THE GENERALIZED BURNSIDE RING AND THE K–THEORY OF A RING WITH ROOTS OF UNITY

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§1. Introduction

Determining the algebraic K-theory of rings of integers in number fields has been the goal of much research. In [10] D. Quillen showed that the Hurewicz map $h : Q_0(S^0) \to B\,\mathrm{GL}(\mathbb{Z})^+$ (see 1.1 for the notation) induces an interesting map on homotopy groups from the stable homotopy groups of spheres to the algebraic K-theory of the ring $\mathbb{Z}$ of rational integers. Quillen observed that if ℓ is an odd prime and if $p \neq \ell$ is another prime generating the ℓ-adic units, then the composite map

$$Q_0(S^0) \xrightarrow{h} B\,\mathrm{GL}(\mathbb{Z})^+ \xrightarrow{\rho} B\,\mathrm{GL}(\mathbb{F}_p)^+$$

induces a surjection

$$\pi_*^s(S^0, \mathbb{Z}/\ell) \to K_*(\mathbb{Z}, \mathbb{Z}/\ell) \to K_*(\mathbb{F}_p, \mathbb{Z}/\ell)$$

on mod ℓ homotopy groups. In geometric terms, one can write $Q_0(S^0) \simeq \mathrm{Im}\,J \times \mathrm{Coker}\,J$ where $\mathrm{Im}\,J$ is the factor of $Q_0(S^0)$ associated to the J-homomorphism $J : \mathrm{O} \to Q_0(S^0)$. The space $\mathrm{Im}\,J$ is then equivalent after localization at ℓ to $B\,\mathrm{GL}(\mathbb{F}_p)^+$ in such a way that the composite

$$\mathrm{Im}\,J \times \mathrm{Coker}\,J \simeq Q_0(S^0) \xrightarrow{h} B\,\mathrm{GL}(\mathbb{Z})^+ \xrightarrow{\rho} B\,\mathrm{GL}(\mathbb{F}_p)^+$$

can be identified after localization at ℓ with the projection $r : \mathrm{Im}\,J \times \mathrm{Coker}\,J \to \mathrm{Im}\,J$.

In [6] the third author complemented Quillen's result by proving that the Hurewicz map h does not in fact detect anything in stable homotopy theory beyond what Quillen

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found. He did this by verifying that the localization of h at ℓ factors through $\operatorname{Im} J$, or, more precisely, that the maps

$$h,\ h\cdot s\cdot r : Q_0(S^0) \to B\,\mathrm{GL}(\mathbb{Z})^+$$

are homotopic after localization at ℓ, where $s : \operatorname{Im} J \to Q_0(S^0)$ is a suitable right inverse after localization at ℓ for the factor projection $r : Q_0(S^0) \to \operatorname{Im} J$. It follows that Quillen's map $h_* : \pi_*^s(S^0, \mathbb{Z}/\ell) \to K_*(\mathbb{Z}, \mathbb{Z}/\ell)$ annihilates the homotopy of $\operatorname{Coker} J$, and that for any ring R the structure of $K_*(R, \mathbb{Z}/\ell)$ as a module over $\pi_*^s(S^0, \mathbb{Z}/\ell)$ extends (uniquely) to a structure of $K_*(R, \mathbb{Z}/\ell)$ as a module over $\pi_*(\operatorname{Im} J, \mathbb{Z}/\ell) = K_*(\mathbb{F}_p, \mathbb{Z}/\ell)$.

Let ζ_a denote a primitive ℓ^a-th root of unity and μ the group of ℓ-primary roots of unity of $\mathbb{Z}[\zeta_a]$. The inclusion $\mu \subset \mathrm{GL}_1(\mathbb{Z}[\zeta_a])$ leads to an enriched analogue

$$h : Q_0(B\mu_+) \to B\,\mathrm{GL}(\mathbb{Z}[\zeta_a])^+$$

of Quillen's Hurewicz map. Let ℓ and p be primes as before (ℓ is odd and p generates the ℓ-adic units) and let $\mathbb{F}_q$ denote $\mathbb{F}_p[\zeta_a]$. In $\mathbb{Z}[\zeta_a]$ there is a unique prime above p with residue class field $\mathbb{F}_q$. The main result of B. Harris and G. Segal [3] implies that the composition

$$Q_0(B\mu_+) \xrightarrow{h} B\,\mathrm{GL}(\mathbb{Z}[\zeta_a])^+ \xrightarrow{\rho} B\,\mathrm{GL}(\mathbb{F}_q)^+$$

induces a surjection of mod ℓ homotopy groups. In fact, for a suitable space X there is after localization at ℓ a product decomposition $Q_0(B\mu_+) \simeq B\,\mathrm{GL}(\mathbb{F}_q)^+ \times X$ in such a way that the (localized) composition

$$B\,\mathrm{GL}(\mathbb{F}_q)^+ \times X \simeq Q_0(B\mu_+) \to B\,\mathrm{GL}(\mathbb{Z}[\zeta_a])^+ \to B\,\mathrm{GL}(\mathbb{F}_q)^+$$

can be identified with the projection $r : B\,\mathrm{GL}(\mathbb{F}_q)^+ \times X \to B\,\mathrm{GL}(\mathbb{F}_q)^+$.

The purpose of this paper is to extend the results of the third author to this wider context. Theorem 4.1 asserts that after localization at ℓ the maps

$$h,\ h\cdot s\cdot r : Q_0(B\mu_+) \to B\,\mathrm{GL}(\mathbb{Z}[\zeta_a])^+$$

are homotopic, where $s : B\,\mathrm{GL}(\mathbb{F}_q)^+ \to Q_0(B\mu_+)$ is a suitable right inverse for the factor projection r. It follows that for any algebra R over $\mathbb{Z}[\zeta_a]$ the structure of $K_*(R, \mathbb{Z}/\ell)$ as a module over $\pi_*^s(B\mu_+, \mathbb{Z}/\ell)$ extends (uniquely) to a structure of $K_*(R, \mathbb{Z}/\ell)$ as a module over $K_*(\mathbb{F}_q, \mathbb{Z}/\ell)$. We also make some further remarks (4.10) about the relationship of our work to [6], and briefly consider the situation of infinite cyclotomic extensions (4.12).

Let G be a finite ℓ-group. To prove Theorem 4.1 we show that h, $h\cdot s\cdot r$ have homotopic compositions with any map $BG \to Q_0(B\mu_+)$. Our technique is to use the generalized Burnside ring $A(G, \mu)$ to study maps from BG to $Q_0(B\mu_+)$ and to use the representation ring $\mathcal{R}_R G$ of G over R to study maps from BG to $B\,\mathrm{GL}(R)^+$. The key algebraic result is Theorem 3.5, which amounts to an algebraic analogue of (4.1) and asserts that the map from $A(G, \mu)$ to $\mathcal{R}_{\mathbb{Z}[\zeta_a]} G$ naturally factors through a surjection $A(G, \mu) \to \mathcal{R}_{\mathbb{F}_q} G$.

We conjecture that, after localization at ℓ, the composition

$$h\cdot s : B\,\mathrm{GL}(\mathbb{F}_q)^+ \to B\,\mathrm{GL}(\mathbb{Z}[\zeta_a])^+$$

extends to an infinite loop map. One can view Proposition 4.9 as evidence that such an extension exists. The reader is referred to [7] for a discussion of this conjecture and related matters.

1.1 Notation and conventions. The symbol "+" as a subscript denotes a disjoint basepoint; as a superscript it denotes Quillen's plus construction. We write $\Sigma^\infty X_+$ for the suspension spectrum of the pointed space X_+ and QX_+ $(= Q(X_+))$ for the corresponding infinite loop space, where as usual $Q = \Omega^\infty\Sigma^\infty$. Similarly $Q_0X_+ \equiv Q_0(X_+)$ denotes the basepoint component of QX_+. We employ the notation $[-,-]$ to denote pointed homotopy classes of maps between pointed spaces or spectra.

§2. Preliminaries on representation rings and the generalized Burnside ring.

Let G be a finite group. We will let AG denote the *Burnside ring* of G, i.e., the Grothendieck ring of finite G-sets, with sum and product induced respectively by disjoint union and cartesian product of G-sets. Suppose that H is another finite group. The *Burnside module* $A(G,H)$ is defined as follows (cf. [5]). A (G,H)-*set* is a set P with a left G-action and a free right H-action such that the two actions commute; equivalently, P is an H-free $(G\times H)$-set, but it is convenient to put the two actions on opposite sides. Then $A(G,H)$ is the free abelian group on the isomorphism classes $[P]$ of finite (G,H)-sets, subject to the relations $[P \amalg Q] = [P] + [Q]$. We now list some properties of the group $A(G,H)$ and describe some ways in which it is related to representation rings.

(2.1). A (G,H)-set is indecomposable if and only if it has the form $P = P_{K,\theta}$ where $K \subset G$, $\theta : K \to H$ is a homomorphism and $P_{K,\theta} = G \times_K H$. There is an obvious conjugation action of $G\times H$ on the set of all pairs (K,θ), and $P_{K,\theta}$ is isomorphic to $P_{K',\theta'}$ if and only if (K,θ) and (K',θ') lie in the same orbit of this action. As an abelian group, $A(G,H)$ is freely generated by the elements $[P_{K,\theta}]$ as (K,θ) runs through a set of orbit representatives.

(2.2). The group $A(G,H)$ is a module over AG, with multiplication $[P][Q] = [P\times Q]$, $[P]\in AG$, $[Q]\in A(G,H)$. Here G acts diagonally on $P\times Q$ and H acts only on the Q-factor. There is a natural map of AG-modules $\sigma : A(G,H)\to AG$ given by $[P]\mapsto [P/H]$; in addition, σ has a canonical right inverse η, given by $\eta([S]) = [S\times H]$ (here H has a trivial G-action). Composing σ with the usual augmentation $\epsilon : AG{\to}\mathbb{Z}$, we get an augmentation $\epsilon' : A(G,H){\to}\mathbb{Z}$. The augmentation ϵ is split by the ring homomorphism $\mathbb{Z}\to A(G)$ which sends an integer $n\ge 0$ to the trivial G-set $[\mathbf{n}]$ with n elements.

(2.3). Let $IG = \operatorname{Ker}\epsilon$ and $I(G,H) = \operatorname{Ker}\epsilon'$. Then if G is an ℓ-group, the IG-adic and ℓ-adic filtrations on $A(G,H)$ are topologically equivalent on $I(G,H)$.

(2.4). Suppose that H is abelian. Then $A(G,H)$ is a commutative ring, with product $[P][Q] = [P\times_H Q]$. The maps η and σ of (2.2) make $A(G,H)$ into an augmented algebra over AG.

Given a (G,H)-set P, define a map of spectra $\alpha_P : \Sigma^\infty BG_+ \to \Sigma^\infty BH_+$ as the composite

$$\Sigma^\infty BG_+ \xrightarrow{\tau} \Sigma^\infty (EG\times_G (P/H)_+) \xrightarrow{\Sigma^\infty h} \Sigma^\infty BH_+$$

where τ is the transfer and h classifies the principal H-bundle $EG\times_G P$ over $EG\times_G(P/H)$. In the special case $P = P_{K,\theta}$, α_P is the familiar composition of transfer $\tau : \Sigma^\infty BG_+ \to \Sigma^\infty BK_+$ and $B\theta : \Sigma^\infty BK_+ \to BH_+$.

(2.5). The correspondence $P \mapsto \alpha_P$ extends to a homomorphism

$$\alpha : A(G,H) \to [\Sigma^\infty BG_+, \Sigma^\infty BH_+]$$

of AG-modules which is a ring homomorphism if H is abelian. Here $[\Sigma^\infty BG_+, \Sigma^\infty BH_+]$ is a module over the stable cohomotopy ring $\pi_s^0(BG_+)$, and so receives an AG-module structure from the ring homomorphism $AG = A(G,*) \to \pi_s^0 BG_+$. Note that $\Sigma^\infty BH_+$ is a commutative augmented ring spectrum if H is abelian.

(2.6). Recall that QBH_+ can be identified with the group completion of the monoid $\coprod_n (\Sigma_n \wr H)$, and Q_0BH_+ can be identified with $B(\Sigma_\infty \wr H)^+$. Now a (G,H)-set P corresponds to a conjugacy class of homomorphisms $G \to \Sigma_n \wr H$, $n = |P/H|$, and so determines a unique homotopy class $BG \to B(\Sigma_n \wr H)$. The map α can then be reinterpreted in an obvious in terms of group completion or in terms of the plus construction.

(2.7). For any commutative ring R, let $\mathcal{R}_R G$ denote the representation ring of G over R, i.e., the Grothendieck group of finitely generated R-projective RG-modules. It is clear that $\mathcal{R}_R G$ is a ring under tensor product. Let $\mathcal{R}'_R G$ denote the Grothendieck group of all finitely generated RG-modules. If R is a Dedekind domain, then the natural map $\mathcal{R}_R G \to \mathcal{R}'_R G$ is an isomorphism. If G is an ℓ-group and E is a field of characteristic ℓ, then the augmentation ideal of EG is nilpotent and every finitely generated module over EG has a finite filtration with each filtration quotient isomorphic to the trivial G-module E; consequently $\mathcal{R}'_E G \cong \mathbb{Z}$, generated by the trivial G-module E.

(2.8). For a commutative ring R there is an obvious ring homomorphism $AG \to \mathcal{R}_R G$ induced by mapping a G-set P to the permutation module RP. More generally, suppose given a homomorphism $j : H \to R^*$. Then there is a natural homomorphism

$$\varphi(j) : A(G,H) \to \mathcal{R}_R G$$

of AG-modules, given by $P \mapsto RP \otimes_{RH} R$. Here R is an RH-module via j. If H is abelian, $\varphi(j)$ is a ring homomorphism. We remark that the image of $\varphi(j)$ lies in the subroup of $\mathcal{R}_R G$ generated by the R-free RG-modules. Observe that if $P = P_{K,\theta}$, then $\varphi(j)(P_{K,\theta})$ is the induced representation $RG \otimes_{RK} R$, where the action of K on R is given by $j \cdot \theta$. Equivalently, $\varphi(j)(P_{K,\theta})$ is a "monomial representation", that is, the corresponding homomorphism $G \to \mathrm{GL}_n R$ maps into the monomial matrices $\Sigma_n \wr R^*$ (up to conjugacy), where $n = |G/K|$. If μ is a subgroup of R^* and $j : \mu \to R^*$ is the inclusion we will write φ_R instead of $\varphi(j)$.

(2.9). There is a natural ring homomorphism

$$\beta : \mathcal{R}_R G \to [BG_+, B\,\mathrm{GL}(R)^+ \times K_0 R]$$

obtained essentially by associating to an R-projective G-module M the map determined by the G-structure map of M, $G \to \mathrm{Aut}(M)$ and viewing $B\,\mathrm{GL}(R)^+ \times K_0R$ as the homotopy-theoretic group completion of $\coprod_M B\,\mathrm{Aut}(M)$. If H is abelian, then $j : H \to R^*$ induces a functor $P \mapsto RP \otimes_{RH} R$ of bipermutative categories (free right H-sets) $\to$ (f.g. projective R-modules) and hence a map of ring spectra $\Sigma^\infty BH_+ \to KR$. The adjoint map

$$\Phi(j) : BH_+ \to BGLR^+ \times K_0R$$

restricted to BH is just the obvious composite $BH \overset{Bj}{\to} B\,\mathrm{GL}_1(R) \to B\,\mathrm{GL}(R)^+ \times \{1\}$. It induces a ring homomorphism

$$\Phi(j)_* : [BG_+, QBH_+] \to [BG_+, B\,\mathrm{GL}(R)^+ \times K_0R].$$

We will write $\Phi_0(j)$ for the restricted map $Q_0BH_+ \to B\,\mathrm{GL}(R)^+$. If μ is a subgroup of R^* and $j : \mu \to R^*$ is the inclusion, we will write Φ_R instead of $\Phi_0(j)$.

(2.10). Let $\hat{A}(G,H)$ and $\hat{\mathcal{R}}_RG$ denote the IG-adic completions of respectively $A(G,H)$ and $\mathcal{R}_RG$. Then there are maps $\hat\alpha$ and $\hat\beta$ which fit into a commutative diagram

$$\begin{array}{ccccc} A(G,H) & \longrightarrow & \hat{A}(G,H) & \xrightarrow{\hat\alpha} & [BG_+, QBH_+] \\ \downarrow{\scriptstyle \varphi(j)} & & \downarrow{\scriptstyle \hat\varphi(j)} & & \downarrow{\scriptstyle \Phi(j)_*} \\ \mathcal{R}_RG & \longrightarrow & \hat{\mathcal{R}}_RG & \xrightarrow{\hat\beta} & [BG_+, B\,\mathrm{GL}(R)^+ \times K_0R] \end{array}$$

in which the composite of the two upper arrows is α (2.5) and the composite of the two lower arrows is β (2.9).

The first assertion of the following theorem is a result of G. Lewis, J.P. May, and J. McClure [4] derived from the affirmation of the Segal Conjecture; the second assertion is a result of D. Rector [8].

2.11 Theorem. ([4], [8]) *Let G, H be finite groups. Then the map $\hat\alpha$ of (2.10) is an isomorphism. Moreover, if R is a finite field, then map $\hat\beta$ of (2.10) is also an isomorphism.*

§3. Monomial representations of finite ℓ-groups

Throughout this section G denotes a finite ℓ-group. If R is a ring, let $\mu(R)$ denote the ℓ-torsion subgroup of R^*, that is, the group of all ℓ^n-th roots of unity, $n \geq 0$, and let $a(R)$ be defined by the equation $|\mu(R)| = \ell^{a(R)}$, $0 \leq a(R) \leq \infty$. The condition $|\mu(R)| > 2$ is equivalent to $a(r) \geq 1$ if ℓ is odd or to $a(R) \geq 2$ if $\ell = 2$.

Our goal in this section is to investigate the map $\varphi(j) : A(G,\mu) \to \mathcal{R}_RG$ (2.8) in the special situation in which μ is a subgroup of $\mu(R)$ and $j : \mu \to R^*$ is the inclusion; recall (2.8) that in this case we denote $\varphi(j)$ by φ_R.

We begin with the following (known) description of the structure of a simple G-module over a field of characteristic different from ℓ. Our discussion follows the analysis of P. Roquette [11].

3.1 Proposition. *Let F be a field of characteristic different from ℓ with $|\mu(F)| > 2$, and let V be a simple right FG module. Then $\mathrm{End}_{FG}\,V$ is a cyclotomic extension field F' of F, and there exists a subgroup $K \subset G$ and homomorphism $\theta : K \to \mu(F')$ such that $F' = F[\theta(K)]$ and $V \cong F' \otimes_{FK} FG$ as an (F', FG)-bimodule.*

Proof. Denote by D the division algebra $\mathrm{End}_{FG} V$. We must show that D is a cyclotomic field extension of F. Clearly, we may assume that G acts faithfully on V. If V admits a non-trivial decomposition $V = W_1 \oplus \cdots \oplus W_k$ as D-modules such that G permutes the W_i, then V is obtained by induction from a proper subgroup of G. Arguing inductively on the order of G, we may assume that V admits no such decomposition. By [11], G must then be a group of "normal rank one" and hence either G is cyclic or $\ell = 2$ and G is dihedral, semidihedral or quaternionic. If G is cyclic, then one readily verifies that D is a cyclotomic extension F' of F and that the action of G on V is given by a homomorphism $\theta : G \to \mathrm{Aut}_{FG} V \cong (F')^*$ such that $F' = F[\theta(G)]$. We proceed to show that other normal rank one groups do not occur.

Assume then that $\ell = 2$, $a(F) \geq 2$, and that G is normal rank one but not cyclic; in this case G fits into an extension $\mathbb{Z}/2^n \to G \to \mathbb{Z}/2$, $n \geq 2$. Let y generate $\mathbb{Z}/2^n$ and let $x \in G$ project to the generator of $\mathbb{Z}/2$. Then either $xyx^{-1} = y^{-1}$ (dihedral and quaternionic cases) or $xyx^{-1} = y^{2^{n-1}-1}$ (semidihedral case). In all cases $xTx^{-1} = T^{-1}$, where $T = y^{2^{n-2}}$ has order four. Consider V as a module over $\langle T \rangle = \mathbb{Z}/4$. Since $\langle T \rangle$ is normal and V is primitive, the restriction of V to $\langle T \rangle$ is isotypical, and nontrivial since V is faithful. Since $a(F) \geq 2$, we conclude that $T\nu = i\nu$ for all ν, where $i \in \mu(F)$ is a fixed element of order 4. But then $T(x\nu) = xT^{-1}\nu = -i \cdot x\nu$, which is a contradiction. □

Proposition 3.1 enables us to prove the following surjectivity result, which asserts that sometimes the representation ring of G is generated by monomial representations.

3.2 Proposition. *Let F be a field of characteristic different from ℓ with $|\mu(F)| > 2$. Let μ denote $\mu(F)$. Then*

$$\varphi_F : A(G, \mu) \to \mathcal{R}_F G$$

is surjective with right inverse $\sigma : \mathcal{R}_F G \to A(G, \mu)$.

Proof. Since FG is semi-simple, $\mathcal{R}_F G$ is freely generated as an abelian group by the set of isomorphism classes of simple FG-modules. Thus, the surjectivity of φ_F as well as the existence of the left inverse σ follows immediately once we verify that every irreducible FG-module lies in the image of φ_F. As discussed in (2.8), any representation of G obtained by induction from a one-dimensional representation $\theta : K \to \mu$ of some subgroup K of G is of the form $\varphi_F(P_{K,\theta})$. Thus, it suffices to prove that every irreducible representation of FG is induced from some one-dimensional representation of a subgroup K of G.

By (3.1), each irreducible FG-module is of the form $V = L \otimes_{FH} FG$, where $H \subset G$ is a subgroup acting on the cyclotomic field extension L/F via a homomorphism $\theta : H \to \mu(L)$ with $L = F[\theta(H)]$. Let K denote $\theta^{-1}(\mu(F)) \subset H$. Since F is preserved by the action of $\theta(K)$, we conclude the existence of a natural homomorphism of FG-modules

$$F \otimes_{FK} FG \to L \otimes_{FH} FG.$$

This map is a surjective homomorphism of F-vector spaces of the same dimension, and thus an isomorphism. (The equality of these dimensions comes down to the observation that since $|\mu(F)| > 2$ and $L = F[\theta(H)]$ is obtained by adjoining to F additional ℓ-primary roots of unity, the degree of L over F is equal to the order of the quotient group $\mu(L)/\mu(F)$, which in turn is equal to the index of K in H.) □

Remarks. (a) Proposition 3.2 is obviously false if $a(F) = 0$, although if ℓ is odd it is possible to recover the conclusion by making an additional assumption about F [6, 1.15]. (b) If $\ell = 2$ and $F = \mathbb{F}_q$ with $q = 3 \bmod 4$, Proposition 3.2 is false for $G = \mathbb{Z}/8$. (c) The assertion of Proposition 3.2 for fields of characteristic 0 is a special case of a theorem of J. Tornehave [12] (which also treats the case $a(F) = 1$ for $\ell = 2$ when $\mathrm{char}(F) = 0$).

3.3 Representation ring calculations. We will now use Morita theory to carry out certain standard representation ring calculations in a very explicit way. As a matter of terminology, if F is a number field with ring of integers R, we will call $R[1/\ell]$ the *ring of ℓ-integers* of F. The situation we are interested in is as follows. We let F be a number field with $|\mu(F) > 2|$, and $V_1, \ldots, V_N$ a complete nonredundant list of isomorphism classes of simple right FG-modules. For $1 \le i \le N$ we denote by F_i the endomorphism field (3.1) of V_i and by $\theta_i : K_i \to \mu(F_i)$ a homomorphism provided by 3.1, so that $V_i \cong F_i \otimes_{FK_i} FG$ as an (F_i, FG)-bimodule. We let S be the ring of ℓ-integers of F, S_i the ring of ℓ-integers of F_i, and $V_i^S \subset V_i$ the (S_i, SG)-bimodule $S_i \otimes_{SK_i} SG$. (Observe that the tensor product defining V_i^S makes sense because $\theta_i(K_i) \subset \mu(S_i) = \mu(F_i)$). Finally, E is a finite residue class field of S such that $\mu(S) \to \mu(E)$ is an isomorphism, E_i is the field $E \otimes_S S_i$, and V_i^E the (E_i, EG)-bimodule $E_i \otimes_{S_i} V_i^S$. (The fact that E_i is a field follows from the hypothesis on $a(F)$, the assumption $\mu(S) \cong \mu(E)$ and the fact (3.1) that F_i is obtained from F by adjoining ℓ-primary roots of unity: together these imply by elementary number theory that the prime of S determining E is inert in the extension F_i/F.)

There are maps $f_i^F : \mathcal{R}_F G \to K_0 F_i$ given by $f_i^F[M] = [V_i \otimes_{FG} M]$, maps $f_i^S : \mathcal{R}_S G \to K_0 S_i$ given by $f_i^S[M] = [V_i^S \otimes_{SG} M]$, and maps $f_i^E : \mathcal{R}_E G \to K_0 E_i$ given by $f_i^E[M] = [V_i^E \otimes_{EG} M]$. (Here $K_0(-)$ is defined to be the Grothendieck group of finitely generated projective *left* modules.)

3.4 Proposition. *In the above situation, there is a commutative diagram*

$$\begin{array}{ccccc}
\mathcal{R}_E G & \longleftarrow & \mathcal{R}_S G & \longrightarrow & \mathcal{R}_F G \\
{\scriptstyle \times f_i^E}\downarrow & & {\scriptstyle \times f_i^S}\downarrow & & \downarrow{\scriptstyle \times f_i^F} \\
\mathbb{Z}^N \cong \prod_{i=1}^N K_0 E_i & \longleftarrow & \prod_{i=1}^N K_0 S_i & \longrightarrow & \prod_{i=1}^N K_0 F_i \cong \mathbb{Z}^N
\end{array}$$

in which the horizontal arrows are induced by the evident ring homomorphisms. The vertical arrows in this diagram are isomorphisms. Moreover, if both of the groups $\prod_{i=1}^N K_0 E_i$ and $\prod_{i=1}^N K_0 F_i$ are identified with $\mathbb{Z}^N$ by using the standard bases provided by rank one free modules, then the two maps $\mathcal{R}_S G \to \mathbb{Z}^N$ given by this diagram are the same.

Proof. The action of G on the various modules involved gives a commutative diagram of rings and homomorphisms

$$\begin{array}{ccccc}
EG & \longleftarrow & SG & \longrightarrow & FG \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i=1}^N \mathrm{End}_{E_i} V_i^E & \longleftarrow & \prod_{i=1}^N \mathrm{End}_{S_i} V_i^S & \longrightarrow & \prod_{i=1}^N \mathrm{End}_{F_i} V_i
\end{array}$$

The Artin-Wedderburn theorem implies that in this diagram the right vertical arrow is an isomorphism. Let $d_i = \dim_{F_i} V_i$. By construction V_i^S is a free module of rank d_i over S_i. Since $|G|$ is invertible in S the ring SG is a maximal S-order in FG and for general reasons $\prod_{i=1}^N \mathrm{End}_{S_i} V_i^S \cong \prod_{i=1}^N M_{d_i}(S_i)$ is a maximal S-order in $\prod_{i=1}^N \mathrm{End}_{F_i} V_i \cong \prod_{i=1}^N M_{d_i}(F_i)$ (see [6, 1.7] and [9]); it follows that in the above diagram of rings the middle vertical arrow is an isomorphism. The left vertical arrow is then also an isomorphism, since it is the tensor product of the middle arrow over S with E.

The commutativity of the diagram of (3.4) is evident, and the isomorphism statements of (3.4) now follow from Morita theory; more precisely, what follows in the case of the map $(\times f_i^F)$ for instance is the fact that for each i the functor given by tensoring over $\mathrm{End}_{F_i} V_i$ with V_i gives an equivalence from the category of left $\mathrm{End}_{F_i} V_i$ modules to the category of F_i modules. The fact that the two indicated maps $\mathcal{R}_S G \to \mathbb{Z}^N$ agree amounts to the observation that both maps assign to a class $[M]$ then vector $(r_0, \ldots, r_N)$, where $r_i = \mathrm{rank}_{S_i}(V_i^S \otimes_{SG} M)$.

The following theorem is a discrete analogue of Theorem 4.1. Theorem 3.5 extends a result of the third author [6, 1.12] from the Burnside ring $A(G)$ to the generalized Burnside ring $A(G, \mu)$. Recall that ζ_a denotes a primitive ℓ^a-th root of unity.

3.5 Theorem. *Let $R = \mathbb{Z}[\zeta_a]$, where $a \geq 1$ if ℓ is odd and $a \geq 2$ if $\ell = 2$. Let E be a residue field of R such that the reduction map $R \to E$ induces an isomorphism from $\mu = \mu(R)$ to $\mu(E)$, and let $\sigma : \mathcal{R}_E G \to A(G, \mu)$ be a right inverse to φ_E (cf. 3.2). Then*

$$\varphi_R \cdot \sigma \cdot \varphi_E = \varphi_R \, .$$

In other words, there is a unique map $h = \varphi_R \cdot \sigma$ making the following triangle[224z commute

$$\begin{array}{ccc} A(G,\mu) & \overset{\varphi_R}{\to} & \mathcal{R}_R G \\ {\scriptstyle \varphi_E}\downarrow & \nearrow h & \\ \mathcal{R}_E G & & \end{array} .$$

3.6 Remark. The maps φ_E and φ_R in (3.5) are maps of modules over AG; since φ_E is a surjection (3.2) the map h, if it exists, is also a map of modules over AG.

Proof of 3.5. Since φ_E is surjective by Proposition 3.2, it suffices to prove that $\ker \varphi_E = \ker \varphi_R$. Let $S = R[1/\ell]$ and let F be the quotient field of R. By (3.4) there is an isomorphism $\mathcal{R}_E G \cong \mathcal{R}_F G$ which fits in the following commutative diagram:

$$\begin{array}{ccc} A(G,\mu) & \to & \mathcal{R}_S G \\ \downarrow & \swarrow & \downarrow \\ \mathcal{R}_E G & \cong & \mathcal{R}_F G \end{array}$$

The localization theorem yields an exact sequence

$$\bigoplus_{\mathcal{P}|\ell} \mathcal{R}'_{R/\mathcal{P}} G \overset{\tau}{\to} \mathcal{R}'_R G (\cong \mathcal{R}_R G) \overset{\eta}{\to} \mathcal{R}'_S G (\cong \mathcal{R}_S G) \to 0 \, .$$

(see 2.7) where $\mathcal{R}_{R/\mathcal{P}}G \cong \mathbb{Z}$, generated by the trivial G-module $[R/\mathcal{P}]$ of dimension one. The transfer map τ sends $[R/\mathcal{P}]$ to $[R] - [\mathcal{P}]$, where R and $\mathcal{P}$ are again regarded as trivial G-modules. Now the map $\mathcal{R}_R G \to K_0 R$ obtained by forgetting the G-action is canonically split by the construction which takes a projective R-module and gives it the trivial G-action. Hence $K_0 R$ is canonically embedded as a direct summand in $\mathcal{R}_R G$, and the preceeding remarks show that $\ker \eta$ is contained in the class group $\widetilde{K}_0 R$, and is in fact equal to the subgroup of $\widetilde{K}_0 R$ generated by the elements $[R] - [\mathcal{P}]$ as $\mathcal{P}$ runs through the non-principal primes of R over ℓ. Since φ_R maps $A(G, \mu)$ into the R-free RG-modules, $\operatorname{im} \varphi_R \cap \ker \eta = 0$. Consequently, in order to show that $\ker \varphi_E = \ker \varphi_R$ it is enough to show that $\ker \varphi_S = \ker \varphi_F$.

We will use the notation and indexing of (3.3) and (3.4). To show that $\ker \varphi_S = \ker \varphi_F$, it is enough to show that for any (G, μ)-set P and integer i between 1 and N the element $f_i^S(\varphi_S(P)) \in K_0 S_i$ is the class of a *free* S_i-module. Observe (2.8, 3.3) that by definition

$$\begin{aligned} f_i^S(\varphi_S(P)) &= V_i^S \otimes_{SG} \varphi_S(P) \\ &= S_i \otimes_{SK_i} SG \otimes_{SG} \varphi_S(P) \\ &= S_i \otimes_{SK_i} \varphi_S(P) \\ &= S_i \otimes_{SK_i} SP \otimes_{S\mu} S \end{aligned}$$

where K_i acts on S_i via $\theta_i : K_i \to \mu(S_i)$. Choose such a P, fix the integer i; the above formula shows that $f_i^S(\varphi_S(P))$ depends only on the structure of P as a (K_i, μ)-set. We may decompose P as a disjoint union of indecomposable (K_i, μ) sets (see 2.1), each of the form $P_{H,\theta}$ where $H \subset K_i$ is a subgroup and $\theta : H \to \mu$ a homomorphism. The summand of $f_i^S(\varphi_S(P))$ corresponding to such an $P_{H,\theta}$ is the module

$$S_i \otimes_{SK_i} SK_i \otimes_{SH} S = S_i \otimes_{SH} S\,.$$

where in the right-hand tensor product H acts on S by θ and on S_i by the composite of θ_i with the inclusion $H \to K_i$. It is clear that $S_i \otimes_{SH} S$ is a quotient module of S_i. Since this module is also a projective module over S_i, it must be either zero or isomorphic to S_i; in particular, it must be free.

§4. The main theorem

Let $\mathbf{L}$ denote the functor [1] on the category of spaces which assigns to each space X its localization $\mathbf{L}X$ at ℓ. If X is a connected loop space or a simply connected space, then the homotopy group $\pi_i \mathbf{L}X$ is isomorphic to $\mathbb{Z}_{(\ell)} \otimes \pi_i X$, where $\mathbb{Z}_{(\ell)}$ is the ring obtained from $\mathbb{Z}$ by inverting all primes except ℓ. There is a natural map $X \to \mathbf{L}X$ which in all of the cases of interest to us nduces an isomorphism on homology or cohomology with coefficients in $\mathbb{Z}_{(\ell)}$. We will sometimes use the same notation for a map $f : X \to Y$ and for the induced map $\mathbf{L}f : \mathbf{L}X \to \mathbf{L}Y$. The notation $\mathbf{L}B\,\mathrm{GL}(R)^+$ stands for $\mathbf{L}(B\,\mathrm{GL}(R)^+)$.

For the definition of the maps Φ_E and Φ_R which appear in the following statement, see (2.9).

4.1 Theorem. *Suppose that R is the ring $\mathbb{Z}[\zeta_a]$, where $a \geq 1$ if ℓ is odd and $a \geq 2$ if $\ell = 2$. Let $\mu = \mu(R)$ and let E denote a residue class field of R with $\mu \cong \mu(E)$. Then there is a map $s : \mathbf{L}B\,\mathrm{GL}(E)^+ \to \mathbf{L}Q_0(B\mu_+)$ such that in the diagram*

$$\mathbf{L}B\,\mathrm{GL}(E) \xrightarrow{s} \mathbf{L}Q_0(B\mu_+) \xrightarrow{\Phi_E} \mathbf{L}B\,\mathrm{GL}(E)^+ \xrightarrow{s} \mathbf{L}Q_0(B\mu_+) \xrightarrow{\Phi_R} \mathbf{L}B\,\mathrm{GL}(R)^+$$

the following two conditions hold

1. *the composite $\Phi_E \cdot s$ is homotopic to the identity map of $\mathbf{L}B\,\mathrm{GL}(E)^+$, and*
2. *the composite $\Phi_R \cdot s \cdot \Phi_E$ is homotopic to the map Φ_R.*

Remark. The existence of infinitely many quotient fields E of R satisfying the condition of (4.1) is guaranteed by the Tchebotarev density theorem.

Remark. Theorem 4.1 implies that up to homotopy the following diagrams commute:

$$\begin{array}{ccc} \mathbf{L}B\,\mathrm{GL}(E)^+ & \xrightarrow{\mathrm{id}} & \mathbf{L}B\,\mathrm{GL}(E)^+ \\ {\scriptstyle s}\downarrow & \nearrow{\scriptstyle \Phi_E} & \\ \mathbf{L}Q_0(B\mu_+) & & \end{array} \qquad \begin{array}{ccc} \mathbf{L}Q_0(B\mu_+) & \xrightarrow{\Phi_R} & \mathbf{L}B\,\mathrm{GL}(R)^+ \\ {\scriptstyle \Phi_E}\downarrow & \nearrow{\scriptstyle \Phi_R \cdot s} & \\ \mathbf{L}B\,\mathrm{GL}(E)^+ & & \end{array}$$

The composite map $\Phi_R \cdot s$ is determined uniquely up to homotopy (4.8) by the requirement that the diagram on the right commute. The map s itself does not appear to have any uniqueness property.

Remark. Since $\mathbf{L}Q_0(B\mu_+)$ and $\mathbf{L}B\,\mathrm{GL}(R)^+$ are infinite loop spaces, Theorem 4.1 implies that up to homotopy there are product decompositions

$$\mathbf{L}Q_0(B\mu_+) \cong \mathbf{L}B\,\mathrm{GL}(E)^+ \times X_1$$
$$\mathbf{L}B\,\mathrm{GL}(R)^+ \cong \mathbf{L}B\,\mathrm{GL}(E)^+ \times X_2$$

with respect to which Φ_R amounts to the projection $\mathbf{L}B\,\mathrm{GL}(E)^+ \times X_1 \to \mathbf{L}B\,\mathrm{GL}(E)^+$ followed by factor inclusion $\mathbf{L}B\,\mathrm{GL}(E)^+ \to \mathbf{L}B\,\mathrm{GL}(E)^+ \times X_2$.

Remark. The results of [2] show that if E is a field of characteristic different from ℓ which is a union of finite fields and $|a(E)| > 2$, then the homotopy type of $\mathbf{L}B\,\mathrm{GL}(E)^+$ depends only on the isomorphism type of the group $\mu(E)$.

For the convenience of the reader we state the following standard propositions. The first one is due originally in a more general form to D. Sullivan, and can be proved using his technique of compact representable functors.

4.2 Proposition. *Let X be a CW-complex and $X_0 \subset X_1 \subset X_2 \subset \cdots X$ a filtration of X by subcomplexes such that $\cup_i X_i = X$. Let Y and Z be loop spaces. Assume that the groups $\tilde{H}^j(X_i, \mathbb{Z}_{(\ell)})$ are finite for all i, j, and that the homotopy groups of Y and Z are finitely generated modules over $\mathbb{Z}_{(\ell)}$. Then, given maps $f : X \to Z$ and $g : Y \to Z$ with $f_n = f|_{X_n}$, there exists $h : X \to Y$ with $g \cdot h \sim f$ iff for each n there exists $h_n : X_n \to Y$ with $g \cdot h_n \sim f_n$.*

4.3 Remark. Proposition 4.2 implies that if X and Z are spaces of the indicated type, then two maps $f, f' : X \to Z$ are homotopic iff for each n their restrictions to X_n are homotopic (this follows from the observation that f and f' are homotopic iff $(f, f') : X \to Z \times Z$ lifts up to homotopy over the diagonal map $Z \to Z \times Z$). We will apply (4.2) in situations in which the spaces Y and Z are derived from $\mathbf{L}Q_0(B\mu_+)$ (μ a finite group), from $\mathbf{L}B\,\mathrm{GL}(E)^+$ (E a finite field) or from $\mathbf{L}B\,\mathrm{GL}(R)^+$ (R the ring of integers in an algebraic number field). In the first case the relevant homotopy groups are finite by stable homotopy theory, in the second case they are finite by the calculation of Quillen, and in the third they are finitely generated, again by a theorem of Quillen.

4.4 Proposition. *Let G be a finite group, H an ℓ-Sylow subgroup of G and $i : H \to G$ the inclusion. Then the map $\mathbf{L}Q(Bi_+) : \mathbf{L}Q(BH_+) \to \mathbf{L}Q(BG_+)$ has a right inverse.*

4.5 Remark. The right inverse referred to in in 4.4 is derived from the transfer (2.5). Proposition 4.4 implies that if G is a finite group with ℓ-Sylow subgroup H, and $g : Y \to Z$ is a map of ℓ-local infinite loop spaces, then a map $f : BG \to Z$ lifts up to homotopy to a map $BG \to Y$ iff the restriction of f to BH lifts up to homotopy to a map $BH \to Y$. Similarly, a map $BG \to Y$ is null homotopic iff its restriction to BH is.

4.6 Proposition. *If Y is a space which is local at ℓ and $f : X_1 \to X_2$ is a map which induces an isomorphism $H_*(X_1, \mathbb{Z}_{(\ell)}) \to H_*(X_2, \mathbb{Z}_{(\ell)})$, then f induces a bijection $[X_2, Y] \to [X_1, Y]$.*

4.7 Proposition. *Suppose that G is a finite ℓ-group and that Y is a connected loop space with finite homotopy groups. Then the natural map $[BG, Y] \to [BG, \mathbf{L}Y]$ is a bijection.*

Remark. This follows from the fact that Y can be expressed up to homotopy as a product $Y' \times \mathbf{L}Y$, where Y' is the localization of Y with respect to the ring $\mathbb{Z}[1/\ell]$; since the homotopy groups of Y' are uniquely ℓ-divisible, the pointed set $[BG, Y']$ is trivial. We will apply this proposition with $Y = Q_0B\mu_+$ (μ a finite group) or $Y = B\,\mathrm{GL}(E)^+$ (E a finite field).

Proof of 4.1. To find a map $s : \mathbf{L}B\,\mathrm{GL}(E)^+ \to \mathbf{L}Q_0(B\mu_+)$ satisfying 4.1(1) it is enough to find

(1) (by 4.6) a map $B\,\mathrm{GL}(E) = \cup_n B\,\mathrm{GL}_n(E) \to \mathbf{L}Q_0(B\mu_+)$ which is a lift of the map $B\,\mathrm{GL}(E) \to \mathbf{L}B\,\mathrm{GL}(E)^+$, or
(2) (by 4.2) for each $n \geq 1$ a map $B\,\mathrm{GL}_n(E) \to \mathbf{L}Q_0(B\mu_+)$ which is a lift of the map $BGL_n(E) \to \mathbf{L}B\,\mathrm{GL}(E)^+$, or
(3) (by 4.4) for each $n \geq 1$ a map $BG_n \to Q_0(B\mu_+)$ which is a lift of the map $BG_n \to B\,\mathrm{GL}(E)^+$, where G_n denotes some chosen ℓ-Sylow subgroup of $\mathrm{GL}_n(E)$.

To accomplish this last observe that the composite

$$BG_n \to B\,\mathrm{GL}_n(E) \to B\,\mathrm{GL}(E)^+$$

corresponds under the map β of (2.10) to $(i_n - [\mathbf{n}])$, where $i_n \in \mathcal{R}_E G_n$ is given by the inclusion $G_n \subset \mathrm{GL}_n(E)$ and $[\mathbf{n}]$ is the dimension n trivial representation. By (3.2) the element $i_n - [\mathbf{n}]$ lifts to some element j_n of $A(G_n, \mu)$. The map $\alpha(j_n)$ then by (2.10) gives the required map $BG_n \to Q_0(B\mu_+)$.

A chain of reasoning parallel to the above (see 4.3) shows that 4.1(2) can be proved by verifying that for each $n \geq 1$ the two maps

$$\Phi_R\,,\ \Phi_R \cdot s \cdot \Phi_E\ :\ \mathbf{L}Q_0(B\mu_+) \to \mathbf{L}B\,\mathrm{GL}(R)^+$$

become homotopic after precomposition with the map

$$BK_n \to B(\Sigma_n \wr \mu) \to Q_0(B\mu_+) \to \mathbf{L}Q_0(B\mu_+)$$

where $K_n \subset \Sigma_n \wr \mu$ is an ℓ-Sylow subgroup. We will denote this map $BK_n \to \mathbf{L}Q_0(B\mu_+)$ by f_n. Consider the map $g_n = s \cdot \Phi_E \cdot f_n : BK_n \to \mathbf{L}Q_0(B\mu_+)$, and let $\tilde{f}_n, \tilde{g}_n : BK_n \to Q_0(B\mu_+)$ be the lifts of f_n and g_n provided by (4.7). By (2.11) there are elements $a_n, b_n \in \hat{A}(K_n, \mu)$ such that $\tilde{f}_n = \hat{\alpha}(a_n)$ and $\tilde{g}_n = \hat{\beta}(b_n)$. Since $(\mathbf{L}\Phi_E) \cdot f_n \sim (\mathbf{L}\Phi_E) \cdot g_n$, it follows from (4.7) that $\Phi_E \cdot \tilde{f}_n \sim \Phi_E \cdot \tilde{g}_n$ and hence (2.11) that $\hat{\varphi}_E(a_n) = \hat{\varphi}_E(b_n)$. The IG-adic completion of (3.5) (see 3.6) now implies that $\hat{\varphi}_R(a_n) = \hat{\varphi}_R(b_n)$, which gives by (2.10) that $\Phi_R \cdot \tilde{f}_n \sim \Phi_R \cdot \tilde{g}_n$ and by naturality of localization that $(\mathbf{L}\Phi_R) \cdot f_n \sim (\mathbf{L}\Phi_R) \cdot g_n$. □

4.8 Dyer-Lashof operations. Let $R = \mathbb{Z}[\zeta_a]$. In [3], the map $B\,\mathrm{GL}(R)^+ \to B\,\mathrm{GL}(E)^+$ induced by the quotient map $R \to E$ is shown to have a right inverse after localization at ℓ, provided that $\mu(R) \cong \mu(E)$ and $a \geq 2$ if $\ell = 2$. Theorem 4.1 gives us a canonical homotopy class of such a right inverse if $a \geq 1$, since (in view of the fact that the localization at ℓ of Φ_E has a right inverse up to homotopy) the map $\lambda = \Phi_R \cdot s$ is clearly determined (up to homotopy) by the condition that $\lambda \cdot \Phi_E = \Phi_R$. This observation leads to a simple proof of the following.

4.9 Proposition. *Let $R = \mathbb{Z}[\zeta_a]$, where $a \geq 1$ and $a \geq 2$ if $\ell = 2$. Let E be a residue field of R with $\mu(R) \cong \mu(E)$, and let $\lambda = \Phi_R \cdot s : \mathbf{L}B\,\mathrm{GL}(E)^+ \to \mathbf{L}B\,\mathrm{GL}(R)^+$ be the map provided by (4.1). Then λ commutes up to homotopy with the Dyer-Lashof action maps of $\mathbf{L}B\,\mathrm{GL}(E)^+$ and $\mathbf{L}B\,\mathrm{GL}(R)^+$ derived from the infinite loop structures of these spaces.*

Proof. We indicate the proof briefly. Let $X = \mathbf{L}Q_o(B\mu(R)_+)$, $Y = \mathbf{L}B\,\mathrm{GL}(E)^+$, and $Z = \mathbf{L}B\,\mathrm{GL}(R)^+$. For any space W let $D_nW = E\Sigma_n \times_{\Sigma_n} W^n$ $(n \geq 1)$. Construct the diagram

$$\begin{array}{ccccc} D_nX & \xrightarrow{D_n\Phi_E} & D_nY & \xrightarrow{D_n\lambda} & D_nZ \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\Phi_E} & Y & \xrightarrow{\lambda} & Z \end{array}$$

in which the vertical maps are Dyer-Lashof action maps arising from the appropriate infinite loop space structures. The left hand square commutes up to homotopy because the map Φ_E is an infinite loop map. The large rectangle obtained by omitting the central column commutes up to homotopy because $\Phi_R = \lambda \cdot \Phi_E$ and the map Φ_R is an infinite loop map. Commutativity of the right hand square now follows from the fact that $D_n\Phi_E$ has a right inverse up to homotopy, namely, D_ns. □

4.10 Fewer roots of unity. Theorem 4.1 easily leads to a proof at odd primes of the original [6] factorization and splitting results of the third author, a proof which in its totality is slightly different in detail from (but identical in concept to) the proof in [6]. Rather than formulate the most general statement we will just give one example.

4.11 Theorem. [6] *Assume that ℓ is odd and that p is a prime which generates the ℓ-adic units. Then there is a map $\lambda : \mathbf{L}B\,\mathrm{GL}(\mathbb{F}_p)^+ \to \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$ such that the composite*

$$\mathbf{L}Q_0(B\{1\}_+) \cong \mathbf{L}Q_0S^0 \xrightarrow{\Phi_{\mathbb{F}_p}} \mathbf{L}B\,\mathrm{GL}(\mathbb{F}_p)^+ \xrightarrow{\lambda} \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$$

is homotopic to the map $\Phi_{\mathbb{Z}} : \mathbf{L}Q_0S^0 \to \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$.

Proof. Let $R = \mathbb{Z}[\zeta_1]$ and let $\mu = \mu(R)$. The hypothesis on p implies that there is a unique prime of R above p, and we will let E be the corresponding residue class field of R, so that $\mu(E) \cong \mu$. Since $\mathbb{Z}[\zeta_1]$ is a free module of rank $\ell - 1$ over $\mathbb{Z}$ and $\ell - 1$ is relatively prime to ℓ, the transfer construction in algebraic K-theory can be used to produce a map $t : \mathbf{L}B\,\mathrm{GL}(R)^+ \to \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$ which is a left inverse up to homotopy to the map $i_{\mathbb{Z}} : \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+ \to \mathbf{L}B\,\mathrm{GL}(R)^+$ induced by the ring map $\mathbb{Z} \to R$. Let $i_{\mathbb{F}_p} : \mathbf{L}B\,\mathrm{GL}(\mathbb{F}_p)^+ \to \mathbf{L}B\,\mathrm{GL}(E)^+$ be the map induced by $\mathbb{F}_p \to E$, and let $\lambda_{\mathbb{Z}} : \mathbf{L}B\,\mathrm{GL}(\mathbb{F}_p)^+ \to \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$ be the composite

$$\mathbf{L}B\,\mathrm{GL}(\mathbb{F}_p)^+ \xrightarrow{i_{\mathbb{F}_p}} \mathbf{L}B\,\mathrm{GL}(E)^+ \xrightarrow{\lambda_R} \mathbf{L}B\,\mathrm{GL}(R)^+ \xrightarrow{t} \mathbf{L}B\,\mathrm{GL}(\mathbb{Z})^+$$

where λ_R is the map $\Phi_R \cdot s$ provided by (4.1). A simple direct calculation which combines (4.1) and naturality shows that $\lambda_{\mathbb{Z}} = \lambda$ has the desired property. □

4.12 Even more roots of unity. It is natural to ask for an analogue of (4.1) in the situation in which *all* ℓ-primary roots of unity are present. We can offer the following somewhat unsatsfactory result along these lines.

4.13 Theorem. *Let R be the ring $Z[\zeta_i|i \geq 1]$, let p be a prime which generates the ℓ-adic units, and let E be the quotient field $R/p \cong \mathbb{F}_p[\zeta_i|i \geq 1]$. Then there exists a map $\lambda_R : \mathbf{L}B\,\mathrm{GL}(E)^+ \to \mathbf{L}B\,\mathrm{GL}(R)^+$, such that, for any finite subgroup μ of $\mu(R)$ the two maps*

$$\Phi_R\ , \lambda_R \cdot \Phi_E : \mathbf{L}Q_0(B\mu_+) \to \ \mathbf{L}B\,\mathrm{GL}(R)^+$$

are homotopic.

Proof. Let $R_n = \mathbb{Z}[\zeta_n]$, $E_n = R_n/p$, $\mu^n = \mu(R_n)$, and let $\lambda_n : \mathbf{L}B\,\mathrm{GL}(E_n)^+ \to \mathbf{L}B\,\mathrm{GL}(R_n)^+$ denote the map provided up to homotopy by (4.1). There are diagrams

$$\begin{array}{ccccc}
\mathbf{L}Q_0(B\mu^n_+) & \xrightarrow{\Phi_{E_n}} & \mathbf{L}B\,\mathrm{GL}(E_n)^+ & \xrightarrow{\lambda_n} & \mathbf{L}B\,\mathrm{GL}(R_n)^+ \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{L}Q_0(B\mu^{n+1}_+) & \xrightarrow{\Phi_{E_{n+1}}} & \mathbf{L}B\,\mathrm{GL}(E_{n+1})^+ & \xrightarrow{\lambda_{n+1}} & \mathbf{L}B\,\mathrm{GL}(R_{n+1})^+
\end{array}$$

in which the left hand square evidently commutes up to homotopy, as does the large rectangle obtained by removing the center column. It thus follows from remarks above (4.8) that the right hand square commutes up to homotopy. After chosing if necessary models for the spaces $\mathbf{L}B\,\mathrm{GL}(E_n)^+$ so that the natural maps $\mathbf{L}B\,\mathrm{GL}(E_n)^+ \to \mathbf{L}B\,\mathrm{GL}(E_m)^+$

($m > n$) are represented by cofibrations, it is now easy to construct by induction on n a commutative ladder (of genuine maps)

$$\begin{array}{ccccccccc}
\mathbf{L}B\,\mathrm{GL}(E_1)^+ & \longrightarrow & \mathbf{L}B\,\mathrm{GL}(E_2)^+ & \longrightarrow & \cdots & \longrightarrow & \mathbf{L}B\,\mathrm{GL}(E_n)^+ & \longrightarrow & \cdots \\
{\scriptstyle\Lambda_1}\downarrow & & {\scriptstyle\Lambda_2}\downarrow & & & & {\scriptstyle\Lambda_n}\downarrow & & \\
\mathbf{L}B\,\mathrm{GL}(R_1)^+ & \longrightarrow & \mathbf{L}B\,\mathrm{GL}(R_2)^+ & \longrightarrow & \cdots & \longrightarrow & \mathbf{L}B\,\mathrm{GL}(R_n)^+ & \longrightarrow & \cdots
\end{array}$$

such that Λ_n is homotopic to λ_n. The mapping telescope of the upper rail of this diagram is equivalent to $\mathbf{L}B\,\mathrm{GL}(E)^+$ and the mapping telescope of the lower rail to $\mathbf{L}B\,\mathrm{GL}(R)^+$. It is easy to see that the induced map between these telescopes is the desired map λ_R. □

The reader may wonder about the restriction to finite subgroups $\mu \subset \mu(R)$ in the above theorem. To remove this restriction it would be natural to try to add a third (upper) rail to the above ladder, a rail of the form

$$\mathbf{L}Q_0(B\mu^1_+) \to \mathbf{L}Q_0(B\mu^1_+) \to \cdots \mathbf{L}Q_0(B\mu^n_+) \to \cdots .$$

This new rail would be connected to the now center one by the maps Φ_{E_n} and to the lower one by the maps Φ_{R_n}. We are unable to construct such an extended ladder for the following reason. Let A_n, B_n and C_n denote the n'th spaces in respectively the upper, middle, and lower rails. The most direct approach to making the desired construction gives inductively, at the n'th stage, a map $g_n : A_{n+1} \coprod_{A_n} B_n \to C_{n+1}$ and it is required to extend g_n over a map $A_{n+1} \coprod_{A_n} B_n \to B_{n+1}$ to a map $B_{n+1} \to C_{n+1}$. The homotopy class of maps $B_{n+1} \to C_{n+1}$ provided by (4.1) extends the restriction of g_n to A_{n+1} and also extends the restriction of g_n to B_n, but this is not enough. The difficulty is that the restriction map $[A_{n+1} \coprod_{A_n} B_n, C_{n+1}] \to [A_{n+1}, C_{n+1}] \times [B_n, C_{n+1}]$ is not necessarily injective.

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