# Smith Theory Revisited

William G. Dwyer
Clarence W. Wilkerson*

## 1 Introduction

In the late 1930's, P. A. Smith began the investigation of the cohomological properties of a group $G$ of prime order $p$ acting by homeomorphisms on a topological space $X$. This thread has continued now for almost fifty years. Smith was successful in calculating the cohomology of the the fixed point sets for involutions on spheres [Smith 1] and projective spaces [Smith 2]. In the 1950's, Smith theory was reformulated by the introduction of the Borel construction $X_G = EG \times_G X$ and equivariant cohomology $H^*(X_G) = H^*_G(X)$. Borel made the key observation [Borel 1] that the cohomology of the fixed point set was closely related to the torsion-free (with respect to $H^*_G(pt)$) quotient of $H^*_G(X)$ . In the 1960's, this was formalized as the "localization theorem" of Borel-Atiyah-Segal-Quillen [Atiyah-Segal],[Quillen]. (The localization theorem is described below.) The localization theorem has previously been used to deduce the actual cohomology of the fixed point set for particular examples in an *ad hoc* fashion, but a general algorithmic computation of the cohomology of the fixed point set has not been provided in the literature.

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The main result of this note is that the localization theorem provides such a description of the *unlocalized* equivariant cohomology of the fixed point set also. One simply has to use the data given by the equivariant cohomology as a module over both $H^*_G(pt)$ and the mod $p$ Steenrod algebra, $\mathcal{A}_p$.

# 2 Statement of Results

The theory is valid for $G$ an elementary abelian $p-$group, and $X$ a finite $G-CW$ complex. By default, cohomology is understood to have coefficients in the field $F_p$ of $p$ elements. For $K$ a subgroup of $G$, define $S = S(K)$ to be the multiplicative subset of $H^*(BG)$ generated by the Bockstein images in $H^2(BG)$ of the elements $x$ in $H^1(BG)$ which restrict non-trivially to $H^1(BK)$.

The general form of the localization theorem appearing in [Hsiang, Chapter ] specializes the following statement.
2.1 *Theorem* (Localization Theorem) [ ]) *The localized restriction map*

$$S^{-1}H^*_G(X) \longrightarrow S^{-1}H^*_G(X^K)$$

*is an isomorphism.*

For geometric reasons both $H^*_G(X)$ and $H^*_G(X^K)$ are unstable $\mathcal{A}_p-$modules. By [Wilkerson] the localization $S^{-1}H^*_G(X)$ and $S^{-1}H^*_G(X^K)$ inherit $\mathcal{A}_p-$module structures themselves; these induced structures, however, do not satisfy the instability condition. For any graded $\mathcal{A}_p-$module $\mathcal{M}$ let $\mathcal{U}n(\mathcal{M})$ denote the subset of "unstable classes", i.e., $\mathcal{U}n(\mathcal{M})$ is the graded $F_p-$vector space defined as follows:

**(1)** if $p = 2$

$$\mathcal{U}n(\mathcal{M}_k = \{x \in \mathcal{M}_k \mid Sq^i(x) = 0, \;\; i > k\}$$

**(2)** if p is odd

$$\mathcal{U}n(\mathcal{M})_{2k} = \{x \in \mathcal{M}_{2k} \mid \mathcal{P}^i(x) = 0 \; (i > k), \; \beta\mathcal{P}^i(x) = 0 \; (i \geq k)\}$$

$$\mathcal{U}n(\mathcal{M})_{2k+1} = \{x \in \mathcal{M}_{2k+1} \mid \mathcal{P}^i(x) = 0 \; (i > k), \; \beta\mathcal{P}^i(x) = 0 \; (i > k)\}$$

It is true [Adams-Wilkerson] (although nontrivial) that $\mathcal{U}n(\mathcal{M})$ is closed under the $\mathcal{A}_p-$action on $\mathcal{M}$ and is thus an unstable $\mathcal{A}_p-$module. The localization map

$$H_G^*(X^K) \longrightarrow S^{-1}H_G^*(X^K)$$

respects the $\mathcal{A}_p-$action and therefore lifts to a map

$$H_G^*(X^K) \longrightarrow \mathcal{U}n(S^{-1}H_G^*(X^K)).$$

2.2 *Theorem* *The above map*

$$H_G^*(X^K) \longrightarrow \mathcal{U}n(S^{-1}H_G^*(X^K)).$$

*is an isomorphism.*

2.3 *Remark* The objects in Theorem 2.2 are algebras, $\mathcal{A}_p-$modules and modules over $H^*(BG)$; the isomorphism respects these additional structures.

The equivariant cohomology $H_G^*(X^G)$ is just the tensor product $H^*(BG)\otimes_{F_p} H^*(X^G)$, so the special case $K = G$ of Theorem 2.2 leads to the following corollary.

2.4 *Corollary* *There is a natural isomorphism*

$$H^*(X^G) \approx F_p \otimes_{H^*(BG)} \mathcal{U}n(S(G)^{-1}H_G^*(X)).$$

2.5 *Remark* Corollary 2.4 gives a direct functorial calculation of the cohomology of the fixed-point set $X^G$ in terms of the Borel cohomology $H_G^*(X)$. It was the question of whether such a calculation was possible which led to the present work. Note that by [Lannes 1,2] there is a result parallel to 2.4 which expresses $H^*(X^G)$ in terms of $H_G^*(X)$ and the functor "T" of Lannes. We intend to pursue some further connections between 2.4 and the work of Lannes in a subsequent paper.

# 3 Some algebraic preliminaries

We will assume here and in the next section that p is odd; the arguments need straight forward indexing changes to work in the case p = 2. Let $\sigma$ be the cyclic group of order p and let $R$ denote the graded commutative ring $H^*(B\sigma)$. Recall that $R$ is the tensor product over $F_p$ of an exterior algebra on an element $a \in R^1$ with a polynomial algebra on the Bockstein element $b \in \beta(a)$. The ring $R$ is also an unstable $\mathcal{A}_p-$module in such a way that multiplication in $R$ satisfies the Cartan formula; given $b = \beta(a)$, the action of $\mathcal{A}_p$ is determined by the equation $\mathcal{P}^1(b) = b^p$.

3.1 *Definition* An *unstable* $\mathcal{A}_p - R$ *module* is a non-negatively graded $F_p-$vector space $\mathcal{M}$ which is both a graded $R-$module and a graded $\mathcal{A}_p-$module in such a way that the multiplication map $R \otimes_{F_p} \mathcal{M} \longrightarrow \mathcal{M}$ satisfies the Cartan formula.

3.2 *Example* If $N$ is an unstable $\mathcal{A}_p-$module, then the Cartan formula itself gives the natural structure of an unstable $\mathcal{A}_p - R$ module to $R \otimes_{F_p} N$.

If $\mathcal{M}$ is an unstable $\mathcal{A}_p - R$ module, then for $x \in \mathcal{M}^k$ define $\Phi(x)$ by the formula

$$\Phi(x) = \sum_{i=0}^{[k/2]} (-b)^{i(p-1)} \mathcal{P}^{[k/2]-i}(x)$$

3.3 *Lemma* *The operation* $\Phi$ *has the following properties*

**(i)** $\Phi(x+y) = \Phi(x) + \Phi(y)$

**(ii)** $\Phi(bx) = 0$

**(iii)** $\Phi(ax) = \begin{cases} a\Phi(x) & \mid x \mid \text{ even} \\ b^{p-1}a\Phi(x) & \mid x \mid \text{ odd} \end{cases}$

**(iv)** *if* $x/b \in \mathcal{U}n(b^{-1}\mathcal{M})$, *then* $b^k\Phi(x) = 0$ *for some* $k \geq 0$.

3.4 *Remark* In part (iv) of 3.3, the implicit $\mathcal{A}_p-$module structure on $b^{-1}\mathcal{M}$ is as always the one provided by [Wilkerson].

*Proof of* 3.3 Part(i) is clear, while (ii) and (iii) follow from the Cartan formula. For (iv), observe that if $x \in \mathcal{M}_k$ then the image of $\Phi(x)$ in $b^{-1}\mathcal{M}$ is exactly $b\mathcal{P}^{[k/2]}(x/b)$; this must vanish in $b^{-1}\mathcal{M}$ if $x/b$ is an unstable class.

If $N$ is an $F_p-$vector space and $x$ is a non-zero element of $R \otimes_{F_p} N$, write $x$ as a (finite!) sum $\sum_{i=0}^{\infty} x^i$ with $x^i \in R \otimes_{F_p} N$ and define the leading term of $x$ to be the unique $x^i$ such that $x^j = 0$ for $j > i$.

3.5 *Lemma* *Suppose that $N$ is an $F_p-$vector space and that $x_1, \ldots, x_j$ are non-zero elements of $R \otimes_{F_p} N$ such that the leading term of each $x_i$ can be written as $b^{k_i} \otimes m_i$, where the $k_i's$ are non-negative integers and the $m_i's$ are elements of $N$ which are linearly independent over $F_p$. Then the elements $x_1, \ldots, x_j$ are linearly independent over $R$, i.e., given a collection $r_1, \ldots, r_j$ of elements of $R$, the linear combination $\sum_{i=1}^{j} r_i x_i$ vanishes in $R \otimes_{F_p} N$ iff each coefficient $r_i$ is zero.*

*Proof* It is necessary only to inspect the leading term of $\sum_{i=1}^{j} r_i x_i$. The lemma depends only on the fact that $b$ is not a zero-divisor in $R$.

3.6 *Proposition* *If $N$ is an unstable $\mathcal{A}_p-$module, then the natural map*

$$R \otimes_{F_p} N \longrightarrow \mathcal{U}n(b^{-1}(R \otimes_{F_p} N))$$

*is an isomorphism.*

*Proof* It is clear that $b$ does not annihilate any non-zero element of $R \otimes_{F_p} N$. By 3.3(iv), then, it is sufficient to show that given $x \in R \otimes_{F_p} N$ with $\Phi(x) = 0$ there exists $y \in R \otimes_{F_p} N$ such that $x = by$. Assume $x \neq 0$, and write

$$x = \sum_{i=1}^{j} a^{\varepsilon_i} b^{k_i} \otimes m_i$$

where $\varepsilon \in \{0, 1\},\ k_i \geq 0$ and the elements $m_1, \ldots, m_j$ are linearly independent over $F_p$. By 3.3 (ii)-(iii)

$$\Phi(x) = \sum_{i=1}^{j} r_i \Phi(m_i)$$

where $r_i = 0$ iff $k_i > 0,\ i = 1, \ldots, j$. However, by 3.5 and the definition of $\Phi$ the elements $\Phi(m_1), \ldots, \Phi(m_j)$ are linearly independent over $R$. Consequently, the vanishing of $\Phi(x)$ implies that each $r_i$ is zero which entails that each $k_i$ is greater than zero and thus that $x$ is divisible by $b$ in $R \otimes_{F_p} N$.

# 4 Completion of the Proof

We will continue to use the notation of §3. A two-dimensional element of $S(K)$ will be called a generator; by definition, every element in $S(K)$ is a product of generators.

4.1 *Lemma* *For each generator $y$ of $S(K)$ there exists a retraction $f : G \to \sigma$ and an unstable $\mathcal{A}_p$−module $N$ such that*

**(i)** $f^*(b) = y$ , *and*

**(ii)** $H^*_G(X^K)$ *with the R-action induced by* $f^*$ *is isomorphic as an unstable* $\mathcal{A}_p - R$ *module to* $R \otimes_{F_p} N$.

*Proof* Write $y = \beta(x)$ for $x \in H^1(BG)$ and let $f : G \longrightarrow \sigma$ be the unique surjection such that $f^*(a) = x$. By the definition of $S(K)$ it is possible to find a section $i : \sigma \longrightarrow G$ of $f$ such that $i(\sigma) \subseteq K$. Use $f$ and $i$ to identify $G$ with the product $\sigma \times J$ where $J = ker(f)$; since $\sigma$ acts trivially on $X^K$ via this identification, the Künneth formula produces an isomorphism

$$H^*_G(X^K) \approx H^*_{\sigma\times J}(X^K) \approx H^*(B\sigma) \otimes_{F_p} H^*_J(X^K)$$

The lemma follows with $N = H^*_J(X^K)$.

*Proof of* 2.2 By 4.1 no generator of $S(K)$ annihilates any non-zero element of $H^*_G(X^K)$, this implies that the map

$$H^*_G(X^K) \longrightarrow \mathcal{U}n(S(K)^{-1}H^*_G(X^K))$$

is injective. To prove surjectivity, note that each element of $S(K)^{-1}H^*_G(X^K)$ can be written as $x/s$ with $x \in H^*_G(X^K)$ and $s \in S(K)$. It is clear that if $x/s$ is an unstable class, then so is $t \cdot (x/s)$ for $t \in S(K)$. Therefore, by induction on the length of an expression for $s$ as a product of generators, it is enough to show that if $x$ belongs to $H^*_G(X^K)$ and $y$ is a generator of $S(K)$, then $x/y$ is unstable iff $x/y$ belongs to $H^*_G(X^K)$. This follows immediately from 4.1 and 3.6.

# 5 Involutions on Cohomology Real Projective Spaces

The classification of the fixed point sets for involutions on cohomology real projective spaces provides an interesting illustration of the previous theory. For actual real projective spaces, the cohomological classification was done by [Smith 2], and the generalization to cohomology projective spaces appears in Su, Bredon - see also the books of Bredon [Bredon 1] and W.Y. Hsiang [Hsiang]. There are versions for higher rank elementary p-groups and complex and quarterionic projective spaces, but the classical case should suffice to demonstrate the possibilities of our description.

The guiding principle is that on the cohomological level, the general case mimics the special case of involutions of real projective space $RP^n$ induced by linear involutions on euclidean space $R^{n+1}$. For such an involution, the eigenspaces for { +1, -1 } in $R^{n+1}$ correspond to the two components of the fixed point set of the action on $RP^n$. Each component is a real projective space of dimension less than or equal to $n$. On the cohomology level, the equivariant cohomology $H^*_G(RP^n)$ is the cohomology of the $RP^n$-bundle over $BG$ associated to the vector bundle arising from the representation of $G$ on $R^{n+1}$ . The Leray-Hirsh theorem gives this explicitly as the free $H^*(BG)$-module on $\{ x^i, 0 \leq i \leq n \}$ with the multiplicative relation that $x^i(x+w)^j = 0$, where $w$ is the generator of $H^*(BG)$, $i$ is the dimension of the $\{+1\}$-eigenspace, and $j$ is the dimension of the {-1}-eigenspace.

In Proposition 5.1 below we show that in the general case, the possibilities for the equivariant cohomology rings all coincide with those already present in this special case. The fact that the equivariant cohomology determines completely the cohomology of the fixed point set ( by Corollary 2.4 ), then concludes the proof. That is, in the general case the cohomology of the fixed point is that of a disjoint union of two real projective spaces, or is empty.

However, for the benefit of algebraically inclined readers, we also calculate directly in Proposition 5.2 the cohomology of the fixed point set from the localized equivariant cohomology ring. Analogues of both Propositions 5.1 and 5.2 could be carried out for actions of elementary $2-$groups with no essentially new information. For complex and quaterionic projective cases, more possibilities for the Steenrod algebra action occur, but otherwise, the reasoning applies in a similar fashion.

Proposition 5.1 : If $G = Z/Z2Z$ and $X$ is a finite $G-CW$-complex with $H^*(X) = F_2[x]/x^{n+1}$, then $H^*_G(X) = F_2[w,y]/(f)$ , where $y$ restricts to $x$ and

$f = y^i(y+w)^j$ for $i+j = n+1$.

Proposition 5.2 : If $R = H^*_G(X)$, then in $S^{-1}R$,
1) for any $\epsilon \geq max(i,j)$ the elements $\nu = (x/w)^{2^\epsilon}$ and $\mu = ((x+w)/w)^{2^\epsilon}$ are idempotent and hence unstable.

2) The elements $\{\nu x^s, 0 \leq s < i, \mu(x+w)^t, 0 \leq t < j\ \}$ are a basis for $S^{-1}R$ over $F_2[w, w^{-1}]$, and a basis for $Un(S^{-1}R)$ over $F_2[w]$.

3) The $F_2-$subalgebra generated by $\nu(x)$ is isomorphic to $H^*(RP^{i-1})$ and similarly for that generated by $\mu(x+w)$.

*Proof of Proposition 5.1 :* The Serre spectral sequence for the fibration $X \to X_G \to BG$ collapses if there is a fixed point, and hence $H^*_G(X)$ is a free module over $H^*(BG)$ on generators $\{1, y, \ldots, y^n\ \}$ , with an multiplicative relation $y^{n+1} = g(x,y)$ where $g$ has degree less than $n+1$ in $y$. Since $w$ and $y$ are each in dimension one, the natural surjection from $F_2[w', y']$ onto $R$ is an $\mathcal{A}_2$ map. Hence $g = (y')^{n+1} + f(w', y')$ generates a principal ideal in $F_2[w', y']$ which is closed under the $\mathcal{A}_2-$ action. By Serre [Serre], or Wilkerson [Wilkerson], such a polynomial is a product of linear combinations of $w'$ and $y'$. However, since $R$ is free over $F_2[w]$, the term $w$ can not occur in the product. In this two-dimensional case, the only other possibilites are $y$ and $(y+w)$.

em Proof of Proposition 5.2 : The fact that $\mu$ and $\nu$ are idempotents follows directly from the observations

$$(y/w)^{2^\epsilon} + ((y+w)/w)^{2^\epsilon} = 1$$

and

$$((y/w)(y+w)/w)^{2^\epsilon} = 0$$

because of the choice of large $\epsilon$. Since these elements have grading zero and are idempotent, the Cartan formula demonstrates that the elements are unstable.

Clearly, $\nu x^i = 0$ and $\mu(x+w)^j = 0$, so $\nu x$ and $\mu(x+w)$ do generate algebras isomorphic to those for projective spaces. The union of these two algebras consists of $(n+1)$ linearly independent elements over $F_2[w, w^{-1}]$, so they form a basis for $S^{-1}R$ . If $M$ is this union, then in fact

$$S^{-1}R = M \otimes_{F_2} F_2[w, w^{-1}]$$

as $\mathcal{A}_2$ - $H^*(BG)$- modules, since both $M$ and $F_2[W, W^{-1}]$ are closed under the $\mathcal{A}_2$ action. Hence the unstable elements in $S^{-1}R$ are just $M \otimes_{F_2} F_2[w]$ by Proposition 3.6.

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