# MOTIVIC CELL STRUCTURES

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Abstract. An object in motivic homotopy theory is called cellular if it can be built out of motivic spheres using homotopy colimit constructions. We explore some examples and consequences of cellularity. We explain why the algebraic $K$-theory and algebraic cobordism spectra are both cellular, and prove some Künneth theorems for cellular objects.

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## 1. Introduction

If $\mathcal{M}$ is a model category, and $\mathcal{A}$ is a set of objects in $\mathcal{M}$, one can consider the class of **$\mathcal{A}$-cellular objects**—things that can be built from the homotopy types in $\mathcal{A}$ by iterative homotopy colimit constructions. In the case of topological spaces such cellular classes have been studied by Farjoun [DF] and others. Another place these ideas have appeared is in the work of Dwyer, Greenlees, and Iyengar [DGI], who imported them into homological algebra. This paper is concerned with cellularity in motivic homotopy theory.

Recall that in the motivic context there is a bi-graded family of 'spheres' $S^{p,q}$; we will take this family as our set $\mathcal{A}$. One gets a slightly different theory depending on whether one works unstably or stably. In this paper we develop the basic theory concerning cellular objects in both contexts, and collect an assortment of results which we've found useful in applications. Specifically:

(1) We describe a collection of techniques for showing that schemes are cellular, and apply these to toric varieties, Grassmannians, Stiefel manifolds, and certain quadrics.
(2) We show that the algebraic $K$-theory spectrum $KGL$ and the motivic cobordism spectrum $MGL$ are stably cellular.

(3) For cellular objects, the usual collection of tools for computing carries over from ordinary stable homotopy theory to motivic stable homotopy theory (see Section 7). If $E$ is a motivic ring spectrum then we use these ideas to construct a convergent, tri-graded, Künneth spectral sequence for $E^{*,*}(X \times Y)$ as long as $X$ or $Y$ satisfies some kind of cellularity condition. See Theorems 8.6 and 8.12.

Our experience has been that this material is a good starting point for understanding some of the inner workings of motivic homotopy theory, and so we have tried to make the paper readable to people who only know the basic definitions from [MV].

1.1. **Notions related to cellularity.** Algebraic geometers have worked with the related notion of a scheme with an *algebraic cell decomposition* [F2, Ex. 1.9.1], and its generalization to that of a *linear variety* (introduced by Totaro [T]). A scheme $X$ has an algebraic cell decomposition if it has a filtration by closed subschemes $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq \emptyset$ such that each complement $X_{i+1} - X_i$ is a disjoint union of affines $\coprod_{n_{ij}} \mathbb{A}^{n_{ij}}$. The linear varieties constitute the smallest class which contains the affine spaces $\mathbb{A}^n$ and has the property that if $Z \hookrightarrow X$ is a closed inclusion and at least two of the varieties $Z$, $X$, and $X - Z$ are in the class, then so is the third.

These notions are useful when studying cohomology theories which have a *localization* (or *Gysin*) sequence, because the cohomology of a linear variety can be understood inductively. In the language of motivic homotopy theory these are the *algebraically oriented* cohomology theories, i.e., the ones that have Thom isomorphisms. This means that the cohomology of the Thom space of a bundle over $Z$ is isomorphic to the cohomology of $Z$ (up to a shift). If $Z \hookrightarrow X$ is a closed inclusion of smooth schemes then there is a homotopy cofiber sequence of the form $X - Z \to X \to \operatorname{Th} N_{X/Z}$ [MV, Thm. 3.2.23], where $\operatorname{Th} N_{X/Z}$ is the Thom space of the normal bundle of $Z$ in $X$. One gets a long exact sequence relating the cohomology of $X - Z$, $X$, and $Z$.

Our class of stably cellular varieties is very close to the class of linear varieties. For every linear variety which we've tried to prove is stably cellular, we've been able to do so (and vice versa); however, proving that something is cellular is often much harder. This is true for the Grassmannians $\mathrm{Gr}_k(\mathbb{A}^n)$, for instance. The Schubert cells give an 'algebraic cell decomposition' showing that the Grassmannian is linear, but to show the variety is cellular it is not enough just to see the cells inside the variety: one has to produce an 'attaching map' showing explicitly how to build up the variety via homotopy colimits. In the Schubert cell approach one runs into some hairy problems in trying to make this work, which we have not been able to resolve. Our proof that Grassmannians are cellular follows a completely different strategy.

If one is only interested in cohomology theories with Thom isomorphism, then perhaps there is no reason for studying cellular varieties as opposed to linear ones. But the notion of 'cellular' seems more familiar and sensible to a topologist, and most of our techniques for understanding the classical stable homotopy category depend in some way on things being built from cells. As those techniques get imported into motivic homotopy theory, the notion of cellularity may become more useful.

1.2. **Non-cellular varieties.** Folklore says most schemes cannot be cellular. This should be a consequence of the theory of weights in the cohomology of algebraic varieties [De]. The spheres $S^{p,q}$ only have even weights in their cohomology, and so it should be impossible to construct something with odd weights (like an elliptic curve, for instance) from the spheres.

Unfortunately, to write down a careful proof that an elliptic curve is not cellular seems to require surmounting some obstacles. One possibility is to work over $\mathbb{C}$ and use mixed Hodge theory, but this requires showing that the mixed Hodge structures are well-defined invariants of the motivic stable homotopy category. This takes at least a little work, due to the presence of infinite objects (like infinite wedges of schemes, etc.) in the stable homotopy category.

Another possibility is to work over a number field $k$, and to use the weights coming from the Galois actions on $l$-adic cohomology (again see [De]). Here, one should show that there is a realization functor from the motivic stable homotopy category to the derived category of $Gal(\bar{k}/k)$-modules. Proposition 9.6 shows that if an elliptic curve $E$ is cellular then it can actually be built from spheres using a finite number of extensions and retracts, and so the argument with weights should work out. We will not pursue these ideas further in this paper.

## 2. Cellular objects

Let $\mathcal{M}$ be a pointed model category, and let $\mathcal{A}$ be a set of objects in $\mathcal{M}$.

**Definition 2.1.** *The class of* ***$\mathcal{A}$-cellular*** *objects is the smallest class of objects of $\mathcal{M}$ such that*

*(1) every object of $\mathcal{A}$ is $\mathcal{A}$-cellular;*
*(2) if $X$ is weakly equivalent to an $\mathcal{A}$-cellular object, then $X$ is $\mathcal{A}$-cellular;*
*(3) if $D\colon I \to \mathcal{M}$ is a diagram such that each $D_i$ is $\mathcal{A}$-cellular, then so is* $\operatorname{hocolim} D$.

The idea is that the $\mathcal{A}$-cellular objects are the ones that can, up to homotopy, be built out of objects in $\mathcal{A}$. This definition is precisely the same as the usual notion of cellularity for the category of pointed topological spaces [DF, Ex. 2.D.2.1].

We recall some useful results about cellularity in general. These properties apply to all possible $\mathcal{M}$ and $\mathcal{A}$. To start with, note that any contractible object of $\mathcal{M}$ (i.e., an object weakly equivalent to the initial and terminal object $*$) is $\mathcal{A}$-cellular because it is the homotopy colimit of an empty diagram.

**Lemma 2.2.** *If $X \to Y \to Z$ is a homotopy cofiber sequence in $\mathcal{M}$ such that $X$ and $Y$ are $\mathcal{A}$-cellular, then so is $Z$.*

*Proof.* The object $Z$ is the homotopy pushout of the diagram $* \leftarrow X \to Y$. □

**Lemma 2.3.** *If $X$ is $\mathcal{A}$-cellular, then so is $\Sigma X$.*

*Proof.* Apply Lemma 2.2 to the homotopy cofiber sequence $X \to * \to \Sigma X$. □

Recall that a pointed model category is **stable** if the suspension and loops functors $\Sigma$ and $\Omega$ form a self-equivalence of the homotopy category. Throughout the paper we will abuse notation and also write $\Sigma$ (resp., $\Omega$) for a chosen derived functor of suspension (resp., loops) on the model category level. For instance, to compute $\Sigma X$ we can factor $X \to *$ as a cofibration followed by trivial fibration $X \rightarrowtail CX \xrightarrow{\sim} *$ and then take $CX/X$.

**Lemma 2.4.** *Suppose that $\mathcal{M}$ is a stable model category. Also suppose that for every object $A$ of $\mathcal{A}$, $\Omega A$ is weakly equivalent to an object of $\mathcal{A}$. An object $X$ in $\mathcal{M}$ is $\mathcal{A}$-cellular if and only if $\Sigma X$ is $\mathcal{A}$-cellular.*

*Proof.* Consider the class $\mathcal{C}$ of objects $X$ of $\mathcal{M}$ such that $\Omega X$ is cellular; we want to show that $\mathcal{C}$ coincides with the class of $\mathcal{A}$-cellular objects. By Lemma 2.3, $\mathcal{C}$ is contained in the class of $\mathcal{A}$-cellular objects. To show that $\mathcal{C}$ contains the class of $\mathcal{A}$-cellular objects, it suffices to check that $\mathcal{C}$ has the three properties listed in Definition 2.1.

Property (1) follows immediately from the hypothesis of the lemma, property (2) follows immediately from the fact that $\Omega$ respects weak equivalences, and property (3) follows from the fact that $\Omega$ respects homotopy colimits in a stable model category. $\square$

**Lemma 2.5.** *Suppose that $\mathcal{M}$ is a stable model category. Also suppose that for every object $A$ of $\mathcal{A}$, $\Omega A$ is weakly equivalent to an object of $\mathcal{A}$. If $X \to Y \to Z$ is a homotopy cofiber sequence in $\mathcal{M}$ such that any two of $X$, $Y$, and $Z$ are $\mathcal{A}$-cellular, then so is the third.*

*Proof.* One case is Lemma 2.2. For the other two cases, first observe from Lemma 2.4 that $\Omega Y$ is $\mathcal{A}$-cellular whenever $Y$ is (and similarly for $\Omega Z$). Now, the object $Y$ is the homotopy pushout of the diagram $* \leftarrow \Omega Z \to X$, and the object $X$ is the homotopy pushout of the diagram $* \leftarrow \Omega Y \to \Omega Z$. $\square$

2.6. **The motivic setting.** Let $\mathcal{MV}$ denote the Morel-Voevodsky model category on simplicial presheaves over $Sm_k$, the site of smooth schemes of finite type over some fixed ground field $k$. In fact, there are at least two versions of this model category, the injective [MV], [Ja, App. B] and projective [Bl]. The identity functors give a Quillen equivalence between these model categories (which have the same class of weak equivalences), and this guarantees that it doesn't matter which model structure is considered for the purposes of cellularity. Thus, in each situation we can choose whichever structure is more convenient. Unless otherwise stated, we will use the injective version because it is convenient to have all objects cofibrant. Actually, we will work with the pointed category $\mathcal{MV}_*$, in which every object is equipped with a map from $\mathrm{Spec}(k)$.

We recall the following two important kinds of weak equivalences in $\mathcal{MV}$. First, if $\{U, V\}$ is a basic Nisnevich cover of $X$ [MV, Defn. 3.1.3], then the map

$$U \amalg_{U \times_X V} V \to X$$

is an $\mathbb{A}^1$-weak equivalence. Second, if $X$ is any object of $Sm_k$, then the map $X \times \mathbb{A}^1 \to X$ is an $\mathbb{A}^1$-weak equivalence. In a certain sense, these two kinds of maps generate all $\mathbb{A}^1$-weak equivalences—cf. [D, Sec. 8.1], especially the paragraph following the proof of (8.1). Every proof that an object is cellular must necessarily come down to using these two facts.

Actually, in this paper we will never have to explicitly use the Nisnevich topology—all our arguments only involve Zariski covers and facts from [MV]. Also, the $\mathbb{A}^1$-weak equivalence for two-fold covers given in the last paragraph implies a corresponding statement for Zariski covers of any size: the $\mathbb{A}^1$-homotopy type of a scheme can be recovered from a Zariski cover by an appropriate gluing construction. This is what we will mostly use (see Lemma 3.6 below for a precise statement).

Recall that $S^{1,1} = \mathbb{A}^1 - 0$, with the point 1 as basepoint; and $S^{1,0}$ is the constant simplicial presheaf whose sections are the simplicial set $\Delta^1/\partial\Delta^1$. For $p \geq q \geq 0$, one defines

$$S^{p,q} = (S^{1,0} \wedge \cdots \wedge S^{1,0}) \wedge (S^{1,1} \wedge \cdots \wedge S^{1,1})$$

where there are $p - q$ copies of $S^{1,0}$ and $q$ copies of $S^{1,1}$.

**Definition 2.7.** *Let $\mathcal{A} = \{S^{p,q} \,|\, p \geq q \geq 0\}$ be the set of spheres in $\mathcal{MV}_*$. An object $X$ in $\mathcal{MV}_*$ is **unstably cellular** if $X$ is $\mathcal{A}$-cellular.*

If $X$ is a pointed (possibly non-smooth) scheme, then the statement "$X$ is unstably cellular" is to be interpreted as saying that the object of $\mathcal{MV}_*$ represented by $X$ is unstably cellular.

2.8. **Motivic spectra.** We let $\mathcal{S}pectra(\mathcal{MV})$ denote the category of symmetric $\mathbb{P}^1$-spectra. Starting with the injective model structure on $\mathcal{MV}_*$, we get an induced model structure on $\mathcal{S}pectra(\mathcal{MV})$ from [Ho2, Defn. 8.7]. This turns out to be identical to the one provided by [Ja, Thm. 4.15]. Note that there is a Quillen pair $\Sigma^\infty \colon \mathcal{MV}_* \rightleftarrows \mathcal{S}pectra(\mathcal{MV}) \colon \Omega^\infty$. We may desuspend the objects $\Sigma^\infty(S^{p,q})$ in both variables, giving spectra which we will denote $S^{p,q}$ for all $p, q \in \mathbb{Z}$.

**Remark 2.9.** One can also use Bousfield-Friedlander $\mathbb{P}^1$-spectra rather than symmetric spectra. The model structure is provided by [Ho2, Defn. 3.3] or [Ja, Thm. 2.9]. Since this model category is Quillen equivalent to that of symmetric $\mathbb{P}^1$-spectra [Ho2, Sec. 10], all our basic results hold in either one. The smash product for symmetric spectra will be important in Sections 7 and 8, however.

**Definition 2.10.** *Let $\mathcal{B} = \{S^{p,q} \,|\, p, q \in \mathbb{Z}\}$ be the class of all spheres in $\mathcal{S}pectra(\mathcal{MV})$. An object $X$ of $\mathcal{S}pectra(\mathcal{MV})$ is **cellular** if $X$ is $\mathcal{B}$-cellular. We say that an object $X$ in $\mathcal{MV}_*$ is **stably cellular** if $\Sigma^\infty X$ is cellular.*

Again, the statement that a (possibly non-smooth) *pointed scheme* is 'stably cellular' is really an abbreviation for the same statement about the object of $\mathcal{MV}_*$ that it represents.

**Example 2.11.** $\mathbb{A}^n - 0$ is unstably cellular, because after choosing any basepoint, the variety $\mathbb{A}^n - 0$ is weakly equivalent to the sphere $S^{2n-1,n}$. This fact is claimed in [MV, Ex. 3.2.20]. For the convenience of the reader, we give a detailed explanation.

For $n = 1$ this is the definition that $S^{1,1}$ equals $\mathbb{A}^1 - 0$. For $n = 2$, we cover $\mathbb{A}^2 - 0$ by the open sets $U = (\mathbb{A}^1 - 0) \times \mathbb{A}^1$ and $V = \mathbb{A}^1 \times (\mathbb{A}^1 - 0)$. Then $U \cap V = (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0)$, and we have an associated homotopy pushout square

$$\begin{array}{ccc} (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \longrightarrow & (\mathbb{A}^1 - 0) \times \mathbb{A}^1 \\ \downarrow & & \downarrow \\ (\mathbb{A}^1 - 0) \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 - 0. \end{array}$$

By projecting away the $\mathbb{A}^1$ factors, we find that $\mathbb{A}^2 - 0$ is weakly equivalent to the homotopy pushout of

$$\mathbb{A}^1 - 0 \xleftarrow{\pi_1} (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \xrightarrow{\pi_2} (\mathbb{A}^1 - 0).$$

In order to compute this homotopy pushout, we look at the diagram

$$\begin{array}{ccccc} * & \longleftarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{A}^1 - 0 & \longleftarrow & (\mathbb{A}^1 - 0) \vee (\mathbb{A}^1 - 0) & \longrightarrow & \mathbb{A}^1 - 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1 - 0 & \longleftarrow & (\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) & \longrightarrow & \mathbb{A}^1 - 0 \end{array}$$

in which the two middle horizontal arrows collapse one summand to a point, and compute the homotopy colimit in two ways. If we first compute the homotopy colimits of the rows and then the homotopy colimit of this new diagram, we get the desired homotopy pushout. On the other hand, if we first compute the homotopy colimits of the vertical columns, then we get

$$* \longleftarrow (\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0) \longrightarrow *,$$

and then the homotopy colimit of this new diagram is $\Sigma((\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0))$. Since we must get the same homotopy type no matter which way we go about computing the homotopy colimit, it must be that the desired homotopy pushout is $\Sigma[(\mathbb{A}^1 - 0) \wedge (\mathbb{A}^1 - 0)]$, which is $S^{3,2}$.

For arbitrary $n$ one proceeds by induction, covering $\mathbb{A}^n - 0$ by $(\mathbb{A}^{n-1} - 0) \times \mathbb{A}^1$ and $\mathbb{A}^n \times (\mathbb{A}^1 - 0)$ and using the same argument to see that $\mathbb{A}^n - 0$ is weakly equivalent to $\Sigma[(\mathbb{A}^{n-1} - 0) \wedge (\mathbb{A}^1 - 0)]$.

**Example 2.12.** According to [MV, Cor. 3.2.18], there is a cofiber sequence

$$\mathbb{P}^{n-1} \to \mathbb{P}^n \to S^{2n,n}$$

in $\mathcal{MV}_*$ after choosing any basepoint for $\mathbb{P}^{n-1}$. This shows inductively that $\mathbb{P}^n$ is stably cellular. We will show below that $\mathbb{P}^n$ is in fact unstably cellular.

For $n \geq 1$, there is a canonical projection map $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$ sending a point $v$ to the line spanned by $v$. Also, there is a natural inclusion $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ coming from the inclusion $\mathbb{A}^{n-1} \hookrightarrow \mathbb{A}^{n-1} \oplus \mathbb{A}^1 = \mathbb{A}^n$.

**Proposition 2.13.** *For $n \geq 1$, there is a homotopy cofiber sequence $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ in $\mathcal{MV}_*$ after choosing any basepoint in $\mathbb{A}^n - 0$. Thus each $\mathbb{P}^n$ is unstably cellular.*

*Proof.* We decompose $\mathbb{A}^{n+1}$ into $\mathbb{A}^n \oplus \mathbb{A}^1$. Let $l$ be the line spanned by the vector $(\mathbf{0}, 1)$ (where the notation is with respect to this decomposition), and let $U = \mathbb{P}^n - \{l\}$. There is an open embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ which sends $\mathbf{v}$ to the line spanned by $(\mathbf{v}, 1)$—call this open subset $V$. Then $U \cap V$ is isomorphic to $\mathbb{A}^n - 0$, and we have a homotopy pushout square

$$\begin{array}{ccc} \mathbb{A}^n - 0 & \longrightarrow & \mathbb{P}^n - \{l\} \\ \downarrow & & \downarrow \\ \mathbb{A}^n & \longrightarrow & \mathbb{P}^n. \end{array}$$

Since $\mathbb{A}^n$ is contractible, this identifies $\mathbb{P}^n$ with the homotopy cofiber of $\mathbb{A}^n - 0 \to \mathbb{P}^n - \{l\}$.

Now, there is a projection map $\mathbb{P}^n - \{l\} \to \mathbb{P}^{n-1}$ induced by the obvious projection $\mathbb{A}^{n-1} \oplus \mathbb{A}^1 \to \mathbb{A}^{n-1}$. This is a line bundle, and is therefore a weak equivalence (in fact, an $\mathbb{A}^1$-homotopy equivalence). The composite $\mathbb{A}^n - 0 \hookrightarrow \mathbb{P}^n - \{l\} \to \mathbb{P}^{n-1}$ is precisely our projection map $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$. So the homotopy cofiber of $\mathbb{A}^n - 0 \to \mathbb{P}^n - \{l\}$ is weakly equivalent to the homotopy cofiber of $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$. □

**Remark 2.14.** Note that the homotopy cofiber sequence $\mathbb{A}^1 - 0 \to * \to \mathbb{P}^1$ identifies $\mathbb{P}^1$ with $\Sigma(\mathbb{A}^1 - 0) \simeq \Sigma(S^{1,1}) \simeq S^{2,1}$.

2.15. **Basepoints.** When working with unstable cellularity one has to be a little careful about basepoints. Here is one case where the issue disappears:

**Proposition 2.16.** *Suppose $X$ is an object of $\mathcal{MV}$ and $a, b\colon * \to X$ are two choices of basepoint. If $a$ and $b$ are weakly homotopic in $\mathcal{MV}$, then $(X, a)$ is weakly equivalent to $(X, b)$ in $\mathcal{MV}_*$ (hence one is unstably cellular if and only if the other is).*

*Proof.* First, one readily reduces to the case where $X$ is fibrant. Let $X'$ be the pushout in $\mathcal{MV}$ of the diagram

$$X \xleftarrow{a} * \xrightarrow{i_0} \mathbb{A}^1$$

where $i_0$ is the inclusion of $\{0\}$; the idea is that $X'$ is $X$ with an 'affine whisker' attached. The homotopy $H : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ that takes $(s,t)$ to $st$ can be used to show that the map $X' \to X$ which collapses the $\mathbb{A}^1$ onto $a$ is a weak equivalence in $\mathcal{MV}$. If we regard $X'$ as pointed via the map $1 \to \mathbb{A}^1$, then the same map is a weak equivalence $(X', 1) \to (X, a)$ in $\mathcal{MV}_*$.

Since $a$ and $b$ are weakly homotopic and $X$ is fibrant, there is a map $H\colon \mathbb{A}^1 \to X$ such that $Hi_0 = a$ and $Hi_1 = b$. This induces a map $X' \to X$ sending 1 to $b$, which is readily checked to be a weak equivalence. So we have a zig-zag of weak equivalences in $\mathcal{MV}_*$ of the form $(X, a) \leftarrow (X', 1) \to (X, b)$. □

The applicability of the above result is limited by the fact that in motivic homotopy theory $\mathrm{Ho}(*, X)$ is often bigger than one would expect. For instance $\mathrm{Ho}(*, \mathbb{A}^1 - 0) = k^*$, which is extremely big if $k$ is the complex numbers. For $\mathbb{A}^1 - 0$ the basepoint doesn't matter for another reason, namely because all choices are equivalent up to automorphism.

It turns out that we will almost always work with stable cellularity, and in this context the basepoint can be ignored:

**Proposition 2.17.** *If $X \in \mathcal{MV}_*$, then $X$ is stably cellular if and only if $X_+$ is stably cellular. As a consequence, if $X \in \mathcal{MV}$ has two basepoints $x_0$ and $x_1$ then $(X, x_0)$ is stably cellular if and only if $(X, x_1)$ is stably cellular.*

*Proof.* The first statement follows from considering the cofiber sequence $\Sigma^\infty S^{0,0} \to \Sigma^\infty(X_+) \to \Sigma^\infty X$ and applying Lemma 2.5. The second statement is true because either condition is equivalent to $X_+$ being stably cellular. □

Because of Proposition 2.17 we can now rephrase the definition of stable cellularity.

**Definition 2.18.** *A (pointed or unpointed) object $X$ of $\mathcal{MV}$ is **stably cellular** if $\Sigma^\infty(X_+)$ is stably cellular in $\mathcal{S}pectra(\mathcal{MV})$.*

This is the definition that we will use from now on. It saves us the trouble of having to worry about basepoints.

## 3. Basic results

The notions of unstable cellularity and stable cellularity are related by the following lemma.

**Lemma 3.1.** *If $X$ is an unstably cellular object of $\mathcal{MV}_*$, then it is also stably cellular.*

*Proof.* The functor $\Sigma^\infty$ is a left Quillen functor, so it respects weak equivalences and homotopy colimits. Thus, it suffices to show that $\Sigma^\infty S^{p,q}$ is stably cellular. But this is isomorphic to the stable sphere $S^{p,q}$, which is stably cellular by definition. □

We will now study the basic constructions that behave well with respect to cellularity.

**Lemma 3.2.** *If each $X_i$ is a stably cellular object of $\mathcal{MV}$, then so is $X = \coprod_i X_i$.*

*Proof.* This follows from the simple calculation that $\Sigma^\infty(X_+)$ is isomorphic to $\bigvee_i \Sigma^\infty(X_{i+})$. □

The set of spheres is closed under smash product. This implies that smash products preserve unstably cellular objects.

**Lemma 3.3.** *If $X$ and $Y$ are unstably cellular objects, then so is $X \wedge Y$.*

*Proof.* Since every object in $\mathcal{MV}_*$ is injective cofibrant, the functor $X \wedge (-)$ is a left Quillen functor from the pointed category $\mathcal{MV}_*$ to itself, so it respects homotopy colimits and weak equivalences. Thus, it suffices to show that $X \wedge S^{p,q}$ is unstably cellular for every $p$ and $q$.

But the functor $(-) \wedge S^{p,q}$ also respects homotopy colimits and weak equivalences, so it suffices to show that $S^{s,t} \wedge S^{p,q}$ is unstably cellular. This is isomorphic to $S^{s+p,t+q}$, which is unstably cellular by definition. □

Note, in particular, that if $X$ is a pointed unstably cellular object, then so is $\Sigma^{p,q} X$.

**Lemma 3.4.** *If $X$ and $Y$ are stably cellular objects of $\mathcal{MV}$, then so are $X \wedge Y$ and $X \times Y$.*

*Proof.* The proof for $X \wedge Y$ works just as in Lemma 3.3, using the facts that $\Sigma^\infty(X \wedge Y)$ is weakly equivalent to $\Sigma^\infty X \wedge \Sigma^\infty Y$ and that every suspension spectrum is cofibrant.

For $X \times Y$, there is an unstable cofiber sequence $X \vee Y \to X \times Y \to X \wedge Y$, so we also have a stable cofiber sequence

$$\Sigma^\infty(X \vee Y) \to \Sigma^\infty(X \times Y) \to \Sigma^\infty(X \wedge Y).$$

We just showed that the third term is stably cellular. The first term is isomorphic to $\Sigma^\infty X \vee \Sigma^\infty Y$, which is a homotopy colimit of stably cellular spectra and thus also stably cellular. Hence, the second term is stably cellular as well. □

**Example 3.5.** By Lemma 3.4, the torus $(\mathbb{A}^1 - 0)^k$ is stably cellular for every $k$. We do not know if tori are unstably cellular, but we suspect not. Throughout the rest of the paper we will find that products arise all over the place, which is why we end up working primarily with stable cellularity.

**Lemma 3.6.** *If $X$ is a scheme and $U_* \to X$ is a hypercover in $\mathcal{MV}$ in the sense of* [SGA4, Defn. 7.3.1.2] *(or* [DHI, Defn. 4.2]*) and each $U_n$ is stably cellular, then $X$ is also stably cellular. If the hypercover is in $\mathcal{MV}_*$ and each $U_n$ is unstably cellular, so is $X$.*

*Proof.* This follows simply from the fact that the map $\operatorname{hocolim}_n U_n \to X$ is a weak equivalence in $\mathcal{MV}$ [DHI, Thm. 6.2]. Then $\operatorname{hocolim}_n \Sigma^\infty(U_{n+}) \to \Sigma^\infty(X_+)$ is also a weak equivalence. □

Note that if $X$ is a scheme and $\{U_i\}$ a Zariski open cover of $X$, then the associated Čech complex is a hypercover in the above sense. It is not necessary that $X$ be smooth here.

**Definition 3.7.** *A Zariski cover $\{U_\alpha\}$ of a scheme $X$ is* ***completely stably cellular*** *if each intersection $U_{\alpha_1\cdots\alpha_n} = U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$ is stably cellular.*

A similar definition can be made with the notion of unstably cellular, but we will not bother with it.

**Lemma 3.8.** *If a variety $X$ has a Zariski cover which is completely stably cellular, then $X$ is also stably cellular.*

*Proof.* Let $\{U_\alpha\}$ be the cover of $X$. Consider the Čech complex $C_*$ of this cover, which is a simplicial diagram of varieties such that $C_n$ is $\coprod U_{\alpha_0\alpha_1\cdots\alpha_n}$. Now $C_*$ is obviously a hypercover of $X$ in $\mathcal{MV}$ (cf. [DHI, 3.4, 4.2]). Each $C_n$ is stably cellular by Lemma 3.2, so Lemma 3.6 applies. □

By an 'algebraic fiber bundle with fiber $F$' we mean a map $E \to B$ which in the Zariski topology is locally isomorphic to a projection $U \times F \to U$.

**Lemma 3.9.** *If $p\colon E \to B$ is an algebraic fiber bundle with fiber $F$ such that $F$ is stably cellular and $B$ has a completely stably cellular cover that trivializes the bundle, then $E$ is also stably cellular.*

*Proof.* Let $\{U_0, \ldots, U_k\}$ be the completely stably cellular cover of $B$. Consider the cover $\{V_0, \ldots V_k\}$ of $E$, where $V_i = p^{-1}U_i$. Each $V_i$ is isomorphic to $F \times U_i$, so it is stably cellular by Lemma 3.4. Moreover, the intersections $V_{i_0\cdots i_n}$ are isomorphic to $F \times U_{i_0\cdots i_n}$, so this cover of $E$ is completely stably cellular. Lemma 3.8 finishes the proof. □

**Corollary 3.10.** *If $p\colon E \to B$ is an algebraic vector bundle such that $B$ has a completely stably cellular cover that trivializes the bundle, then the Thom space $\mathrm{Th}(p)$ is also stably cellular.*

*Proof.* Let $s\colon B \to E$ be the zero section of the vector bundle. From the definition of a Thom space [MV, Defn. 3.2.16], we have an unstable cofiber sequence

$$E - s(B) \to E \to \mathrm{Th}(p),$$

where $s(B)$ is the zero section of the bundle. So we just have to show that the first two terms in this sequence are stably cellular. First, $E$ is weakly equivalent to $B$ (because $E$ can be fiberwise linearly contracted onto its zero section), and $B$ is stably cellular by Lemma 3.8.

Next, $E - s(B) \to B$ is an algebraic fiber bundle with fiber $\mathbb{A}^n - 0$. By Lemma 3.9, we know that $E - s(B)$ is stably cellular provided that $\mathbb{A}^n - 0$ is. Recall that $\mathbb{A}^n - 0$ is weakly equivalent to $S^{2n-1,n}$, so it is stably cellular. □

**Lemma 3.11.** *If $x$ is a closed point of a smooth variety $X$, then $X$ is stably cellular if and only if $X - \{x\}$ is stably cellular.*

*Proof.* The homotopy purity theorem [MV, Thm. 3.2.23] tells us that there is a cofiber sequence in $\mathcal{MV}$ of the form

$$X - \{x\} \to X \to \mathrm{Th}(p),$$

where $p$ is the normal bundle of $\{x\}$ in $X$. Now $\mathrm{Th}(p)$ is just $\mathbb{A}^n/(\mathbb{A}^n - 0) \simeq S^{2n,n}$, where $n$ is the dimension of $X$. Thus we have a cofiber sequence

$$X - \{x\} \to X \to S^{2n,n},$$

in which the third term is stably cellular. Lemma 2.5 finishes the proof. □

**Remark 3.12.** Suppose that $Z \hookrightarrow X$ is a closed inclusion of smooth schemes. The homotopy purity theorem gives a cofiber sequence $X - Z \to X \to \mathrm{Th}\, N_{X/Z}$. It is tempting to conclude that $\mathrm{Th}\, N_{X/Z}$ is cellular if $Z$ is cellular, and so $X$ is stably cellular if both $Z$ and $X - Z$ are stably cellular. Unfortunately we haven't been able to prove the first implication, only the weaker version in Corollary 3.10. This weakness is the main reason that proving a variety is cellular often feels like more work than it should be; in particular it is what causes trouble with Grassmannians in the next section.

## 4. Grassmannians and Stiefel varieties

**Proposition 4.1.** *The variety $GL_n$ is stably cellular for every $n \geq 1$.*

*Proof.* The proof is by induction on $n$. Let $l$ be the line spanned by $(1, 0, \ldots, 0)$ in $\mathbb{A}^n$. There is a fibre bundle $GL_n \to \mathbb{P}^{n-1}$ that takes $A$ to the line $A(l)$ (where we have $GL_n$ acting on $\mathbb{A}^n$ from the left). The fiber over $[1, 0 \ldots, 0]$ is the parabolic subgroup $P$ consisting of all invertible $n \times n$ matrices $(a_{ij})$ such that $a_{j1} = 0$ for $j \geq 2$. As a variety (but not as a group), $P$ is isomorphic to $(\mathbb{A}^1 - 0) \times \mathbb{A}^{n-1} \times GL_{n-1}$, which is stably cellular by induction and Lemma 3.4. The usual cover of $\mathbb{P}^n$ by affines $\mathbb{A}^n$ is a completely cellular trivializing cover for the bundle, so Lemma 3.8 applies. □

Recall that the Grassmannian $\mathrm{Gr}_k(\mathbb{A}^n)$ is the variety of $k$-planes in $\mathbb{A}^n$. Also, the Stiefel variety $V_k(\mathbb{A}^n)$ consists of all ordered sets of $k$ linearly independent vectors in $\mathbb{A}^n$. These objects are connected by a fiber bundle $V_k(\mathbb{A}^n) \to \mathrm{Gr}_k(\mathbb{A}^n)$ that takes a set of $k$ linearly independent vectors to the $k$-plane that it spans. The fiber of this bundle is $GL_k$.

**Proposition 4.2.** *For all $n \geq k \geq 0$, the Stiefel variety $V_k(\mathbb{A}^n)$ is stably cellular.*

*Proof.* Consider the fiber bundle $p : V_k(\mathbb{A}^n) \to \mathbb{A}^n - 0$ that takes the first vector in an ordered set of $k$ linearly independent vectors. The fiber of this bundle is $\mathbb{A}^{k-1} \times V_{k-1}(\mathbb{A}^{n-1})$, which we may assume by induction is stably cellular. Because of Lemma 3.9, it suffices to find a completely stably cellular cover of $\mathbb{A}^n - 0$ that trivializes the bundle.

For $1 \leq i \leq n$, let $U_i$ be the open set of $\mathbb{A}^n - 0$ consisting of all vectors $(x_1, \ldots, x_n)$ such that $x_i \neq 0$. The intersections of these open sets are of the form $(\mathbb{A}^1 - 0)^k \times \mathbb{A}^{n-k}$, so they are stably cellular.

It remains to show that the bundle is trivial over $U_i$. Without loss of generality, it suffices to consider $U_1$. We regard $\mathbb{A}^{n-1}$ as the subset of $\mathbb{A}^n$ consisting of vectors whose first coordinate is zero. Let $f\colon U_1 \times V_{k-1}(\mathbb{A}^{n-1}) \times \mathbb{A}^{k-1} \to p^{-1}U_1$ be the map

$$(\mathbf{x}, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}, t_1, \ldots, t_{k-1}) \mapsto (\mathbf{x}, t_1\mathbf{x} + \mathbf{v}_1, t_2\mathbf{x} + \mathbf{v}_2, \ldots, t_{k-1}\mathbf{x} + \mathbf{v}_{k-1}).$$

One readily checks that this is an isomorphism. □

Let $G$ be an algebraic group. By a **principal $G$-bundle** we mean an algebraic fiber bundle $\pi\colon E \to B$ together with an action $E \times G \to E$, such that $\pi(eg) = \pi(e)$ and the induced map $E \times G \to E \times_B E$ sending $(e, g) \to (e, eg)$ is an isomorphism.

**Proposition 4.3.** *If $E \to B$ is a principal $G$-bundle where both $E$ and $G$ are stably cellular, then so is $B$.*

*Proof.* Let $C_*$ be the Čech complex of the bundle. This means that $C_m$ is the fiber product $E \times_B E \times_B \cdots \times_B E$ ($m+1$ copies of $E$). Because fiber bundles are locally split, $C_*$ is a hypercover of $B$ (cf. [DHI, 3.4, 4.2]). Using Lemma 3.6, we just need to show that each $C_m$ is stably cellular.

From the definition of a principal bundle, $C_m$ is isomorphic to $E \times G^m$, which is stably cellular by Lemma 3.4. □

**Proposition 4.4.** *For all $n \geq k \geq 0$, the Grassmannian variety $Gr_k(\mathbb{A}^n)$ is stably cellular.*

*Proof.* Consider the fiber bundle $V_k(\mathbb{A}^n) \to \mathrm{Gr}_k(\mathbb{A}^n)$. This is a principal $G$-bundle with $G = GL_k$. Thus, Proposition 4.3 applies because of Propositions 4.1 and 4.2. □

**Remark 4.5.** One might also try to prove that Grassmannians are cellular by using the Schubert cell decomposition. There are various approaches to this, and as far as we know all of them run into unpleasant problems. One possibility, for instance, is to consider the standard open cover of $\mathrm{Gr}_k(\mathbb{A}^n)$ by affines $\mathbb{A}^{k(n-k)}$—these are precisely the top-dimensional open Schubert cells. If the finite intersections of these opens are all cellular, then so is $\mathrm{Gr}_k(\mathbb{A}^n)$ by Lemma 3.8. Unfortunately these finite intersections become complicated, and we have only managed to prove they are cellular for $\mathrm{Gr}_1(\mathbb{A}^n)$ and $\mathrm{Gr}_2(\mathbb{A}^n)$. The general case remains an intriguing open question.

Our proof that Grassmannians are cellular generalizes easily to the case of flag varieties. Given integers $0 \leq d_1 < d_2 < \cdots < d_k \leq n$, let $\mathrm{Fl}(d_1, \ldots, d_k; n)$ denote the variety of flags $V_1 \subseteq \cdots \subseteq V_k \subseteq \mathbb{A}^n$ such that $\dim V_i = d_i$.

**Proposition 4.6.** *The flag variety $Fl(d_1, \ldots, d_k; n)$ is stably cellular.*

*Proof.* Write $\mathrm{Fl} = \mathrm{Fl}(d_1, \ldots, d_k; n)$. There is an algebraic fiber bundle $V_{d_k}(\mathbb{A}^n) \to \mathrm{Fl}$ taking an ordered set of $k$ linearly independent vectors to the flag whose $i$th space is spanned by the first $d_i$ vectors. This is a principal $G$-bundle, where $G$ is the subgroup of $GL_n$ consisting of matrices in block form

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,k+1} \\ 0 & A_{22} & \cdots & A_{1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{k+1,k+1} \end{bmatrix}$$

where $A_{11} \in GL_{d_1}$, $A_{ii} \in GL_{d_i - d_{i-1}}$, and $A_{k+1,k+1} \in GL_{n-d_k}$. As a variety $G$ is isomorphic to $GL_{d_1} \times GL_{d_2-d_1} \times \cdots \times GL_{d_k-d_{k-1}} \times GL_{n-d_k} \times \mathbb{A}^N$ for some number $N$. So $G$ is stably cellular, hence Fl is stably cellular by Proposition 4.3. □

## 5. Other examples of cellular varieties

5.1. **Toric varieties.** We will show that every toric variety is weakly equivalent in $\mathcal{MV}$ to a homotopy colimit of copies of tori $T^m = (\mathbb{A}^1 - 0)^m$. Since the tori are stably cellular by Lemma 3.4, so are toric varieties. This result is almost trivial in the smooth case, since all smooth affine toric varieties have the form $\mathbb{A}^n \times (\mathbb{A}^1 - 0)^m$. The singular case takes a tiny bit more work. Recall that a singular variety, when regarded as an element of $\mathcal{MV}$, is really the presheaf that it represents.

For background definitions and results, see [F1]. Let $N$ denote an $n$-dimensional lattice; it is isomorphic to $\mathbb{Z}^n$, but we work in a coordinate-free context. Let $V$ be the corresponding $\mathbb{R}$-vector space $N \otimes \mathbb{R}$. Recall that if $\sigma$ is a strongly convex rational polyhedral cone in $V$, then one gets a finitely-generated semigroup $S_\sigma = \sigma^\vee \cap \operatorname{Hom}(N, \mathbb{Z}) \subseteq \operatorname{Hom}(N, \mathbb{R})$, where $\sigma^\vee$ is the dual cone of $\sigma$. The affine toric variety $X(\sigma)$ is defined to be $\operatorname{Spec} k[S_\sigma]$.

**Lemma 5.2.** *If $\sigma$ generates $V$ as an $\mathbb{R}$-vector space, then the corresponding toric variety $X(\sigma)$ is simplicially $\mathbb{A}^1$-contractible.*

*Proof.* We need to construct a homotopy $H\colon X(\sigma) \times \mathbb{A}^1 \to X(\sigma)$ such that $H_0\colon X(\sigma) \times 0 \to X(\sigma)$ is constant and $H_1\colon X(\sigma) \times 1 \to X(\sigma)$ is the identity. To do this, we first choose nonzero generators $u_1, \ldots, u_t$ of $S_\sigma$. Then we can write $X(\sigma)$ as $\operatorname{Spec} k[Y_1, \ldots, Y_t]/I$ where $I$ is generated by all polynomials of the form

$$Y_1^{a_1} Y_2^{a_2} \cdots Y_t^{a_t} - Y_1^{b_1} Y_2^{b_2} \cdots Y_t^{b_t}$$

such that $a_1 u_1 + a_2 u_2 + \cdots + a_t u_t = b_1 u_1 + b_2 u_2 + \cdots + b_t u_t$ (see [F1, Exercise, p.19]).

We claim that because $\sigma$ generates $V$, there is a vector $w$ in $V$ such that $\langle u_i, w \rangle > 0$ for all $i$ (here $\langle -, - \rangle$ is the pairing between $V^*$ and $V$). Accepting this for the moment, define

$$H(y_1, y_2, \ldots, y_t, s) = (s^{\langle w, u_1 \rangle} y_1, s^{\langle w, u_2 \rangle} y_2, \ldots, s^{\langle w, u_t \rangle} y_t).$$

To see that this map is well-defined, suppose that $u = \Sigma a_i u_i = \Sigma b_i u_i$ and that $(y_1, \ldots, y_t)$ satisfies the equation $y_1^{a_1} y_2^{a_2} \cdots y_t^{a_t} - y_1^{b_1} y_2^{b_2} \cdots y_t^{b_t} = 0$. Then

$$(s^{\langle w, u_1 \rangle} y_1)^{a_1} (s^{\langle w, u_2 \rangle} y_2)^{a_2} \cdots (s^{\langle w, u_t \rangle} y_t)^{a_t} - (s^{\langle w, u_1 \rangle} y_1)^{b_1} (s^{\langle w, u_2 \rangle} y_2)^{b_2} \cdots (s^{\langle w, u_3 \rangle} y_t)^{b_t}$$

equals

$$s^{\langle w, u \rangle} y_1^{a_1} y_2^{a_2} \cdots y_t^{a_t} - s^{\langle w, u \rangle} y_1^{b_1} y_2^{b_2} \cdots y_t^{b_t},$$

which is still equal to zero. Note that $H_0$ is the constant map with value $(0, 0, \ldots, 0)$, and that $H_1$ is the identity.

We have only left to produce the vector $w$. If we had $\langle u_i, v \rangle = 0$ for all $v \in \sigma \cap N$, then the fact that $\sigma$ generates $V$ and is rational would imply that $u_i = 0$. Therefore, for each $i$ there exists a $w_i \in \sigma \cap N$ such that $\langle u_i, w_i \rangle \neq 0$. Since $u_i \in \sigma^\vee$, we must in fact have $\langle u_i, w_i \rangle > 0$. Let $w = w_1 + \cdots + w_t$. Again using the fact that $u_i \in \sigma^\vee$ and $w_j \in \sigma$, we know that $\langle u_i, w_j \rangle \geq 0$ for $i \neq j$. Hence $\langle u_i, w \rangle > 0$ for all $i$. □

**Proposition 5.3.** *Let $\sigma$ be a strongly convex polyhedral rational cone. Then $X(\sigma)$ is $\mathbb{A}^1$-homotopic to $(\mathbb{A}^1 - 0)^m$, where $m$ is the codimension of $\mathbb{R} \cdot \sigma$ in $V$.*

*Proof.* Split $N$ as $N' \oplus N''$ so that $\mathbb{R} \cdot \sigma = V' = N' \otimes \mathbb{R}$ and $(\mathbb{R} \cdot \sigma) \cap V'' = 0$, where $V'' = N'' \otimes \mathbb{R}$ (cf. [F1, p. 29]). Then $\sigma$ equals $\sigma' \times 0$ in $V' \times V''$, where $\sigma'$ is the same cone as $\sigma$ except that it lies in $V'$. Now $X(\sigma) \cong X(\sigma') \times (\mathbb{A}^1 - 0)^m$ by [F1, p. 5] and [F1, p. 19]. The above lemma shows that $X(\sigma')$ is simplicially $\mathbb{A}^1$-contractible, so $X(\sigma)$ is simplicially $\mathbb{A}^1$-homotopic to $(\mathbb{A}^1 - 0)^m$. □

**Theorem 5.4.** *Every toric variety can be obtained via homotopy colimits from the tori* $(\mathbb{A}^1 - 0)^m$, *and hence is stably cellular.*

*Proof.* Given a fan $\Delta$, the toric variety $X(\Delta)$ has an open cover consisting of affine toric varieties $X(\sigma)$ for cones $\sigma$ belonging to $\Delta$. Since $X(\sigma) \cap X(\tau)$ is equal to $X(\sigma \cap \tau)$, this cover of $X(\Delta)$ has the property that every intersection of pieces of the cover is $\mathbb{A}^1$-homotopic to a torus. The usual argument with Čech complexes tells us that $X(\Delta)$ is a homotopy colimit of the objects $X(\sigma)$. □

5.5. **Quadrics.** If $q(x_1, \ldots, x_n)$ is a quadratic form then one can look at the affine quadric $AQ(q) \hookrightarrow \mathbb{A}^n$ defined by $q(x_1, \ldots, x_n) = 0$, as well as the corresponding projective quadric $Q(q) \in \mathbb{P}^{n-1}$. Note that if $q$ is nondegenerate then $AQ(q)$ is singular (but only at the origin) whereas $Q(q)$ is nonsingular. The varieties $\mathbb{P}^{n-1} - Q(q)$ play a central role in [DI].

**Proposition 5.6.** *If $q$ is hyperbolic and non-degenerate, then* $\mathbb{A}^n - AQ(q)$, $AQ(q) - 0$, $Q(q)$, *and* $\mathbb{P}^{n-1} - Q(q)$ *are all stably cellular.*

*Proof.* If $q$ is hyperbolic then $n$ is even and after a change of basis we have $q(a_1, b_1, \ldots, a_k, b_k) = a_1 b_1 + \cdots + a_k b_k$ (where $n = 2k$). We will abbreviate $AQ(q)$ as just $AQ$, etc. The map $\mathbb{A}^{2k} - AQ \to \mathbb{A}^k - 0$ given by $(a_i, b_i) \mapsto (a_i)$ is an algebraic fiber bundle with fiber $(\mathbb{A}^1 - 0) \times \mathbb{A}^{k-1}$. If $U_i \hookrightarrow \mathbb{A}^k - 0$ is the open subscheme of vectors whose $i$th coordinate is nonzero, then $\{U_i\}$ is a completely stably cellular trivializing cover for the bundle. So Lemma 3.8 tells us that $\mathbb{A}^{2k} - AQ$ is stably cellular.

We consider the closed subscheme $AQ - 0 \hookrightarrow \mathbb{A}^{2k} - 0$. The homotopy purity sequence has the form $\mathbb{A}^{2k} - AQ \hookrightarrow \mathbb{A}^{2k} - 0 \to \operatorname{Th} N$ where $N$ is the normal bundle. But the normal bundle is easily checked to be trivial, so $\operatorname{Th} N \simeq S^{2,1} \wedge (AQ - 0)_+$. Since we know $\mathbb{A}^{2k} - AQ$ is stably cellular, the cofiber sequence shows us that $AQ - 0$ is also stably cellular.

For $Q$, we consider the principal $(\mathbb{A}^1 - 0)$-bundle $\mathbb{A}^1 - 0 \to AQ - 0 \to Q$ and apply Proposition 4.3. For $\mathbb{P}^{n-1} - Q$ we apply the same proposition to the principal bundle $\mathbb{A}^1 - 0 \to \mathbb{A}^{2k} - AQ \to \mathbb{P}^{n-1} - Q$. □

## 6. Algebraic $K$-theory and algebraic cobordism

We show that the motivic spectra $KGL$ and $MGL$, representing algebraic $K$-theory and algebraic cobordism respectively, are stably cellular.

In this section it will be more convenient to work in the category of naive spectra (a.k.a. Bousfield-Friedlander spectra). We use the model structure on this category, induced by that on $\mathcal{MV}$, that is described in [Ho2, Defn. 3.3] and [Ja, Thm. 2.9] (the two turn out to be equal).

Several times in the following proofs we will use the fact that in $\mathcal{MV}$ filtered colimits are homotopy colimits. This is inherited from the corresponding property of $s\mathcal{S}et$, using the fact that homotopy colimits for simplicial presheaves can be computed objectwise.

First recall a standard idea from stable homotopy theory:

**Lemma 6.1.** *Let $\{E_n, \Sigma^{2,1}E_n \to E_{n+1}\}$ be a motivic spectrum. Then $E$ is weakly equivalent to the homotopy colimit of the diagram*

$$\Sigma^\infty E_0 \to \Sigma^{-2,-1}\Sigma^\infty E_1 \to \Sigma^{-4,-2}\Sigma^\infty E_2 \to \cdots .$$

*Proof.* One model for $\Sigma^{-2n,-n}\Sigma^\infty E_n$ is given by the formulas $(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = \Sigma^{2(k-n),k-n}E_n$ if $k \geq n$ and $(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = E_k$ otherwise. Thus, for every $k$, $(\Sigma^{-2n,-n}\Sigma^\infty E_n)_k = E_k$ for sufficiently large $n$. Since homotopy colimits of spectra are computed degreewise in $k$, this shows that the $k$th term of the homotopy colimit is $E_k$, as desired. □

**Theorem 6.2.** *The algebraic $K$-theory spectrum $KGL$ is stably cellular.*

*Proof.* Recall that $KGL_n$ equals the object $\mathbb{Z} \times BGL$ of $\mathcal{MV}_*$ [V1, Ex. 2.8]. By Lemma 6.1, it suffices to show that $\mathbb{Z} \times BGL$ is stably cellular, or equivalently by Lemma 3.2 that $BGL$ is stably cellular. Now $BGL$ is weakly equivalent to $\operatorname{colim}_k \operatorname{colim}_n \operatorname{Gr}_k(\mathbb{A}^n)$ by [V1, Ex. 2.8]. Proposition 4.4 finishes the proof because these colimits are filtered colimits and hence homotopy colimits. □

**Remark 6.3.** In the above result one can avoid the use of Grassmannians by observing that $BGL$ is the homotopy colimit of the usual bar construction—i.e., of the simplicial diagram $[n] \mapsto GL^n$. Here $GL = \operatorname{colim}_k GL_k$, as usual; note that this is a filtered colimit, hence a homotopy colimit. Proposition 4.1 shows that each $GL_k$ is stably cellular, and so $GL$ is as well. Then so is each $GL^n$, hence $BGL$ is stably cellular.

**Theorem 6.4.** *The algebraic cobordism spectrum $MGL$ is stably cellular.*

*Proof.* Let $p_{n,k}\colon E_{n,k} \to \operatorname{Gr}_k(\mathbb{A}^n)$ be the tautological $k$-dimensional bundle, and let $E^0_{n,k}$ be the complement of the zero section. By Lemma 6.1, we need to show that each $MGL_k$ is stably cellular. From [V1, Ex. 2.10], the object $MGL_k$ is $\operatorname{colim}_n \operatorname{Th}(p_{n,k})$, which is equal to $\operatorname{colim}_n E_{n,k}/E^0_{n,k}$. Since the colimit is a filtered colimit, it is also a homotopy colimit. Therefore, we only have to show that $E_{n,k}/E^0_{n,k}$ is stably cellular. By Lemma 2.5, this reduces to showing that $E_{n,k}$ and $E^0_{n,k}$ are stably cellular. The first is weakly equivalent to $\operatorname{Gr}_k(\mathbb{A}^n)$, so it is stably cellular by Proposition 4.4. The following proposition shows that the second is also stably cellular. □

**Proposition 6.5.** *Let $p_{n,k}\colon E_{n,k} \to Gr_k(\mathbb{A}^n)$ be the tautological $k$-dimensional bundle, and let $E^0_{n,k}$ be the complement of the zero section. The variety $E^0_{n,k}$ is stably cellular.*

*Proof.* To simplify notation, let $V = V_k(\mathbb{A}^n)$, $G = \operatorname{Gr}_k(\mathbb{A}^n)$, and $E^0 = E^0_{n,k}$. The map $E^0 \times_G V \to E^0$ is a principal bundle with group $GL_k$. By Proposition 4.3, we just need to show that $E^0 \times_G V$ is stably cellular. Now $E^0 \times_G V$ is the variety of ordered sets of $k$ linearly independent vectors together with a non-zero vector in the span of these vectors, which is isomorphic to $V \times (\mathbb{A}^k - 0)$. This variety is stably cellular by Proposition 4.2 and Lemma 3.4. □

**Remark 6.6.** It is known, at least in characteristic 0, that the Lazard ring $\mathbb{Z}[x_1, x_2, \ldots]$ sits inside of $MGL_{2*,*}$ as a retract (here $x_i$ has degree $(2i, i)$). Hopkins and Morel have announced a proof that these two rings are actually equal. In this case the $x_i$'s form a regular sequence, and one can inductively start forming homotopy cofibers of $MGL$-module spectra: $MGL/(x_1)$ is defined to be the homotopy cofiber of $\Sigma^{2,1}MGL \to MGL$, then $MGL/(x_1, x_2)$ is the homotopy cofiber of a map $\Sigma^{4,2}MGL/(x_1) \to MGL/(x_1)$, etc. According to Hopkins and Morel the spectrum $MGL/(x_1, x_2, \ldots)$ is weakly equivalent to the motivic cohomology spectrum $H\mathbb{Z}$ (just as happens in classical topology). From this of course it would follow that $H\mathbb{Z}$ is cellular: we already know $MGL$ is cellular, and $H\mathbb{Z}$ would be built inductively from suspensions of $MGL$ via homotopy cofibers.

## 7. Cellularity and stable homotopy

The material in this section is completely formal, but at the same time surprisingly powerful. If $E$ is a motivic spectrum, write $\pi_{p,q}E$ for the set of maps $\operatorname{Ho}(S^{p,q}, E)$ in the stable homotopy category. First we will prove that stable $\mathbb{A}^1$-weak equivalences between cellular objects are detected by $\pi_{p,q}$. Then we'll produce a pair of spectral sequences for computing $\pi_{p,q}$ of smash products and function spectra that are applicable only under certain cellularity assumptions.

**Proposition 7.1.** *If $E$ is cellular and $\pi_{p,q}E = 0$ for all $p$ and $q$ in $\mathbb{Z}$, then $E$ is contractible.*

*Proof.* We may as well assume that $E$ is both cofibrant and fibrant. Consider the class of all spectra $A$ such that the pointed simplicial set $\operatorname{Map}(\tilde{A}, E)$ is contractible, where $\tilde{A} \to A$ is a cofibrant-replacement for $A$. This class is closed under weak equivalences and homotopy colimits, and our assumptions imply that it contains the spheres $S^{p,q}$. Therefore the class contains every cellular object; in particular, it contains $E$. But if $\operatorname{Map}(E, E)$ is contractible, then the identity map is null in the stable homotopy category (because $E$ is cofibrant and fibrant), and this implies that $E$ is contractible. □

**Corollary 7.2.** *Let $E \to F$ be a map between cellular spectra, and assume it induces isomorphisms on $\pi_{p,q}$ for all $p$ and $q$ in $\mathbb{Z}$. Then the map is a weak equivalence.*

This corollary is proved in [H, Thm. 5.1.5], but we include a proof for completeness and because it's short.

*Proof.* Let $K$ be the homotopy fiber of $E \to F$. Since we are in a stable category, it is enough to prove that $K$ is contractible. But our assumptions imply that $K$ is cellular, and that $\pi_{*,*}K = 0$. So the proposition gives us $K \simeq *$. □

**Proposition 7.3.** *If $E$ is any motivic spectrum, there is a zig-zag $A \to \hat{E} \leftarrow E$ in which $E \to \hat{E}$ is a weak equivalence, $A \to \hat{E}$ induces isomorphisms on bi-graded homotopy groups, and $A$ is cellular.*

The following proof is an adaptation of the usual construction in ordinary topology of cellular approximations to any space.

*Proof.* First let $E \to \hat{E}$ be a fibrant-replacement. Consider all possible maps $f\colon S^{p,q} \to \hat{E}$ as $p$ and $q$ range over all integers, and let $C_0 = \oplus_f S^{p,q}$. There is a canonical map $C_0 \to \hat{E}$.

Factor this map as $C_0 \overset{\sim}{\rightarrowtail} \hat{C}_0 \to \hat{E}$, where the first map is a trivial cofibration and the second is a fibration. Next consider all possible maps $f\colon S^{p,q} \to \hat{C}_0$ which become zero in $\pi_{p,q}(\hat{E})$. We get a map $\oplus_f S^{p,q} \to \hat{C}_0$, and let $C_1$ be the mapping cone. There exists a commutative triangle of the form

$$\begin{array}{ccc} \hat{C}_0 & \longrightarrow & \hat{E} \\ \downarrow & \nearrow & \\ C_1. & & \end{array}$$

We again factor $C_1 \to \hat{E}$ as $C_1 \overset{\sim}{\rightarrowtail} \hat{C}_1 \to \hat{E}$, and repeat the procedure to get $C_2$. Continuing, we get a sequence of cofibrations between fibrant objects

$$\hat{C}_0 \rightarrowtail \hat{C}_1 \rightarrowtail \hat{C}_2 \to \cdots$$

all with maps down to $\hat{E}$. We let $C$ denote the homotopy colimit, and note that there is a natural map $C \to \hat{E}$. The map $C \to \hat{E}$ is surjective on homotopy groups because of the way in which $\hat{C}_0$ was defined. To show that $C \to \hat{E}$ is *injective* on homotopy groups, we need a technical result (see Corollary 9.1) that tells us that $\pi_{p,q} C \cong \operatorname{colim}_n \pi_{p,q} \hat{C_n}$. From this observation, injectivity follows immediately. Finally, note that each $\hat{C}_i$ is cellular and therefore $C$ is cellular. □

**Remark 7.4.** The above proof actually shows that the full subcategory of $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the cellular spectra is the same as the smallest full triangulated subcategory which contains the spheres and is closed under infinite direct sums. It might seem like we also need to include mapping telescopes in this statement, but we get these for free—see [BN].

**Definition 7.5.** *Given any motivic spectrum $E$, let* $\mathrm{Cell}(E)$ *be the corresponding cellular spectrum constructed in Proposition 7.3.*

It is easy to see from the proof of Proposition 7.3 that Cell is a functor. It's slightly inconvenient that we don't obtain a natural map $\mathrm{Cell}(E) \to E$. However, because $E \to \hat{E}$ is a weak equivalence, we do obtain a natural weak homotopy class $\mathrm{Cell}(E) \to E$.

The following simple lemma will be important later in the proof of Theorem 8.12.

**Lemma 7.6.** *The functor* Cell *takes homotopy cofiber sequences to homotopy cofiber sequences.*

*Proof.* Let $A \to B \to C$ be a homotopy cofiber sequence, and let $D$ denote the homotopy cofiber of $\mathrm{Cell}(A) \to \mathrm{Cell}(B)$. Then $D$ is cellular, and the induced homotopy class $D \to C$ is an isomorphism on $\pi_{*,*}$ by the five-lemma. Now the map $\mathrm{Cell}(B) \to \mathrm{Cell}(C)$ induces $D \to \mathrm{Cell}(C)$, and this map is also an isomorphism on $\pi_{*,*}$. By Proposition 7.2, this last map is a weak equivalence. □

If $E$ is a motivic ring spectrum, one can talk about $E$-modules, smash products over $E$ (denoted $\wedge_E$), and function spectra $F_E(-,-)$. The definitions are formal, given a symmetric monoidal model category of spectra (see [EKMM, Ch. III], for example). As in [EKMM] we will blur the distinction between these constructions and their derived versions—to the seasoned homotopy-theorist it will always be clear which we mean (and it's almost always the derived one).

We will need the following basic tool from the algebra of ring spectra:

**Proposition 7.7.** *Let $E$ be a motivic ring spectrum, $M$ be a right $E$-module, and $N$ be a left $E$-module. Assume $E$ and $M$ are cellular. Then there there is a strongly convergent tri-graded spectral sequence of the form*

$$E^2_{a,(b,c)} = \operatorname{Tor}^{\pi_{*,*}E}_{a,(b,c)}(\pi_{*,*}M, \pi_{*,*}N) \Rightarrow \pi_{a+b,c}(M \wedge_E N),$$

*and a conditionally-convergent tri-graded spectral sequence of the form*

$$E_2^{a,(b,c)} = \operatorname{Ext}^{a,(b,c)}_{\pi_{*,*}E}(\pi_{*,*}M, \pi_{*,*}N) \Rightarrow \pi_{a+b,c}F_E(M, N).$$

Some kind of cellularity hypothesis is essential for this proposition in order to guarantee convergence.

The proof of this result is almost exactly the same as the one of [EKMM, Thm. IV.4.1]; we will record some consequences before spelling out exactly what changes need to be made. In the notation, $a$ is the homological grading on the Tor and $(b, c)$ is the internal grading coming from the bi-graded homotopy groups. The differentials in the first spectral sequence have the form $d_r \colon E^r_{a,(b,c)} \to E^r_{a-r,(b+r-1,c)}$. The edge homomorphism of the spectral sequence is the obvious map

$$[\pi_{*,*}M \otimes_{\pi_{*,*}E} \pi_{*,*}N]_{(b,c)} \to \pi_{b,c}(M \wedge_E N).$$

Similar remarks apply to the Ext spectral sequence.

**Corollary 7.8.** *Let $M$ be a cellular motivic spectrum, and let $N' \to N$ be an arbitrary map. If $N' \to N$ induces isomorphisms on bi-graded homotopy groups, then $M_{*,*}(N') \to M_{*,*}(N)$ and $(N')^{*,*}(M) \to N^{*,*}(M)$ are both isomorphisms.*

*Proof.* We apply the spectral sequences of the theorem in the case where $E$ is the sphere spectrum. □

Here is one immediate consequence of Proposition 7.3 and the above corollary: if $X$ is any scheme, then there is a cellular spectrum $A$ and a zig-zag of maps between $A$ and $\Sigma^\infty X_+$ which induce isomorphisms on $KGL$- and $MGL$- homology. We don't know whether the same statement can be made for *cohomology*.

*Proof of Proposition 7.7.* We follow the method explained in [EKMM, IV.5]. First, we set $K_{-1} = M$ and inductively build a sequence of homotopy cofiber sequences $K_i \to F_i \to K_{i-1}$ with the property that $F_i$ is a free right $E$-module and $\pi_{*,*}F_i \to \pi_{*,*}K_{i-1}$ is surjective. The resulting chain complex

$$\cdots \to \pi_{*,*}F_2 \to \pi_{*,*}F_1 \to \pi_{*,*}F_0 \to \pi_{*,*}M$$

is a free resolution over $\pi_{*,*}E$.

We now have a tower of homotopy cofiber sequences of the form

(7.9)
$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & M & \longrightarrow & M & \longrightarrow & \Sigma^{1,0}K_0 & \longrightarrow & \Sigma^{2,0}K_1 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & & * & & F_0 & & \Sigma^{1,0}F_1 & & \Sigma^{2,0}F_2 & & \cdots \end{array}$$

(the tower is trivial as it extends left). We apply $(-) \wedge_E N$ to this tower, and consider the resulting homotopy spectral sequence. Note that for each fixed $q$ we will get a homotopy spectral sequence for $\pi_{*,q}(-)$, with only the first variable changing: so we really have a family of spectral sequences, one for each $q$. We are free to think of this as a 'tri-graded' spectral sequence.

Note that $F_i$ is a wedge of various suspensions of $E$, indexed by a free set of generators for $\pi_{*,*}F_i$ as a $\pi_{*,*}E$-module. So $F_i \wedge_E N$ is a wedge of suspensions of $N$ indexed by the same set. Therefore $\pi_{*,*}(F_i \wedge_E N)$ is a direct sum of copies of $\pi_{*,*}N$ (with the grading shifted appropriately), and the identification of the $E_2$-term as Tor falls out immediately. This is all the same as the argument in [EKMM].

The place where we have to be careful is in convergence. By [Bd, Thm. 6.1(b)] we only have to show that $\operatorname{colim} \pi_{*,*}(\Sigma^{n,0}K_n \wedge_E N) = 0$. Let $K_\infty$ be the homotopy colimit of the sequence $M \to \Sigma^{1,0}K_0 \to \Sigma^{2,0}K_1 \to \cdots$. From a technical result proved at the end of the paper (see Corollary 9.1), we have $\pi_{p,q}(K_\infty) = \operatorname{colim}_n \pi_{p,q}\Sigma^{n+1,0}K_n$. The tower was constructed in such a way that each $K_i \to \Sigma^{1,0}K_{i+1}$ induces the zero map on $\pi_{*,*}$; so $\pi_{*,*}K_\infty = 0$. In ordinary topology this would tell us that $K_\infty$ is contractible, and therefore that $K_\infty \wedge_E N$ is contractible—and we would be done. However, our assumptions imply inductively that all the $K_i$ are cellular, and therefore so is $K_\infty$. By Proposition 7.1 the vanishing of $\pi_{*,*}$ tells us that $K_\infty$ is contractible.

The proof for the Ext case follows the same ideas. For conditional convergence [Bd, Defn. 5.10], we need to show that $\lim \pi_{*,*}F_E(\Sigma^{n,0}K_n, N)$ and $\lim^1 \pi_{*,*}F_E(\Sigma^{n,0}K_n, N)$ are both zero. This follows from the usual short exact sequence for homotopy groups of the homotopy limit of a tower [BK, IX.3.1] and the fact that $\operatorname{holim} F_E(\Sigma^{n,0}K_n, N)$ is weakly equivalent to $F_E(K_\infty, N)$, which is contractible. □

7.10. **Cellularity for $E$-modules.** If $E$ is a motivic ring spectrum, we can consider the model category of right $E$-modules [SS]. We define the ***E*-cellular modules** to be the smallest class which contains the modules $S^{p,q} \wedge E$ and is closed under weak equivalence and homotopy colimits.

Most of the results of the previous section carry over to $E$-cellularity. The key observation is that $\operatorname{Ho}_E(S^{p,q} \wedge E, X)$ is isomorphic to $\pi_{p,q}(X)$.

It follows as in Proposition 7.2 that an $E$-module map between $E$-cellular spectra is a weak equivalence if it induces isomorphisms on $\pi_{p,q}$ for all $p$ and $q$ in $\mathbb{Z}$. For any $E$-module $X$, the construction of Proposition 7.3 gives us a zig-zag $\operatorname{Cell}_E(X) \to \hat{X} \leftarrow X$ of $E$-module maps in which $X \to \hat{X}$ is a weak equivalence, $\operatorname{Cell}_E(X) \to \hat{X}$ induces isomorphisms on bi-graded homotopy groups, and $\operatorname{Cell}_E(X)$ is $E$-cellular. As in Definition 7.5, we obtain natural weak homotopy classes $\operatorname{Cell}_E(X) \to X$, but not actual maps. We can also prove that $\operatorname{Cell}_E$ takes homotopy cofiber sequences of $E$-modules to homotopy cofiber sequences.

We will need the following improvement on Proposition 7.7.

**Proposition 7.11.** *The spectral sequences of Proposition 7.7 have the indicated convergence properties as long as $M$ is $E$-cellular (but without any other assumptions on $E$ and $M$).*

*Proof.* The basic setup is the same as in the proof of Proposition 7.7, and for the Tor-spectral sequence we again only need to show that $\operatorname{colim}_n \pi_{*,*}(\Sigma^{n,0}K_n \wedge_E N) = 0$. Because $M$ is $E$-cellular, so is each $K_i$ and so is $K_\infty$. We know that $\pi_{*,*}K_\infty = 0$. From the $E$-cellular version of Proposition 7.2, we conclude that $K_\infty$ is contractible.

The Ext case is again similar; see the end of the proof of Proposition 7.7 for an outline of the differences. □

Note that if $X$ is a cellular spectrum then $X \wedge E$ is $E$-cellular. This lets us apply the spectral sequences in the case where $M$ has the form $X \wedge E$, but without any assumptions on the spectrum $E$.

## 8. Finite cell complexes and Künneth theorems

**Definition 8.1.** *The **finite cell complexes** are the smallest class of objects in $\mathcal{S}pectra(\mathcal{MV})$ with the following properties:*

*(1) the class contains the spheres $S^{p,q}$;*
*(2) the class is closed under weak equivalence;*
*(3) if $X \to Y \to Z$ is a homotopy cofiber sequence and two of the objects are in the class, then so is the third.*

The full subcategory of $\mathrm{Ho}\,(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the finite cell complexes is the smallest full triangulated subcategory containing all the spheres $S^{p,q}$ (which will necessarily be closed under finite direct sums, but not infinite ones).

**Remark 8.2.** It is worth mentioning that a finite homotopy colimit of finite cell complexes need not be a finite cell complex. This happens even in ordinary topology: for example, the homotopy co-invariants of $\mathbb{Z}/2$ acting on a point is $\mathbb{R}P^{\infty}$.

**Remark 8.3.** If a scheme $X$ has a finite Zariski cover $\{U_i\}$ such that each intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is a finite cell complex, then so is $X$. Our arguments from Section 4 therefore show that $GL_n$ and $V_k(\mathbb{A}^n)$ are finite cell complexes. The argument from Proposition 4.4 does *not* show that Grassmannians are finite cell complexes, however. It turns out that they are, but the proof is much more elaborate. We have omitted it because for us the *linear* spectra (see below) are almost as good as finite cell complexes, and Grassmannians are obviously linear.

For any two motivic spectra $A$ and $B$ and any motivic ring spectrum $E$, there is a natural map

$$F(A,E) \wedge F(B,E) \to F(A \wedge B, E \wedge E) \to F(A \wedge B, E). \tag{8.4}$$

In particular, using the identification $F(S^0,E) \cong E$ one finds that $F(X,E)$ is a bimodule over the ring spectrum $E$, and that the above map factors as

$$F(A,E) \wedge F(B,E) \to F(A,E) \wedge_E F(B,E) \xrightarrow{\eta_{A,B}} F(A \wedge B, E).$$

**Proposition 8.5.** *If $A$ (or $B$) is a finite cell complex, then the map $\eta_{A,B}$ induces isomorphisms on all $\pi_{p,q}(-)$.*

*Proof.* Fix $B$, and consider the class of objects $A$ such that $\eta_{A,B}$ induces isomorphisms on bi-graded homotopy groups. One easily checks that this class is closed under weak equivalences, and has the two-out-of-three property for homotopy cofiber sequences. To see that the class contains $S^{p,q}$, use the fact that $F(S^{p,q},E) \simeq S^{-p,-q} \wedge E$ as $E$-modules. □

**Theorem 8.6.** *Suppose that $X$ and $Y$ are two motivic spectra such that $X$ is a finite cell complex. Let $E$ be a motivic ring spectrum. Then there exists a strongly-convergent tri-graded Künneth spectral sequence of the form*

$$\mathrm{Tor}^{E^{*,*}}_{a,(b,c)}(E^{*,*}X, E^{*,*}Y) \Rightarrow E^{a+b,c}(X \wedge Y).$$

The grading conventions are the same as in Proposition 7.7.

*Proof.* First note that $F(S^{p,q}, E)$ is $E$-cellular (being equivalent to $S^{-p,-q} \wedge E$). Using this together with the fact that $X$ is a finite cell complex, it follows that $F(X, E)$ is $E$-cellular. We now apply Proposition 7.11 with $M = F(X, E)$ and $N = F(Y, E)$. The groups $\pi_{*,*}(M \wedge_E N)$ are identified with $\pi_{*,*}F(X \wedge Y, E)$ by Proposition 8.5. □

The necessity of some kind of finiteness hypothesis in the above result is well known in ordinary topology—see [A, Lect. 1]. The result is often applied when $X = \Sigma^\infty A_+$ and $Y = \Sigma^\infty B_+$, where $A$ and $B$ are schemes, in which case $X \wedge Y = \Sigma^\infty (A \times B)_+$.

**Remark 8.7.** Note that the higher Tor's vanish if $E^{*,*}(X)$ is free as a module over $E^{*,*}$, in which case we obtain a Künneth isomorphism

$$E^{*,*}(X) \otimes_{E^{*,*}} E^{*,*}(Y) \cong E^{*,*}(X \wedge Y).$$

**Remark 8.8.** In [J], Joshua produced a similar Künneth spectral sequence. His result was stated only for algebraic $K$-theory and motivic cohomology, and assumed that $X$ and $Y$ were *schemes* (rather than arbitrary objects of $\mathcal{MV}$). Also, his spectral sequence was only bi-graded rather than tri-graded: this is essentially because he was applying the results of [EKMM] rather than reproving them in the bi-graded context, and so his motivic cohomology rings were graded by total degree.

Our proof is essentially the same as Joshua's (which in turn is the same as the modern proof in stable homotopy theory) although we were able to streamline things by using the language of motivic spectra.

Joshua's result assumes that one of the schemes $X$ and $Y$ is *linear*, as opposed to being a finite cell complex in our sense. If one assumes the ring spectrum $E$ satisfies a Thom isomorphism theorem, then one can make our result encompass Joshua's by expanding the class of finite cell complexes so as to be closed under the process of 'taking Thom spaces':

**Definition 8.9.** *The* ***linear*** *motivic spectra are the smallest class of objects in* $\mathcal{S}pectra(\mathcal{MV})$ *with the following properties:*

*(1) the class contains the spheres* $S^{p,q}$*;*
*(2) the class is closed under weak equivalence;*
*(3) if* $X \to Y \to Z$ *is a homotopy cofiber sequence and two of the objects are in the class, then so is the third;*
*(4) if* $\xi\colon E \to X$ *is an algebraic vector bundle over a smooth scheme* $X$*, then* $\Sigma^\infty \operatorname{Th}\xi$ *belongs to the class if and only if* $\Sigma^\infty X_+$ *belongs to the class.*

**Remark 8.10.** If $Z \hookrightarrow X$ is a closed inclusion of smooth schemes, recall that there is a stable homotopy cofiber sequence $X - Z \to X \to \operatorname{Th} N_{X/Z}$ [MV, 3.2.23]. It follows that if two of the three objects $\Sigma^\infty Z_+$, $\Sigma^\infty X_+$, and $\Sigma^\infty (X - Z)_+$ are linear, then so is the third.

Let $E$ be a motivic ring spectrum, and let $\xi \to X$ be an algebraic vector bundle of rank $n$ over a smooth scheme $X$. We'll say that $E$ **satisfies Thom isomorphism for** $\boldsymbol{\xi}$ if there is a class $u \in E^{2n,n}(\operatorname{Th}\xi)$ such that multiplication by $u$ gives an isomorphism $E^{*,*}(X) \to \tilde{E}^{*+2n,*+n}(\operatorname{Th}\xi)$. To be more precise, note that we have a map of motivic spaces

$$\frac{\xi}{\xi - 0} \xrightarrow{\Delta} \frac{\xi \times \xi}{(\xi - 0) \times \xi} \cong \frac{\xi}{\xi - 0} \wedge \xi_+ \simeq \frac{\xi}{\xi - 0} \wedge X_+.$$

This map induces $\alpha\colon F(\mathrm{Th}\,\xi \wedge X_+, E) \to F(\mathrm{Th}\,\xi, E)$. If we write $u$ as a homotopy class $S^{-2n,-n} \to F(\mathrm{Th}\,\xi, E)$, we can consider the composite

(8.11)

$$S^{-2n,-n} \wedge F(X_+, E) \to F(\mathrm{Th}\,\xi, E) \wedge F(X_+, E) \to F(\mathrm{Th}\,\xi \wedge X_+, E) \xrightarrow{\alpha} F(\mathrm{Th}\,\xi, E),$$

where the second map is the same as in (8.4). The claim is that this composite is an isomorphism on $\pi_{*,*}$.

We say that $E$ **satisfies Thom isomorphism** if it does so for every algebraic vector bundle over a smooth scheme. The spectra $H\mathbb{Z}$, $KGL$, and $MGL$ are all known to satisfy Thom isomorphism. The reader may wish to compare the above discussion to the notion of algebraically orientable spectrum from [HK].

**Theorem 8.12.** *Suppose $E$ is a ring spectrum satisfying Thom isomorphism. If $X$ and $Y$ are motivic spectra such that $X$ is linear, then there is a strongly convergent Künneth spectral sequence as in Theorem 8.6.*

In the following proof, we continue our sloppiness about distinguishing various functors and their derived versions. It should be clear from context that we almost always mean the derived version.

*Proof.* The proof requires a little more care than the similar things we've done so far. If $Z$ is a pointed motivic space, we abbreviate $F(\Sigma^\infty Z, E)$ as just $F(Z, E)$.

Recall from Section 7.10 that $\mathrm{Cell}_E$ is a functorial $E$-cellular approximation. Let $\mathcal{C}$ denote the class of all motivic spectra $A$ such that for all motivic spectra $Y$, the composite

$$\pi_{p,q}(\mathrm{Cell}_E(F(A,E)) \wedge_E F(Y,E)) \to \pi_{p,q}(F(A,E) \wedge_E F(Y,E)) \to \pi_{p,q}F(A \wedge Y, E)$$

is an isomorphism for all $p$ and $q$. The first map above makes sense because we have a homotopy class $\mathrm{Cell}_E(F(A,E)) \to F(A,E)$ even though we don't have an actual map.

The class $\mathcal{C}$ clearly is closed under weak equivalences and contains the spheres $S^{p,q}$. The $E$-cellular version of Lemma 7.6 and the five-lemma show that it also has property (3) of Definition 8.9. To show that $\mathcal{C}$ contains every linear spectrum, we must check that if $\xi$ is a vector bundle over a smooth scheme $X$, then $\mathrm{Th}\,\xi$ belongs to $\mathcal{C}$ if and only if $\Sigma^\infty X_+$ belongs to $\mathcal{C}$.

However, from (8.11) we have the map $u\colon F(X_+, E) \to S^{2n,n} \wedge F(\mathrm{Th}\,\xi, E)$ which is an isomorphism on $\pi_{*,*}$. It follows that $\mathrm{Cell}_E(F(X_+, E))$ is weakly equivalent to $S^{2n,n} \wedge \mathrm{Cell}_E(F(\mathrm{Th}\,\xi, E))$. We now look at the diagram

$$\begin{array}{ccc} \pi_{p,q}(\mathrm{Cell}_E(F(X_+,E)) \wedge_E F(Y,E)) & \longrightarrow & \pi_{p,q}F(X_+ \wedge Y, E) \\ \downarrow & & \downarrow \\ \pi_{p,q}(S^{2n,n} \wedge \mathrm{Cell}_E(F(\mathrm{Th}\,\xi,E)) \wedge_E F(Y,E)) & \longrightarrow & \pi_{p,q}(S^{2n,n} \wedge F(\mathrm{Th}\,\xi \wedge Y, E)). \end{array}$$

Both vertical maps induce isomorphisms on $\pi_{*,*}$—the one on the left is a weak equivalence, and for the one on the right this is the 'generalized Thom isomorphism' from Lemma 8.13 below. So the top horizontal map is a $\pi_{*,*}$-isomorphism if and only if the bottom map is one. This is equivalent to what we needed to prove.

We have shown that for every linear spectrum $X$, the map

$$\pi_{p,q}(\mathrm{Cell}_E(F(X,E)) \wedge_E F(Y,E)) \to \pi_{p,q}F(X \wedge Y, E)$$

is an isomorphism. By Proposition 7.11 there is a strongly convergent spectral sequence of the form

$$\operatorname{Tor}^{E^{*,*}}_{a,(b,c)}(\pi_{*,*}\operatorname{Cell}_E(F(X,E)), E^{*,*}(Y)) \to E^{*,*}(X \wedge Y).$$

But $\operatorname{Cell}_E(F(X,E)) \to F(X,E)$ is an isomorphism on bi-graded homotopy groups, so this completes the proof. □

**Lemma 8.13.** *Let $E$ be a ring spectrum satisfying Thom isomorphism, and let $\xi$ be a vector bundle of rank $n$ over a smooth scheme $X$. Then for every motivic spectrum $Y$, the map*

$$\eta_Y : F(X_+ \wedge Y, E) \to S^{2n,n} \wedge F(\operatorname{Th}\xi \wedge Y, E)$$

*induces isomorphisms on $\pi_{*,*}$. In other words, multiplication by the Thom class gives an isomorphism*

$$E^{*,*}(X_+ \wedge Y) \cong E^{*+2n,*+n}(\operatorname{Th}\xi \wedge Y).$$

The construction of the map $\eta_Y$ is analogous to the discussion preceding Theorem 8.12.

*Proof.* Let $\mathcal{C}$ be the class of all spectra $Y$ such that $\eta_Y$ induces isomorphisms on $\pi_{*,*}$. Clearly the class is closed under weak equivalences, homotopy colimits, and $S^{p,q}$-suspensions. Since every motivic spectrum is a homotopy colimit of desuspensions of suspension spectra (see Lemma 6.1), it suffices to show that suspension spectra belong to $\mathcal{C}$. For $F \in \mathcal{MV}_*$, the homotopy cofiber sequence

$$S^{0,0} \to \Sigma^\infty F_+ \to \Sigma^\infty F$$

tells us that we can just prove the lemma in the case $Y = \Sigma^\infty F_+$.

But every motivic space $F$ is a homotopy colimit of smooth schemes [D, Sec. 2.6], so it actually suffices to show that $\Sigma^\infty Z_+$ belongs to $\mathcal{C}$ for every pointed smooth scheme $Z$. But in this case the statement is just the Thom isomorphism for the bundle $\pi^*\xi$, where $\pi : X \times Z \to X$ is the projection. □

## 9. Stable homotopy groups of filtered colimits

In this section we prove the following basic result about the category of motivic spectra, needed several times in the course of the paper. The phrase 'directed system', as used in the present section, refers to sequences $E_0 \to E_1 \to \cdots$ indexed by the natural numbers.

**Proposition 9.1.** *Let $i \mapsto E_i$ be a directed system of motivic spectra. Then $\operatorname{colim}_i \pi_{p,q} E_i \to \pi_{p,q}(\operatorname{hocolim} E_i)$ is an isomorphism for all $p$ and $q$ in $\mathbb{Z}$.*

9.2. **Preliminary remarks.** Let $\mathcal{T}$ be a triangulated category with infinite direct sums, as in [BN, Def. 1.2]. An object $X \in \mathcal{T}$ is called **compact** if $\mathcal{T}(X, \oplus_i E_i) \cong \oplus_i \mathcal{T}(X, E_i)$ for every countable collection of objects $\{E_i\}$. The full subcategory of $\mathcal{T}$ consisting of compact objects is readily seen to be a triangulated subcategory (and therefore closed under finite sums).

Recall from [BN] that if $E_0 \to E_1 \to \cdots$ is a directed system in $\mathcal{T}$ then one can form what we'll call the **triangulated homotopy colimit** $\operatorname{thocolim} E_i$, defined as the cofiber of $\oplus E_i \xrightarrow{1-sh} \oplus E_i$. If $\mathcal{M}$ is a stable model category and $E_0 \to E_1 \to \cdots$ is a directed system in $\mathcal{M}$, one can check that $\operatorname{hocolim} E_i$ (as defined in the model category sense) is isomorphic to the object $\operatorname{thocolim} E_i$ in $\operatorname{Ho}(\mathcal{M})$.

**Lemma 9.3.** *Let $\mathcal{M}$ be a stable model category. An object $X \in Ho(\mathcal{M})$ is compact if and only if the natural map $\operatorname{colim}_i Ho(X, E_i) \to Ho(X, \operatorname{thocolim} E_i)$ is an isomorphism, for every directed system $\{E_i\}$ in $\mathcal{M}$.*

*Proof.* The 'only if' part is immediate, and works for any triangulated category. For the 'if' part, note that if $\{E_i\}$ is any collection of cofibrant objects then the hocolim of

$$E_0 \to E_0 \vee E_1 \to E_0 \vee E_1 \vee E_2 \to \cdots$$

is isomorphic to $\oplus E_i$ in $\operatorname{Ho}(\mathcal{M})$. This fact is easily proven using facts about model categories, but it is unclear to us if it works for a general triangulated category. □

9.4. **Compact objects in motivic spectra.** Since the model categories of motivic symmetric spectra and naive spectra are Quillen equivalent [Ho2, Sec. 10], we can prove our theorems in either setting. It is easier to work with naive spectra.

If $W \in \mathcal{MV}_*$ then $\Sigma^\infty W$ denotes the usual suspension spectrum of $W$. If $E$ is a spectrum and $n \geq 0$, we let $\Sigma^{2n,n}E$ denote the naive spectrum with $[\Sigma^{2n,n}E]_i = E_{n+i}$. Likewise, when $n < 0$ let $\Sigma^{2n,n}E$ denote the spectrum with $[\Sigma^{2n,n}E]_i = *$ for $i < -n$ and $[\Sigma^{2n,n}E]_i = E_{i+n}$ for $i \geq -n$. Finally, we abbreviate $\Sigma^{2n,n}(\Sigma^\infty W)$ to $\Sigma^{(2n,n)+\infty}W$. Note that these objects are cofibrant spectra by [Ho2, Prop. 1.14].

Our aim is the following:

**Theorem 9.5.** *If $X$ is any pointed smooth scheme and $n \in \mathbb{Z}$, then $\Sigma^{(2n,n)+\infty}X$ is a compact object of the motivic stable homotopy category.*

From this one readily deduces the result about $\pi_{p,q}$ of directed hocolims:

*Proof of Proposition 9.1.* The $S^{p,q}$'s are in the smallest triangulated subcategory which contains $\Sigma^{(2n,n)+\infty}X$ for every $X$ and $n$. By Theorem 9.5 and the observation that the compact objects form a triangulated subcategory, the $S^{p,q}$'s are therefore compact. Now use Lemma 9.3. □

Recall from Definition 8.1 the notion of a finite cell complex. One consequence of Theorem 9.5 is that we never really need infinite homotopy colimits to build stably cellular schemes.

**Proposition 9.6.** *Let $X$ be a smooth scheme, and suppose $X$ is stably cellular. Then $\Sigma^\infty X_+$ is a retract, in $Ho(\mathcal{S}pectra(\mathcal{MV}))$, of a finite cell complex.*

*Proof.* Let $\mathcal{T}$ be the full subcategory of $\operatorname{Ho}(\mathcal{S}pectra(\mathcal{MV}))$ consisting of the cellular objects. Then the spheres $S^{p,q}$ are a set of weak generators for $\mathcal{T}$: that is, the smallest full triangulated subcategory of $\mathcal{T}$ containing the $S^{p,q}$'s and closed under infinite direct sums is $\mathcal{T}$ itself (see Remark 7.4). A result of Neeman, recounted in [K, 5.3], shows that any compact object in $\mathcal{T}$ is a retract of something that can be built from the generators via finitely many extensions. The fact that $\Sigma^\infty X_+$ is compact therefore finishes the proof. □

We now turn to the proof of Theorem 9.5. A fundamental difficulty with the injective model structure on $\mathcal{MV}$ (cf. [MV] or [Ja, App. B]) is that a directed colimit of fibrant objects need not be fibrant. The projective structure [Bl] doesn't have this problem, but it has other difficulties instead. We will stick with the injective structure, but to get around the problem with colimits we need a new definition.

**Definition 9.7.** *An object $F \in \mathcal{MV}$ is **almost-fibrant** if the following conditions are satisfied:*

(1) *It is flasque, in the sense of* [Ja, Sec. 1.4]*;*
(2) *It is objectwise-fibrant;*
(3) *For every elementary Nisnevich cover* $\{U, V \to X\}$*, the natural map* $F(X) \to F(U) \times_{F(U \times_X V)} F(V)$ *is a weak equivalence.*
(4) *For every smooth scheme* $X$*, the map* $F(X) \to F(X \times \mathbb{A}^1)$ *is a weak equivalence.*

Note that the flasque condition implies in particular that for any inclusion of schemes $W \to X$ the map $F(X) \to F(W)$ is a Kan fibration. It follows that the pullback $F(U) \times_{F(U \times_X V)} F(V)$ appearing in (3) is actually a homotopy pullback as well.

The following two results follow immediately from the ideas of [Bl], [DHI], and [Ja, Sec. 1.4], so we only sketch the proofs. The first one explains the name 'almost-fibrant':

**Proposition 9.8.** *(a) Every fibrant object of* $\mathcal{MV}$ *is almost-fibrant.*
*(b) If* $F$ *and* $F'$ *are almost-fibrant, then a map* $F \to F'$ *is a weak equivalence in* $\mathcal{MV}$ *if and only if it is an objectwise weak equivalence.*
*(c) If* $F$ *is almost-fibrant and* $X$ *is a smooth scheme then* $Ho\,(X, F) = \pi_0 F(X) = \pi_0 \operatorname{Map}(X, F)$*.*

*Proof.* Part (a) follows from the definitions. For part (b), observe that almost-fibrant objects are fibrant in the $\mathbb{A}^1$-local projective model structure of [Bl]. A weak equivalence between fibrant objects in this structure is necessarily an objectwise weak equivalence, as this is a general property of Bousfield localizations.

For part (c), let $F \to F'$ be a fibrant-replacement in $\mathcal{MV}$. Then $\operatorname{Ho}(X, F) \cong \pi_0 \operatorname{Map}(X, F') \cong \pi_0 F'(X)$, and by (b) the latter is isomorphic to $\pi_0 F(X) \cong \pi_0 \operatorname{Map}(X, F)$. □

Now we record some nice properties of almost-fibrant objects. Recall that if $F \in \mathcal{MV}_*$ then $\Omega^{2,1} F$ is the simplicial presheaf whose value at $X$ is the pullback of $* \to F(X) \leftarrow F(X \times \mathbb{P}^1)$.

**Proposition 9.9.** *(a) A directed colimit of almost-fibrant objects is almost-fibrant.*
*(b) Suppose* $\{E_i\}$ *and* $\{F_i\}$ *are two directed systems of almost-fibrant objects, and* $\{E_i \to F_i\}$ *is a levelwise weak equivalence of systems. Then* $\operatorname{colim} E_i \to \operatorname{colim} F_i$ *is also a weak equivalence.*
*(c) If* $F \in \mathcal{MV}_*$ *is almost-fibrant, then* $\Omega^{2,1} F$ *is almost-fibrant. If* $F \to F'$ *is a weak equivalence between pointed almost-fibrant objects, then* $\Omega^{2,1} F \to \Omega^{2,1} F'$ *is an objectwise weak equivalence.*
*(d) If* $i \mapsto F_i$ *is a directed system in* $\mathcal{MV}_*$*, then the canonical map* $\operatorname{colim}_i \Omega^{2,1} F_i \to \Omega^{2,1}(\operatorname{colim}_i F_i)$ *is an isomorphism.*
*(e) If* $F \to F'$ *is a weak equivalence between almost-fibrant objects of* $\mathcal{MV}_*$ *and* $X$ *is a pointed smooth scheme, then* $\operatorname{Map}_{\mathcal{MV}_*}(X, F) \to \operatorname{Map}_{\mathcal{MV}_*}(X, F')$ *is a weak equivalence of simplicial mapping spaces.*

*Proof.* Part (a) easily follows from the definitions, together with the fact that filtered colimits preserve fibrations and weak equivalences of simplicial sets. Part (b) follows from Proposition 9.8(b) and the observation that colimits preserve objectwise weak equivalences.

For the first statement in part (c), the only hard step is to check the flasque condition; but this is done in [Ja, Cor. 1.9, Cor. 1.10]. The second statement follows from the first statement and Proposition 9.8(b). Part (d) is immediate from the definitions. For part (e), note that if $X$ is a pointed smooth scheme and $F \in \mathcal{MV}_*$ then the simplicial mapping space $\mathrm{Map}_{\mathcal{MV}_*}(X,F)$ is the pullback of $* \to \mathrm{Map}_{\mathcal{MV}}(*,F) \leftarrow \mathrm{Map}_{\mathcal{MV}}(X,F)$. When $F$ is almost-fibrant the right-hand map is a fibration. Using Proposition 9.8(b) and the right properness of $s\mathcal{S}et$, it follows that $\mathrm{Map}_{\mathcal{MV}_*}(X,F) \to \mathrm{Map}_{\mathcal{MV}_*}(X,F')$ is a weak equivalence. □

Recall that a motivic spectrum $\{E_i\}$ is fibrant if each $E_i$ is fibrant in $\mathcal{MV}$ and the maps $E_i \to \Omega^{2,1}E_{i+1}$ are all weak equivalences.

**Definition 9.10.** *A motivic spectrum $\{E_i\}$ is* ***almost-fibrant*** *if each $E_i$ is almost-fibrant in $\mathcal{MV}$ and the maps $E_i \to \Omega^{2,1}E_{i+1}$ are weak equivalences.*

**Proposition 9.11.** *(a) Every fibrant motivic spectrum is almost-fibrant.*
*(b) A directed colimit of almost-fibrant spectra is almost-fibrant.*
*(c) If $E$ and $E'$ are almost-fibrant spectra, a map $E \to E'$ is a stable weak equivalence if and only if for every $n \geq 0$ the map $E_n \to E'_n$ is an objectwise weak equivalence in $\mathcal{MV}$.*
*(d) If $i \mapsto E_i$ is a directed system of almost-fibrant motivic spectra, then the natural map $\mathrm{hocolim}\, E_i \to \mathrm{colim}\, E_i$ is a weak equivalence.*

*Proof.* Part (a) follows from Proposition 9.9(a). Part (b) is routine, using Proposition 9.9(a-d). Part (c) is well-known for fibrant spectra, so it suffices to produce a fibrant-replacement $E \to \hat{E}$ which is an objectwise weak equivalence in all levels (as the same can be done for $E'$). This can be accomplished by considering naive spectra with levelwise weak equivalences and taking a fibrant replacement in this setting, and then referring to Proposition 9.9(c) to see that this gives an $\Omega$-spectrum.

For (d), it suffices to show that if $\{E_i\}$ and $\{F_i\}$ are directed systems of almost-fibrant objects and each $E_i \to F_i$ is a stable weak equivalence, then $\mathrm{colim}\, E_i \to \mathrm{colim}\, F_i$ is a stable weak equivalence. By part (c), the map of $n$th spaces $[E_i]_n \to [F_i]_n$ is an objectwise weak equivalence in $\mathcal{MV}_*$. It follows that $[\mathrm{colim}_i\, E_i]_n \to [\mathrm{colim}_i\, F_i]_n$ is still an objectwise weak equivalence in $\mathcal{MV}_*$, and this implies that $\mathrm{colim}\, E_i \to \mathrm{colim}\, F_i$ is a stable weak equivalence. □

Recall that naive spectra form a *simplicial* model category, with the simplicial action being the level-wise one inherited from $\mathcal{MV}_*$. We write $\mathrm{Map}(-,-)$ for the simplicial mapping space, both for motivic spectra and in $\mathcal{MV}_*$.

**Lemma 9.12.** *If $E$ is an almost-fibrant spectrum then $Ho\,(\Sigma^{(2n,n)+\infty}X, E)$ is isomorphic to $\pi_0\,\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E)$, for any pointed smooth scheme $X$.*

*Proof.* Let $E \to \hat{E}$ be a fibrant-replacement. If $n \geq 0$ then

$$\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E) \cong \mathrm{Map}_{\mathcal{MV}_*}(X \wedge \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1, E_0) \cong \mathrm{Map}_{\mathcal{MV}_*}(X, \Omega^{2n,n}E_0).$$

Here $\Omega^{2n,n}E_0 = \Omega^{2,1}\Omega^{2,1}\cdots\Omega^{2,1}E_0$, and both isomorphisms come from various adjunctions. Note that the same chain of isomorphisms exists with $E$ replaced by $\hat{E}$. By Proposition 9.9(c), $\Omega^{2n,n}E_0 \to \Omega^{2n,n}\hat{E}_0$ is an objectwise weak equivalence between almost-fibrant objects. So Proposition 9.9(e) implies that $\mathrm{Map}_{\mathcal{MV}_*}(X, \Omega^{2n,n}E_0) \simeq \mathrm{Map}_{\mathcal{MV}_*}(X, \Omega^{2n,n}\hat{E}_0) \cong \mathrm{Map}(\Sigma^{(2n,n)+\infty}X, \hat{E})$. The set of components of the latter space is of course $\mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, E)$ (using the fact that $\Sigma^{(2n,n)+\infty}X$ is cofibrant).

The case $n < 0$ is similar, except one instead starts with the isomorphism $\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E) \cong \mathrm{Map}_{\mathcal{MV}_*}(X, E_{-n})$. □

**Lemma 9.13.** *Let $F_0 \to F_1 \to \cdots$ be a directed system in $\mathcal{MV}_*$, and let $X$ be a pointed smooth scheme. Then $\mathrm{Map}_{\mathcal{MV}_*}(X, \mathrm{colim}\, F_i) \cong \mathrm{colim}\, \mathrm{Map}_{\mathcal{MV}_*}(X, F_i)$.*

*Proof.* Note that the result is obvious for unpointed mapping spaces, since $\mathrm{Map}_{\mathcal{MV}}(X, F) \cong F(X)$. In the pointed case $\mathrm{Map}_{\mathcal{MV}_*}(X, \mathrm{colim}\, F_i)$ equals the pullback of the diagram $* \to \mathrm{Map}_{\mathcal{MV}}(*, \mathrm{colim}\, F_i) \leftarrow \mathrm{Map}_{\mathcal{MV}}(X, \mathrm{colim}\, F_i)$, which is the same as the colimit of the pullbacks of diagrams $* \to \mathrm{Map}_{\mathcal{MV}}(*, F_i) \leftarrow \mathrm{Map}_{\mathcal{MV}}(X, F_i)$. □

Finally we can prove the theorem:

*Proof of Theorem 9.5.* By Lemma 9.3 we must show that if $E_0 \to E_1 \to \cdots$ is a directed system of motivic spectra then $\mathrm{colim}_i \,\mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, E_i) \to \mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, \mathrm{hocolim}\, E_i)$ is an isomorphism. By taking a functorial fibrant replacement for each $E_i$, we can assume that the $E_i$'s are fibrant.

Since each $E_i$ is (in particular) almost-fibrant, it follows from Proposition 9.11(d) that $\mathrm{hocolim}_i\, E_i \to \mathrm{colim}_i\, E_i$ is a stable weak equivalence. Thus, we have

$$\mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, \mathrm{hocolim}_i\, E_i) = \mathrm{Ho}\,(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}_i\, E_i).$$

This in turn is isomorphic to $\pi_0\,\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}_i\, E_i)$ by Lemma 9.12 because the spectrum $\mathrm{colim}_i\, E_i$ is almost-fibrant. So, we are reduced to showing that

$$\mathrm{colim}\, \pi_0\, \mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E_i) \to \pi_0\, \mathrm{Map}(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}\, E_i)$$

is an isomorphism. The idea is to prove that the mapping spaces themselves are isomorphic, using adjointness to reduce to mapping spaces in $\mathcal{MV}_*$.

When $n < 0$ the mapping space $\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}\, E_i)$ is equal to $\mathrm{Map}_{\mathcal{MV}_*}(X, \mathrm{colim}[E_i]_{-n})$. By Lemma 9.13 we can pull the colimit outside, and then adjointness gives us $\mathrm{colim}\, \mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E_i)$.

When $n > 0$ one has

$$\mathrm{Map}(\Sigma^{(2n,n)+\infty}X, \mathrm{colim}\, E_i) \cong \mathrm{Map}_{\mathcal{MV}_*}(X, \Omega^{2n,n}(\mathrm{colim}[E_i]_0)),$$

where $\Omega^{2n,n}(-)$ is shorthand for $\Omega^{2,1} \cdots \Omega^{2,1}(-)$. By Proposition 9.9(d) we can commute the $\Omega^{2n,n}$ past the colimit, and then Lemma 9.13 lets us take the colimit outside. Using adjointness again, we get $\mathrm{colim}\, \mathrm{Map}(\Sigma^{(2n,n)+\infty}X, E_i)$. This completes the proof. □

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