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Fibrewise Localization

and

Unstable Adams Spectral Sequences

by

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0. Introduction.

Let LX be localization with respect to some homology theory and let $X \to LX$ be the localization map. If $E \to B$ is a fibration, then fibrewise localization KE is defined to be a homotopy pullback

$$\begin{array}{ccc} KE & \to & LE \\ \downarrow & & \downarrow \\ B & \to & LB\,. \end{array}$$

This definition of fibrewise localization is completely general but, to be useful, it should have three properties.

First, we do not know in general that $LE \to LB$ has LF as its homotopy theoretic fibre, where F is the fibre of $E \to B$. Hence, we do not know that the homotopy theortic fibre of $KE \to B$ is LF and this, of course, should be true of anything called fibrewise localization. Suppose L is localization with respect to homology with coefficients in a ring R where R is either Z/nZ or a subring of Q. Then a satisfactory solution is given by, among others, Bousfield-Kan [2]. They construct a natural transformation $X \to R_\infty X$ such that, at least for nilpotent X, $R_\infty X$ is the localization of X with respect to $H(\,;R)$. In addition, if $\pi_1 B$ acts nilpotently on $H_i(F;R)$ for all i, then the homotopy theoretic fibre of $R_\infty E \to R_\infty B$ is $R_\infty F$. For example, this is the case if p is a prime, π_1 is a p-group, and $R = F_p =$ the field with p elements.

Second, if Y is a space over B, we would like to get some hold on the homotopy groups of the space $\Gamma(Y, KE)$ of maps $Y \to KE$ over B. The Bousfield-Kan construction of $F_{p\infty} X$ gives an unstable Adams spectral sequence for computing the homotopy groups of the s-pace $\mathrm{map}_*(Y, F_{p\infty} X)$ of pointed maps. We are led to imitate this in order to compute the homotopy groups of $\Gamma(Y, KE)$. Furthermore, the E_2 term of the Bousfield-Kan unstable Adams spectral sequence can be identified by nonabelian homological algebra. Bousfield-Kan do give a construction of fibrewise localization, denoted $\dot{F}_{p\infty} E$, and their construction gives an unstable Adams spectral sequence. But their construction does not seem to permit an algebraic description of the E_2 term. We give in this paper a new construction of fibrewise localization, denoted $BR_\infty(E)$, which gives when $R = F_p$ a spectral sequence with an algebraically identifiable E_2 term.

Third, this spectral sequence gives information about $\Gamma(Y, BF_{p\infty}(E))$ but most often one really wants to know about $\Gamma(Y, E)$. We have not resolved the question of what are the properties of the natural maps

$\Gamma(Y,E) \to \Gamma(Y,BF_{p\infty}(E))$ but a special case has been settled by the first author. Suppose π is a p-group, X is a simply connected space with a π-action, and $E \to B$ is the Borel construction $E\pi \times_\pi X \to B\pi$. Then, if $Y \to B$ is the identity $B\pi \to B\pi$, the map $\Gamma(B\pi,E) \to \Gamma(B\pi,BF_{p\infty}(E))$ is a mod p homotopy isomorphism, that is, the fibres are simple spaces with homotopy groups which are uniquely p-divisible, [3].

We apply our unstable Adams spectral sequence to give another proof of the generalized Sullivan conjecture. This proof follows closely a proof by Lannes [4] using the Bousfield-Kan Adams spectral sequence.

We would like to express our gratitude to A.K. Bousfield for his many letters to us describing unstable Adams spectral sequences, bicosimplicial spaces, the role of derivations, and convergence criteria.

Throughout this paper, our basic reference is Bousfield-Kan [2]. Many of the ideas there are also expounded in [5].

1. A construction of fibrewise localization.

We work in the category of simplicial sets and adopt the convention that "space" means "simplicial set." The Bousfield-Kan construction of the localization $R_\infty X$ is done in two stages. In the first stage, a triple denoted R is used to construct a canonical resolution $X \to R^\bullet X$ which is an augmented cosimplicial space. In the second stage, a functor tot from cosimplicial spaces to spaces is applied to $X^\bullet$. The result tot $X^\bullet$ is $R_\infty X$. Since our construction of fibrewise localization fits into this framework, we describe this process in more detail.

Let T be a triple with triple structure $\eta : I \to T$ and $\mu : T^2 \to T$. Given an object X, the canonical resolution of X is the augmented cosimplicial object $X \to T^\bullet X$ defined by:

$$
\begin{aligned}
&(T^\bullet X)^s = T^{s+1}X \\
&d^i = T^i\eta(T^{s-i}X) : (T^\bullet X)^{s-1} \to (T^\bullet X)^s \\
&s^i = T^i\mu(T^{s-i}X) : (T^\bullet X)^{s+1} \to (T^\bullet X)^s \\
&\text{where } i = 0, \dots, s
\end{aligned}
$$

and the augmentation is $d^0 = \eta : X \to (T^\bullet X)^0$.

It is common to write $A^\bullet, B^\bullet, C^\bullet, \dots$ for cosimplicial objects and $A^{-1} \to A^\bullet$ or $X \to A^\bullet$ if $A^\bullet$ is augmented.

It is useful to be aware of the following equivalent formulation of an augmented cosimplicial object $X \to A^\bullet$. Write $\underline{X}$ for the constant cosimplicial object, i.e. $\underline{X}^s = X$ for all $s \geq 0$ and all coface operators d^i and codegeneracy operators s^i are identity maps. An augmentation $X \to A^\bullet$ is equivalent to a cosimplicial map $\underline{X} \to A^\bullet$.

The first step of the Bousfield-Kan construction of $R_\infty X$ is the canonical resolution $X \to R^\bullet X$ associated to the triple R defined as follows.

If X is a space and R is a commutative ring, then RX is the space of finite formal sums $\Sigma r[x]$ where r is in R and x is in X. The triple structure $\eta(X) : X \to RX$ and $\mu(X) : R^2X \to RX$ is: $\eta(X) = [x]$ and $\mu(\Sigma s[\Sigma r[x]]) = \Sigma sr[x]$. The second stage of the Bousfield-Kan construction is to assign to any cosimplicial space $Y^\bullet$ a space tot $Y^\bullet$ and then $R_\infty X$ is tot $R^\bullet X$. The functor tot is described as follows:

Let $\Delta^\bullet$ be the cosimplicial space with $(\Delta^\bullet)^s = \Delta^s$ and with d^i and s^i the standard coface and codegeneracy maps. Given cosimplicial spaces $A^\bullet$ and $B^\bullet$, let $\hom(A^\bullet, B^\bullet)$ be the cosimplicial mapping space where the n-simplices $\hom(A^\bullet, B^\bullet)_n$ are the set of cosimplicial morphisms $\Delta^n \times A^\bullet \to B^\bullet$. Here, $(\Delta^n \times A^\bullet)^s = \Delta^n \times A^s$, $d^i = 1 \times d^i$, and $s^i = 1 \times s^i$. The face operators $d_i : \hom(A^\bullet, B^\bullet)_n \to \hom(A, B)_{n-1}$ and degeneracy operators $s_i : \hom(A^\bullet, B^\bullet)_n \to \hom(A^\bullet, B^\bullet)_{n+1}$ are $d_i(f) = f(d^i \times 1)$ and $s_i(f) = f(s^i \times 1)$. Let $\mathrm{tot}\ Y^\bullet = \hom(\Delta^\bullet, Y^\bullet)$.

It is not difficult to check that, if $\underline{X}$ is constant, then $\mathrm{tot}\ \underline{X} = X$. Hence, if $X \to Y^\bullet$ is an augmented cosimplicial space, applying tot to $\underline{X} \to Y^\bullet$ gives a map $X \to \mathrm{tot}\ Y^\bullet$.

If T is a triple defined on any category which admits a forgetful functor to spaces, then applying tot to the canonical resolution $X \to T^\bullet X$ gives a map of spaces $X \to \mathrm{tot}\ T^\bullet X = T_\infty X$.

We construct fibrewise localization by applying this procedure to the following triple BR defined on the category of spaces over a fixed space B.

Given a space E over B with map $p : E \to B$, let $BR(E) = B \times RE$ and regard $BR(E)$ as a space over B by projecting on the first factor. The triple structure $\eta : 1 \to BR$ and $\mu : (BR)^2 \to BR$ is: $\eta(E) = (p, \eta(E)) : E \to B \times RE$ and $\mu(E) = 1 \times (\mu(E)R(\pi_2)) : B \times R(B \times RE) \to B \times RE$ where $\pi_2 : B \times RE$ is projection on the second factor.

Our candidate for fibrewise localization is the resulting space $BR_\infty(E) = \mathrm{tot}\ BR^\bullet(E)$. Since there is a map $BR^\bullet(E) \to \underline{B}$, $BR_\infty(E)$ is a space over B and $E \to BR_\infty(E)$ is a map over B.

If E and B are fibrant, i.e. Kan complexes, we show in section 7 that $BR_\infty(E)$ is a homotopy pullback of

$$\begin{array}{ccc} & & R_\infty E \\ & & \downarrow \\ B & \to & R_\infty B \end{array}.$$

Hence, $BR_\infty(E)$ is a fibrewise localization of E.

2. Derived functors of derivations.

Let $\underline{CA}$ denote the category of graded commutative unstable coalgebras with unit over the mod p Steenrod algebra. If B is an object in $\underline{CA}$, let $\underline{CA}/B$ be the category of objects over B.

Let $S^t = H_*(S^t; F_p)$ in $\underline{CA}$ and let X be an object in $\underline{CA}/B$. The projection $S^t \otimes X \to X$ makes $S^t \otimes X$ into an object in $\underline{CA}/B$ so that this projection is a map in $\underline{CA}/B$. If $t > 0$, there is a unique augmentation $F_p \to S^t$. If $t = 0$, pick one. This gives a map $X \to S^t \otimes X$.

If $\psi : X \to E$ is a map in $\underline{CA}/B$, then a map $\varphi : S^t \otimes X \to E$ in $\underline{CA}/B$ is called a derivation of degree t with respect to ψ if

$$\begin{array}{ccc} X & \xrightarrow{\psi} & E \\ \downarrow & \nearrow \varphi & \\ S^t \otimes X & & \end{array}$$

is commutative. Of course, φ is determined by its restriction to $\overline{H}_n(S^t; F_p) \otimes X$. Hence, when $t = 0$, a derivation is just a map in $\underline{CA}/B$ and, when $t \geq 1$, a derivation is just the usual definition of a coalgebra derivation over the Steenrod algebra. Thus, if $\mathrm{Der}^t_{\underline{CA}/B}(X, E)_\psi$ is the set of all such derivations, then $\mathrm{Der}^0_{\underline{CA}/B}(X, E)_\psi = \mathrm{Hom}_{\underline{CA}/B}(X, E)$ and, if $t \geq 1$, $\mathrm{Der}^t_{\underline{CA}/B}(X, E)_\psi$ is an F_p module.

Let $\underline{F}_p$ be the category of positively graded F_p modules. The forgetful functor $J : \underline{CA} \to \underline{F}_p$ has a right adjoint $G : \underline{F}_p \to \underline{CA}$. We also write G for the resulting triple $G = GJ : \underline{CA} \to \underline{CA}$ with triple structure $\eta : I \to G$ and $\mu : G^2 \to G$. Now define a triple $B - G : \underline{CA}/B \to \underline{CA}/B$ as follows: If E is in $\underline{CA}/B$ with map $p : E \to B$, let $(B - G)(E) = B \otimes G(E)$, $\eta(E) = (p \otimes \eta)\Delta : E \to E \otimes E \to B \otimes G(E)$, and $\mu(E) = (1 \otimes \mu)(1 \otimes G(\pi_2)) : B \otimes G(B \otimes G(E)) \to B \otimes G(G(E)) \to B \otimes G(E)$ where $\pi_2 : B \otimes G(E) \to G(E)$ is the projection given by the unit $B \to F_p$.

If $A^\bullet$ is any cosimplicial group, the cohomotopy group $\pi^s A^\bullet$ is the cohomology group $H^s(A^\bullet, d)$ where $d = \Sigma(-1)^i d^i$. If $A^\bullet$ is a cosimplicial set, we may still define $\pi^0 A^\bullet$ as the coequalizer of d^0 and $d^1 : A^0 \to A^1$.

If $\psi : X \to E$ is a map in $\underline{CA}/B$, then the augmentation $E \to (B - G)^\bullet(E)$ gives unique maps $X \to (BG)^{n+1}(E)$ which we also denote by ψ. The right derived functors of $\mathrm{Der}^t_{\underline{CA}/B}(X, E)_\psi$ are defined, for all $s \geq 0$ if $t \geq 1$ and for $s = 0$ if $t = 0$, by $\mathrm{Ext}^{s,t}_{\underline{CA}/B}(X, E)_\psi = \pi^s(\mathrm{Der}^t_{\underline{CA}/B}(X, (B - G)^\bullet(E))_\psi)$.

Since coalgebras have units $\epsilon : E \to F_p$, $\underline{CA} = \underline{CA}/F_p$ and we have [4].

LEMMA 2.1. *Let E be an object in $\underline{CA}$ with augmentation $\eta : F_p \to E$. If E is connected, then* $\text{Ext}\,^{s,t}_{\underline{CA}}(F_p, E)_\eta = 0$ *for $t - s \leq 0$ unless $s = t = 0$.*

If B and F are objects in $\underline{CA}$, let $B \otimes F$ be the object in $\underline{CA}/B$ with map $\pi_1 = 1 \otimes \epsilon : B \otimes F \to B$. Given a map $\psi : X \to B \otimes F$ in $\underline{CA}/B$, it is easy to verify that:

LEMMA 2.2. $Der_{\underline{CA}/B}(X, B \otimes F)_\psi = Der^s_{\underline{CA}}(X, F)_{\pi_2\psi}$ *where* $\pi_2 = \epsilon \otimes 1$. *More generally,* $\text{Ext}\,^{s,t}_{\underline{CA}/B}(X, B \otimes F)_\psi = \text{Ext}\,^{s,t}_{\underline{CA}}(X, F)_{\pi_2\psi}$.

All of the preceding dualizes to algebras as follows: Let $\underline{A}$ be the category of unstable algebras over the mod p Steenrod algebras. If B^* is an object in $\underline{A}$, let $\underline{A} \setminus B^*$ be the category of objects under B^*, that is, of objects E^* in $\underline{A}$ together with maps $p^* : B^* \to E^*$. The forgetful functor $J : \underline{A} \to \underline{F}_p$ has a left adjoint G^* and we also write G^* for the cotriple $G^* = G^*J$. There is a cotriple $(B^* - G^*)$ on $\underline{A} \setminus B$ defined by $(B^* - G^*)(E^*) = B^* \otimes G^*(E^*)$. If $\psi^* : E^* \to X^*$ is a map in $\underline{A} \setminus B^*$, we define $\text{Der}_{\underline{A}\setminus B^*} E^*, X^*)_{\psi^*}$ and $\text{Ext}\,^{s,t}_{\underline{A}\setminus B^*}(E^*, X^*)_{\psi^*}$ by the obvious dual procedure. Then, if X, E, B are finite type objects in $\underline{CA}$ with dual objects X^*, E^*, B^* in $\underline{A}$, we have

$$\text{Der}^s_{\underline{CA}/B}(X, E)_\psi = \text{Der}^s_{\underline{A}\setminus B^*}(E^*, X^*)_{\psi^*} \text{ and}$$
$$\text{Ext}\,^{s,t}_{\underline{CA}/B}(X, E)_\psi = \text{Ext}\,^{s,t}_{\underline{A}\setminus B^*}(E^*, X^*)_{\psi^*}\,.$$

If $\epsilon : E^* \to F_p$ is an augmented object in $\underline{A}$, let $E^*_c = F^p \otimes_{E^0} E^*$ be the connected component with induced augmentation $\overline{\epsilon} : E^*_c \to F_p$.

LEMMA 2.3. *If $p^* : B^* \to E^*$ is a map of augmented objects, then* $Der_{\underline{A}\setminus B^*}(E^*, F_p)_\epsilon = Der^s_{\underline{A}\setminus B^*_c}(E^*_c, F_p)_{\overline{\epsilon}}$ *and* $\text{Ext}\,^{s,t}_{\underline{A}\setminus B^*}(E^*, F_p)_\epsilon = \text{Ext}\,^{s,t}_{\underline{A}\setminus B^*_c}(E^*_c, F_p)_{\overline{\epsilon}}$.

The proof of the above is essentially the same as the proof for the special case $\underline{A} \setminus B^* = \underline{A}$ in [4].

Let $\pi = Z/pZ$ and $H = H^*B\pi$. Recall that Lannes [4] has given a functor $T : \underline{A} \to \underline{A}$ defined as a left adjoint, $\text{Hom}\,_{\underline{A}}(B^*, H \otimes C^*) = \text{Hom}\,_{\underline{A}}(TB^*, C^*)$. It has the following remarkable properties:

a) T preserves free objects,

b) T preserves tensor products, $T(A \otimes B) = TA \otimes TB$,
c) T is exact, that is, T preserves simplicial resolutions, and
d) if A is finite dimensional, the natural map $F_p \to H$ induces an isomorphism $TH \to H$.

From these properties, it follows easily that e) T preserves derivations, $\mathrm{Der}^s_{\underline{A}\backslash B^*}(E^*, H \otimes X^*)_{\psi^*} = \mathrm{Der}^s_{\underline{A}\backslash TB^*}(TE^*, X^*)_{\psi^*}$ where $\underline{\psi}^* : TE^* \to X^*$ is the morphism corresponding to $\psi^* : E^* \to H \otimes X^*$, and f) T preserves derived functors, $\mathrm{Ext}\,^{s,t}_{\underline{A}\backslash B^*}(E^*, H \otimes X^*)_{\psi^*} = \mathrm{Ext}\,^{s,t}_{\underline{A}\backslash TB^*}(TE^*, X^*)_{\underline{\psi}^*}$.

3. An unstable Adams spectral sequence.

In this section, we describe in our context the extended homotopy spectral sequence of Bousfield-Kan and a variation of it due to Bousfield.

If W is a space, let $W^{[s]}$ denote its s skeleton. If $X^\bullet$ is a cosimplicial space, let $\text{tot}_s X^\bullet = \text{hom}(\Delta^{\bullet[s]}, X^\bullet)$ for $s \geq 0$ and let $\text{tot}_{-1} X^\bullet =$ a point. Then $\text{tot}_0 X^\bullet = X^0$ and the natural maps $\text{tot}_s X^\bullet \to \text{tot}_{s-1} X^\bullet$ form an inverse system with $\lim\limits_{\leftarrow} \text{tot}_s X^\bullet = \text{tot}_\infty X^\bullet = \text{tot} X^\bullet$.

If $X^\bullet$ is pointed, let $N^\bullet X^\bullet$ be the normalized object defined by $N^0 X^\bullet = X^0$ and $N^n X^\bullet = \bigcap_{i=0}^{n-1} \ker s^i : X^n \to X^{n-1}$ for $n \geq 1$. The fibre of $\text{tot}_s X^\bullet \to \text{tot}_{s-1} X^\bullet$ is the pointed mapping space $\Omega^s N^s X^\bullet = \text{map}_*(S^s, N^s X^\bullet)$.

If $X^\bullet$ is a cosimplicial group, then the differential $d = \Sigma(-1)^i d^i$ acts on $N^\bullet X^\bullet$ with $H^s(N^\bullet X^\bullet, d)$ isomorphic to $\pi^s X^\bullet$.

Let $\psi : Y \to E$ be a map of spaces over B and let $\Gamma(Y, E)$ be the space of all such maps with ψ as the basepoint. If $E \to X^\bullet$ is an augmented cosimplicial space over B, then ψ defines unique maps $Y \to X^n$ for $n \geq 0$ and hence a map $Y \to \text{tot}\ X^\bullet$. These maps are also denoted by ψ. Hence, $\Gamma(Y, X^n)$, $\Gamma(X, \text{tot}\ X^\bullet)$, and $\Gamma(Y, E) \to \Gamma(Y, X^\bullet)$ are all pointed.

Exercise. $\text{tot}\ \Gamma(Y, X^\bullet) = \Gamma(Y, \text{tot}\ X^\bullet)$.

In particular, $\text{tot}\ \Gamma(Y, BR^\bullet(E)) = \Gamma(Y, BR_\infty(E))$. Furthermore, s-ince $\Gamma(Y, BR^\bullet(E))$ is grouplike [2], these spaces are fibrant.

Write tot_s for $\text{tot}_s \Gamma(Y, BR^\bullet(E))$. The tower of maps $\text{tot}_s \to \text{tot}_{s-1}$ is a tower of R-principal fibrations which are not necessarily surjective. Since $\lim\limits_{\leftarrow} \text{tot}_s = \Gamma(Y, BR_\infty(E))$, the exact homotopy sequences of these fibrations fit together, as in Bousfield-Kan , into an unstable Adams spectral sequence. That is, $E_r^{s,t} = E_r^{s,t}(Y, E, \psi)$ is defined for $t \geq s \geq 0$ and there are differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. The abutment is $\pi_{t-s}\Gamma(Y, BR_\infty(E))$ and $E_1^{s,t} = \pi_{t-s}(\Omega^s N^s \Gamma(Y, BR^\bullet(E)) = \pi_t\ N^s \Gamma(Y, BR^\bullet(E)) = N^s \pi_t\, \Gamma(Y, BR^\bullet(E))$. Furthermore, d_1 is $d = \Sigma(-1)^i d^i$ and therefore $E_2^{s,t} = \pi^s \pi_t\ \Gamma(Y, BR^\bullet(E))$.

The following convergence results follow from Bousfield-Kan . If $k \geq 0$ and $E_2^{s,t}(Y, E, \psi) = 0$ for $0 \leq t-s \leq k$, then $\Gamma(Y, BR_\infty(E))$ is k-connected. If $\theta : E \to E'$ is a map over B, $E_2^{s,t}(Y, E, \psi) = 0$ for $t-s = 0$, and $E_2^{s,t}(Y, E, \psi) \to E_2^{s,t}(Y, E'\theta\psi)$ is an isomorphism for $t-s \geq 0$, then

$\Gamma(Y, BR_\infty(E)) \to \Gamma(Y, BR_\infty(E'))$ is a homotopy equivalence.

Now, let $R = F_p$ and $H_*X = H_*(X; F_p)$. The topological triple BF_p and the algebraic triple $H_*(B) - G$ are related by a natural isomorphism $\theta : H_*(BF_p(E)) \to (H_*B - G)(H_*E)$ which respects the triple structures. That is, the following diagrams commute.

$$\begin{array}{ccc} H_*E & \xrightarrow{H_*(\eta)} & H_*(BF_p(E)) \\ & & \downarrow \theta \\ & \eta \searrow & \\ & & (H_*B - G)(H_*E) \end{array}$$

$$\begin{array}{ccc} H_*(BF_p^2(E)) & \xrightarrow{H_*(\mu)} & H_*(BF_p(E)) \\ \theta & & \\ (H_*B - G)(H_*(BF_p(E))) & & \downarrow \theta \\ (H_*B - G)(\theta) & & \\ (H_*B - G)^2(H_*E) & \xrightarrow{\mu} & (H_*B - G)(H_*E) \end{array}$$

Regarding $S^t \to \Gamma(Y, E)$ as a map $S^t \times Y \to E$ gives a map $\pi_t\Gamma(Y, E) \to \mathrm{Der}^t_{\underline{CA}/H_*B}(H_*Y, H_*E)_{H_*\psi}$ which is an isomorphism when $E = BF_p(E') = B \times F_p(E')$.

Hence, $E_1^{s,t}(Y, E, \psi) = N^s\pi_t\Gamma(Y, BF_p^\bullet(E)) = N^s\mathrm{Der}^t_{\underline{CA}/H_*B}(H_*Y, H_*BF_p^\bullet(E))_{H_*\psi} = N^s\mathrm{Der}^t_{\underline{CA}/H_*B}(H_*Y, (H_*B{-}G)^\bullet(H_*E))_{H_*\psi}$ and $E_2^{s,t}(Y, E, \psi) = \mathrm{Ext}^{s,t}_{\underline{CA}/H_*B}(H_*Y, H_*E)_{H_*\psi}$.

Bousfield points out that the following variation occurs when one considers only the component $\Gamma(Y, BF_{p\infty}(E))_\psi$ of ψ. One gets a spectral sequence $E_r^{s,t}(Y, E)_\psi$ with $E_2^{s,t}(Y, E)_\psi = \mathrm{Ext}^{s,t}_{\underline{CA}/A_*B}(H_*Y, H_*E)_{H_*\psi}$ for $1 \le t \ge s \ge 0$ and $= 0$ otherwise. If $\varprojlim{}^1 E_r^{s,t}(Y, E)_\psi = 0$ for all $s \ge 0$ and $t > s$, this spectral sequence satisfies the following strong convergence condition: When $t > s$, $\pi_{t-s}\Gamma(Y, BF_{p\infty}(E))_\psi$ is the inverse limit of Q_s = image of $\pi_{t-s}\Gamma(Y, BF_{p\infty}(E)) \to \pi_{t-s}\ \mathrm{tot}_s$. The kernels $e^{s,t}$ of the maps $Q_s \to Q_{s-1}$ are $E_\infty^{s,t}(Y, E)_\psi$.

When $t = s$, let $\tilde{\pi}_0 =$ the set of φ in $\pi_0\Gamma(Y, E)$ such that $H_*\varphi = H_*\psi$. Then the map $\tilde{\pi}_0 \to \varprojlim Q_s$ is a surjection with trivial kernel and the kernel $e^{s,s}$ injects into $E_\infty^{s,s}(Y, E)_\psi$ for $s \ge 0$.

Bousfield also proves the following result. Let $\alpha : H_*Y \to H_*E$ be a map in $\underline{CA}/H_*B$ such that $\mathrm{Ext}^{s,s-1}_{\underline{CA}/H_*B}(H_*Y, H_*E)_\alpha = 0$ for all $s \ge 2$.

Then there exists a map $\varphi : Y \to BR_\infty(E)$ over B with $H_*\varphi = \alpha$. Moreover, the canonical injections $e^{s,s} \to E_\infty^{s,t}(Y,E)_\psi$ are bijections. In particular, if $\mathrm{Ext}\,{}^{s,s}_{\underline{CA}/H_*B}(H_*Y, H_*E)_\alpha = 0$ for $s \geq 0$, then φ is unique up to homotopy over B.

Now, we give an application. Let $\pi = Z/pZ$, $X =$ a finite CW complex with a cellular π-action, and $X^\pi =$ the fixed point set. Let $E_\pi(X) = E\pi \times_\pi X$ be the Borel construction and note $E_\pi(X^\pi) = B_\pi \times X^\pi \to E_\pi(X)$. As usual, spaces may be identified with their singular complexes, so that we may apply $BR_{p\infty}$ and $BF_{p\infty}(E_\pi(X) = E_\pi(F_{p\infty}X)$.

Proposition 3.1 (Generalized Sullivan conjecture). *There is a weak homotopy equivalence from $F_{p\infty}(X^\pi)$ to the space $\Gamma(E_\pi(F_{p\infty}X))$ of sections of $E_\pi(F_{p\infty}X)$.*

Proof: Since X^π is finite dimensional, the Sullivan conjecture [5] implies that $\Gamma(E_\pi(F_{p\infty}X^\pi)) = \mathrm{map}(B\pi, F_{p\infty}X^\pi) \simeq F_{p\infty}X$. Hence, 3.1 reduces to showing that $\Gamma(E_\pi(F_{p\infty}X^\pi)) \to \Gamma(E_\pi(F_{p\infty}X))$ is a weak equivalence of spaces of sections. Letting $B\pi \to B\pi$ be the identity, we must show that

$$\Gamma(B\pi, E_\pi(F_{p\infty}X^\pi)) \to \Gamma(B\pi, E_\pi(F_{p\infty}X))$$

is a weak equivalence.

If $H = H^*B\pi$, a basic computation of Lannes [4] is: $TH = \coprod_{\varphi\epsilon\,\mathrm{hom}(\pi,\pi)} H$ with $1 : H \to H$ corresponding to the projection on the degree 0 component of the $\varphi = 1$ component, $TH \to H \to F_p$. Another computation of Lannes [4] is: if $E = E_\pi(X), TH^*E = \left(\coprod_{\varphi\neq 0} H \otimes H^*X^\pi\right) \coprod H^*E$ with the map $TH \to TH^*E$ being given as the natural inclusion $H \to H \otimes H^*X^\pi$ on the $\varphi \neq 0$ components and as $p^* : H \to H^*E$ on the $\varphi = 0$ component.

For any homomorphism $\psi^* : H \otimes H^*X^\pi \to H$ under H, $\mathrm{Ext}^{s,t}_{\underline{A}\backslash H}(H \otimes H^*X^\pi, H)_{\psi^*} = \mathrm{Ext}^{s,t}_{\underline{A}}(H^*X^\pi, H)_{\psi^*\iota} = \mathrm{Ext}^{s,t}_{\underline{A}}(TH^*X^\pi, F_p)_{\underline{\psi^*\iota}} = \mathrm{Ext}^{s,t}_{\underline{A}}(H^*X^\pi, F_p)_{\underline{\psi^*\iota}} = \mathrm{Ext}^{s,t}_{\underline{A}}(H^*X^\pi_c, F_p)_{\overline{\underline{\psi^*\iota}}}$, where $\iota : H^*X^\pi \to H \otimes H^*X^\pi$ is the natural inclusion. In particular, 2.1 implies that this $= 0$ for $t - s \leq 0$ unless $s = t = 0$.

For any homomorphism $\varphi^* : H^*E \to H$ under H, $\mathrm{Ext}^{s,t}_{\underline{A}\backslash H}(H^*E, H)_{\varphi^*} = \mathrm{Ext}^{s,t}_{\underline{A}\backslash TH}(TH^*E, F_p)_{\underline{\varphi^*}} = \mathrm{Ext}^{s,t}_{\underline{A}\backslash TH_c}(TH^*E_c, F_p)_{\overline{\underline{\varphi}}^*} = \mathrm{Ext}^{s,t}_{\underline{A}\backslash H}(H \otimes H^*X^\pi, F_p)_{\overline{\underline{\varphi}}^*} =$

$\mathrm{Ext}^{s,t}_{\underline{A}}(H^*X^\pi, F_p)_{\underline{\varphi}^*\iota}$. Hence, if $\varphi^* = \psi^* j^* : H^*E \to H \otimes H^*X^\pi \to$H, then $\mathrm{Ext}^{s,t}_{\underline{A}\backslash H}(H \otimes H^*X^\pi, H)_{\psi^*} \to \mathrm{Ext}^{s,t}_{\underline{A}\backslash H}(H^*E, H)_{\varphi^*}$ is an isomorphism.

Setting $s = t = 0$ gives that $\mathrm{hom}_{\underline{A}\backslash H}(H \otimes H^*X^\pi, H) \to \mathrm{hom}_{\underline{A}\backslash H}(H^*E, H)$ is an isomorphism and, by Bousfield, $\pi_0\Gamma(B\pi, B\pi \times X^\pi) = \pi_0\Gamma(B\pi, B\pi \times F_{p\infty}X) \to \pi_0\Gamma(B\pi, E_\pi(F_{p\infty}X))$ is a bijection. Since the unstable Adams spectral sequences for each component map isomorphically at E_2, $\Gamma(B\pi, B\pi \times F_{p\infty}X) \to \Gamma(B\pi, E_\pi(F_{p\infty}X))$ is a weak equivalence. ∎

4. Fibrations and weak equivalences.

If $X^\bullet$ is a cosimplicial space, then Bousfield-Kan defines for $n \geq -1$ the matching spaces $M^n X^\bullet$ as follows: If $n = -1$, $M^n X^\bullet$ is a point $*$ and if $n \geq 0$, $M^n X^o$ is the subset of the $n+1$ fold product $X^n \times \cdots \times X^n$ consisting of those $(x^o, \ldots, x^n)$ such that $s^i x^j = s^{j-1} x^i$ whenever $0 \leq i \leq j < n$. If $n \geq -1$, there are natural maps

$$(*)_n : X^{n+1} \to M^n X^\bullet$$

given by $x \mapsto (s^0 x, \ldots, s^n x)$ if $n \geq 0$.

These matching spaces are used by Bousfield-Kan to define fibrations in the category of cosimplicial spaces. A map $f : X^o \to Y^o$ is called a fibration if, for all $n \geq -1$, the natural maps into the fibre product

$$(**)_n : X^{n+1} \to Y^{n+1} \times_{M^n Y^\bullet} M^n X^\bullet$$

are fibrations. A cosimplicial space $X^\bullet$ is called fibrant if $X^\bullet \to \underline{*}$ is a fibration, equivalently, if the maps $(*)_n$ are fibrations for all $n \geq -1$.

It is useful to note that these definitions of fibration and of fibrant object do not depend on the coface operators.

Fibrations have the following properties. Products of fibrations are fibrations and hence products of fibrant cosimplicial spaces are fibrant. Pullbacks of fibrations are fibrations and hence fibres of fibrations are fibrant. The more difficult proposition below is proved by Bousfield-Kan.

Proposition 4.1. *If $E^\bullet \to B^\bullet$ is a fibration with fibre $F^\bullet$, then tot $E^\bullet \to$ tot $B^\bullet$ is a fibration with fibre tot $F^\bullet$. In particular, if $X^\bullet$ is fibrant, then tot $X^\bullet$ is a fibrant space, that is, a Kan complex.*

We now list seven useful examples.

0) Pullbacks of fibrations are fibrations.
1) If B is a fibrant space, then the constant cosimplicial space $\underline{B}$ is fibrant. The maps $(*)_n$ are $B \to *$ if $n = -1$ and isomorphisms if $n \geq 0$.
2) Any isomorphism $X^\bullet \to Y^\bullet$ is a fibration.
3) If $Y^\bullet$ is fibrant, then the projection $X^\bullet \times Y^\bullet \to X^\bullet$ is a fibration.
4) A cosimplicial space $X^\bullet$ is called grouplike if, for all $n \geq 0$, X^n is a simplicial group and the coface and codegeneracy operators except possibly for d^0 are all homomorphisms. Bousfield-Kan show

that grouplike objects are fibrant. For example, the canonical resolutions $R^{\bullet}Y$ are grouplike. More generally, they show that any surjective homomorphism of grouplike objects is a fibration.

5) Let $p : E \to B$ be a space over B with E and B fibrant. Consider the triple $(B \times -)(E) = B \times E$ where $B \times E$ is regarded as a space over B by projecting on the first factor and where the triple structure $\eta(E) : E \to B \times E$ and $\mu(E) : B \times B \times E \to B \times E$ is given by $\eta(E) = (p, 1)$ and $\mu(E) =$ projection on the first and third factors. The canonical resolution $E \to (B \times -)^{\bullet}(E)$ is called the Rector complex. The natural map $(B \times -)^{\bullet}(E) \to \underline{B}$ is a fibration since the map $(**)_n$ is $B \times E \to B$ if $n = -1$, and an isomorphism if $n \geq 1$.
6) Let $p : E \to B$ be a space over B and consider the resolution $E \to BR^{\bullet}(E)$ introduced in section 1. We claim that $q : BR^{\bullet}(E) \to \underline{B}$ is a fibration. Pick a basepoint in B and let $F^{\bullet}$ be the fibre of q. If we forget the coface operators, then $BR^{\bullet}(E) \to \underline{B}$ is the projection $\underline{B} \times F^{\bullet} \to \underline{B}$. Since $F^{\bullet}$ is grouplike, $F^{\bullet}$ is fibrant. Hence, $BR^{\bullet}(E) \to \underline{B}$ is a fibration. In particular, if B is fibrant, then so is $BR^{\bullet}(E)$.

Bousfield-Kan call a map $f : X^{\bullet} \to Y^{\bullet}$ a weak equivalence if $f^n : X^n \to Y^n$ is a weak equivalence for all $n \geq 0$. The following result of Bousfield-Kan relates this notion to fibrant objects.

Proposition 4.2. *If $f : X^{\bullet} \to Y^{\bullet}$ is a weak equivalence with $X^{\bullet}$ and $Y^{\bullet}$ fibrant, then tot $f :$ tot $X^{\bullet} \to$ tot $Y^{\bullet}$ is a homotopy equivalence.*

5. Bicosimplicial spaces.

If $A^{\bullet\bullet}$ is a bicosimplicial object, then denote the horizontal operators by $d_h^i : A^{n-1,m} \to A^{n,m}$ and $s_h^i : A^{n+1,m} \to A^{n,m}$ and the vertical operators by $d_v^i : A^{n,m-1} \to A^{n,m}$ and $s_v^i : A^{n,m+1} \to A^{n,m}$. For example, if $X^\bullet$ and $Y^\bullet$ are cosimplicial spaces, then $X^\bullet \hat{\times} Y^\bullet$ is the bicosimplicial space with $(X^\bullet \hat{\times} Y^\bullet)^{n,m} = X^n \times Y^m$ and $d_h^i = d^i \times 1$, $s_h^i = s^i \times 1$, $d_v^i = 1 \times d^i$, $s_v^i = 1 \times s^i$.

If $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$ are two bicosimplicial spaces, then define the bicosimplicial mapping space $\hom(A^{\bullet\bullet}, B^{\bullet\bullet})$ to be the space with n-simplices $\hom(A^{\bullet\bullet}, B^{\bullet\bullet})_n$ equal to the set of bicosimplicial maps $\Delta^n \times A^{\bullet\bullet} \to B^{\bullet\bullet}$ and with face and degeneracy operators $d_i(f) = f(d^i \times 1)$ and $s_i(f) = f(s^i \times 1)$.

The following lemma, which relates bicosimplicial and cosimplicial mapping spaces, is left as an exercise.

Lemma 5.1. *If $X^\bullet$ and $Y^\bullet$ are cosimplicial spaces and $C^{\bullet\bullet}$ is a bicosimplicial space, then* $\hom(X^\bullet \hat{\times} Y^\bullet, C^{\bullet\bullet}) = \hom(X^\bullet, \hom(Y^\bullet, C^{\bullet\bullet}))$ *where* $\hom(Y^\bullet, C^{\bullet\bullet})_n$ *is the cosimplicial space with* $\hom(Y^\bullet, C^{\bullet\bullet})^n = \hom(Y^\bullet, C^{n\bullet})$.

For any bicosimplicial space $A^{\bullet\bullet}$, define tot $A^{\bullet\bullet}$ to be the space $\hom(\Delta^\bullet \hat{\times} \Delta^\bullet, A^{\bullet\bullet})$. Also, define horizontal and vertical tot functors by: $\mathrm{tot}_h A^{\bullet\bullet}$ is the cosimplicial space with $(\mathrm{tot}_h A^{\bullet\bullet})^m = \mathrm{tot} A^{\bullet m}$ and $\mathrm{tot}_v A^{\bullet\bullet}$ is the cosimplicial space with $(\mathrm{tot}_v A^{\bullet\bullet})^n = \mathrm{tot}\ A^{n\bullet}$. The preceding lemma has the following corollary.

Corollary 5.2. *If $A^{\bullet\bullet}$ is a bicosimplicial space, then tot $A^{\bullet\bullet}$ = tot $\mathrm{tot}_h A^{\bullet\bullet}$ = tot $\mathrm{tot}_v A^{\bullet\bullet}$.*

Note. One also has tot $A^{\bullet\bullet}$ = tot (diag $A^{\bullet\bullet}$) where diag $A^{\bullet\bullet}$ is the cosimplicial space with $(\mathrm{diag}\ A^{\bullet\bullet})^n = A^{nn}$ and with coface and codegeneracy operators, $d^i = d_h^i d_v^i = d_v^i d_h^i$, $s^i = s_h^i s_v^i = s_v^i s_h^i$.

If $A^{\bullet\bullet}$ is a bicosimplicial space, we define horizontal and vertical cosimplicial mapping spaces by $(M_h^n A^{\bullet\bullet})^m = M^n A^{\bullet m}$ and $(M_v^m A^{\bullet\bullet})^n = M^m A^{n\bullet}$.

Definition 5.3. A map $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ of bicosimplicial spaces is a fibration if $A^{n+1,\bullet} \to B^{n+1,\bullet} \times_{M_h^n B^{\bullet\bullet}} M_h^n A^{\bullet\bullet}$ is a fibration of cosimplicial spaces for all $n \geq -1$.

Remark. The above definition is due to Bousfield. He points out that this is part of the definition of a closed model category in which a map $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a weak equivalence if and only if $g^{nm} : A^{nm} \to B^{nm}$ is a homotopy equivalence for all $n, m \geq 0$. In addition, he observes the surprising fact that this definition is equivalent to requiring that $A^{\bullet,m+1} \to B^{\bullet,m+1} \times_{M_v^m B^{\bullet\bullet}} M_v^m A^{\bullet\bullet}$ be a fibration for all $n \geq -1$.

As before, $A^{\bullet\bullet}$ is called fibrant if $A^{\bullet\bullet} \to \underline{\underline{*}}$ is a fibration where $\underline{\underline{*}}$ is the bicosimplicial space which is a constant point.

Note that tot_h and tot_v preserve matching spaces, that is, $M^m(\mathrm{tot}_h A^{\bullet\bullet}) = \mathrm{tot}\,(M_v^m A^{\bullet\bullet})$ and $M^n(\mathrm{tot}_v A^{\bullet\bullet}) = \mathrm{tot}\,(M_h^n A^{\bullet\bullet})$. Furthermore, since tot preserves fibre products, it follows from 4.1 that:

Lemma 5.4. *If $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a fibration, then $\mathrm{tot}_h g$, $\mathrm{tot}_v g$, and $\mathrm{tot}\, g$ are all fibrations. In particular, if $A^{\bullet\bullet}$ is fibrant, then $\mathrm{tot}_h A^{\bullet\bullet}$, $\mathrm{tot}_v A^{\bullet\bullet}$, and $\mathrm{tot}\, A^{\bullet\bullet}$ are all fibrant.*

This lemma and 4.2 imply:

Proposition 5.5. *If $g : A^{\bullet\bullet} \to B^{\bullet\bullet}$ is a map of bicosimplicial spaces with $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$ fibrant and if either $\mathrm{tot}_h g$ or $\mathrm{tot}_v g$ is a weak equivalence, then $\mathrm{tot}\, g$ is a homotopy equivalence.*

Let $X^\bullet \to A^{\bullet\bullet}$ be an augmented bicosimplicial space via a cosimplicial map $X^\bullet \to A^{0\bullet}$ (or $X^\bullet \to A^{\bullet 0}$). The above proposition specializes to:

Corollary 5.6. *If $X^\bullet \to A^{\bullet\bullet}$ is an augmented bicosimplicial space with $X^\bullet$ and $A^{\bullet\bullet}$ fibrant and if $X^\bullet \to \mathrm{tot}_h A^{\bullet\bullet}$ (or $X^\bullet \to \mathrm{tot}_v A^{\bullet\bullet}$) is a weak equivalence, then $\mathrm{tot}\, X^\bullet \to \mathrm{tot}\, A^{\bullet\bullet}$ is a homotopy equivalence.*

The cosimplicial examples of section 4 have analogues for bicosimplicial spaces. In particular, pullbacks of fibrations are fibrations, if B is a fibrant space then the constant bicosimplicial space $\underline{\underline{B}}$ is fibrant, isomorphisms are fibrations, and if $B^{\bullet\bullet}$ is fibrant then the projection $A^{\bullet\bullet} \times B^{\bullet\bullet} \to A^{\bullet\bullet}$ is a fibration.

Definition 5.7. A bicosimplicial space $A^{\bullet\bullet}$ is grouplike if each fixed row $A^{\bullet m}$ and each fixed column $A^{n\bullet}$ is a grouplike cosimplicial space.

Proposition 5.8. *A surjective homomorophism $A^{\bullet\bullet} \to B^{\bullet\bullet}$ of grou-*

plike bicosimplicial spaces is a fibration. In particular, a grouplike bicosimplicial space is fibrant.

PROOF: As in Bousfield-Kan , page 276, $A^{n+1\,\bullet} \to B^{n+1\,\bullet} \times_{M_h^n B^{\bullet\bullet}} M_h^n A^{\bullet\bullet}$ is a surjective homomorphism of grouplike cosimplicial spaces, therefore, a fibration. ∎

6. Contractions.

Let $X^\bullet$ be a cosimplicial object augmented by $d^0 : X^{-1} \to X^0$. The augmented cosimplicial object $X^{-1} \to X^\bullet$ is said to admit a left contraction if for $n \geq -1$ there are maps $s^{-1} : X^{n+1} \to X^\bullet$ such that the usual cosimplicial identities are satisfied, that is:

$$d^i d^j = d^j d^{i-1} \text{ if } i > j$$
$$s^i d^j = d^j s^{i-1} \text{ if } i > j, = 1 \text{ if } i = j \text{ or } j-1, = d^{j-1} s^{i-1} \text{ if } i < j$$
$$s^k s^j = s^{j-1} s^i \text{ if } i < j\,.$$

Similarly, it said to admit a right contraction if for $n \geq -1$ there are maps $s^{n+1} : X^{n+1} \to X^n$ such that the usual cosimplicial identities are satisfied.

Let $d : X^{-1} \to \text{ tot } X^\bullet$ be the map induced by the augmentation. A left (or right) contraction gives a map $\overline{s} : X^\bullet \to \underline{X}^{-1}$ defined by $\overline{s}^n = s^{-1} \cdots s^{-1} : X^n \to X^{-1}$ (or $\overline{s}^n = s^0 \cdots s^n$). If $s = \text{ tot } \overline{s} : \text{ tot } X^\bullet \to X^{-1}$, then it is easy to see that $sd : X^{-1} \to \text{ tot } X^\bullet \to X^{-1}$ is the identity. We shall show that $ds : \text{ tot } X^\bullet \to X^{-1} \to \text{ tot } X^\bullet$ is homotopic to the identity. Thus, $d : X^{-1} \to \text{ tot } X^\bullet$ is a homotopy equivalence whenever $X^{-1} \to X^\bullet$ admits a contraction.

The lemma below is left as an exercise.

Lemma 6.1. *If W is a space and $X^\bullet$ and $Y^\bullet$ are cosimplicial spaces, then map $(W, \text{hom}(X^\bullet, Y^\bullet)) = \text{hom}(W \times X^\bullet, Y^\bullet)$ where map denotes the simplicial mapping space.*

Hence, a homotopy between ds and 1 is a map

$$H : \text{hom}(\Delta^\bullet, X^\bullet) \to \text{hom}(\Delta^1 \times \Delta^\bullet, X^\bullet)\,.$$

This homotopy is defined as follows. Let $\pi : \Delta^1 \times \Delta^{n-1} \to \Delta^n$ be the map defined by: $\pi(0,i) = 0$, $\pi(1,i) = i{+}1$, and $\pi((a_0,\ldots,a_n),(b_0,\ldots,b_n)) = (\pi(a_0,b_0),\ldots,\pi(a_n,b_n))$. Similarly, let $\sigma : \Delta^1 \times \Delta^{n-1} \to \Delta^n$ be defined by $\sigma(0,i) = i$, $\sigma(1,i) = n$, and $\sigma((a_0,\ldots,a_n),(b_0,\ldots,b_n)) = (\sigma(a_0,b_0),\ldots,\sigma(a_n,b_n))$. If $X^{-1} \to X^\bullet$ has a left contraction, define H on an m-simplex $f : \Delta^m \times \Delta^\bullet \to X^\bullet$ by $(Hf)^n = s^{-1} f^{n+1}(1 \times \pi) : \Delta^m \times \Delta^1 \times \Delta^n \to \Delta^m \times \Delta^{n+1} \to X^{n+1} \to X^n$. If it has a right contraction, define H by $(Hf)^n = s^{n+1} f^{n+1}(1 \times \sigma)$.

Check that $\pi(1 \times d^{j-1}) = d^j \pi$, $\pi(1 \times s^{j-1}) = s^j \pi$, $\sigma(1 \times d^j) = d^j \sigma$, and $\sigma(1 \times s^j) = s^j \sigma$.

Then it is easy to see that Hf is in $\hom(\Delta^1 \times \Delta^\bullet, X^\bullet)_n$. It is also apparent that H is a simplicial map. In the case of a left contraction, $H_0 = ds$ and $H_1 = 1$, and in the case of a right contraction, $H_0 = 1$ and $H_1 = ds$.

The standard examples of contractions are as follows. Let T be a triple and $X \to T^\bullet X$ the canonical resolution.

1) A left contraction for $TX \to T(T^\bullet X)$ is defined for $n \geq -1$ by $s^{-1} = \mu(T^{n+1}x) : T(T^{n+2}X) \to T(T^{n+1}X)$.
2) A right contraction for $TX \to T^\bullet(TX)$ is defined for $n \geq -1$ by $s^{n+1} = T^{n+1}\mu(X) : T^{n+2}(TX) \to T^{n+1}(TX)$.
3) A constant object $X \to \underline{X}$ admits both left and right contractions given by $s^{-1} = 1$ and $s^{n+1} = 1$.
4) Let E be a space over B with E and B fibrant. The Rector complex $E \to (B \times -)^\bullet(E)$ of example 4 in section 2 admits a left contraction. For $n \geq -1$, define it by letting $s^{-1} : (B \times -)^{n+2}(E) \to (B \times -)^{n+1}(E)$ be projection on the last $n+2$ factors. Hence, $E \to (B \times -)_\infty(E)$ is a map over B which is a homotopy equivalence where the target is a fibration. Notice that the maps s^{-1} are not maps over B and hence the homotopy inverse $(B \times -)_\infty(E) \to E$ is not a map over B.

Definition 6.2. Let $T : \underline{\underline{C}} \to \underline{\underline{C}}$ be a triple with structure maps $\eta : 1 \to T$ and $\mu : T^2 \to T$. A right T-module is a covariant functor $M : \underline{\underline{C}} \to \underline{\underline{B}}$ together with a natural transformation $\varphi : MT \to M$ such that the following commute.

$$\begin{array}{ccc} MT & \xrightarrow{\varphi} & M \\ M\eta\uparrow & & \nearrow 1 \\ M & & \end{array} \qquad \begin{array}{ccc} MTT & \xrightarrow{M\varphi} & MT \\ \downarrow M\mu & & \downarrow \varphi \\ MT & \xrightarrow{\varphi} & M \end{array}$$

If M is a right T-module and $S : \underline{\underline{B}} \to \underline{\underline{A}}$ is a covariant functor, then SM is also a right T-module via $S\varphi$.

Of course, one can define a left T-module in the obvious way and one has:

Proposition 6.3. *If M is a right (left) T-module and $X \to T^\bullet X$ is the canonical resolution, then $MX \to MT^\bullet X$ ($MX \to T^\bullet MX$) admits a left (right) contraction.*

Proof: Define $s^{-1} : MT^{n+1}X \to MT^nX$ by $s^{-1} = \varphi(T^nX)$ and $s^n : T^{n+1}MX \to T^nMX$ by $s^n = T^n\varphi(x)$. ∎

Notice that examples 1 and 2 above are special cases of 6.3 As a less trivial example, we use 6.3 to show that two constructions of Bousfield-Kan are equivalent.

Let $\overline{R}X$ be the set of formal sums $\Sigma r[x]$ where $r\epsilon R$, $x\epsilon X$, and $\Sigma r = 1$. Then $\overline{R}X$ is contained in RX and it is clear that the triple structure of R restricts to one for $\overline{R}$. Furthermore, R is a right $\overline{R}$ module via $\varphi : R\overline{R} \to R$ defined by $\varphi(\Sigma r[\Sigma s[x]]) = \Sigma rs[x]$. We claim that there is a homotopy equivalence $\overline{R}_\infty X \to R_\infty X$.

Consider the bicosimplicial space $R^\bullet(\overline{R}^\bullet X)$ with $R^\bullet(\overline{R}^\bullet X)^{nm} = R^{n+1}(\overline{R}^{m+1}X)$. It has augmentations $R^\bullet X \to R^\bullet(\overline{R}^\bullet X)$ and $\overline{R}^\bullet X \to R^\bullet(\overline{R}^\bullet X)$. Furthermore, $R^\bullet X, \overline{R}^\bullet X$, and $R^\bullet(\overline{R}^\bullet X)$ are all grouplike, hence fibrant. (To make $\overline{R}^\bullet X$ grouplike, choose a basepoint in X and convert the affine spaces $\overline{R}^{m+1}X$ into groups.) Since each R^{n+1} is a right $\overline{R}$-module, $R^\bullet X \to \text{tot}_v R^\bullet(\overline{R}^\bullet X)$ is a weak equivalence and hence $R_\infty X \to \text{tot}\ R^\bullet(\overline{R}^\bullet X)$ is a homotopy equivalence. On the other hand, by Bousfield-Kan , $Y \to R_\infty(Y)$ is a homotopy equivalence if Y is a simplicial R-module. Hence, $\overline{R}^\bullet X \to R_\infty(\overline{R}^\bullet X)$ is a weak equivalence and $\overline{R}_\infty X \to \text{tot}\ R^\bullet(\overline{R}^\bullet X)$ is a homotopy equivalence.

7. The homotopy pullback property.

In this section, we prove:

Proposition 7.1. *Let E be a space over B with E and B fibrant. Then $BR_\infty(E)$ is a homotopy pullback of*

$$\begin{array}{ccc} & & R_\infty E \\ & & \downarrow \\ B & \to & R_\infty B\,. \end{array}$$

The proof of 7.1 consists of two stages. First, we show that, in the case of a product bundle $B \times F \to B$, there is a homotopy equivalence $B \times R_\infty(R) \to BR_\infty(B \times F)$. Second, we apply the fact that BR_∞ is the fibrewise localization on product bundles to establish 7.1.

The natural map $BR(B\times F) = B\times R(B\times F) \to B\times RF$ is compatible with the triple structures of BR and R. Hence, it defines a map $\theta : BR^\bullet(B\times F) \to B\times R^\bullet(F)$ and a map $\theta_\infty : BR_\infty(B\times F) \to B\times R_\infty(F)$ over B.

Lemma 7.2. *If B is fibrant, then $\theta_\infty : BR_\infty(B \times F) \to B \times R_\infty(F)$ is a homotopy equivalence.*

Proof: It follows from example 6 in section 4 that θ_∞ is a map of fibrations over B. In order to show that θ_∞ is a homotopy equivalence, it is sufficient to show that for every choice of a basepoint in B the resulting map of fibres is a homotopy equivalence. (This requires that B be fibrant.) Inspection shows that the fibre of $BR^\bullet(B \times F) \to \underline{B}$ is the canonical resolution $R(B\times -)^\bullet(F)$ associated to the triple $F \mapsto R(B\times F)$ with evident structure maps $F \to R(B \times F)$ and $R(B \times R(B \times F)) \to R(B\times F)$. Furthermore, θ induces the map of fibres $\alpha : R(B\times -)^\bullet(F) \to R^\bullet(F)$ coming from the map of triples $R(B\times F) \to RF$. Hence, we need to show that $\alpha_\infty : R(B\times -)_\infty(F) \to R_\infty(F)$ is a homotopy equivalence.

Consider the bicosimplicial spaces $A^{\bullet\bullet} = R(B\times -)^\bullet(R^\bullet F)$ and $R^\bullet(R^\bullet F)$ with $A^{nm} = R(B \times -)^{n+1}(R^{m+1}F)$ and $R^\bullet(R^\bullet F)^{nm} = R^{n+1}(R^{m+1}F)$. There are augmentations $\beta = R(B \times -)^\bullet(\eta) : R(B \times -)^\bullet(F) \to A^{\bullet\bullet}$, $\gamma = \eta : R^\bullet F \to A^{\bullet\bullet}$, $\epsilon = R^\bullet(\eta) : R^\bullet F \to R^\bullet(R^\bullet F)$, and $\delta = \eta : R^\bullet F \to$

$R^\bullet(R^\bullet F)$ which fit into a commutative diagram.

$$\begin{array}{ccccc} F & \longrightarrow & R(B\times -)^\bullet(F) & \xrightarrow{\alpha} & R^\bullet F \\ \downarrow & & \downarrow\beta & & \downarrow\epsilon \\ R^\bullet F & \xrightarrow{\gamma} & A^{\bullet\bullet} & \xrightarrow{\kappa} & R^\bullet(R^\bullet F)\ . \\ & & \delta & & \end{array}$$

That α_∞ is a homotopy equivalence follows from the fact that the induced maps ϵ_∞, δ_∞, γ_∞ and β_∞ are homotopy equivalences. Since $R^\bullet F$, $R(B\times -)^\bullet(F)$, $A^{\bullet\bullet}$ and $R^\bullet(R^\bullet F)$ are all grouplike, they are fibrant and we can apply Corollary 5.6.

Example 1 in section 6 shows that $R^\bullet F \to \ \mathrm{tot}_v R^\bullet(R^\bullet F)$ is a weak equivalence and hence $\epsilon_\infty : R_\infty F \to \ \mathrm{tot}\ R^\bullet(R^\bullet F)$ is a homotopy equivalence. Example 2 shows that $R^\bullet F \to \ \mathrm{tot}_h R^\bullet(R^\bullet F)$ is a weak equivalence and hence $\delta_\infty : R_\infty F \to \ \mathrm{tot}\ R^\bullet(R^\bullet F)$ is a homotopy equivalence.

To show that $R^{m+1}R \to R(B\times -)^\bullet(R^{m+1}F)$ admits a contraction, it suffices to treat the case $m=0$. But $\varphi = \mu(F)R(\pi_2) : R(B\times RF) \to R(RF) \to RF$ makes R into a left $R(B\times -)-$ module. Hence, there is a right contraction and γ_∞ is a homotopy equivalence.

The left contraction in example 1 of section 6 is a linear contraction of $R \to R(R^\bullet F)$. Since $R(B\times R^\bullet F) = RB \otimes R(R^\bullet F)$, we get a left contraction of $R(B\times -)(F) \to R(B\times -)(R^\bullet F)$ and, by induction, a left contraction of $R(B\times -)^{n+1}(F) \to R(B\times -)^{n+1}(R^\bullet F)$ for all $n \geq 0$. Hence, β_∞ is a homotopy equivalence. ∎

The proof of 7.2 being complete, we recast 1.2 in a more convenient form. Define a map $\psi : B\times R^\bullet(F) \to BR^\bullet(B\times F)$ by $\psi(b, \Sigma r_0[\Sigma r_1[\cdots[\Sigma r_n[f]]\cdots]]) = (b, \Sigma r_0[b, \Sigma r_1[b, \cdots[\Sigma r_n[b,f]]\cdots]])$. Since the composition $\theta\psi : B\times R^\bullet(F) \to BR^\bullet(B\times F) \to B\times R^\bullet(F)$ is the identity, we get:

Lemma 7.3. *If B is fibrant, then $\psi_\infty : B\times R_\infty(F) \to BR_\infty(B\times F)$ is a homotopy equivalence.*

Consider what a homotopy pullback of

$$\begin{array}{ccc} & & E \\ & & \downarrow \\ X & \longrightarrow & B \end{array}$$

is. The Rector complex gives a homotopy equivalence $E \to (B\times -)_\infty(E)$ and the target is a fibration whenever E and B are fibrant. Let

$$\begin{array}{ccc} (B\times -)_\infty(E)_{|X} & \longrightarrow & (B\times -)_\infty(E) \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

be a pullback diagram. Then a homotopy pullback is any fibration Y over X which is homotopy equivalent to $(B\times -)_\infty(E)_{|X}$ via maps over B.

Form the bicosimplicial space $B^{\bullet\bullet} = BR^\bullet(B\times -)^\bullet(E)$ with $B^{nm} = BR^{n+1}(B\times -)^{m+1}(E)$. If we ignore coface operators, $B^{\bullet\bullet}$ is the product of $\underline{\underline{B}}$ and a grouplike object. Hence, $B^{\bullet\bullet}$ is fibrant.

Form the bicosimplicial space $C^{\bullet\bullet}$ with $C^{nm} = B\times R^{n+1}((B\times -)^m(E))$ with horizontal operators coming from R^{n+1} and vertical operators coming from embedding $C^{n\bullet}$ into $R^{n+1}((B\times -)^\bullet(E))$. If we forget coface operators, then $C^{\bullet\bullet}$ is the product of $\underline{\underline{B}}$ and a grouplike object, hence, $C^{\bullet\bullet}$ is fibrant.

The map ψ in 7.3 gives a map $\psi : C^{\bullet\bullet} \to B^{\bullet\bullet}$ over B and a weak equivalence $\mathrm{tot}_h\psi : \mathrm{tot}_h C^{\bullet\bullet} \to \mathrm{tot}_h B^{\bullet\bullet}$ where $\mathrm{tot}_h C^{\bullet m} = B\times R_\infty((B\times -)^m(E))$. Hence, $\mathrm{tot}\ \psi : \mathrm{tot}\ C^{\bullet\bullet} \to \mathrm{tot}\ B^{\bullet\bullet}$ is a homotopy equivalence.

By Bousfield-Kan , the map $R_\infty(X\times Y) \to R_\infty X \times R_\infty Y$ is always a homotopy equivalence. Hence, there is a weak equivalence $\mathrm{tot}_h C^{\bullet\bullet} \to E^\bullet$ over B where $E^m = B\times (R_\infty B\times -)^m(R_\infty E)$. There is a pullback diagram

$$\begin{array}{ccc} E^\bullet & \to & (R_\infty B\times -)^\bullet(R_\infty E) \\ \downarrow & & \downarrow \\ \underline{B} & \to & \underline{R_\infty B} \end{array}$$

Therefore, $E^\bullet$ is fibrant, tot $E^\bullet$ is the pullback $(R_\infty B\times -)_\infty(R_\infty E)_{|B}$, and there is a homotopy equivalence

$$\mathrm{tot}\ C^{\bullet\bullet} \to (R_\infty E\times -)_\infty(R_\infty B)_{|B}\,.$$

Since $E \to (B\times -)^\bullet(E)$ admits a contraction, so does each $BR^{n+1}(E) \to BR^{n+1}(B\times -)^\bullet(E)$, $BR^\bullet(E) \to \mathrm{tot}_v B^{\bullet\bullet}$ is a weak equivalence, and $BR_\infty(E) \to \mathrm{tot}\ B^{\bullet\bullet}$ is a homotopy equivalence.

That $BR_\infty(E)$ is a homotopy pullback of

$$\begin{array}{ccc} & & R_\infty E \\ & & \downarrow \\ B & \longrightarrow & R_\infty B \end{array}$$

is a consequence of these homotopy equivalences which are maps over B, $BR_\infty(E) \to \text{ tot } B^{\bullet\bullet} \leftarrow \text{ tot } C^{\bullet\bullet} \to (R_\infty B \times -)_\infty (R_\infty E)_{|B}$.

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