EXOTIC COHOMOLOGY FOR $\mathrm{GL}_n(\mathbf{Z}[1/2])$

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§1. Introduction

Let Λ denote the ring $\mathbf{Z}[1/2]$, G_n the group $\mathrm{GL}_n(\Lambda)$ of invertible $n \times n$ matrices over Λ, and D_n the group of diagonal matrices in G_n. The inclusion $D_n \to G_n$ induces a classifying space map $\iota_n : \mathrm{B}D_n \to \mathrm{B}G_n$ and a cohomology homomorphism

$$\iota_n^* : \mathrm{H}^*(\mathrm{B}G_n; \mathbf{F}_2) \to \mathrm{H}^*(\mathrm{B}D_n; \mathbf{F}_2) \ .$$

Say that the mod 2 cohomology of G_n is *detected on diagonal matrices* if ι_n^* is injective. In [15, 14.7] Quillen made a conjecture which specializes in the case of the ring Λ to the following statement (see [9, p. 51]):

1.1 Conjecture. *For any $n \geq 1$ the mod 2 cohomology of G_n is detected on diagonal matrices.*

There is some evidence for this conjecture. Mitchell [14] and Henn [10] have proved it for $n \leq 3$. Voevodsky has announced a proof of the mod 2 Quillen-Lichtenbaum Conjecture for $\mathbf{Z}$, and from [5] and [14] it follows that ι_n^* is injective on the image of $\mathrm{H}^*(\mathrm{BGL}(\Lambda); \mathbf{F}_2) \to \mathrm{H}^*(\mathrm{B}G_n; \mathbf{F}_2)$. In particular, 1.1 is true in the stable range.

The aim of this paper, though, is to give a *disproof* of Conjecture 1.1.

1.2 Theorem. *The mod 2 cohomology of G_{32} is* **not** *detected on diagonal matrices.*

Given the remarks above, it is a consequence of 1.2 that there exists an element in the cohomology of G_{32} which is not in the image of $\mathrm{H}^*(\mathrm{BGL}(\Lambda); \mathbf{F}_2)$. In fact, 1.2 is proved by a technique distantly related to the one which Quillen uses in [15, p. 592] to show that for various other number rings S and primes p the restriction map $\mathrm{H}^*(\mathrm{BGL}(S); \mathbf{F}_p) \to \mathrm{H}^*(\mathrm{BGL}_n(S); \mathbf{F}_p)$ is not surjective.

Further developments. Very recently, Henn and Lannes have improved upon 1.2 by showing that the mod 2 cohomology of G_{14} is not detected on diagonal matrices. Moreover, Henn proves in [11, 0.6] that the Poincaré series $p_n(t)$ of the kernel of ι_n^* has a pole at $t = 1$ of order $n - n_0 + 1$, where n_0 is the smallest natural number such that $\iota_{n_0}^*$ is not injective. Combining these results shows that for *any* $n \geq 14$ the cohomology of G_n is far from being detected on diagonal matrices.

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A generalization. Theorem 1.2 is a consequence of a result which is slightly more general. Recall that if P and G are groups and $\alpha, \beta : P \to G$ are homomorphisms, then α is said to be *conjugate* to β if there is an element $g \in G$ such that $g\alpha g^{-1} = \beta$. Let $\rho_{\mathbf{R}} : G_n \to \mathrm{GL}_n(\mathbf{R})$ and $\rho_{\mathbf{F}_3} : G_n \to \mathrm{GL}_n(\mathbf{F}_3)$ be the obvious homomorphisms. Two homomorphisms $\alpha, \beta : P \to G_n$ are said to *become conjugate over* $\mathbf{R}$ (resp. *become conjugate over* $\mathbf{F}_3$) if $\rho_{\mathbf{R}}\alpha$ and $\rho_{\mathbf{R}}\beta$ (resp. $\rho_{\mathbf{F}_3}\alpha$ and $\rho_{\mathbf{F}_3}\beta$) are conjugate.

1.3 Theorem. *Suppose that the mod 2 cohomology of G_n is detected on diagonal matrices. Let P be a finite 2-group with homomorphisms $\alpha, \beta : P \to G_n$. Then α is conjugate to β if and only if α becomes conjugate to β over $\mathbf{R}$ and over $\mathbf{F}_3$.*

To obtain 1.2 from 1.3, let μ_n denote the group of 2^n-th roots of unity. The smallest n with the property that the ideal class group of $\Lambda(\mu_n)$ is nontrivial is 6, and the degree of $\Lambda(\mu_6)$ over Λ (equivalently, the rank of $\Lambda(\mu_6)$ as a Λ-module) is $\phi(2^6) = 2^5 = 32$. Let $P = \mu_6$, considered as a subgroup of $\Lambda(\mu_6)^\times$, and let $\mathcal{I}$ be a nonprincipal ideal in $\Lambda(\mu_6)$. It is then not hard to use the multiplicative actions of P on $\mathcal{I}$ and on $\Lambda(\mu_n)$ itself to construct two nonconjugate homomorphisms $P \to G_{32}$ which become conjugate over $\mathbf{R}$ and over $\mathbf{F}_3$.

The proof of 1.3 is homotopy theoretic. If X and Y are spaces, let $[X, Y]$ denote the set of (unpointed) homotopy classes of maps $X \to Y$; if P and G are groups, let $\{P, G\}$ denote the set of conjugacy classes of homomorphisms $P \to G$. The classifying space functor gives a bijection $\{P, G\} \cong [\mathrm{B}P, \mathrm{B}G]$. For each $n \geq 1$ we construct a space X_n together with a map

$$\chi_n : \mathrm{B}G_n \to X_n$$

such that the following three statements hold.

1.4 Proposition. *If the mod 2 cohomology of G_n is detected on diagonal matrices, then $\chi_n^* : \mathrm{H}^*(X_n; \mathbf{F}_2) \to \mathrm{H}^*(\mathrm{B}G_n; \mathbf{F}_2)$ is an isomorphism.*

1.5 Proposition. *If χ_n^* is an isomorphism, then for any finite 2-group P the map $\chi_n \cdot (\text{-}) : [\mathrm{B}P, \mathrm{B}G_n] \to [\mathrm{B}P, X_n]$ is a bijection.*

1.6 Proposition. *Let P be a finite 2-group with homomorphisms $\alpha, \beta : P \to G_n$. Then $\chi_n \cdot (\mathrm{B}\alpha)$ is homotopic to $\chi_n \cdot (\mathrm{B}\beta)$ if and only if α becomes conjugate to β over $\mathbf{R}$ and over $\mathbf{F}_3$.*

Together these imply 1.3.

Section 2 contains a description of the machinery which is used to construct the space X_n, §3 has proofs of the above three propositions, and §4 contains the derivation of 1.2 from 1.3. Throughout the paper, unspecified homology and cohomology is to be taken with mod 2 coefficients. The symbol $\hat{\mathrm{B}}G$ denotes the 2-completion [1] of the classifying space of G. The usual topological groups $\mathrm{GL}_n(\mathbf{R})$ and $\mathrm{GL}_n(\mathbf{C})$ are denoted $\mathrm{GL}_n^{\mathrm{top}}(\mathbf{R})$ and $\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C})$.

The idea for this paper originated in conversations with S. Mitchell, and the approach depends heavily on his calculations from [14]. The construction of X_n goes back to work with E. Friedlander [4]. Some of the arguments below can be generalized, but we have decided to concentrate on proving 1.2.

§2. Étale homotopy theory

The space X_n promised in §1 is constructed using étale homotopy theory [8], which gives a covariant mechanism for assigning a space (more accurately a pro-space) $A_{\text{ét}}$ to any reasonable scheme or simplicial scheme A. We build X_n as a certain space of maps between two étale homotopy types (2.6), in imitation of the way in which $\mathrm{GL}_n(\Lambda)$ can be described as a certain set of maps between schemes.

Étale homotopy types. Here are some examples of étale homotopy types. In the examples, the symbol ℓ stands for a prime number.

2.1 Fields and complete local rings. If k is a field, then $\mathrm{Spec}(k)_{\text{ét}}$ is a pro-space of type $K(\pi, 1)$, where π is the Galois group over k of the separable algebraic closure of k. If S is a complete local ring with residue class field k, then the map $\mathrm{Spec}(k)_{\text{ét}} \to \mathrm{Spec}(S)_{\text{ét}}$ induced by $S \to k$ is an equivalence. For instance, $\mathrm{Spec}(\mathbf{C})_{\text{ét}}$ is contractible, $\mathrm{Spec}(\mathbf{R})_{\text{ét}}$ is equivalent to $\mathrm{B}\mathbf{Z}/2$, and both $\mathrm{Spec}(\mathbf{F}_3)_{\text{ét}}$ and $\mathrm{Spec}(\mathbf{Z}_3)_{\text{ét}}$ are equivalent to the profinite completion of a circle.

If S is a commutative ring, let $\mathrm{GL}_{n,S}$ denote the rank n general linear group scheme over S. Applying the usual bar construction to $\mathrm{GL}_{n,S}$ gives a classifying object $\mathrm{BGL}_{n,S}$ [8, 1.2], which is a simplicial scheme. See [8, §8] for the following three examples.

2.2 General linear groups over algebraically closed fields.. If k is an algebraically closed field of characteristic zero, the pro-space $(\mathrm{BGL}_{n,k})_{\text{ét}}$ is equivalent to the profinite completion of the space $\mathrm{BGL}_n^{\text{top}}(\mathbf{C})$. If k is the algebraic closure of a finite field, then the ℓ-completion of $(\mathrm{BGL}_{n,k})_{\text{ét}}$ at any prime ℓ not equal to $\mathrm{char}(k)$ is equivalent to the Bousfield-Kan ℓ-completion tower $\{(\mathbf{Z}/\ell)_s\mathrm{BGL}_n^{\text{top}}(\mathbf{C})\}_s$.

2.3 General linear groups over other fields. If k is a field of characteristic zero with algebraic closure $\bar{k}$, the natural sequence

$$(\mathrm{BGL}_{n,\bar{k}})_{\text{ét}} \to (\mathrm{BGL}_{n,k})_{\text{ét}} \to \mathrm{Spec}(k)_{\text{ét}}$$

is a fibration sequence of pro-spaces. In effect, if π is the Galois group of $\bar{k}$ over k, $(\mathrm{BGL}_{n,k})_{\text{ét}}$ is the Borel construction of the natural action of π on $(\mathrm{BGL}_{n,\bar{k}})_{\text{ét}}$. If k is a finite field with algebraic closure $\bar{k}$ and $\ell \neq \mathrm{char}(k)$, there is a similar fibration sequence

$$\{(\mathbf{Z}/\ell)_s(\mathrm{BGL}_{n,\bar{k}})_{\text{ét}}\}_s \to \{(\mathbf{Z}/\ell)_s^{\bullet}(\mathrm{BGL}_{n,k})_{\text{ét}}\}_s \to \mathrm{Spec}(k)_{\text{ét}}\ ,$$

where $\{(\mathbf{Z}/\ell)_s^{\bullet}(-)\}_s$ denotes fibrewise ℓ-completion over $\mathrm{Spec}(k)_{\text{ét}}$.

2.4 General linear groups over number rings. Suppose that R is a number ring, and that S is the ring $R[1/\ell]$. Let $\bar{k}$ denote either the algebraic closure of the quotient field of S, or the algebraic closure of one of the residue class fields of S (note that none of these residue fields have characteristic ℓ). Then the natural sequence

$$\{(\mathbf{Z}/\ell)_s(\mathrm{BGL}_{n,\bar{k}})_{\text{ét}}\}_s \to \{(\mathbf{Z}/\ell)_s^{\bullet}(\mathrm{BGL}_{n,S})_{\text{ét}}\}_s \to \mathrm{Spec}(S)_{\text{ét}}\ .$$

is a fibration sequence. There are identical fibration sequences if S is replaced by a complete local ring of residue characteristic different from ℓ.

2.5 Number rings. The étale homotopy type of a number ring is not easy to pin down; see [5, 2.1] for a description of its untwisted "integral" homology. We will settle for a partial description of $\mathrm{Spec}(\Lambda)_{\text{ét}}$, where, as usual, $\Lambda = \mathbf{Z}[1/2]$. Choose an embedding $\mathbf{Z}_3 \to \mathbf{C}$. There are induced commutative diagrams of rings and of pro-spaces

$$\begin{array}{ccc} \mathbf{C} & \longleftarrow & \mathbf{R} \\ \uparrow & & \uparrow \\ \mathbf{Z}_3 & \longleftarrow & \Lambda \end{array} \qquad \begin{array}{ccc} \mathrm{Spec}(\mathbf{C})_{\text{ét}} & \longrightarrow & \mathrm{Spec}(\mathbf{R})_{\text{ét}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{Z}_3)_{\text{ét}} & \longrightarrow & \mathrm{Spec}(\Lambda)_{\text{ét}} \end{array} .$$

Since $\mathrm{Spec}(\mathbf{C})_{\text{ét}}$ is contractible, the diagram induces a map

$$\mathrm{Spec}(\mathbf{Z}_3)_{\text{ét}} \vee \mathrm{Spec}(\mathbf{R})_{\text{ét}} \to \mathrm{Spec}(\Lambda)_{\text{ét}} .$$

Pick an equivalence $\mathrm{B}\mathbf{Z}/2 \to \mathrm{Spec}(\mathbf{R})_{\text{ét}}$, and a map $S^1 \to \mathrm{Spec}(\mathbf{Z}_3)$ which sends the generator of $\pi_1 S^1$ to the Frobenius automorphism of $\bar{\mathbf{F}}_3$ over $\mathbf{F}_3$. What results is a map

$$S^1 \vee \mathrm{B}\mathbf{Z}/2 \to \mathrm{Spec}(\Lambda)_{\text{ét}} .$$

By [5], this map induces an isomorphism on mod 2 cohomology (cf. [6, §2]).

2.6 Étale approximations to $\mathrm{BGL}_n(S)$. Let S be an algebra over Λ. The homomorphism $\Lambda \to S$ induces a map $\mathrm{Spec}(S)_{\text{ét}} \to \mathrm{Spec}(\Lambda)_{\text{ét}}$. We set $\ell = 2$ and let $\mathrm{BGL}_n^{\text{ét}}(S)$ denote the basepoint component of the space which in [4, 2.3] is called $\mathrm{Hom}_\ell(\mathrm{Spec}(S)_{\text{ét}}, (\mathrm{BGL}_{n,\Lambda})_{\text{ét}})_\Lambda$; in other words, $\mathrm{BGL}_n^{\text{ét}}(S)$ is the basepoint component of the function space of maps over $\mathrm{Spec}(\Lambda)_{\text{ét}}$ from $\mathrm{Spec}(S)_{\text{ét}}$ to the fibrewise 2-completion of $(\mathrm{BGL}_{n,\Lambda})_{\text{ét}}$. (The function space is pointed because the map $\mathrm{BGL}_{n,\Lambda} \to \mathrm{Spec}(\Lambda)$ has a natural section provided by the basepoint of the fibrewise bar construction.) Since $\mathrm{BGL}_n(S)$ can be identified as the basepoint component of the space of maps $\mathrm{Spec}(S) \to \mathrm{BGL}_{n,\Lambda}$ over $\mathrm{Spec}(\Lambda)$ [4, 4.2], functoriality gives a map

$$\chi_{n,S} : \mathrm{BGL}_n(S) \to \mathrm{BGL}_n^{\text{ét}}(S)$$

(see [4, 2.5, pf. of 4.2]). We will describe this map below in some particular cases.

2.7 Remark. Suppose that B is a space (i.e. a trivial pro-space) together with a map $B \to \mathrm{Spec}(\Lambda)_{\text{ét}}$. Let "holim" denote the homotopy inverse limit functor from pro-spaces to spaces [8, §6], and E be the fibration over B obtained by the following homotopy pullback diagram

$$\begin{array}{ccc} E & \longrightarrow & \mathrm{holim}\{(\mathbf{Z}/2)^\bullet_s(\mathrm{BGL}_{n,\Lambda})_{\text{ét}}\}_s \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathrm{holim}\,\mathrm{Spec}(\Lambda)_{\text{ét}} \end{array} .$$

The homotopy fibre of $E \to B$ is $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{C})$. Essentially by definition, the space $\mathrm{Hom}_\ell(B, (\mathrm{BGL}_{n,\Lambda})_{\text{ét}})_\Lambda$ is equivalent to the space of sections of the map $E \to B$.

It follows from a spectral sequence argument [4, 2.11] that if S is a Λ-algebra and $f : B \to \mathrm{Spec}(S)_{\text{ét}}$ is a map of pro-spaces which induces an isomorphism on mod 2 cohomology, then f induces an equivalence

$$\mathrm{BGL}_n^{\text{ét}}(S) \xrightarrow{\simeq} \mathrm{Hom}_\ell(B, (\mathrm{BGL}_{n,\Lambda})_{\text{ét}})_\Lambda \ .$$

2.8 The complex numbers. Since $\mathrm{Spec}(\mathbf{C})_{\text{ét}}$ is contractible, it follows from 2.7 that $\mathrm{BGL}_n^{\text{ét}}(\mathbf{C})$ is equivalent to $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{C})$. The same calculation works for any separably closed field of characteristic not 2. The map $\chi_{n,\mathbf{C}}$ is essentially the composite of the usual map $\mathrm{BGL}_n(\mathbf{C}) \to \mathrm{BGL}_n^{\text{top}}(\mathbf{C})$ with the 2-completion map.

2.9 The real numbers. Since $\mathrm{Spec}(\mathbf{R})_{\text{ét}}$ is equivalent to $\mathrm{B}\mathbf{Z}/2$, it follows from 2.7 that $\mathrm{BGL}_n^{\text{ét}}(\mathbf{R})$ is equivalent to the basepoint component of the space of sections of a fibration over $\mathrm{B}\mathbf{Z}/2$ with $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{C})$ as the fibre. By naturality, this fibration is the one associated to the action of $\mathbf{Z}/2$ on $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{C})$ by complex conjugation. It follows from [7] that the natural map from $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{R})$ to this space of sections is an equivalence. The map $\chi_{n,\mathbf{R}}$ is the essentially the composite of the usual map $\mathrm{BGL}_n(\mathbf{R}) \to \mathrm{BGL}_n^{\text{top}}(\mathbf{R})$ with the 2-completion map.

2.10 The field $\bar{\mathbf{F}}_3$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{BGL}_n(\bar{\mathbf{F}}_3) & \longrightarrow & \mathrm{colim}_n\, \mathrm{BGL}_n(\bar{\mathbf{F}}_3) = \mathrm{BGL}(\bar{\mathbf{F}}_3) \\
{\scriptstyle \chi_{n,\bar{\mathbf{F}}_3}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{colim}_n(\chi_{n,\bar{\mathbf{F}}_3})} \\
\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3) & \longrightarrow & \mathrm{colim}_n\, \mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)
\end{array}$$

in which the colimit maps are induced by the usual matrix block inclusions. The right hand vertical arrow induces an isomorphism on mod 2 cohomology (see [4, 4.5, 8.6]). The horizontal maps induce surjections on mod 2 cohomology; for the upper one see [16] and for the lower one 2.7. By diagram chasing, $\chi_{n,\bar{\mathbf{F}}_3}$ induces a surjection on mod 2 cohomology. Since $\mathrm{BGL}_n(\bar{\mathbf{F}}_3)$ and $\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ have mod 2 cohomology rings which are abstractly isomorphic ([16], 2.7) and finite in each dimension, $\chi_{n,\bar{\mathbf{F}}_3}$ must be an isomorphism on mod 2 cohomology. It follows that $\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ is equivalent to $\hat{\mathrm{B}}\mathrm{GL}_n(\mathbf{F}_3)$.

2.11 The field $\mathbf{F}_3$. There is a map $S^1 \to \mathrm{Spec}(\mathbf{F}_3)_{\text{ét}}$ which sends a generator of $\pi_1(S^1)$ to Frobenius automorphism ψ of $\bar{\mathbf{F}}_3$ over $\mathbf{F}_3$. This map is an isomorphism on cohomology with finite coefficients, in particular, on mod 2 cohomology. It follows from 2.7 that $\mathrm{BGL}_n^{\text{ét}}(\mathbf{F}_3)$ is equivalent to the space of sections of a fibration over S^1 with fibre $\hat{\mathrm{B}}\mathrm{GL}_n^{\text{top}}(\mathbf{C}) \simeq \mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ (2.8). By naturality, this is the fibration associated to the action of ψ on $\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ and so its space of sections is (by definition) the homotopy fixed point set $(\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$. Consider the commutative diagram of spaces with an action of ψ (the action in the left hand column is trivial):

$$\begin{array}{ccc}
\mathrm{BGL}_n(\mathbf{F}_3) & \xrightarrow{u} & \mathrm{BGL}_n(\bar{\mathbf{F}}_3) \\
{\scriptstyle \chi_{n,\mathbf{F}_3}}\big\downarrow & & \big\downarrow{\scriptstyle \chi_{n,\bar{\mathbf{F}}_3}} \\
\mathrm{BGL}_n^{\text{ét}}(\mathbf{F}_3) & \xrightarrow{v} & \mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)
\end{array} \ .$$

The space $\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ is 2-complete (2.7). The map $\chi_{n,\bar{\mathbf{F}}_3}$ induces an equivalence $\hat{\mathrm{B}}\mathrm{GL}_n(\bar{\mathbf{F}}_3) \to \mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3)$ (2.10). Quillen [16] shows that the map $\mathrm{BGL}_n(\mathbf{F}_3) \to (\hat{\mathrm{B}}\mathrm{GL}_n(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$ induced by the map u gives an isomorphism on mod 2 cohomology. It follows that the map

$$\mathrm{BGL}_n(\mathbf{F}_3) \xrightarrow{\chi_{n,\mathbf{F}_3}} \mathrm{BGL}_n^{\text{ét}}(\mathbf{F}_3) \xrightarrow{\simeq} (\mathrm{BGL}_n^{\text{ét}}(\bar{\mathbf{F}}_3))^{\mathrm{h}\psi}$$

also induces an isomorphism on mod 2 cohomology, and that $\mathrm{BGL}_n^{\text{ét}}(\mathbf{F}_3)$ is equivalent to $\hat{\mathrm{B}}\mathrm{GL}_n(\mathbf{F}_3)$.

Remark. The conjecture that $\chi_{n,S}$ induces an isomorphism on mod 2 cohomology is a very strong unstable analogue of the mod 2 Quillen-Lichtenbaum Conjecture for the ring S. This conjecture is true for a finite field (cf. 2.11) or the algebraic closure of a finite field (cf. 2.10). It is unknown whether or not it is true for the fields $\mathbf{R}$ and $\mathbf{C}$. The results in this paper show that it is *false* for the ring Λ.

§3. The space X_n and its properties

We define X_n to be the space $\mathrm{BGL}_n^{\text{ét}}(\Lambda)$ from 2.6, and $\chi_n : \mathrm{B}G_n \to X_n$ the map $\chi_{n,\Lambda} : \mathrm{BGL}_n(\Lambda) \to \mathrm{BGL}_n^{\text{ét}}(\Lambda)$. In this section we prove 1.4, 1.5 and 1.6.

(Co)homology of X_n. Let $\mathrm{B}D_\bullet$, $\mathrm{B}G_\bullet$ and $X_\bullet$ denote respectively the spaces $\coprod \mathrm{B}D_n$, $\coprod \mathrm{B}G_n$ and $\coprod X_n$. The index n in these coproducts runs over all non-negative integers, where for $n = 0$ the spaces involved are contractible. There are maps

$$\mathrm{B}D_\bullet \xrightarrow{\iota} \mathrm{B}G_\bullet \xrightarrow{\chi} X_\bullet \ .$$

Under matrix block sum all three of these spaces are homotopy associative H-spaces; $\mathrm{B}G_\bullet$ and $X_\bullet$ are homotopy commutative. The maps ι and χ respect the multiplications.

There is a natural identification

$$\mathrm{B}D_1 \simeq \mathrm{B}(\mathbf{Z} \times \mathbf{Z}/2) \simeq S^1 \times \mathrm{B}\mathbf{Z}/2 \ .$$

Let $e \in \mathrm{H}_1\, S^1$ and $\beta_k \in \mathrm{H}_k\, \mathrm{B}\mathbf{Z}/2$ be generators. We denote the classes $e \otimes \beta_{k-1}$ and $1 \otimes \beta_k$ in $\mathrm{H}_k\, \mathrm{B}D_1$ by a_k and b_k respectively. Let $a_k^G = \iota_*(a_k)$, $b_k^G = \iota_*(b_k)$, $a_k^X = (\chi\iota)_*(a_k)$, $b_k^X = (\chi\iota)_*(b_k)$. Since $\mathrm{H}_*\, \mathrm{B}D_\bullet$ is the free $\mathbf{F}_2$-algebra on the elements a_k ($k \geq 1$) and b_k ($k \geq 0$), the image of ι_* is the subalgebra of $\mathrm{H}_*\, \mathrm{B}G_\bullet$ generated by the classes a_k^G and b_k^G, while the image of $(\chi\iota)_*$ is generated by the classes a_k^X and b_k^X.

3.1 Proposition. *The algebra $\mathrm{H}_*\, X_\bullet$ is the free commutative $\mathbf{F}_2$-algebra on the classes a_k^X ($k \geq 1$) and b_k^X ($k \geq 0$) subject to the following relations:*

1. $(a_k^X)^2 = 0$ *for k odd, and*
2. $a_k^X b_0^X + a_{k-1}^X b_1^X + \cdots + a_1^X b_{k-1}^X = 0$ *for k even.*

3.2 Lemma. *There is a homotopy fibre square*

$$\begin{array}{ccc} X_n & \longrightarrow & \mathrm{BGL}_n^{\text{ét}}(\mathbf{R}) \\ \downarrow & & \downarrow \\ \mathrm{BGL}_n^{\text{ét}}(\mathbf{Z}_3) & \longrightarrow & \mathrm{BGL}_n^{\text{ét}}(\mathbf{C}) \end{array} .$$

Proof. This is a consequence of 2.5 and 2.7. $\square$

3.3 Remark. The square from 3.2 can up to homotopy be rewritten in the following way:

$$\begin{array}{ccc} X_n & \longrightarrow & \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{R}) \\ \downarrow & & {\scriptstyle u}\downarrow \\ \hat{\mathrm{B}}\mathrm{GL}_n(\mathbf{F}_3) & \xrightarrow{v} & \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C}) \end{array} .$$

The rewriting is justified by 2.9, 2.8, 2.1 and 2.7. By naturality the map u is the usual one induced by the map $\mathbf{R} \to \mathbf{C}$. The map v is a little more problematical, but it is clear from 2.5 that the restriction of v to the diagonal matrices in $\mathrm{GL}_n(\mathbf{F}_3)$ is induced by the ordinary diagonal inclusion $\{\pm 1\}^n \to \mathrm{BGL}_n(\mathbf{C})$. The fact that the mod 2 cohomology of $\mathrm{BGL}_n(\mathbf{F}_3)$ is detected on diagonal matrices [14, §3] [16] means that it is easy to compute the map on mod 2 homology or cohomology induced by v.

Proof of 3.1. This is essentially given by Mitchell in [14, 4.6]; his space $JK(\mathbf{Z})$ can be identified with $\mathrm{colim}_n\, X_n$, where the maps in the colimit are the block inclusions given by product with a point in X_1. The role of diagram [14, 4.1] is played by the homotopy fibre square from 3.3 above. $\square$

3.4 Proposition. *The classes a_k^G, $k \geq 1$ and b_k^G, $k \geq 0$ in $\mathrm{H}_*\,\mathrm{B}G_\bullet$ satisfy the analogs of relations* (1) *and* (2) *from 3.1.*

Proof. See [14, pf. of 7.1]. The place to look for these relations is in $\mathrm{H}_*\,\mathrm{BGL}_2(\Lambda)$, and Mitchell computes this homology explicitly [14, §6]. $\square$

Proof of 1.4. By 3.1 and 3.4, the homology map $(\chi_n\iota_n)_*$ is surjective and its kernel is equal to the kernel of $(\iota_n)_*$. By duality, the cohomology map $(\chi_n\iota_n)^*$ is injective and its image is equal to the image of ι_n^*. This implies that ι_n^* is injective if and only if χ_n^* is an isomorphism. $\square$

Maps into X_n. Let P be a finite 2-group. We are interested in studying the set $[\mathrm{B}P, X_n]$ of unbased homotopy classes of maps from $\mathrm{B}P$ to X_n. The homotopy fibre square from 3.3 gives a map

$$[\mathrm{B}P, X_n] \to [\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n(\mathbf{F}_3)] \times [\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{R})] \; . \tag{3.5}$$

3.6 Lemma. *The map 3.5 is injective.*

Proof. It follows from 3.3 that there is a homotopy fibre square of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}(\mathrm{B}P, X_n) & \longrightarrow & \mathrm{Map}(\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{R})) \\ \downarrow & & \downarrow \\ \mathrm{Map}(\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n(\mathbf{F}_3)) & \longrightarrow & \mathrm{Map}(\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C})) \end{array} .$$

By an elementary argument, it is enough to show that each component of the space $\mathrm{Map}(\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C}))$ is 1-connected. By [7] the space $\mathrm{Map}(\mathrm{B}P, \hat{\mathrm{B}}\mathrm{GL}_n^{\mathrm{top}}(\mathbf{C}))$ is equivalent to the 2-completion of the disjoint union $\coprod_\rho \mathrm{B}C(\rho(P))$, where ρ runs through a set of representatives for the conjugacy classes of homomorphisms $P \to \mathrm{GL}_n(\mathbf{C})$, and $C(\rho(P))$ is the centralizer of $\rho(P)$. By elementary representation theory each one of these centralizers is isomorphic to a product of complex general linear groups, and so is connected. □

The following theorem is derived in [13, §1] from Carlsson's work in [2] and [3]. We state it only for the prime 2, although it holds for any prime.

3.7 Theorem. *Let Γ be a group of virtually finite cohomological dimension, and let P be a finite 2-group. Then the natural map*

$$\{P, \Gamma\} \cong [\mathrm{B}P, \mathrm{B}\Gamma] \to [\mathrm{B}P, \hat{\mathrm{B}}\Gamma]$$

is a bijection.

Proof of 1.5. The space X_n is 2-complete (3.3), so the fact that χ_n^* is an isomorphism implies that X_n is equivalent to $\hat{\mathrm{B}}G_n$. Since G_n is a group of virtually finite cohomological dimension [17, p. 124] the result follows from 3.7. □

Proof of 1.6. Consider the commutative diagram

$$\begin{array}{ccccc} [\mathrm{B}P, \mathrm{BGL}_n(\mathbf{R})] & \xleftarrow{\mathrm{B}\rho_{\mathbf{R}}} & [\mathrm{B}P, \mathrm{B}G_n] & \xrightarrow{\mathrm{B}\rho_{\mathbf{F}_3}} & [\mathrm{B}P, \mathrm{BGL}_n(\mathbf{F}_3)] \\ \downarrow & & \downarrow & & \downarrow \\ [\mathrm{B}P, \mathrm{BGL}_n^{\text{ét}}(\mathbf{R})] & \longleftarrow & [\mathrm{B}P, X_n] & \longrightarrow & [\mathrm{B}P, \mathrm{BGL}_n^{\text{ét}}(\mathbf{F}_3)] \end{array} .$$

By 2.9, 2.11, 3.7 and [7], the right and left vertical arrows are bijections. The result follows from 3.6. □

§4. Exploiting the class group

In this section we derive 1.1 from 1.3 by showing how class groups of cyclotomic extensions of Λ can be used to construct the necessary homomorphisms from finite 2-groups into $\mathrm{GL}_{32}(\Lambda)$.

Recall that μ_n denotes the multiplicative group of 2^n-th roots of unity. We make a distinction between $\Lambda[\mu_n]$, which is the group ring of μ_n over Λ, and $\Lambda(\mu_n)$, which is the integral closure of Λ in the field obtained from $\mathbf{Q}$ by adjoining μ_n. There is a surjection $\Lambda[\mu_n] \to \Lambda(\mu_n)$ (cf. [18, 2.6]) which is not an isomorphism unless

$n = 0$. Because of this surjection, though, two modules over $\Lambda(\mu_n)$ are isomorphic as $\Lambda(\mu_n)$-modules if and only if they are isomorphic as $\Lambda[\mu_n]$-modules.

4.1 Remark. If M is a Λ-module and $\alpha : \mu_n \to \mathrm{Aut}(M)$ is a homomorphism, let M_α denote the $\Lambda[\mu_n]$-module obtained by letting μ_n act on M via α. Given two homomorphisms $\alpha, \beta : \mu_n \to \mathrm{Aut}(M)$, it is clear that α is conjugate to β if and only if the $\Lambda[\mu_n]$-modules M_α and M_β are isomorphic.

4.2 Lemma. *Suppose that $\mathcal{I}$ is a nonzero ideal in $\Lambda(\mu_n)$. Then there are isomorphisms of $\Lambda(\mu_n)$-modules*

$$\begin{aligned} \mathbf{R} \otimes_\Lambda \mathcal{I} &\cong \mathbf{R} \otimes_\Lambda \Lambda(\mu_n) \\ \mathbf{F}_3 \otimes_\Lambda \mathcal{I} &\cong \mathbf{F}_3 \otimes_\Lambda \Lambda(\mu_n) \end{aligned} .$$

Proof. Since the quotient $\Lambda(\mu_n)/\mathcal{I}$ is a torsion group, the first isomorphism results from taking the inclusion $\mathcal{I} \to \Lambda(\mu_n)$ and tensoring with $\mathbf{R}$. For the second, note that by the Chebotarev density theorem there are an infinite number of prime ideals in $\Lambda(\mu_n)$ isomorphic (as modules) to $\mathcal{I}$. Up to isomorphism, then, $\mathcal{I}$ can be taken to be a prime ideal of residue characteristic different from 3. Tensoring the inclusion $\mathcal{I} \to \Lambda(\mu_n)$ with $\mathbf{F}_3$ again gives the required isomorphism. □

Recall that $\Lambda(\mu_n)$ is free as a module over Λ of rank $\phi(2^n) = 2^{n-1}$ [18, 2.5]. The same is true of any nonzero ideal in $\Lambda(\mu_n)$.

4.3 Lemma. *If $\Lambda(\mu_n)$ has a nonzero ideal which is not principal, then there exist two nonconjugate homomorphisms $\alpha, \beta : \mu_n \to G_{\phi(2^n)}$ which become conjugate both over $\mathbf{R}$ and over $\mathbf{F}_3$.*

Proof. Let $\mathcal{I}$ be such a nonzero ideal. Choosing bases for $\mathcal{I}$ and for $\Lambda(\mu_n)$ allows both to be identified (as Λ-modules) with $\Lambda^{\phi(2^n)}$. The multiplicative actions of μ_n on $\mathcal{I}$ and on $\Lambda(\mu_n)$ thus give two homomorphisms $\alpha, \beta : \mu_n \to \mathrm{GL}_{\phi(2^n)}(\Lambda)$. These homomorphisms are not conjugate, because $\mathcal{I}$ and $\Lambda(\mu_n)$ are not isomorphic as $\Lambda[\mu_n]$-modules (equivalently, as $\Lambda(\mu_n)$-modules). The homomorphisms are conjugate over $\mathbf{R}$ or over $\mathbf{F}_3$, because (4.2) the two modules become isomorphic when tensored with $\mathbf{R}$ or with $\mathbf{F}_3$. □

Proof of 1.2. According to 4.3, it is enough to show that $\Lambda(\mu_6)$ is not a principal ideal domain. By [18, p. 353] the ideal class group of $\mathbf{Z}(\mu_6)$ is cyclic of order 17. Since the ideal class group of $\Lambda(\mu_6)$ is the quotient of the ideal class group of $\mathbf{Z}(\mu_6)$ by the subgroup generated by the prime ideals of $\mathbf{Z}(\mu_6)$ which lie above 2, it is enough to show that there is only one prime $\mathcal{P}$ in $\mathbf{Z}(\mu_6)$ above 2, and that $\mathcal{P}$ is principal. This is a standard calculation; see [18, 1.4] or [12, p. 73]. □

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