ and that the Grothendieck topology comes with a specified basis of covering families. Starting with a map $U \to G$, we know by the local-lifting property (3.1) that there is a covering family $\{V_a \to U_0\}$ with liftings $s_a \colon V_a \to F$. In general, $s_a|_{V_{ab}}$ and $s_b|_{V_{ab}}$ are not equal, but the two liftings become homotopic after projecting down to G. We can lift this homotopy to F by passing to a suitable covering family of V_{ab}, again using the fact that $F \to G$ is a local weak equivalence. Next we move on to consider patching on the triple intersections. Once again, we can patch up to homotopy after refining the triple intersections by a covering family. In this way we build a hypercover V over which a lifting is defined. The work in this section is just a precise way of saying all this.

The proof involves an inductively constructed hypercover, and the following lemma is the core of the induction step:

Lemma 5.4. *Let F and G be presheaves of sets, and let $F \to G$ be a generalized cover. If J is a presheaf of sets with a map $J \to G$, then there exists a generalized cover $Z \to J$ such that Z is a coproduct of representables and such that the diagram*

$$\begin{tikzcd} & & F \arrow[d] \\ Z \arrow[r] \arrow[urr, dotted] & J \arrow[r] & G \end{tikzcd}$$

has a lifting.

Proof. For each map $f \colon X \to J$ from a representable, choose a covering sieve R_f of X so that the composites $U \to X \to J \to G$ lift to F for every $U \to X$ in the sieve. Let Z denote the coproduct

$$Z = \coprod_{X \xrightarrow{f} J} \left(\coprod_{\substack{U \to X \\ \text{in } R_f}} U \right).$$

The obvious map $Z \to J$ is a generalized cover, and the composite $Z \to J \to G$ lifts to F. □

Proposition 5.5. *Let $F \to G$ be a local acyclic fibration, and let $U \to G$ be a map where $U \to X$ is a hypercover. Let $n \geq 0$, and suppose that there is an n-truncated hypercover $V \to X$ refining $\mathrm{sk}_n U$ such that the diagram*

$$\begin{tikzcd} & & F \arrow[d] \\ V \arrow[r] \arrow[urr] & U \arrow[r] & G \end{tikzcd} \tag{5.6}$$

commutes. Then there is an $(n+1)$-truncated hypercover $W \to X$ refining U and a map $W \to F$ making the corresponding diagram commute, and such that on n-skeleta the diagram is equal to (5.6).

Proof. The core of the proof is just an Artin-Mazur argument [AM, Ch. 8]. First form the pullback $F' = U \times_G F$. The map $F' \to U$ is still a local acyclic fibration, and we need only produce an $(n+1)$-truncated hypercover W and a lifting into F'. In other words, we can reduce to the case where $U = G$ (and $F = F'$). Note that in this case G is locally fibrant—the representable X is locally fibrant for trivial reasons, and $U \to X$ is a local fibration. Moreover, since $F \to G$ is a local fibration, F is locally fibrant as well.

Now $F^{\Delta^{n+1}} \to F^{\partial\Delta^{n+1}}$ is a local fibration by Proposition 3.3, so the map in the 0th level is a generalized cover by (3.4). When $n > 0$ this map is precisely $F_{n+1} \to M_{n+1}F$ (the $n = 0$ case being only slightly different). Our initial diagram gives a map $V^{\partial\Delta^{n+1}} \to F^{\partial\Delta^{n+1}}$, and the 0th level has the form $M_{n+1}V \to M_{n+1}F$. So Lemma 5.4 says that there is a generalized cover $Z \to M_{n+1}V$, where Z is a coproduct of representables, such that the composite $Z \to M_{n+1}F$ lifts through F_{n+1}. We take W to be the $(n+1)$-truncated hypercover with $\mathrm{sk}_n W = \mathrm{sk}_n V$ and $W_{n+1} = Z \amalg L_{n+1}V$. □

Proof of Proposition 5.1. Given a square as in the statement of the proposition, it may be interpreted as a map

$$U \to F^K \times_{G^K} G^L.$$

We are trying to produce a hypercover $V \to X$ refining $U \to X$ and a lifting

$$\begin{tikzcd} & & F^L \arrow[d] \\ V \arrow[r] \arrow[urr, dotted] & U \arrow[r] & F^K \times_{G^K} G^L. \end{tikzcd}$$

The vertical map is a local acyclic fibration by Proposition 3.3, so the hypercover can be produced inductively using Proposition 5.5. □

If we have a map $F \to G$ which is a local weak equivalence but not necessarily a fibration, we can say the following:

Proposition 5.7. *Let $F \to G$ be a local weak equivalence between locally fibrant simplicial presheaves. Then given any diagram as in (5.2), there exists a hypercover $V \to X$ refining $U \to X$ and relative-homotopy-liftings in the diagram (5.3).*

Recall that relative-homotopy-liftings were defined in Proposition 3.1, and discussed extensively in [DI2].

Proof. Given a diagram as in (5.2), we need to produce a hypercover $V \to X$ refining $U \to X$ together with liftings in the diagram

$$\begin{tikzcd}
& & K \otimes V \arrow[r] \arrow[dll] \arrow[d] & F \arrow[d] \\
L \otimes V \arrow[d, "i_0"'] \arrow[urrr, dotted] & & L \otimes V \arrow[r] \arrow[dll, "i_1"'] & G \\
RH \otimes V. \arrow[urrr, dotted] & & &
\end{tikzcd}$$

Here RH denotes the pushout of $L \times \Delta^1 \longleftarrow K \times \Delta^1 \xrightarrow{\pi} K$, and the maps i_0 and i_1 are the obvious inclusions $L \hookrightarrow RH$.

Consider the square

$$\begin{tikzcd}
F^{RH} \arrow[r] \arrow[d, "i_1^*"'] & F^L \times_{G^L} G^{RH} \arrow[d] \\
F^L \arrow[r] & F^K \times_{G^K} G^L.
\end{tikzcd}$$

By [DI2, Cor. 7.5], the fact that $F \to G$ is a local weak equivalence between locally fibrant objects implies that the horizontal maps are also local weak equivalences. By the same result, the fact that $i_1 \colon L \to RH$ is a weak equivalence of simplicial sets implies that the left vertical map is a local weak equivalence. So we conclude that the same is true of the right vertical map. Even more, the right vertical map is a local fibration by Lemma 5.8 below.

Our initial data from (5.2) was a map $U \to F^K \times_{G^K} G^L$, so by Proposition 5.1 (for $n = 0$ and $K \to L$ equal to $\emptyset \to *$) there is a hypercover $V \to X$ refining $U \to X$ for which the composite lifts through $F^L \times_{G^L} G^{RH}$. This provides the necessary relative-homotopy-lifting. $\square$

Lemma 5.8. *Let $F \to G$ be a map between locally fibrant simplicial presheaves. Assume we have a square of simplicial sets*

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & M \arrow[d] \\
L \arrow[r] & N
\end{tikzcd}$$

such that both $K \to M$ and $M \amalg_K L \to N$ are cofibrations. Then the induced map

$$F^M \times_{G^M} G^N \to F^K \times_{G^K} G^L$$

is a local fibration.

Proof. The hypotheses imply that both $F^M \to F^K$ and $G^N \to G^{(M \amalg_K L)} = G^M \times_{G^K} G^L$ are local fibrations, using [J2, Cor. 1.5]. Now we observe that there are pullback squares

$$\begin{tikzcd}
F^M \times_{G^M} G^N \arrow[r] \arrow[d] & G^N \arrow[d] \\
F^M \times_{G^K} G^L \arrow[r] & G^M \times_{G^K} G^L
\end{tikzcd}
\qquad
\begin{tikzcd}
F^M \times_{G^K} G^L \arrow[r] \arrow[d] & F^M \arrow[d] \\
F^K \times_{G^K} G^L \arrow[r] & F^K,
\end{tikzcd}$$

and the pullback of a local fibration is again a local fibration. Finally, the map we want is just the composite $F^M \times_{G^M} G^N \to F^M \times_{G^K} G^L \to F^K \times_{G^K} G^L$. □

6. Hypercovers and localizations

In this section we prove the main theorem, that Jardine's model category can be obtained by localizing the injective structure $sPre(\mathcal{C})$ at the hypercovers. This lets us identify the fibrant objects in the model structure. Similar results are proven for the projective version $U\mathcal{C}_{\mathcal{L}}$.

We start with a definition:

Definition 6.1. *A collection of hypercovers S is called* ***dense*** *if every hypercover $U \to X$ in $sPre(\mathcal{C})$ can be refined by a hypercover $V \to X$ which belongs to S. The collection is called* ***split*** *if every hypercover in S can be refined by a split hypercover which also belongs to S.*

For instance, Theorem 8.6 shows that when $\mathcal{C}$ is a Verdier site the collection of basal hypercovers is both split and dense.

The following is our main goal.

Theorem 6.2. *Let S be a collection of hypercovers which contains a set that is dense. Then the localization $sPre(\mathcal{C})/S$ exists and coincides with Jardine's model structure $sPre(\mathcal{C})_{\mathcal{L}}$. Similarly, the localization $U\mathcal{C}/S$ exists and coincides with $U\mathcal{C}_{\mathcal{L}}$.*

Our notation is that if $\mathcal{M}$ is a model category and S is a collection of maps, then $\mathcal{M}/S$ denotes the left Bousfield localization of $\mathcal{M}$ at S (if it exists)—see [D] for a summary treatment or [H] for complete details. The fibrations and weak equivalences in $\mathcal{M}/S$ are called S-fibrations and S-equivalences, while the cofibrations are the same as those in $\mathcal{M}$.

The hypothesis of the theorem is a little stronger than just assuming that S is dense, because S itself may not be a set. For the same reason, the existence of the localization is not automatic. One of the things we will do is apply this theorem in the case where S is the collection of *all* hypercovers, and this is not a set: in our definition of hypercover one can have arbitrarily large coproducts of representables appearing. So we'll need to verify that S contains a dense *set*, and this can be done by making use of the fact that our site is small. We choose a suitably large regular cardinal, and then we only consider hypercovers in which the number of summands in each level is bounded by our cardinal. For now we can ignore this point, but see Section 6.6.

The corollary below follows easily from the theorem, and in fact the two are equivalent.

Corollary 6.3. *Let S be a collection of hypercovers which contains a set that is dense.*

(i) A simplicial presheaf F is fibrant in $sPre(\mathcal{C})_{\mathcal{L}}$ if and only if F is injective-fibrant and satisfies descent for all hypercovers in S.
(ii) F is fibrant in $U\mathcal{C}_{\mathcal{L}}$ if and only if it is objectwise fibrant and satisfies descent for all hypercovers in S.

In particular, a simplicial presheaf F satisfies descent for all hypercovers if and only if it satisfies descent for all elements of S.

Proof. First observe that the fibrant objects in $sPre(\mathcal{C})/S$ are the injective-fibrant objects F such that $\mathrm{Map}(X,F) \to \mathrm{Map}(U,F)$ is a weak equivalence for every $U \to X$ in S [H, Thm. 4.1.1(2)]. (Since everything is cofibrant in $sPre(\mathcal{C})$, one doesn't have to take cofibrant-replacements for U and X.) Lemma 4.4(i) says the latter condition is the same as F satisfying descent for U. By Theorem 6.2 the model structure $sPre(\mathcal{C})_{\mathcal{L}}$ is the same as $sPre(\mathcal{C})/S$, so this proves (i).

The last statement in the corollary now follows as well. The point is that Theorem 6.2 applies not only to S, but also to the collection of *all* hypercovers—so these two collections give the same localization. Thus, an injective-fibrant object satisfies descent for the hypercovers in S if and only if it satisfies descent for all hypercovers. Now use that a simplicial presheaf F satisfies descent for a hypercover U precisely when an injective-fibrant-replacement for F does.

The proof of (ii) is the same as (i), but any hypercover U must be replaced by a cofibrant object before it appears in a mapping space, and Lemma 4.4(ii) is used instead of Lemma 4.4(i). $\square$

To prove Theorem 6.2, we need a general criterion for checking whether two localizations are identical:

Lemma 6.4. *Let $\mathcal{M}$ be a model category, and let $S \subseteq T$ be two classes of maps for which the localizations $\mathcal{M}/S$ and $\mathcal{M}/T$ exist. For the two localizations to be the same it suffices to check the following: if an S-fibration $X \twoheadrightarrow Y$ between S-fibrant objects is a T-equivalence, then it is an S-equivalence.*

Proof. We must show the hypothesis implies that every T-equivalence $A \to B$ is an S-equivalence. Let L denote a fibrant-replacement functor in $\mathcal{M}/S$, and consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\sim"', "S"] & B \arrow[d, "\sim"', "S"] \\ LA \arrow[r] & LB. \end{tikzcd}$$

Since $S \subseteq T$ the two vertical maps are T-equivalences, and the top map is a T-equivalence by assumption—so the bottom map is one as well. Now factor the bottom map in $\mathcal{M}/S$ as an S-acyclic cofibration followed by an S-fibration:

$$LA \overset{S}{\underset{\sim}{\rightarrowtail}} X \twoheadrightarrow LB.$$

Note that X is S-fibrant, because LB is. Also, since both the first map and the composite are T-equivalences, so is the second map.

Therefore the map $X \to LB$ is a T-equivalence and an S-fibration, and the domain and codomain are S-fibrant. Our hypothesis then says that $X \to LB$ is an S-equivalence. Applying the two-out-of-three property (twice) shows that $A \to B$ is an S-equivalence. $\square$

For the moment let S be a *set* of hypercovers that is dense. Because S is a set, we know that the model structure $sPre(\mathcal{C})/S$ exists (by [H], using that $sPre(\mathcal{C})$ is left proper and cellular). The fibrant objects in $sPre(\mathcal{C})/S$ (called S-fibrant objects) are the injective-fibrant objects which satisfy descent for all hypercovers in S. Since every hypercover is a local weak equivalence by definition, $sPre(\mathcal{C})_{\mathcal{L}}$ is a localization of $sPre(\mathcal{C})/S$. To show that the two structures coincide, we now check the criterion from the above lemma:

Lemma 6.5. *Let F and G be S-fibrant objects, and let $f : F \to G$ be an S-fibration that is also a local weak equivalence. If X is a representable, then every square*

$$\begin{tikzcd} \partial\Delta^n \otimes X \arrow[r] \arrow[d] & F \arrow[d] \\ \Delta^n \otimes X \arrow[r] & G \end{tikzcd}$$

has a lifting. In particular, f is actually an objectwise acyclic fibration and therefore an S-equivalence.

Proof. The second claim follows from the first by adjointness and because acyclic fibrations of simplicial sets are detected by the right lifting property with respect to the maps $\partial\Delta^n \to \Delta^n$.

Now we prove the first claim. First, f is an objectwise fibration since every S-fibration is an injective-fibration and also a projective-fibration. This implies that f is also a local fibration. Because f is both a local fibration and a local weak equivalence, Proposition 5.1 guarantees us a hypercover $U \to X$ such that the diagram

$$\begin{tikzcd} \partial\Delta^n \otimes U \arrow[r] \arrow[d] & \partial\Delta^n \otimes X \arrow[r] & F \arrow[d] \\ \Delta^n \otimes U \arrow[r] \arrow[urr, dotted] & \Delta^n \otimes X \arrow[r] & G \end{tikzcd}$$

has a lifting. In applying Proposition 5.1, we have used that X is (trivially) a hypercover of itself. Since S is dense, we may refine U and assume that $U \to X$ belongs to S. We now write down the following diagram of simplicial mapping spaces:

$$\begin{tikzcd} \mathrm{Map}(X, F^{\Delta^n}) \arrow[r, "\sim"] \arrow[d, two heads] & \mathrm{Map}(U, F^{\Delta^n}) \arrow[d, two heads] \\ \mathrm{Map}(X, G^{\Delta^n} \times_{G^{\partial\Delta^n}} F^{\partial\Delta^n}) \arrow[r, "\sim"] & \mathrm{Map}(U, G^{\Delta^n} \times_{G^{\partial\Delta^n}} F^{\partial\Delta^n}). \end{tikzcd}$$

All the model categories we have been considering are simplicial model categories, and this implies that $F^{\Delta^n} \to G^{\Delta^n} \times_{G^{\partial\Delta^n}} F^{\partial\Delta^n}$ is an S-fibration between S-fibrant objects. It is a local weak equivalence by Proposition 3.3. Therefore, the vertical maps above are fibrations of simplicial sets because both X and U are S-cofibrant. Likewise, the horizontal maps are weak equivalences because $U \to X$ is an S-equivalence between S-cofibrant objects and both F^{Δ^n} and $G^{\Delta^n} \times_{G^{\partial\Delta^n}} F^{\partial\Delta^n}$ are S-fibrant.

We are given a 0-simplex x in the lower left corner in the above diagram, and we want to find a lift in the upper left corner. We have already shown that the image of x in the lower right corner lifts to the upper right corner. Since the horizontal maps are weak equivalences, there is another 0-simplex y belonging to the connected component of x such that y has a lift in the upper left corner. But fibrations of simplicial sets are surjective onto the components in their images, so x also has a lift. □

Proof of Theorem 6.2. We first consider the claim for $sPre(\mathcal{C})_{\mathcal{L}}$. For the case when our collection of hypercovers S is itself a *set*, we have already done all the work.

Since hypercovers are local weak equivalences we know $S \subseteq \mathcal{L}$, and so we are in the situation of Lemma 6.4. The necessary condition was verified in Lemma 6.5.

In the general case, let S' be a dense set of hypercovers contained in S. As shown in the previous paragraph, $sPre(\mathcal{C})/S'$ is equal to $sPre(\mathcal{C})_{\mathcal{L}}$. So every local weak equivalence is a weak equivalence in $sPre(\mathcal{C})/S'$, and in particular every hypercover in S is an S'-equivalence. This shows that $sPre(\mathcal{C})/S$ exists and is equal to $sPre(\mathcal{C})/S'$.

The argument for $U\mathcal{C}/S$ is basically the same. Assume first that S is a set of hypercovers which is dense. One reproves the analog of Lemma 6.5 for the projective model structure, assuming that F and G are fibrant objects of $U\mathcal{C}/S$ and that $F \to G$ is an objectwise fibration. The only difference in the proof is that one replaces U by a cofibrant object before dealing with simplicial mapping spaces. The rest of the argument is exactly the same, as is the generalization to the case where S need not be a set. □

6.6. **Cardinality considerations.** Early in this section we mentioned that the collection of all hypercovers is not a set, but contains a subset that is dense. We will now give the proof. Recall from (4.14) that to any hypercover $U \to X$ one can attach an indexing simplicial set K, where K_n is the set of representable summands of U_n. The **size** of the hypercover is the cardinality of $\coprod_n K_n$, i.e., the number of representable summands that appear in U. The main point is that in the arguments from Proposition 5.5 and Lemma 5.4, one can control the size of the constructed hypercover.

Proposition 6.7. *The class of all hypercovers has a subset which is dense.*

Proof. Choose a regular cardinal λ sufficiently large compared to the cardinality of the set of morphisms in $\mathcal{C}$, and let S denote the set of all hypercovers of size less than λ. We will show that any hypercover $U \to X$ can be refined by one in S.

Since $U_0 \to X$ is a generalized cover, there is a covering sieve R of X such that every $W \to X$ in R lifts through U_0. Let $V_0 = \coprod_{W \to X} W$, where the coproduct ranges over all maps $W \to X$ in R. The number of summands in V_0 is clearly bounded by λ.

Now assume by induction that we have constructed an n-truncated hypercover $V \to X$ which refines U, and such that the number of summands in V is less than λ. To extend V we use the argument from Proposition 5.5, where we must show that Z does not have too many representable summands. Inspecting the construction of Z given in Lemma 5.4, it suffices to show that there aren't too many maps from a representable into $M_{n+1}V$. This can be deduced from Proposition 4.15. □

6.8. **A short example about fibrant replacement.** We end this section with a simple (and well-known) example demonstrating the use of Corollary 6.3. Let A be a presheaf of abelian groups on the site $\mathcal{C}$, and let I_* denote an injective resolution of the sheafification $\tilde{A}$ in the category of sheaves. We will explain how to use I_* to construct a fibrant replacement for the simplicial presheaf $K(A, n)$.

Let $\mathcal{I}$ denote the chain complex of presheaves which has I_k in dimension $n - k$ when $k < n$, and has the presheaf of n-boundaries B_n in dimension 0. The Dold-Kan correspondence lets us identify presheaves of (non-negatively graded) chain complexes with the abelian group objects in $sPre(\mathcal{C})$, and so $\mathcal{I}$ can be regarded as a simplicial presheaf. Since right now we are only dealing with abelian things, it's easier just to think about chain complexes, though.

The map $A \to I_0$ induces a map $K(A,n) \to \mathcal{I}$ (and recall that as a chain complex, $K(A,n)$ has A in dimension n and 0 everywhere else). This map is a local weak equivalence because it induces isomorphisms on homology group sheaves. We claim that $\mathcal{I}$ satisfies descent for all hypercovers, and so is a fibrant object in $U\mathcal{C}_{\mathcal{L}}$. This shows that one can identify weak homotopy classes of maps $\mathrm{Ho}\,(\Delta^k/\partial\Delta^k \otimes X, K(A,n))$ with $H_{n-k}(\mathcal{I}(X))$, which is just the sheaf cohomology group $H^{n-k}_{\mathcal{C}}(X, \tilde{A})$. The connection with sheaf cohomology is also explained in [J1, Section 2].

If $U \to X$ is a hypercover of X, let $\mathbb{Z}[U]$ denote the chain complex of presheaves obtained by applying the free abelian group functor to the presheaves U_n. It is known that after sheafification $\mathbb{Z}[U]$ becomes a resolution of $\mathbb{Z}[X]$ (this is basically the 'Illusie Conjecture'—see [J1, Thm 2.5] for a proof). The mapping space $\mathrm{Map}(U, \mathcal{I})$ may be identified with $\mathrm{Map}(\mathbb{Z}[U], \mathcal{I})$ using adjointness, and this is just the total complex associated to the bicomplex $(p,q) \to \mathcal{I}_q(U_p)$. By running the spectral sequence for the homology of this bicomplex, making use of the fact that the $\mathcal{I}_k$'s are injective sheaves ($k \geq 1$) and $\mathbb{Z}[U]^{\sim}$ is a resolution of $\mathbb{Z}[X]^{\sim}$, one finds that the spectral sequence collapses and the homology is just that of $\mathrm{Map}(X, \mathcal{I})$. In other words, $\mathrm{Map}(X, \mathcal{I}) \to \mathrm{Map}(U, \mathcal{I})$ is a weak equivalence.

7. Other applications

Our main application for studying hypercovers is to produce realization functors on $\mathbb{A}^1$-homotopy theory [DI1, Is]. In this section we consider a few other applications to the homotopy theory of simplicial presheaves.

7.1. **Change of site.**

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck sites, and $f \colon \mathcal{C} \to \mathcal{D}$ is a functor. The direct image functor $f_* \colon sPre(\mathcal{D}) \to sPre(\mathcal{C})$ has a left adjoint f^*. One is interested in conditions on f which imply that these adjoint functors are well-behaved in relation to the homotopy theory of simplicial presheaves. Here is a general result which is now easy to prove:

Proposition 7.2. *Suppose that there is a dense set S of hypercovers in $\mathcal{C}$ such that f^* takes elements of S to hypercovers in $\mathcal{D}$. Then the adjoint functors (f^*, f_*) give a Quillen map $U\mathcal{C}_{\mathcal{L}} \to U\mathcal{D}_{\mathcal{L}}$. (Recall that a **Quillen map** is just a Quillen pair* [H, Defn. 8.6.1] *regarded as a map of model categories in the direction of the left adjoint.)*

In this result one *cannot* replace $U\mathcal{C}_{\mathcal{L}}$ by $sPre(\mathcal{C})_{\mathcal{L}}$. The functor f^* usually does not preserve monomorphisms, which are the cofibrations in $sPre(\mathcal{C})_{\mathcal{L}}$.

Proof. Using general facts about the universal model category $U\mathcal{C}$ [D, Prop. 2.3], the functors (f^*, f_*) are a Quillen map from $U\mathcal{C}$ to $U\mathcal{D}$. If T denotes the collection of hypercovers in $\mathcal{D}$, then we have assumed that f^* maps S into T. Therefore, by general considerations [D, Section 5] one gets a Quillen pair between $U\mathcal{C}/S$ and $U\mathcal{D}/T$. But we have already seen in Theorem 6.2 that these localizations are just $U\mathcal{C}_{\mathcal{L}}$ and $U\mathcal{D}_{\mathcal{L}}$. □

Suppose that f^* preserves finite limits; then $M_n(f^*U) \cong f^*(M_nU)$. Suppose also that f is continuous, in the sense that $\{f(U_a) \to f(X)\}$ generates a covering sieve of $f(X)$ if $\{U_a \to X\}$ is a covering sieve. Then f^* preserves generalized covers. So under these two hypotheses one finds that f^* preserves hypercovers.

The hypothesis that f^* preserves finite limits is not always satisfied in examples of interest (see [MV, Ex. 1.19, p. 103]), and so here is a slightly different criterion which is useful:

Corollary 7.3. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Verdier sites (see Section 8). Assume the functor $f\colon \mathcal{C} \to \mathcal{D}$ preserves finite limits for diagrams of basal maps, and takes covering families $\{U_a \to X\}$ in $\mathcal{C}$ to covering families $\{f(U_a) \to f(X)\}$ in $\mathcal{D}$. Then (f^*, f_*) give a Quillen map $U\mathcal{C}_{\mathcal{L}} \to U\mathcal{D}_{\mathcal{L}}$.*

Proof. Preserving finite limits implies that f^* preserves matching objects (use Proposition 4.15 and the material in section 8). So the condition about preserving covering families shows that f^* takes basal hypercovers in $\mathcal{C}$ to basal hypercovers in $\mathcal{D}$. Thus, Proposition 7.2 applies. □

As an example, let $S \to T$ be a map of schemes and consider the base-change functor $f\colon Sm/T \to Sm/S$ from the category of smooth schemes over T to the category of smooth schemes over S. This functor satisfies the properties of the above proposition for any of the standard topologies (such as Zariski, étale, or Nisnevich) on Sm/S and Sm/T. So one gets a Quillen pair $U(Sm/T)_{\mathcal{L}} \to U(Sm/S)_{\mathcal{L}}$, by the above corollary. Compared to the discussion in [MV], this approach is much simpler.

7.4. **Computing homotopy classes of maps.**
Given a simplicial presheaf F, we will use $\mathbb{H}F$ to denote a fibrant-replacement in $sPre(\mathcal{C})_{\mathcal{L}}$ (or in $U\mathcal{C}_{\mathcal{L}}$, depending on the context). In some sense the ultimate goal of sheaf theory is to compute the simplicial sets $\mathbb{H}F(X)$. For instance, if A is a presheaf of abelian groups and $F = K(A, n)$ is the associated Eilenberg-MacLane simplicial presheaf, then $\pi_i \mathbb{H}F(X) = H^{n-i}(X, \tilde{A})$ (see Section 6.8) . If F is a presheaf of chain complexes then $\mathbb{H}F(X)$ computes the hypercohomology of X with coefficients in F, and this is where the notation $\mathbb{H}F$ comes from (in this context it goes back to [Th]).

There is no known method for computing $\mathbb{H}F$ in general—one can use the small object argument, but this is not very computable. For 'nice' sites one can use the Godement resolution [J2, Prop. 3.3], but this is also not so computable. In this section we give analogs of the Verdier hypercovering theorem, which show how to compute some invariants of $\mathbb{H}F(X)$ using hypercovers.

We'll write $\mathrm{Ho}\,(F, G)$ for the set of weak homotopy classes of maps from F to G in the homotopy category of $sPre(\mathcal{C})_{\mathcal{L}}$. Likewise, $\pi(F, G)$ denotes the set $sPre(\mathcal{C})(F, G)/\sim$, where the equivalence relation is generated by *simplicial* homotopy.

Given an object X in $\mathcal{C}$, let HC_X denote the full subcategory of $sPre(\mathcal{C})$ consisting of all hypercovers of X. We let πHC_X denote the category with the same objects, but where $\pi HC_X(U, V)$ equals $\pi(U, V)$.

Proposition 7.5. *The category πHC_X is filtered.*

This proposition is proved in [SGA4, Exposé V, 7.3.2] and also in [AM, Section 8] with a slightly different notion of hypercover (see Section 9). We prove it again here because it is straightforward with the techniques that we have already developed.

Proof. If $U \to X$ and $V \to X$ are both hypercovers, then so is $U \times_X V \to X$. Thus, we only need show that two parallel arrows $V \rightrightarrows U$ in πHC_X can be equalized.

The two maps from $V \to X$ to $U \to X$ can be assembled into the square

$$\begin{tikzcd} \partial\Delta^1 \otimes V \arrow[r] \arrow[d] & U \arrow[d] \\ \Delta^1 \otimes V \arrow[r] & X \end{tikzcd}$$

in which the bottom map factors through $V \to X$. The right vertical arrow is a local acyclic fibration by definition. Therefore, we apply Proposition 5.1 and obtain another hypercover $W \to X$ that refines V, together with a diagram

$$\begin{tikzcd} \partial\Delta^1 \otimes W \arrow[r] \arrow[d] & \partial\Delta^1 \otimes V \arrow[r, shift left] \arrow[r, shift right] & U \arrow[d] \\ \Delta^1 \otimes W \arrow[r] \arrow[urr] & \Delta^1 \otimes V \arrow[r] & X. \end{tikzcd}$$

The two compositions $W \to U$ are simplicially homotopic and hence equal in πHC_X. ☐

The following is a generalization of the Verdier hypercovering theorem [SGA4, Exposé V, 7.4.1(4)]. The case $K = *$ of part (b) appeared in [B], and is cited several times in Jardine's papers (see [J2, p. 83], for instance). It can be deduced from general considerations about the category of locally fibrant simplicial presheaves being a 'category with fibrant objects'. The generalization to arbitrary K, as well as to the relative setting in (c), doesn't seem to follow from these considerations, however. The case of arbitrary K can be deduced from $K = *$ using [DI2, Cor. 7.5], but the material in Section 5 makes it just as easy to give a proof which handles all cases at once.

Theorem 7.6. *Let F be a locally-fibrant simplicial presheaf and let X belong to $\mathcal{C}$. Let $F \to \mathbb{H}F$ be a fibrant replacement for F in $sPre(\mathcal{C})_{\mathcal{L}}$. Then*

(a) Given a 0-simplex p of $\mathbb{H}F(X)$, there is a hypercover $V \to X$ and a map $v : V \to F$ such that the following square commutes up to simplicial homotopy:

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[d] & F \arrow[d] \\ X \arrow[r, "p"] & \mathbb{H}F. \end{tikzcd}$$

We say that 'p is represented by the map v'.

(b) Given a finite simplicial set K, there is an isomorphism

$$Ho(K \otimes X, F) \cong \operatorname*{colim}_{U \to X} \pi(K \otimes U, F)$$

where the colimit is taken over (the opposite category of) πHC_X.

(c) Given p and $V \to F$ as in (a), there is an isomorphism

$$\pi_n(\mathbb{H}F(X), p) \cong \operatorname*{colim}_{U \to V} \pi_n(\mathrm{Map}(U, \mathbb{H}F), p|_U) \cong \operatorname*{colim}_{U \to V} \pi(\Delta^n/\partial\Delta^n \otimes U, F)_{v|_U}.$$

Here $v|_U$ denotes the map $U \to V \to F$, and $p|_U$ denotes the map $U \to V \to X \to \mathbb{H}F$. The colimits are taken over the overcategory $\pi HC_X \downarrow V$ of hypercovers refining V, and $\pi(\Delta^n/\partial\Delta^n \otimes U, F)_{v|_U}$ denotes the set of all maps $f \colon \Delta^n/\partial\Delta^n \otimes U \to F$ such that $f\ |_{*\otimes U}$ is the given map $v|_U \colon U \to V \to F$, modulo simplicial homotopy relative to $* \otimes U$.

Proof. Part (a) is a direct consequence of Proposition 5.7 because F and $\mathbb{H}F$ are both locally-fibrant.

For surjectivity in (b), note that any element α of $\mathrm{Ho}\,(K \otimes X, F)$ is represented by an actual map $K \otimes X \to \mathbb{H}F$. From Proposition 5.7 again, we get a hypercover $U \to X$ and a diagram

$$\begin{tikzcd} K \otimes U \arrow[r, "f"] \arrow[d] & F \arrow[d] \\ K \otimes X \arrow[r] & \mathbb{H}F \end{tikzcd}$$

commuting up to simplicial homotopy. The map f has image α in $\mathrm{Ho}\,(K \otimes X, F)$.

For injectivity, suppose given two maps $K \otimes U \to F$ that have the same image in $\mathrm{Ho}\,(K \otimes X, F)$. Since $K \otimes U \to K \otimes X$ is a local weak equivalence, this means that the two compositions $K \otimes U \to \mathbb{H}F$ are simplicially homotopic. Hence we have a diagram

$$\begin{tikzcd} (\partial\Delta^1 \times K) \otimes V \arrow[r] \arrow[d] & (\partial\Delta^1 \times K) \otimes U \arrow[r] & F \arrow[d] \\ (\Delta^1 \times K) \otimes V \arrow[r] \arrow[urr, dotted] & (\Delta^1 \times K) \otimes U \arrow[r] & \mathbb{H}F \end{tikzcd}$$

for some refinement V of U, where the lift is a relative-homotopy-lifting. In particular, the upper left triangle commutes on the nose, so the two maps $K \otimes U \to F$ are equal in $\mathrm{colim}_{U \to X}\, \pi(K \otimes U, F)$.

For (c), note that the natural map $\mathrm{Map}(X, \mathbb{H}F) \to \mathrm{Map}(U, \mathbb{H}F)$ is a weak equivalence. So it induces an isomorphism $\pi_n(\mathbb{H}F(X), p) \xrightarrow{\cong} \pi_n(\mathrm{Map}(U, \mathbb{H}F), p|_U)$, and after taking the colimit over all U we get the first isomorphism in the theorem.

For the second isomorphism, observe that composing with $F \to \mathbb{H}F$ induces maps $\pi(\Delta^n/\partial\Delta^n \otimes U, F)_{v|_U} \to \pi_n(\mathrm{Map}(U, \mathbb{H}F), p|_U)$. As in the proof of part (b) above, the fact that these maps give an isomorphism after passing to the colimit is a direct consequence of Proposition 5.7. □

Note that if S is a dense set of hypercovers then the colimits in the above results can just as well be taken over the full subcategory of πHC_X whose objects belong to S. It would be interesting to construct an explicit model for the simplicial set $\mathbb{H}F(X)$ using hypercovers, but we haven't been able to do this.

7.7. The coconnected case.

Definition 7.8. *A locally-fibrant simplicial presheaf F is said to be* ***locally n-coconnected*** *if it has the following property: for any X in $\mathcal{C}$ and any 0-simplex x in $F(X)$, the homotopy group sheaves $\pi_k(F, x)$ on $\mathcal{C} \downarrow X$ vanish for all $k \geq n$.*

Using techniques from [DI2], a locally-fibrant simplicial presheaf is locally n-coconnected if and only if it has the local lifting property with respect to the maps $\partial\Delta^k \otimes X \to \Delta^k \otimes X$ for $k > n$.

Not surprisingly, for n-coconnected presheaves one can calculate homotopy classes of maps by only using *bounded* hypercovers. This is what we'll prove next.

If $n \geq 0$, let $HC_X(n)$ denote the category of bounded hypercovers $U \to X$ of height at most n (see (4.10)). Let $\pi HC_X(n)$ denote the category with the same objects but with simplicial homotopy classes of maps. Arguments similar to Proposition (7.5) show that $\pi HC_X(n)$ is filtered.

Proposition 7.9. *Suppose that F is locally fibrant and locally n-coconnected. Then given a finite simplicial set K, there is an isomorphism*

$$Ho\,(K \otimes X, F) \cong \operatorname*{colim}_{U \to X} \pi(K \otimes U, \operatorname{cosk}_n F)$$

where the colimit is taken over the category $\pi HC_X(n)$.

Proof. First, the map $F \to \operatorname{cosk}_n F$ is a local weak equivalence between locally fibrant objects. So we can say that

$$\operatorname{Ho}(K \otimes X, F) \cong \operatorname{Ho}(K \otimes X, \operatorname{cosk}_n F) \cong \operatorname*{colim}_{U \to X} \pi(K \otimes U, \operatorname{cosk}_n F)$$

where the colimit runs over the full category πHC_X; the second isomorphism comes from Theorem 7.6. We need to show that

$$\operatorname*{colim}_{U \in \pi HC_X(n)} \pi(K \otimes U, \operatorname{cosk}_n F) \to \operatorname*{colim}_{U \in \pi HC_X} \pi(K \otimes U, \operatorname{cosk}_n F)$$

is an isomorphism. Observe that for any simplicial set L, a map $L \otimes U \to \operatorname{cosk}_n F$ factors through $\operatorname{cosk}_n(L \otimes U)$, and the map $L \otimes U \to \operatorname{cosk}_n(L \otimes U)$ factors as $L \otimes U \to L \otimes \operatorname{cosk}_n U \to \operatorname{cosk}_n(L \otimes U)$. Applying this when $L = K$ shows surjectivity, and from $L = K \times \Delta^1$ one can deduce injectivity. ☐

Proposition 7.10. *Let S be a Noetherian scheme, and let $\mathcal{C}$ be Sm/S (or $\mathcal{S}ch/S$) with either the ètale or Nisnevich topology. Let X be an object in $\mathcal{C}$ with the property that every finite set of points is contained in an affine open. Then every bounded hypercover of X can be refined by a Čech complex.*

Proof. In the case of the étale topology, this is essentially the content of [Ar, Thm. 4.1]. Since the result is trivial for hypercovers of height 0, we'll suppose by induction that it works for hypercovers of height at most n. Let $U \to X$ be a hypercover of height $n+1$. By Theorem 8.6, U can be refined by a basal hypercover $U' \to X$ (see Section 8 below). Let $V = \operatorname{cosk}_n U'$, which is a hypercover of height at most n. By induction, there is an étale covering family $\{W_i \to X\}$ such that $\check{C}W$ refines V (where $W = \coprod W_i$). Consider the induced map $\check{C}W_{n+1} \to V_{n+1} = M_{n+1}U'$. The map $U'_{n+1} \to M_{n+1}U'$ is an étale cover, which pulls back to an étale cover $E \to \check{C}W_{n+1}$. Theorem 4.1 of [Ar] (applied to the case of no geometric points) says that there is a refinement Z of W such that the map $\check{C}Z_{n+1} \to \check{C}W_{n+1}$ factors through E. In particular, this means that $\check{C}Z$ refines U' (and therefore U) up through dimension $n+1$; since the height of U is $n+1$, this means it automatically refines U in all dimensions. This completes the proof.

For the Nisnevich topology it is essentially the same argument, only using a revised version of [Ar, Thm 4.1]—see [MV, Prop. 1.9, p. 99]. ☐

Corollary 7.11. *Let $\mathcal{C}$ and X be as above, and suppose F is a locally fibrant simplicial presheaf which is locally n-coconnected. If K is a finite simplicial set, there is an isomorphism*

$$Ho\,(K \otimes X, F) \cong \operatorname*{colim}_{U \to X} \pi(K \otimes U, \operatorname{cosk}_n F)$$

where the colimit is taken over the category $\pi HC_X(0)$ consisting of the Čech complexes.

In particular, if A is a presheaf of abelian groups and we take $K = *$ and $F = K(A,n)$, then the above corollary gives the isomorphism between Čech cohomology and sheaf cohomology established in [Ar] and [MV, Prop. 1.9, p. 99].

Proof. This is a direct consequence of the previous two propositions. The subcategory $\pi HC_X(0)$ is final in $\pi HC_X(n+1)$. ☐

Remark 7.12. The above proposition and its corollary are not true for the Zariski topology, and therefore not for the open covering topology on an arbitrary topological space. We repeat the example of [MV, Ex. 1.10, p. 99]: Let $X = \operatorname{Spec} R$ be the semi-localization of $\mathbb{A}^2_k$ at the points $(0,0)$ and $(0,1)$. As a topological space X has exactly two closed points x_1 and x_2 (of codimension 2), infinitely many points of codimension 1 (corresponding to the irreducible closed curves in $\mathbb{A}^2$ passing through both $(0,0)$ and $(0,1)$), and a generic point of codimension 0. Any open cover of X can be refined by a cover with exactly two elements: take any of the pieces containing x_1 and x_2, respectively.

Let $U_1 = X - \{x_1\}$ and $U_2 = X - \{x_2\}$. Pick two of the codimension 1 points f and g which specialize to both x_1 and x_2. Let $W_1 = (U_1 \cap U_2) - \{f\}$ and $W_2 = (U_1 \cap U_2) - \{g\}$. Let $\Omega_0 = U_1 \amalg U_2$ and $\Omega_1 = (U_1 \amalg U_2) \amalg W_1 \amalg W_2$ (the first part is degenerate). Consider the hypercover $\operatorname{cosk}_1 \Omega$. This hypercover cannot be refined by a Čech complex.

8. Verdier sites

The definition of hypercover we've adopted so far in this paper is extremely broad. It has the advantage of working for any Grothendieck site, but it is so broad that it can sometimes be cumbersome. One is often in the position of having to check that something works for all hypercovers, and so it is important to have—whenever possible—a smaller collection of objects to deal with. This is the subject of the present section.

To see the basic problem, look at the site of topological spaces with the Grothendieck topology given by open covers. Under Definition 4.2, to give a hypercover of a space X basically corresponds to giving a simplicial space U_* such that each matching map $U_n \to M_n U$ is locally split. This allows for an incredible amount of freedom in what a hypercover can look like, so much so that it's very difficult to say anything concrete about it. To make things easier, it is reasonable that one should be able to look just at the 'open hypercovers', where the maps $U_n \to M_n U$ all have the form $\amalg_a W_a \to M_n U$ for some open covering $\{W_a\}$ of the target. These are much more manageable objects.

The notion of a Verdier site—introduced in the following definition—is just an axiomatization of the above situation. It is a Grothendieck site with enough extra data that one can talk about a special kind of 'basal hypercover' rather than the more general notion we have been working with. A Verdier site is almost just a Grothendieck site with a basis, but we need to throw in one extra property.

Definition 8.1. *A **Verdier site** is a category $\mathcal{C}$ together with a given collection of covering families $\{U_a \to X\}$ satisfying the properties below. A map $U \to X$ in $\mathcal{C}$ is **basal** if it belongs to one of these covering families. With this terminology, the properties can be stated as follows:*

(i) Any single isomorphism $\{Z \to X\}$ forms a covering family.

(ii) *If $\{U_a \to X\}$ is a covering family and $Y \to X$ is a map, then the pullbacks $Y \times_X U_a$ all exist, and $\{Y \times_X U_a \to Y\}$ is a covering family.*
(iii) *If $\{U_a \to X\}$ is a covering family and one is given a collection of covering families $\{V_{ab} \to U_a\}$, then the collection of compositions $\{V_{ab} \to U_a \to X\}$ is also a covering family.*
(iv) *If $U \to X$ is a basal map then the diagonal $U \to U \times_X U$ is also basal.*

Conditions (i)–(iii) say that the collection of covering families serves as a basis for a Grothendieck topology on $\mathcal{C}$ in the usual way. Most of the familiar geometric Grothendieck sites satisfy the above axioms: these include topological spaces, where the covering families are open covers, as well as the Zariski, Nisnevich, and étale topologies on schemes. The reason for not assuming that $\mathcal{C}$ has all pullbacks is so that our results apply to the Grothendieck topologies on *smooth* schemes which are used in $\mathbb{A}^1$-homotopy theory [MV].

Observe that pullbacks along any basal map always exist (part (ii)), and that any composition of basal maps is again basal (part (iii)). It follows that if $\{V_a \to X\}$ is a finite collection of basal maps and $\{U_a \to V_a\}$ is another collection of basal maps, then the induced map $\prod_X U_a \to \prod_X V_a$ of fibre-products is again basal.

For the following definition, note that to give a map $f\colon \coprod_i rX_i \to \coprod_j rY_j$ between coproducts of representables one must choose, for every index i, a prescribed value of j and a map $X_i \to Y_j$.

Definition 8.2.
(a) *A map $f\colon X \to Y$ is **basal** if X is a coproduct $\coprod_i rX_i$ of representables, Y is also a coproduct $\coprod_j rY_j$ of representables, and the various maps $X_i \to Y_j$ determining f are all basal, in the sense of Definition 8.1.*
(b) *A **basal hypercover** $U \to X$ is a hypercover such that the matching maps $U_n \to M_nU$ are all basal.*

The second part of this definition only makes sense if one knows that the matching objects M_nU are all coproducts of representables, but we will see in Proposition 8.5 that this is the case. First, an easy lemma:

Lemma 8.3. *Let $F \to H \leftarrow G$ be maps between coproducts of representables, where $G \to H$ is basal. Then the pullback is also a coproduct of representables, and the map from the pullback to F is basal.*

Proof. Use the fact that the Yoneda embedding preserves whatever limits exist and that coproducts commute with fibre-products. The necessary pullbacks in $\mathcal{C}$ exist because pullbacks along basal maps always exist. □

Lemma 8.4. *Let $U \to X$ be an n-truncated basal hypercover, and let K be a finite simplicial set of dimension at most n. Then $\hom_+(K, U)$ is a coproduct of representables.*

Proof. We proceed by induction on the dimension of K. When K is empty, $\hom_+(K, U)$ is just X, which is a coproduct of representables by assumption.

Now assume that the lemma has been proven for simplicial sets of dimension at most $k-1$, and let K be obtained from a $(k-1)$-dimensional simplicial set L by attaching finitely many k-simplices. By repeating the following argument, we may assume that only one k-simplex was attached. It follows that $\hom_+(K, U)$ is the pullback of the diagram

$$\hom_+(\Delta^k, U) \to \hom_+(\partial\Delta^k, U) \leftarrow \hom_+(L, U).$$

All three objects are coproducts of representables, the first because U_k is a coproduct of representables and the last two by the induction hypothesis. Since the left map above is basal, Lemma 8.3 tells us that $\hom_+(K, U)$ is also a coproduct of representables. □

Let us return momentarily to Definition 8.2(b). If $U \to X$ is a hypercover and $U_0 \to X$ is basal, then the above proposition specialized to $\partial\Delta^1$ shows that M_1U is a coproduct of representables. So we may ask that $U_1 \to M_1U$ be basal, which in turn forces M_2U to be a coproduct of representables. This shows that our definition of basal hypercover makes sense, in a recursive sort of way.

Proposition 8.5. *Let $K \to L$ be any map of finite simplicial sets whose dimensions are at most k, and let $U \to X$ be a k-truncated basal hypercover. Then the map $\hom_+(L, U) \to \hom_+(K, U)$ is basal.*

Proof. Consider the class C of all maps of finite simplicial sets having the property stated in the lemma. By definition of basal hypercovers, C contains the generating cofibrations $\partial\Delta^n \to \Delta^n$. Cobase changes preserve C by Lemmas 4.7(v), 8.3, and 8.4. Also, finite compositions preserve C because basal maps are closed under finite composition. This shows that C contains all inclusions of finite simplicial sets.

In particular, $\emptyset \to \Delta^n$ belongs to C. This means that $U_n \to X$ is basal for every basal hypercover $U \to X$. By the definition of Verdier sites, the map $U_n \to U_n \times_X U_n$ is also basal. In other words, C contains the codiagonal $\Delta^n \amalg \Delta^n \to \Delta^n$ for every n.

Every surjection can be built from the above codiagonals with finitely many compositions and cobase changes. Thus, every surjection belongs to C. But every map is a composition of a surjection with an inclusion, so every map belongs to C. □

The proposition below is the main thing we need about basal hypercovers. See [AM, Lem. 8.8] for the same result without reference to basal maps. Unfortunately, dealing with these basal maps definitely increases the technical complications.

Theorem 8.6. *In a Verdier site, any hypercover may be refined by a split, basal hypercover. In particular, the basal hypercovers are dense.*

Proof. Let $U \to X$ be any hypercover. The fact that $U_0 \to X$ is a generalized cover means there is a covering sieve R of X such that every map in R lifts through U_0. But our Grothendieck topology was generated by a basis, so there is a covering family $\{W_a \to X\}$ for which every element belongs to R. Setting $V_0 = \coprod_a rW_a$, we have that $V_0 \to X$ is basal and refines $U_0 \to X$.

Continuing by induction, we may assume we have built a split, basal, n-truncated hypercover V which refines U (up through dimension n). Our job is to define V_{n+1}. We consider the maps

$$\begin{tikzcd} & U_{n+1} \arrow[d] \\ M_{n+1}V \arrow[r] & M_{n+1}U, \end{tikzcd}$$

where all the objects are coproducts of representables. Using the same reasoning as in the first paragraph, there is a map $W \to M_{n+1}V$ that is basal, that is a generalized cover, and that fits in the upper left corner of this diagram, i.e., it refines the pullback generalized cover $U_{n+1} \times_{M_{n+1}U} M_{n+1}V \to M_{n+1}V$. Set $V_{n+1} = W \amalg L_{n+1}V$. Now V is a split, $(n+1)$-truncated hypercover; the question is whether $V_{n+1} \to M_{n+1}V$ is basal. Because of the way W was constructed, we need only show that the map $L_{n+1}V \to M_{n+1}V$ is basal.

Recall from Section 4.12 that there is a natural map $\mathrm{dgn}_n V \to \mathrm{cosk}_n V$. In dimension n this is the identity map on V_n, and in dimension $n+1$ it's the map $L_{n+1}V \to M_{n+1}V$. Picking any degeneracy s_i from level n to $n+1$, we get a diagram

$$\begin{tikzcd} L_{n+1}V \arrow[r] & M_{n+1}V \\ V_n \arrow[u, "s_i"] \arrow[r, equal] & V_n. \arrow[u, "s_i"'] \end{tikzcd}$$

Every representable summand of $L_{n+1}V$ is of the form $s_i(rU)$ for some i and some representable summand rU of V_n, so it suffices to show that the right-hand map $s_i : V_n \to M_{n+1}V$ is basal. But this degeneracy is induced by the corresponding collapse map $\partial\Delta^{n+1} \to \Delta^n$, i.e., the composition $s\colon \partial\Delta^{n+1} \hookrightarrow \Delta^{n+1} \xrightarrow{s_i} \Delta^n$. In other words, s_i coincides with $\hom_+(\Delta^n, V) \to \hom_+(\partial\Delta^{n+1}, V)$. The fact that this is basal follows from Proposition 8.5. $\square$

Remark 8.7. Suppose there is a regular cardinal λ with the property that every covering family in $\mathcal{C}$ has size less than λ. By tracing through the above proof, following similar observations to those in Proposition 6.7, one can show that the split, basal hypercover can be constructed so that in each level it has fewer than λ summands. This is needed in the next section.

9. Internal hypercovers

In this final section we give a slight modification of Theorem 6.2 which is useful in applications—for instance, it is needed in [Is]. This involves once again tweaking the definition of hypercover in a certain way.

What sometimes happens is that the Grothendieck site $\mathcal{C}$ is rich enough that one can talk about hypercovers as elements of $s\mathcal{C}$ rather than $sPre(\mathcal{C})$, and this is usually a convenience. For example this is the approach taken in [AM], and it is also used in [DI1] in the context of simplicial spaces. Handling this involves only a slight difference from what we have done, mostly caused by the fact that the coproduct in $\mathcal{C}$ (which we will denote by $\cup$) is not the same as the coproduct of presheaves: i.e., $r(X \cup Y)$ is not the same as $rX \amalg rY$.

Throughout this section we work with a Verdier site for which there exists a regular cardinal λ such that:

(1) Every covering family $\{U_i \to X\}$ has cardinality less than λ.
(2) Coproducts of size less than λ exist in $\mathcal{C}$.
(3) If $\{X_i\}$ is a set of objects whose cardinality is less than λ, then the map of presheaves $\coprod_i rX_i \to r(\bigcup_i X_i)$ becomes an isomorphism after sheafification.

For example, if Sm_k denotes the category of smooth schemes of finite type over a fixed ground field k, we may give it the structure of a Verdier site by saying that the covering families are finite collections $\{U_i \to X\}$ such that $\coprod U_i \to X$ is an étale (or Zariski or Nisnevich) cover. This generates the usual Grothendieck topology, and satisfies the above properties with $\lambda = \aleph_0$.

Definition 9.1. *Given an object X of $\mathcal{C}$, an **internal hypercover** of X is a simplicial object U in $s\mathcal{C}$ which is augmented by X, with the property that each matching map $U_n \to M_nU$ is isomorphic over M_nU to a map of the form $\coprod_i V_i \to M_nU$, for some basal maps $\{V_i \to M_nU\}$ which generate a covering sieve.*

Of course one has to worry about whether the matching object M_nU exists, since the site $\mathcal{C}$ need not have arbitrary limits. But we shall see that the condition on $U_k \to M_kU$ for $k \leq n-1$ guarantees that M_nU does in fact exist. Even though $\mathcal{C}$ is not necessarily complete, the conclusions of Lemma 4.7 are still valid when the limits $\hom_+(K, W)$ do exist in $\mathcal{C}$. For example, if $\hom_+(L, W)$, $\hom_+(K, W)$, and $\hom_+(M, W)$ all exist, and the pullback of

$$\hom_+(L, W) \to \hom_+(K, W) \leftarrow \hom_+(M, W)$$

also exists, then $\hom_+(L \amalg_K M, W)$ exists and is isomorphic to the above pullback.

Proposition 9.2. *If $U \to X$ is an n-truncated internal hypercover then the object $\hom_+(K, U)$ exists whenever K is a simplicial set of dimension at most n. In particular, the matching object $M_nU = \hom_+(\partial\Delta^n, U)$ exists.*

Proof. The proof follows the same lines as the proof of Lemma 8.4. □

We continue our notational convention of writing U for a simplicial object of $\mathcal{C}$ and also for the simplicial presheaf that it represents.

Theorem 9.3. *The model category $sPre(\mathcal{C})_{\mathcal{L}}$ of simplicial presheaves may be obtained as the localization of $sPre(\mathcal{C})$ at the following collection of maps $\mathcal{I}$:*

(i) Maps of the form $\coprod W_i \to (\bigcup W_i)$, for collections $\{W_i\}$ in $\mathcal{C}$ of size less than λ.

(ii) The maps $rU \to rX$, for all internal hypercovers $U \to X$.

Proof. Let $sPre(\mathcal{C})_{\mathcal{I}}$ denote the localization we're considering. First note that all the maps in $\mathcal{I}$ are local weak equivalences. For maps of type (ii), this is Theorem 6.2. For maps of type (i) it follows from assumption (3) at the beginning of this section, because every simplicial presheaf is locally weakly equivalent to its sheafification. So $sPre(\mathcal{C})_{\mathcal{L}}$ is a stronger localization than $sPre(\mathcal{C})_{\mathcal{I}}$. To see that the localizations coincide, it will suffice to show that if $V \to X$ is a basal hypercover in which the number of summands in each level is smaller than λ, then $V \to X$ is a weak equivalence in $sPre(\mathcal{C})_{\mathcal{I}}$. This is by virtue of Theorem 6.2, Theorem 8.6, and Remark 8.7.

Each presheaf V_n may be decomposed as a coproduct of representables in an essentially unique way: $V_n = \coprod_\alpha V_{n\alpha}$. We define an object U of $s\mathcal{C}$ by $U_n = \bigcup_\alpha V_{n\alpha}$, and with face and degeneracy maps lifted from those in V. For the rest of the proof we will be careful to distinguish U from the simplicial presheaf rU. Observe that there is a canonical map $V \to rU$, commuting with the augmentations down to X.

We claim that U is an internal hypercover of X. Assuming this for the moment, relation (i) in our definition of $sPre(\mathcal{C})_{\mathcal{I}}$ shows that $V_n \to rU_n$ is an $\mathcal{I}$-weak equivalence for each n. Since every simplicial presheaf F is the homotopy colimit $\operatorname{hocolim}_n F_n$ (see Remark 2.1), it follows that $V \to rU$ is also an $\mathcal{I}$-weak equivalence. Using that $U \to X$ is an internal hypercover, relation (ii) gives that $rU \simeq X$; so one concludes that $V \simeq X$ as well.

It remains only to verify that U is an internal hypercover. First note that $\hom_+(K, V) \to \hom_+(K, rU)$ induces an isomorphism on sheafifications for every finite simplicial set K. When K has dimension 0 this follows from property (3) that we assumed at the beginning of the section. For higher dimensional K one proceeds by induction on the number of non-degenerate simplices in K, using the same pullback square from Lemma 8.4 and the fact that sheafification preserves finite limits.

So taking $K = \partial\Delta^n$ we have that $M_nV \to M_n(rU)$ induces an isomorphism on sheafifications, and in particular is a generalized cover. This, together with the fact that $V_n \to M_nV$ is a generalized cover, shows immediately that the same is true of $rU_n \to M_n(rU)$. So rU is a hypercover of X.

Finally, to see that U is an internal hypercover one just uses that the Yoneda embedding preserves all limits that exist: so $M_n(rU)$ is isomorphic to $r(M_nU)$. $\square$

Appendix A. Čech localizations

This appendix is a bit of an aside from the main body of the paper. Here we investigate how descent for Čech complexes compares to descent for all hypercovers. These are not equivalent notions in general—see Example A.10—although in some cases they turn out to agree. Unlike hypercover descent, Čech descent is often a reasonably straightforward thing to verify; so it's useful to know how strong a notion it is. In this section we show that Čech descent actually implies descent for all *bounded* hypercovers, and we give some related results of interest.

If $\xi\colon F \to G$ is a map of presheaves of sets, the **Čech complex** of ξ is the simplicial presheaf $\check{C}\xi$ (often denoted $\check{C}F$ by abuse) given by

$$[n] \mapsto F \times_G F \times_G \cdots \times_G F \quad (n+1 \text{ factors}).$$

A simplicial presheaf F is said to have **Čech descent** if it satisfies descent for $\check{C}U$ whenever $U \to X$ is a generalized cover in which X is representable and U is a coproduct of representables.

Here is a short proposition we will need to use often:

Proposition A.1. *Let $\{U_a \to X\}$ be any set of maps in $\mathcal{C}$, and let $R \hookrightarrow X$ be the sieve generated by these maps. Let $U = \coprod_a rU_a$. Then there is a natural map $\check{C}U \to R$, and this map is an objectwise weak equivalence.*

Proof. If $\xi\colon K \to L$ is any map of simplicial sets, then the Čech complex $\check{C}\xi$ is fibrant and homotopy discrete. This shows that the natural map $\check{C}U \to \pi_0\check{C}U$ is an objectwise weak equivalence. The presheaf R is equal to the presheaf $\pi_0\check{C}U$, i.e., $R(Y) = \pi_0\check{C}U(Y)$ for all Y in $\mathcal{C}$. $\square$

Let $\check{\mathcal{C}}$ denote the set of maps $\{R \hookrightarrow X\}$, where X runs over all objects in $\mathcal{C}$ and R runs over all covering sieves (this is a *set* because $\mathcal{C}$ is small). Let $sPre(\mathcal{C})_{\check{\mathcal{C}}}$ denote the Bousfield localization of $sPre(\mathcal{C})$ at the set $\check{\mathcal{C}}$. We'll refer to this model category as the **Čech localization** of $sPre(\mathcal{C})$, for reasons which will shortly become apparent (see Corollary A.3).

Given a covering sieve $R \hookrightarrow X$, let $\check{C}(R)$ denote the Čech complex corresponding to the cover $\coprod U_a \to X$ where the coproduct ranges over all maps $U_a \to X$ in the sieve. The above proposition implies that $\check{C}R \to X$ factors through R, and $\check{C}R \to R$ is an objectwise weak equivalence. So localizing at the set $\{R \hookrightarrow X\}$ is equivalent to localizing at $\{\check{C}R \to X\}$. We will see in a minute that this is actually equivalent to localizing at $\{\check{C}U \to X\}$, for *all* generalized covers $U \to X$, and so our Čech localization is analogous to the hypercover localization of Theorem 6.2. The advantage of starting with just the sieves rather than the generalized covers is that these form a set, and so the localization automatically exists.

Proposition A.2. *Given any simplicial presheaf F, the map $F \to \tilde{F}$ from F to its (levelwise) sheafification is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$.*

Unfortunately the proof of this result is somewhat involved, so we'll postpone it until the end of the section.

Corollary A.3. *Let $F \to G$ be any generalized cover of presheaves of sets. Then the map $\check{C}F \to G$ is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$.*

Proof. The map $\check{C}F \to G$ factors as $\check{C}F \to \pi_0\check{C}F \to G$. As in the proof of Proposition A.1, the first map is an objectwise weak equivalence. The second map is a monomorphism of presheaves, and the fact that $F \to G$ is a generalized cover shows that it is a local epimorphism. Hence, the map becomes an isomorphism upon sheafification. Proposition A.2 then shows that it is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$, and so we can conclude the same for the composite $\check{C}F \to G$. □

We now derive the connection with hypercovers. Recall from Definition 4.10 that a hypercover $U \to X$ is bounded if $U = \operatorname{cosk}_n U$ for some n.

Proposition A.4. *Given a bounded hypercover U of X, the map $U \to X$ is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$.*

Proof. We proceed by induction, starting from the fact that bounded hypercovers of height 0 are just Čech complexes and therefore are handled by Corollary A.3.

Suppose that $U \to X$ is a bounded hypercover of height $n+1$. Define V to be $\operatorname{cosk}_n U$, so V is a bounded hypercover of height at most n. Therefore, we may assume by induction that $V \to X$ is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$. The canonical map $U \to V$ gives a generalized cover $U_{n+1} \to V_{n+1}$, by the very definition of what it means for U to be a hypercover. Lemma A.5 below shows that in fact $U_k \to V_k$ is a generalized cover for all k.

Consider the following bisimplicial object, augmented horizontally by V:

$$V \longleftarrow U \leftleftarrows U \times_V U \;\Lleftarrow\; \cdots$$

The kth row is the (augmented) Čech complex for the generalized cover $U_k \to V_k$. Note that for $0 \leq k \leq n$ the kth row is the constant simplicial object with value U_k because $U_k \to V_k$ is the identity. Call this bisimplicial object (without the horizontal augmentation) W_{**}. There is an obvious map $\operatorname{hocolim} W_{**} \to X$.

One may compute $\operatorname{hocolim} W_{**}$ by first taking the homotopy colimit of the rows, and then taking the homotopy colimit of the resulting simplicial object. But in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$ the homotopy colimit of the kth row is just V_k by Corollary A.3. Also, we have assumed by induction that $V \simeq \operatorname{hocolim}_k V_k$ is weakly equivalent to X. So $\operatorname{hocolim} W_{**} \to X$ is a weak equivalence.

Let D denote the diagonal of W_{**}. Standard homotopy theory tells us that $D = \operatorname{hocolim}_k D_k$ is weakly equivalent to $\operatorname{hocolim} W_{**}$. We claim that U is a retract over X of D. Note first that one has, in complete generality, a map $U \to D$; in dimension k it is the unique horizontal degeneracy $W_{0k} \to W_{kk}$.

To produce a map $D \to U$ it is enough to give $\operatorname{sk}_{n+1} D \to \operatorname{sk}_{n+1} U$, because $U = \operatorname{cosk}_{n+1} U$. But note that $\operatorname{sk}_n D = \operatorname{sk}_n U$, and choosing any face map $[0] \to [n+1]$

gives a map $D_{n+1} \to U_{n+1}$, inducing a corresponding map $\mathrm{sk}_{n+1} D \to \mathrm{sk}_{n+1} U$ as desired.

It is straightforward to check that $U \to D \to U$ is the identity (because $U = \mathrm{cosk}_{n+1} U$ one only has to check it on $(n+1)$-skeleta), and all the maps commute with the augmentations down to X. We have already shown that $D \to X$ is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$. Since $U \to X$ is a retract of $D \to X$, it must also be a weak equivalence. $\square$

Lemma A.5. *If U is a hypercover of height $n+1$, then the map $U \to \mathrm{cosk}_n U$ is a generalized cover in every dimension.*

Proof. First note that for $k \leq n$, the map $U_k \to [\mathrm{cosk}_n U]_k$ is the identity, so it is a generalized cover. For any k, the map $U_k \to [\mathrm{cosk}_n U]_k$ may be rewritten as

$$\hom_+(\Delta^k, U) \to \hom_+(\mathrm{sk}_n \Delta^k, U).$$

But $U = \mathrm{cosk}_{n+1} U$, so the domain may be written as

$$\hom_+(\Delta^k, U) = \hom_+(\Delta^k, \mathrm{cosk}_{n+1} U) = \hom_+(\mathrm{sk}_{n+1} \Delta^k, U).$$

So we are interested in the map $\hom_+(\mathrm{sk}_{n+1} \Delta^k, U) \to \hom_+(\mathrm{sk}_n \Delta^k, U)$ induced by the inclusion $\mathrm{sk}_n \Delta^k \to \mathrm{sk}_{n+1} \Delta^k$. Recall from Lemma 4.11 the pullback square

$$\begin{tikzcd}
\hom_+(\mathrm{sk}_{n+1} \Delta^k, U) \arrow[r] \arrow[d] & \prod_X \hom_+(\Delta^{n+1}, U) \arrow[d] \\
\hom_+(\mathrm{sk}_n \Delta^k, U) \arrow[r] & \prod_X \hom_+(\partial\Delta^{n+1}, U).
\end{tikzcd}$$

The map $\hom_+(\Delta^{n+1}, U) \to \hom_+(\partial\Delta^{n+1}, U)$ is just the matching map $U_{n+1} \to M_n U$, and is therefore a generalized cover. So the right vertical map in the above square is a finite product of generalized covers, which is again a generalized cover. Finally, we see that the left vertical map is a pullback of a generalized cover, hence also a generalized cover. $\square$

If $R \hookrightarrow X$ is a covering sieve, let $\mathcal{I}_R$ denote the full subcategory of $\mathcal{C} \downarrow X$ consisting of all maps in R. Consider the diagram $\mathcal{I}_R \to sPre(\mathcal{C})$ sending $(U \to X)$ to U. The colimit of this diagram is R, and we will write the homotopy colimit as $\mathrm{hocolim}_R U$. The natural map from the homotopy colimit to the colimit gives $\mathrm{hocolim}_R U \to R$, and this turns out to be an objectwise weak equivalence by [D, Lemma 2.7]. This fact has nothing to do with sieves, and is true in a slightly generalized form for arbitrary simplicial presheaves.

Theorem A.6. *The following classes of maps give the same localization of $sPre(\mathcal{C})$:*

(a) The set of all covering sieves $R \hookrightarrow X$;
(b) The set of all maps $\mathrm{hocolim}_R U \to X$, where $R \hookrightarrow X$ is a covering sieve;
(c) The class of all hypercovers of height 0, i.e., the Čech complexes $\check{C}U \to X$;
(d) The class of all bounded hypercovers $U \to X$;
(e) The class of maps $F \to \tilde{F}$ from simplicial presheaves to their sheafifications.

If the topology on $\mathcal{C}$ is given by a basis of covering families, then one can also add

(a') The set of all covering sieves $R_U \hookrightarrow X$ where R_U is the sieve generated by the covering family $\{U_a \to X\}$.

It may seem surprising that the localization in (e) does not give the usual notion of local weak equivalence, but only the weaker Čech version. Example A.10 shows that it really is weaker. Also, note that the above theorem could just as well have been stated for $U\mathcal{C}$ rather than $sPre(\mathcal{C})$—the proofs are essentially the same.

Proof. The fact that $\operatorname{hocolim}_R U \to R$ is an objectwise weak equivalence immediately shows that the localizations in (a) and (b) are the same. And we have seen in Proposition A.1 that the localizations in (a) and (c) are the same.

The localization in (d) is *a priori* stronger than that in (a); Proposition A.4 shows that the two localizations actually agree. Likewise, the localization in (e) is stronger than the one in (a), because $R \hookrightarrow X$ becomes an isomorphism upon sheafification. Proposition A.2 shows that they agree.

Finally, if our topology is given by a basis of covering families then the proof that the localizations in (a') and (e) coincide follows the proof of Proposition A.2 more or less verbatim. □

It would be interesting to know more about $sPre(\mathcal{C})_{\check{\mathcal{C}}}$, for instance to have an explicit characterization of the weak equivalences. Perhaps this wouldn't be so useful, since the chief interest in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$ is that it is sometimes a more convenient version of $sPre(\mathcal{C})_{\mathcal{L}}$ (see Example A.11).

Corollary A.7. *Let F be a simplicial presheaf. Then F satisfies descent for all Čech complexes if and only if it satisfies descent for all bounded hypercovers.*

Proof. Let F' be a fibrant-replacement for F in $sPre(\mathcal{C})$. The statement of the corollary for F' requires that F' be local with respect to the Čech complexes $\check{C}V \to X$ if and only if it is local with respect to the bounded hypercovers $U \to X$. This is true by the above proposition (parts (c) and (d)). But of course F has descent for a certain class of objects precisely when F' has descent for that same class, because $F \to F'$ is an objectwise weak equivalence. □

Corollary A.8. *Suppose the Grothendieck topology on $\mathcal{C}$ is given by a basis of covering families. Then a simplicial presheaf F satisfies Čech descent if and only if it satisfies descent for all the Čech complexes $\check{C}U \to X$ in which X is a representable and $U = \coprod_a U_a$ for some covering family $\{U_a \to X\}$ in the basis.*

Proof. This is similar to the proof of Corollary A.7, using Theorem A.6 (parts (a') and (c)) and Proposition A.1. □

The following result can be useful for verifying hypercover descent.

Corollary A.9. *Let F be an objectwise-fibrant simplicial presheaf with the property that $F(X)$ has no homotopy in dimension n or higher, for every X in $\mathcal{C}$. Then F satisfies descent for all hypercovers if and only if it satisfies descent for all Čech complexes.*

Proof. First we need to consider the localization $U\mathcal{C}/S$, where S is the set of maps $\{\partial\Delta^{n+1} \otimes X \to \Delta^{n+1} \otimes X | X \in \mathcal{C}\}$. It is easy to check that the fibrant objects in $U\mathcal{C}/S$ are the simplicial presheaves G such that each $G(X)$ is fibrant and has no homotopy above dimension $n-1$. Given an objectwise fibrant simplicial presheaf G, one can construct the localization $L_S G$ via the small object argument applied to the maps in S. By thinking about this, one sees that the maps of simplicial sets $G(X) \to L_S G(X)$ are isomorphisms up through simplicial dimension n. So

$L_S G(X)$ has the same homotopy groups as $G(X)$ up through dimension $n-1$, but no homotopy groups in higher dimensions. Even if G is not objectwise fibrant, $L_S G$ is objectwise weakly equivalent to $L_S(\mathrm{Ex}^\infty G)$, and so it is still true that $G(X)$ and $L_S G(X)$ have the same $(n-1)$-type for all $X \in \mathcal{C}$.

General localization theory says that a map $G \to H$ is an S-equivalence if and only if $L_S G \to L_S H$ is an objectwise equivalence, and so this is the same as saying that $G(X) \to H(X)$ induces isomorphisms on all homotopy groups up through dimension $n-1$, for every X. In particular, the map $G \to \operatorname{cosk}_n G$ is an S-equivalence.

Now consider the localization $U\mathcal{C}/T$, where T is the union of S and the set of all covering sieves $R \hookrightarrow X$. A simplicial presheaf F is fibrant in $U\mathcal{C}/T$ precisely if it is objectwise fibrant, has Čech descent, and each $F(X)$ has no homotopy in dimension n or higher.

Suppose that F is as in the statement of the corollary, and that F satisfies descent for all Čech complexes. Then we know that F is fibrant in $U\mathcal{C}/T$. To verify descent for all hypercovers, it suffices by Corollary 6.3 to verify it for all split hypercovers $U \to X$. But note that $U \to \operatorname{cosk}_n U$ is necessarily a T-equivalence (because it is an S-equivalence). Yet $\operatorname{cosk}_n U$ is a bounded hypercover of X, and hence $\operatorname{cosk}_n U \to X$ is a T-equivalence as well (using the $U\mathcal{C}$ version of Proposition A.4). Hence $U \to X$ is also a T-equivalence. Since F is fibrant in $U\mathcal{C}/T$ and X and U are cofibrant in $U\mathcal{C}/T$, the morphism $\mathrm{Map}(X, F) \to \mathrm{Map}(U, F)$ is a weak equivalence; so F satisfies descent for $U \to X$ by Lemma 4.4. $\square$

Here is an example showing that the Čech localization can be strictly weaker than the localization at all hypercovers. In other words, we exhibit a simplicial presheaf which has descent for all Čech complexes but does not have descent for all hypercovers. The example is a slight modification of one suggested to us by Carlos Simpson.

Example A.10. Let $X = X_0$ be the open interval $(0,1)$. Now let $U_0 = (0, \frac{2}{3})$, $V_0 = (\frac{1}{3}, 1)$, and $X_1 = U_0 \cap V_0$. Note that $X_1 \cong X$, and let $U_1 = (\frac{1}{3}, \frac{5}{9})$, $V_1 = (\frac{4}{9}, \frac{2}{3})$, and $X_2 = U_1 \cap V_1$. Again one has $X_2 \cong X$, and we define U_2, V_2, and X_3 in the expected way. Continue. Our site $\mathcal{C}$ consists of the spaces $\{X_i, U_i, V_i \mid i \geq 0\}$ with the inclusions between them, and equipped with the usual notion of open cover. The category $\mathcal{C}$ is depicted as

$$\begin{tikzcd}
 & U_0 \arrow[dl] & & U_1 \arrow[dl] & & \cdots \arrow[dl] \\
X_0 & & X_1 \arrow[ul] \arrow[dl] & & X_2 \arrow[ul] \arrow[dl] & \\
 & V_0 \arrow[ul] & & V_1 \arrow[ul] & & \cdots . \arrow[ul]
\end{tikzcd}$$

Define a presheaf of spaces on our site in the following way:

$$F(X_0) = \emptyset, \quad F(U_n) = D^n_+, \quad F(V_n) = D^n_-, \quad \text{and} \quad F(X_{n+1}) = S^n \quad (n \geq 0).$$

Here D^n_+ and D^n_- denote the upper and lower hemispheres of S^n. The restriction maps $F(U_n) \to F(X_{n+1})$ and $F(V_n) \to F(X_{n+1})$ are the inclusions of the hemispheres in S^n, while the maps $F(X_n) \to F(U_n)$ and $F(X_n) \to F(V_n)$ are the inclusions of the boundaries of the hemispheres. Define the simplicial presheaf G

by $G(W) = \mathbb{Z}Sing F(W)$—that is, $G(W)$ is the result of applying the free abelian group functor to the singular complex of $F(W)$. Using the Dold-Kan correspondence, one can regard G as a presheaf of chain complexes; then $G(W)$ is the usual complex for computing the singular homology of $F(W)$.

Now G has Čech descent: this can be checked using Corollary A.8, and so the main point is that for every n the square

$$\begin{tikzcd} G(U_n \cup V_n) \arrow[r] \arrow[d] & G(U_n) \arrow[d] \\ G(V_n) \arrow[r] & G(U_n \cap V_n) \end{tikzcd}$$

is a homotopy pullback. On the other hand, we will construct a hypercover for which G does not have descent. The combinatorics of this construction are slightly complicated, but the idea is this: Start with $\Omega_0 = U_0 \amalg V_0$, then consider $\operatorname{cosk}_0 \Omega$ except replace each non-degenerate occurrence of X_1 with $U_1 \amalg V_1$. Next take $\operatorname{cosk}_1 \Omega$, replace each non-degenerate occurrence of X_2 with $U_2 \amalg V_2$, and continue. This gives the hypercover $\Omega \to X$.

Now we will be more precise. Let P_n be the category of all nonempty subsets of $\{0, 1, \ldots, n\}$, with inclusions. Note that the objects of P_n can be identified with the sub-simplices of Δ^n, and that $[n] \mapsto P_n$ forms a simplicial category. Let S_n denote the set of all functors $J \colon P_n^{op} \to \mathcal{C}$ with the following properties:

(1) All the values of J belong to $\{U_i, V_i \mid i \geq 0\}$;
(2) Given a subset $\sigma = \{i_0, \ldots, i_k\}$ in P_n, if $\bigcap_j J(\{i_0, \ldots, \hat{i_j}, \ldots, i_k\}) = U_m$ (resp. V_m) then $J(\sigma) = U_m$ (resp. V_m).
(3) If $\bigcap_j J(\{i_0, \ldots, \hat{i_j}, \ldots, i_k\}) = X_m$ then $J(\sigma)$ is either U_m or V_m (and this includes the case $k = 0$).

Let Ω denote the simplicial presheaf defined by

$$\Omega_n := \coprod_{J \in S_n} J(\{0, 1, \ldots, n\}),$$

with simplicial structure induced by that of P. Intuitively, each summand of Ω_n corresponds to an n-simplex together with a certain labelling of its simplices given by J: the labelling is such that smaller simplices are labelled by larger opens, and such that the above properties are satisfied. The reader is encouraged to work out what these properties say for small values of n, and to verify that Ω is a hypercover of X_0 (use Proposition 4.15). Check that $\Omega_0 = U_0 \amalg V_0$ and $\Omega_1 = U_0 \amalg (U_1 \amalg V_1) \amalg (U_1 \amalg V_1) \amalg V_0$.

We claim that $\operatorname{holim}_n G(\Omega_n)$ is not connected, whereas $G(X) = 0$. To calculate $\pi_0(\operatorname{holim}_n G(\Omega_n))$ we can just work in the category of chain complexes. The cosimplicial object $[n] \mapsto G(\Omega_n)$ corresponds to a double complex, and we are trying to compute the 0th homology of the total complex (the one called Tot^{Π} in [W], rather than $\operatorname{Tot}^{\oplus}$). But observe that each $G(\Omega_n)$ has homology only in dimension 0 because each $G(U_i)$ and $G(V_i)$ is contractible. Therefore, the E_1-term of the spectral sequence for the homology of the bicomplex is concentrated in a line. Thus, the bicomplex's 0th homology is the kernel of $d_0 - d_1 \colon H_0 G(\Omega_0) \to H_0 G(\Omega_1)$, which is $\mathbb{Z}$. This completes the verification that G does not satisfy descent for the hypercover Ω.

Example A.11. Sometimes the localizations $U\mathcal{C}_{\check{\mathcal{C}}}$ and $U\mathcal{C}_{\mathcal{L}}$ *do* coincide. Let S be a Noetherian scheme of finite dimension and let $\mathcal{C}$ be the site Sm/S with either the Zariski or Nisnevich topology (one can also take $\mathcal{S}ch/S$ here). For these sites the localizations $U\mathcal{C}_{\check{\mathcal{C}}}$ and $U\mathcal{C}_{\mathcal{L}}$ agree. For the Zariski topology this is a direct consequence of the 'Brown-Gersten Theorem', which identifies the fibrant objects in $U\mathcal{C}_{\mathcal{L}}$ with the objectwise-fibrant simplicial presheaves satisfying Čech descent for all two-fold Zariski covers $\{U, V \to X\}$. This is essentially proven in [BG], although one has to translate their proof into our more modern setting. See also [Bl, Lemmas 4.1, 4.3].

For the Nisnevich topology we have to explain a little more. Given an elementary distinguished square $\{U \hookrightarrow X, p\colon V \to X\}$ [MV, Def. 1.3, p.96], let $P(U,V)$ denote the simplicial presheaf which has $U \amalg V$ in dimension 0, $U \amalg p^{-1}(U) \amalg V$ in dimension 1, and is degenerate in higher dimensions. The Brown-Gersten Theorem in this context is [MV, Lemma 1.6, p.98]; together with [Bl, Lemma 4.3], it implies that $U\mathcal{C}_{\mathcal{L}}$ is the localization of $U\mathcal{C}$ at the maps $P(U,V) \to X$ for all elementary distinguished squares. We already know that $U\mathcal{C}_{\mathcal{L}}$ is a stronger localization than $U\mathcal{C}_{\check{\mathcal{C}}}$, so we just need to show that the maps $P(U,V) \to X$ are weak equivalences in $U\mathcal{C}_{\check{\mathcal{C}}}$.

To see that $P(U,V) \to X$ is a weak equivalence in $U\mathcal{C}_{\check{\mathcal{C}}}$, first note that the sections $P(U,V)(Z)$ are simplicial sets with non-degenerate simplices only in dimensions 0 and 1. Each component is a star, centered at a 0-simplex corresponding to a map $Z \to U$ (because every map $Z \to V$ can be an endpoint of at most one 1-simplex). Therefore each component is contractible, so $P(U,V) \to \pi_0 P(U,V)$ is an objectwise weak equivalence. The codomain is just the presheaf $U \amalg_{p^{-1}U} V$, so we are reduced to showing that the map $U \amalg_{p^{-1}U} V \to X$ is a weak equivalence in $U\mathcal{C}_{\check{\mathcal{C}}}$. By the $U\mathcal{C}$ version of Theorem A.6(e) it suffices to show that this map induces an isomorphism on sheafifications, and this is routine (use Nisnevich stalks, or look at [MV, Lemma 3.1.6]).

A.12. **A leftover proof.** The final goal of this section is to give the proof of Proposition A.2: if F is a simplicial presheaf we need to show that $F \to \tilde{F}$ is a weak equivalence in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$. In fact it will suffice to do this when F is a discrete simplicial presheaf, since a simplicial presheaf F can be recovered as a homotopy colimit of the discrete presheaves F_n (Remark 2.1). Unfortunately, even to prove our claim for discrete simplicial presheaves seems to require a wrestling match with the small object argument.

So from now on F is just a presheaf of sets. We introduce two constructions: First, $\mathcal{A}F$ is the presheaf defined by $\mathcal{A}F(X) = F(X)/\sim$, where two sections s and t are equivalent if there is a covering sieve $R \hookrightarrow X$ such that $s|_U = t|_U$ for every $U \to X$ in R. Secondly, $\mathcal{B}F$ is defined to be the pushout

$$\begin{tikzcd} \coprod R \arrow[r] \arrow[d] & F \arrow[d] \\ \coprod X \arrow[r] & \mathcal{B}F \end{tikzcd}$$

where the coproduct is indexed over all objects X in $\mathcal{C}$, all covering sieves $R \hookrightarrow rX$, and all maps $R \to F$. One may check that $\mathcal{A}\mathcal{B}F$ is what is usually denoted F^+, and so $\mathcal{A}\mathcal{B}\mathcal{A}\mathcal{B}F$ is the sheafification $\tilde{F}$.

We will show that the maps $F \to \mathcal{B}F$ and $F \to \mathcal{A}F$ are Čech weak equivalences, for any presheaf F. The first claim is very easy: Since $\coprod R \to \coprod X$ is an acyclic cofibration in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$, its cobase change $F \to \mathcal{B}F$ is also an acyclic cofibration. Unfortunately the second claim is much more difficult. The idea is to build up a Čech weak equivalence $F \to L_\infty F$ by brute force, in such a way that there is an objectwise weak equivalence $L_\infty F \to \mathcal{A}F$.

Given a covering sieve $R \hookrightarrow X$, let $J^n R$ be the pushout

$$\begin{tikzcd} \partial\Delta^n \otimes R \arrow[r] \arrow[d] & \partial\Delta^n \otimes X \arrow[d] \\ \Delta^n \otimes R \arrow[r] & J^n R. \end{tikzcd}$$

The natural map $J^n R \to \Delta^n \otimes X$ is a Čech acyclic cofibration, by SM7. Note that to give a map $J^n R \to G$ is the same as giving a map $\partial\Delta^n \to G(X)$ together with a compatible family of extensions $\Delta^n \to G(U)$ for all maps $U \to X$ in R.

Let $L_0 F = F$, and let $L_{n+1}F$ be obtained from $L_n F$ as the pushout

(A.13)
$$\begin{tikzcd} \coprod J^{n+1}R \arrow[r] \arrow[d, tail, "\sim"'] & L_n F \arrow[d, tail, "\sim"] \\ \coprod \Delta^{n+1} \otimes X \arrow[r] & L_{n+1}F \end{tikzcd}$$

where the coproducts run over all X in $\mathcal{C}$, all covering sieves $R \hookrightarrow X$, and all maps $J^{n+1}R \to L_n F$. Let $L_\infty F$ be the colimit of the chain $L_0 F \to L_1 F \to L_2 F \to \cdots$. Since each $L_n F \to L_{n+1} F$ is a Čech acyclic cofibration, the composite $F \to L_\infty F$ is also a Čech acyclic cofibration.

To get a feel for what's happening here, let's look just at $L_1 F$. A map $J^1 R \to F$ corresponds to giving a map $\partial\Delta^1 \to F(X)$ together with a compatible family of extensions $\Delta^1 \to F(U)$ for all $U \to X$ in the sieve. Since F is discrete, this means that we are giving two elements of $F(X)$ which *agree* when restricted to pieces of the sieve. When we form the pushout $\Delta^1 \otimes X \leftarrow J^1 R \to F$ we are adding a 1-simplex into $F(X)$ which will identify these elements in π_0. So it follows that $\pi_0 L_1 F(Y) = \mathcal{A}F(Y)$, for all Y.

When we pass from $L_n F$ to $L_{n+1}F$ something similar is happening—we will see it boils down to killing off all the higher homotopy, in the end because the objects $F(X)$ were all discrete and so had no higher homotopy to begin with. So the goal is to show that each $L_\infty F(X)$ is fibrant and homotopy discrete, and that $\pi_0 L_\infty F(X) = \mathcal{A}F(X)$. This will imply that the natural map $L_\infty F \to \pi_0 L_\infty F$ is an objectwise weak equivalence, and the target is identified with $\mathcal{A}F$. We will then have $F \xrightarrow{\sim} L_\infty F \xrightarrow{\sim} \mathcal{A}F$ in $sPre(\mathcal{C})_{\check{\mathcal{C}}}$.

The argument will proceed by establishing the following properties: Given any Y in $\mathcal{C}$,

(i) The map of simplicial sets $L_n F(Y) \to L_{n+1}F(Y)$ is an isomorphism on n-skeleta.

(ii) The simplicial set $L_n F(Y)$ has dimension at most n, i.e., it is degenerate in degrees greater than n.

(iii) Given any n-simplex σ in $L_n F(Y)$, there is a covering sieve $R \hookrightarrow Y$ such that $\sigma|_U$ is in the image of $L_{n-1}F(U) \to L_n F(U)$ for every $U \to Y$ in R. In particular, $\sigma|_U$ is a degenerate n-simplex.

(iv) Given any n-simplex σ in $L_nF(Y)$, there is a covering sieve $R \hookrightarrow Y$ such that $\sigma|_U$ is in the image of $F(U) \to L_nF(U)$ for every $U \to Y$ in R.
(v) For $n \geq 2$, any map $\partial\Delta^n \to L_{n-1}F(Y)$ extends to a map $\Delta^n \to L_nF(Y)$.
(vi) Any map $\Lambda^{2,k} \to L_1F(Y)$ extends to $\partial\Delta^2 \to L_2F(Y)$.

Granting these for the moment, let us show they imply the desired result. To show that $L_\infty F(X)$ is fibrant and homotopy discrete, it is enough to verify that it has the extension property with respect to the maps $\partial\Delta^n \to \Delta^n$ ($n \geq 2$) and the maps $\Lambda^{2,k} \to \Delta^2$. These are easy consequences of parts (i), (v), and (vi). Also, part (i) tells us that $\pi_0 L_1 F \to \pi_0 L_\infty F$ is an isomorphism, and we have already remarked that $\pi_0 L_1 F \cong \mathcal{A}F$. This finishes the proof, granting the statements outlined above.

Claim (i) follows from the fact that $J^{n+1}R(Y) \to (\Delta^{n+1} \otimes X)(Y)$ is an isomorphism on n-skeleta. Part (ii) follows from an induction, using that $(\Delta^n \otimes X)(Y)$ has dimension n and that $F(Y)$ has dimension 0 (since we assumed that F is a presheaf of sets). Part (iii) is a straightforward analysis of diagram (A.13), and (iv) follows from (iii) by induction. We will show that (v) is a consequence of (iv), and a similar argument proves (vi).

Suppose we have a map $\sigma\colon \partial\Delta^n \to L_{n-1}F(Y)$. By (iv), for each face $d_i\sigma$ there is a covering sieve $R_i \hookrightarrow Y$ such that $d_i\sigma|_U$ is in the image of $F(U) \to L_{n-1}F(U)$, for every $U \to Y$ in R_i. There is of course a covering sieve R which refines all the R_i. So for each $U \to Y$ in R and each i, there is $(n-1)$-simplex $\alpha_{U,i}$ in $F(U)$ which maps to $(d_i\sigma)|_U$.

Now, it is not clear that as i varies the $(n-1)$-simplices $\alpha_{U,i}$ fit together to give a map $\alpha_U\colon \partial\Delta^n \to F(U)$. However, we know they fit together in $L_{n-1}F(U)$, and the map $F(U) \to L_{n-1}F(U)$ is a cofibration of simplicial sets, hence a monomorphism. So the $\alpha_{U,i}$ must fit together in $F(U)$ as well.

Secondly, it is not immediately clear that the α_U patch together over the covering sieve R: that is, we must check that given maps $U \to V \to X$ where $V \to X$ is in R, then α_U coincides with the restriction of α_V to U. Again, this follows from the fact that everything patches together in $L_{n-1}F$ and the fact that $F \to L_{n-1}F$ is an objectwise cofibration.

So we have constructed a map $\alpha\colon \partial\Delta^n \otimes R \to F$ such that the composite map $\partial\Delta^n \otimes R \to F \to L_{n-1}F$ coincides with $\sigma|_R$. Now we use the fact that F is a *discrete* simplicial presheaf, from which it follows that α can be extended to a map $\bar{\alpha}\colon \Delta^n \otimes R \to F$. Composing this with $F \to L_{n-1}F$ and patching with σ gives a map $J^nR \to L_{n-1}F$, and this will extend to $\Delta^n \otimes Y$ once we pass to L_nF. The upshot is that we've shown σ extends to Δ^n under the map $L_{n-1}F(Y) \to L_nF(Y)$.

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