HOMOLOGY APPROXIMATIONS FOR CLASSIFYING SPACES OF FINITE GROUPS

W. G. Dwyer

University of Notre Dame

§1. Introduction

Suppose that p is a prime number and that G is a finite group, a compact Lie group, or even a p-compact group [12]. Recently there has been a lot of interest in finding ways to construct the classifying space $\mathrm{B}G$, at least up to mod p homology, by gluing together classifying spaces of subgroups of G. In practice this means finding a mod p homology isomorphism

$$\operatorname{hocolim} F \xrightarrow[p]{\sim} \mathrm{B}G \tag{1.1}$$

where $\mathbf{D}$ is some small category, F is a functor from $\mathbf{D}$ to the category of spaces, and, for each object d of $\mathbf{D}$, $F(d)$ has the homotopy type of $\mathrm{B}H$ for some subgroup H of G. An expression like 1.1 is sometimes called a *homology approximation to* $\mathrm{B}G$ or a *homology decomposition of* $\mathrm{B}G$, and can be used either to make calculations with $\mathrm{B}G$ or to prove general theorems about $\mathrm{B}G$ by induction. (Of course an induction is likely to work only if the values of F are of the form $\mathrm{B}H$ for H a *proper* subgroup of G!) For example, Jackowski and McClure [15] approximate $\mathrm{B}G$ by classifying spaces of centralizers of non-trivial elementary abelian p-subgroups of G. Their result had been anticipated for SU(2) (see [10]) and used to prove a homotopy uniqueness theorem. The p-compact group version of their result [11] is exploited in [7]. Jackowski, McClure and Oliver [16] approximate $\mathrm{B}G$ (G compact Lie) by classifying spaces of p-stubborn subgroups of G, and then use the approximation to make beautiful calculations about the space of self-maps of $\mathrm{B}G$. Benson-Wilkerson [4] and Benson [3] use homology approximations to $\mathrm{B}G$, where G is respectively the Mathieu group M_{12} or Conway's group CO_3, to obtain maps from $\mathrm{B}G$ to classifying spaces of 2-compact groups.

One goal of this paper is to describe many different homology decomposition formulas (including the ones mentioned above) in terms of a single invariant: an associated poset of subgroups of G. Although we expect to extend the results in a future paper to compact Lie groups and p-compact groups, we concentrate here on finite groups because there are fewer technicalities to get in the way of the basic ideas. We also obtain what seems to be a new homology decomposition for finite groups; this decomposition generalizes a classical theorem of Swan (see 1.21).

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Many of the results in this paper were previously known, but the author's intention is to gather them into a systematic account of how the usual homology decompositions arise.

1.2 Homology decompositions. Suppose that G is a finite group. We will call a set $\mathcal{C}$ of subgroups of G *a collection* if it is closed under the process of taking conjugates in G. Let $\mathcal{C}$ be such a collection, $\mathbf{S}_{\mathcal{C}} = (\mathcal{C}, \subseteq)$ the poset given by elements of $\mathcal{C}$ with the inclusion relation, and $K_{\mathcal{C}}$ the associated simplicial complex [2, 6.2]. The n-simplices of $K_{\mathcal{C}}$, for $n \geq 0$, are the subsets $\{H_i\} \subset \mathcal{C}$ of cardinality $(n+1)$ which are totally ordered by inclusion. The group G acts on $\mathcal{C}$ by conjugation, and since this action preserves inclusion relationships it passes to an action of G on $K_{\mathcal{C}}$.

Let $\mathrm{E}G$ be the universal cover of $\mathrm{B}G$; if X is a G-space, the *Borel construction* or *homotopy orbit space* $X_{\mathrm{h}G}$ is defined to be the quotient $(\mathrm{E}G \times X)/G$. Let $*$ denote the one-point space with trivial G action.

1.3 Definition. The collection $\mathcal{C}$ is said to be *ample* if the map

$$q^K_{\mathcal{C}} : (K_{\mathcal{C}})_{\mathrm{h}G} \to (*)_{\mathrm{h}G} = \mathrm{B}G \tag{1.4}$$

given by $K_{\mathcal{C}} \to *$ induces an isomorphism on mod p homology.

One of the main conclusions of this paper is that giving a homology decomposition of $\mathrm{B}G$ amounts in practice to describing an ample collection of subgroups of G. In fact, there is a many-to-one correspondence: every ample collection provides at least three homology decompositions. We will give a brief description of each of these decompositions, with details to follow later on. Assume as usual that $\mathcal{C}$ is a collection of subgroups of G.

1.5 The centralizer decomposition. The *$\mathcal{C}$-conjugacy category* $\mathbf{A}_{\mathcal{C}}$ is the category in which the objects are pairs (H, Σ), where H is a group and Σ is a conjugacy class of monomorphisms $i : H \to G$ with $i(H) \in \mathcal{C}$. A morphism $(H, \Sigma) \to (H', \Sigma')$ is a group homomorphism $j : H \to H'$ which under composition carries Σ' into Σ. One should probably restrict H in some way, for instance by requiring H to be a subgroup of G, in order to force $\mathbf{A}_{\mathcal{C}}$ to be small; in any case, $\mathbf{A}_{\mathcal{C}}$ as defined is equivalent to a small category. If $H \subset G$ is a subgroup, let $C_G(H)$ denote the centralizer of H in G. There is a natural functor

$$\alpha_{\mathcal{C}} : (\mathbf{A}_{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Spaces}$$

which assigns to each object (H, Σ) a space which has the homotopy type of $\mathrm{B}C_G(i(H))$ for any $i \in \Sigma$ (see 3.1). There is also a natural map

$$a_{\mathcal{C}} : \operatorname{hocolim} \alpha_{\mathcal{C}} \to \mathrm{B}G \ .$$

1.6 Theorem. *The map $a_{\mathcal{C}}$ induces an isomorphism on mod p homology (that is, $a_{\mathcal{C}}$ gives a homology decomposition of $\mathrm{B}G$) if and only if $\mathcal{C}$ is an ample collection of subgroups of G.*

1.7 The subgroup decomposition. The *$\mathcal{C}$-orbit category* $\mathbf{O}_{\mathcal{C}}$ is the category whose objects are the G-sets G/H, $H \in \mathcal{C}$, and whose morphisms are G-maps. There is an inclusion functor $\mathcal{J}$ from $\mathbf{O}_\mathcal{C}$ to the category of G-spaces. Composing $\mathcal{J}$ with the Borel construction $(-)_{\mathrm{h}G}$ gives a functor

$$\beta_\mathcal{C} : \mathbf{O}_\mathcal{C} \to \mathbf{Spaces}$$

whose value $(G/H)_{\mathrm{h}G}$ at an object G/H has the homotopy type of $\mathrm{B}H$. The natural maps $\beta_\mathcal{C}(G/H) \to \mathrm{B}G$ are compatible as G/H varies and induce a map

$$b_\mathcal{C} : \operatorname{hocolim} \beta_\mathcal{C} \to \mathrm{B}G\ .$$

1.8 Theorem. *The map $b_\mathcal{C}$ induces an isomorphism on mod p homology (that is, $b_\mathcal{C}$ gives a homology decomposition of* $\mathrm{B}G$*) if and only if $\mathcal{C}$ is an ample collection of subgroups of G.*

1.9 The normalizer decomposition. Let $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ be the category of "orbit simplices" for the action of G on $K_\mathcal{C}$. The objects of $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ are the orbits $\bar\sigma$ of the action of G on the simplices of $K_\mathcal{C}$, and there is one morphism $\bar\sigma \to \bar\sigma'$ if for some simplices $\sigma \in \bar\sigma$ and $\sigma' \in \bar\sigma'$, σ' is a face of σ. If H is a subgroup of G, let $N_G(H)$ denote the normalizer of H in G. There is a natural functor

$$\delta_\mathcal{C} : \bar{\mathrm{s}}\mathbf{S}_\mathcal{C} \to \mathbf{Spaces}$$

which assigns to the orbit of a simplex $\sigma = \{H_i\}$ a space which has the homotopy type of $\mathrm{B}(\cap_i N_G(H_i))$ (see 3.3). There is also a map $d_\mathcal{C} : \operatorname{hocolim} \delta_\mathcal{C} \to \mathrm{B}G$.

1.10 Theorem. *The map $d_\mathcal{C}$ induces an isomorphism on mod p homology (that is, $d_\mathcal{C}$ gives a homology decomposition of* $\mathrm{B}G$*) if and only if $\mathcal{C}$ is an ample collection of subgroups of G.*

1.11 Examples of ample collections. Let G as above be a finite group. There are quite a few ample collections of subgroups of G.

1.12 Trivial examples. If $\mathcal{C}$ is any collection of subgroups of G which contains the trivial subgroup $\{e\}$, then $\mathcal{C}$ is ample. This follows from the fact that the poset $\mathbf{S}_\mathcal{C}$ has the trivial subgroup as a minimal element, so that $K_\mathcal{C}$ is contractible (2.5, 2.6) and $q^K_\mathcal{C}$ (1.4) is a weak equivalence. Similar remarks apply if $\mathcal{C}$ contains G itself. The decomposition formulas associated to these collections are usually not interesting, since in one way or another the formulas are circular, i.e., $\mathrm{B}G$ itself is hidden in the homotopy colimit on the left hand side.

1.13 Nontrivial p-subgroups. Let $\mathcal{C} = \mathcal{P}(G)$ be the collection of all nontrivial p-subgroups of G. If p divides the order of G then $\mathcal{C}$ is ample. This can be proved in several ways (see 6.4). We give our own proof of a sharper result (6.3) due originally to Jackowski-McClure-Oliver.

1.14 Nontrivial elementary abelian p-subgroups. Recall that an abelian group is said to be an *elementary abelian p-group* if it is a module over $\mathbf{F}_p$. Let $\mathcal{C}$ be the collection of all nontrivial elementary abelian p-subgroups of G. It is a theorem of Quillen that the inclusion map $K_\mathcal{C} \to K_{\mathcal{P}(G)}$ is a homotopy equivalence [2, 6.6.1]. This implies that $\mathcal{C}$ has the same ampleness properties as $\mathcal{P}(G)$ (1.13). The centralizer decomposition associated to $\mathcal{C}$ is the Jackowski-McClure decomposition [15].

1.15 Distinguished elementary abelian p-subgroups. Call a nontrivial elementary abelian p-subgroup V of G *distinguished* if V is equal to the group of elements of exponent p in the center of $C_G(V)$. Let $\mathcal{C}$ be the collection of distinguished elementary abelian p subgroups of G. The inclusion $K_{\mathcal{C}} \to K_{\mathcal{P}(G)}$ is a homotopy equivalence [2, p. 231, exercise], so this collection also has the same ampleness properties as $\mathcal{P}(G)$. The associated centralizer decomposition is a more economical form of the Jackowski-McClure decomposition.

1.16 The Benson collection. Let P be a Sylow p-subgroup of G, and $\mathcal{E}$ the smallest subset of G which contains the elements of exponent p in the center of P, is closed under conjugation in G, and is closed under the process of taking products of commuting elements. Let $\mathcal{C}$ be the collection of all nontrivial elementary abelian p-subgroups of G which are subsets of $\mathcal{E}$. If p divides the order of G, then $\mathcal{C}$ is ample [3, 3.2].

1.17 p-stubborn subgroups. Recall that a p-subgroup P of G is said to be *p-stubborn* if the quotient $N_G(P)/P$ has no nontrivial normal p-subgroups. Any collection $\mathcal{C}$ of p-subgroups of G which contains all p-stubborn subgroups is ample. This follows from 1.6 and the theorem of Jackowski-McClure-Oliver [16] that for such a collection the subgroup decomposition map $d_{\mathcal{C}}$ (1.8) is a mod p homology isomorphism. We prove something slightly sharper in §7. These p-stubborn collections seem to be of limited usefulness for finite groups. In many cases, e.g. if G is simple, the trivial subgroup is p-stubborn (1.12); see also 7.3.

1.18 The Bouc collection. Let $\mathcal{C}$ be the collection of all *nontrivial* p-stubborn subgroups of G. It is a theorem of Bouc [2, 6.6.6] that the inclusion $K_{\mathcal{C}} \to K_{\mathcal{P}(G)}$ is a homotopy equivalence. Therefore $\mathcal{C}$ has the same ampleness properties as $\mathcal{P}(G)$ does (1.13).

1.19 p-centric subgroups. A p-subgroup P of G is said to be *p-centric* if the centralizer $C_G(P)$ is the product of the center of P and a group of order prime to p. This is equivalent to the condition that the center of P be a Sylow p-subgroup of $C_G(P)$. The collection $\mathcal{C}$ of all p-centric subgroups of G is ample; a slight generalization of this is proved in §8.

1.20 Subgroups which are both p-stubborn and p-centric. The collection $\mathcal{C}$ of all subgroups of G which are both p-stubborn and p-centric is also ample; see 8.10.

1.21 An example illustrating the three decompositions. Suppose that G has an abelian Sylow p-subgroup P. In this case the p-centric subgroups of G are exactly the conjugates of P. By 1.19, the collection $\mathcal{C}$ of conjugates of P is ample. The three associated homology decompositions are easy to figure out. Up to equivalence of categories the $\mathcal{C}$-conjugacy category has only one object, the inclusion $H \to G$; the automorphisms of this object are $N_G(P)/C_G(P)$ and the space assigned to the object by the functor $\alpha_{\mathcal{C}}$ is equivalent to $\mathrm{B}C_G(P)$. Let $Q = N_G(P)/C_G(P)$. The centralizer decomposition (1.6) gives a mod p homology isomorphism

$$\operatorname{hocolim}_{\mathbf{Q}} \mathrm{B}C_G(P) \simeq \mathrm{B}C_G(P)_{\mathrm{h}Q} \simeq \mathrm{B}N_G(P) \to \mathrm{B}G$$

(see 2.16). Up to equivalence of categories the $\mathcal{C}$-orbit category $\mathbf{O}_\mathcal{C}$ also has just one object, represented by the orbit G/P. The self-maps of this object are $N_G(P)/P$, and the space assigned to the object by $\beta_\mathcal{C}$ is equivalent to $\mathrm{B}P$. Let $W = N_G(P)/P$. The subgroup decomposition (1.8) gives a mod p homology isomorphism

$$\operatorname{hocolim}_{\mathbf{W}} \mathrm{B}P \simeq (\mathrm{B}P)_{\mathrm{h}W} \simeq \mathrm{B}N_G(P) \to \mathrm{B}G \ .$$

The category $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ of 1.9 has only one object, which has no nonidentity self-maps. The space assigned to this object by $\delta_\mathcal{C}$ is equivalent to $\mathrm{B}N_G(P)$, and so the normalizer decomposition degenerates into a mod p homology isomorphism

$$\mathrm{B}N_G(P) \to \mathrm{B}G \ .$$

All three decompositions say the same thing, but they express it in different ways. In each case the content of the message is the theorem of Swan stating that if G has an abelian Sylow p-subgroup P, then the map $\mathrm{B}N_G(P) \to \mathrm{B}G$ is an isomorphism on mod p homology. Both 1.19 and 1.20 can be viewed as extensions of this theorem to cases in which the Sylow p-subgroups are not abelian.

1.22 Spectral sequences and sharp decompositions. Consider a homology decomposition of the form 1.1. Bousfield and Kan [5, XII.5.7] give a first quadrant homology spectral sequence

$$E^2_{i,j} = \operatorname{colim}_i \mathrm{H}_j(F; \mathbf{F}_p) \Rightarrow \mathrm{H}_{i+j}(\mathrm{B}G; \mathbf{F}_p) \ ,$$

where colim_i denotes the i'th left derived functor of the colimit construction on the category of functors from $\mathbf{D}$ into abelian groups. Call the homology decomposition *sharp* if this spectral sequence has $E^2_{i,j} = 0$ for $i > 0$. A sharp homology decomposition thus gives an isomorphism $\mathrm{H}_*(\mathrm{B}G; \mathbf{F}_p) \cong \operatorname{colim} \mathrm{H}_*(F; \mathbf{F}_p)$; in other words, it gives a formula for $\mathrm{H}_*(\mathrm{B}G; \mathbf{F}_p)$ in terms of the homology of certain subgroups of G. For example, the Jackowski-McClure decomposition is sharp [15]. In a future paper we intend to look at the question of which of the homology decompositions discussed above are sharp.

Organization of the paper. Section 2 describes some homotopical category theory; this is used in §3 to prove the three decomposition theorems. Sections 4 and 5 have brief discussions of, respectively, G-spaces and finite p-groups. The final three sections contain proofs that various collections of subgroups of a finite group G are ample, or even M-ample for some G-module M (6.1).

Motivation. This paper was originally motivated by a study of [16] and a subsequent attempt to find some common ground between sections 2 and 5 of that paper. In a sense, in the last three sections of this paper we prove decomposition theorems like the one that follows from [16, 2.14] with algebraic techniques like the ones from [16, §5].

1.23 Notation and terminology. This paper is written with the convention that the word "space" by itself means "simplicial set" [17] [5, VIII]. For instance, for the rest of the paper $\mathrm{B}G$ stands for the usual simplicial classifying space of the group G [17,

§21]. A map between spaces is an *equivalence* or *weak equivalence* if it becomes a weak equivalence of topological spaces upon passing to geometric realizations [17].

Throughout the paper, p is a fixed prime number and $\mathbf{F}_p$ is the field with p elements. A space is $\mathbf{F}_p$-*acyclic* if it has the $\mathbf{F}_p$-homology of a point, and a map is an $\mathbf{F}_p$-*equivalence* if it induces an isomorphism on $\mathbf{F}_p$-homology. The results of the paper are stated for finite groups, but some of them, for instance the decomposition results from §3, hold unchanged for infinite discrete groups.

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§2. Homotopy colimits and the Grothendieck construction

One of the main techniques of this paper is to use categories as models for spaces; this makes it possible, for instance, to prove that two maps are homotopic by finding a natural transformation between associated functors. The advantage of this is that it is easier to understand a natural transformation between functors than to understand the formidable amount of data that goes into the construction of an explicit simplicial homotopy [17, §5].

Nerves. The fundamental space associated to a (small) category $\mathbf{D}$ is its *nerve* $|\mathbf{D}|$. See [5, XI §2] or 4.7 for the definition; there are some examples below. Let **Cat** denote the category whose objects are small categories and whose morphisms are functors between them.

2.1 Proposition. [5, XI §2] *The nerve construction gives a functor*

$$|-| : \mathbf{Cat} \to \mathbf{Spaces}.$$

This construction carries a natural transformation between two functors f and f' into a (simplicial) homotopy between $|f|$ and $|f'|$.

Proposition 2.1 has the following immediate consequence.

2.2 Proposition. *If $f : \mathbf{D} \to \mathbf{D}'$ is an equivalence of categories, then $|f| : |\mathbf{D}| \to |\mathbf{D}'|$ is a weak equivalence of spaces.*

Proof. If $f' : \mathbf{D}' \to \mathbf{D}$ is an inverse equivalence, then the composites ff' and $f'f$ are naturally equivalent to the appropriate identity functors. □

2.3 The nerve of a groupoid. Suppose that G is a group. Associated to G is a category $\mathbf{G}$ with one object $*$,and with the monoid of self-maps of $*$ isomorphic to G. A functor from $\mathbf{G}$ to some category $\mathbf{M}$ amounts to an object of $\mathbf{M}$ together with an action of G on it. The nerve of $\mathbf{G}$ is isomorphic to the classifying space $\mathrm{B}G$.

Suppose that $\mathbf{D}$ is a *groupoid*, that is, a small category in which every morphism is invertible. For any object x of $\mathbf{D}$ let G_x denote the group of self-maps of x in $\mathbf{D}$; the associated category $\mathbf{G}_x$ can be identified with the full subcategory of $\mathbf{D}$ generated by the object x. If $\mathbf{D}$ is *connected* in the sense that any two objects can be joined by an arrow, then for any object x of $\mathbf{D}$ the inclusion $\mathbf{G}_x \to \mathbf{D}$ is an equivalence of categories. In this case it follows from 2.2 that the induced map $\mathrm{B}G_x = |\mathbf{G}_x| \to |\mathbf{D}|$ is a weak equivalence. In general $|\mathbf{D}|$ is weakly equivalent to a disjoint union of spaces $\mathrm{B}G_x$, where x ranges over a set of representatives for isomorphism classes of objects in $\mathbf{D}$.

2.4 The nerve of a poset. Let $\mathbf{P} = (P, \leq)$ be a partially ordered set, for short a poset. Associated to $\mathbf{P}$ is a simplicial complex $K_{\mathbf{P}}$ with vertex set P, in which the simplices are the finite subsets of P which are totally ordered by the relation $\leq$. The poset $\mathbf{P}$ can be viewed as a category with object set P, with one morphism $x \to y$ whenever $x \leq y$, and with no other morphisms.

2.5 Proposition. *For any poset $\mathbf{P}$, the topological space $K_{\mathbf{P}}$ is homeomorphic in a natural way to the geometric realization of $|\mathbf{P}|$.*

Proof. Check that the nondegenerate simplices of the simplicial set $|\mathbf{P}|$ correspond exactly to the (geometric) simplices of $K_{\mathbf{P}}$. □

2.6 Proposition. *If the poset $\mathbf{P} = (P, \leq)$ has a minimal object or a maximal object, then $|\mathbf{P}|$ is contractible.*

Proof. Let x be a minimal (resp. maximal) object. The unique maps $x \to y$ (resp. $y \to x$) for $y \in P$ give a natural transformation between the identity functor of $\mathbf{P}$ and the constant functor with value x. Now apply 2.1. □

The homotopy orbit space functor (1.2) can be defined either simplicially or topologically. In both forms it preserves weak equivalences, and the geometric realization functor carries one form to the other. Thus 2.5 allows the notion of ampleness to be reformulated in terms of the nerve of the poset $\mathbf{S}_{\mathcal{C}}$.

2.7 Proposition. *Suppose that G is a finite group, and that $\mathcal{C}$ is a collection of subgroups of G. Then $\mathcal{C}$ is ample if and only if the map*

$$q_{\mathcal{C}} : |\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G} \to (*)_{\mathrm{h}G} = \mathrm{B}G$$

induced by $|\mathbf{S}_{\mathcal{C}}| \to *$ is an $\mathbf{F}_p$-equivalence.

Categorical models for homotopy colimits. By definition [5, XII §2], the nerve of a category $\mathbf{D}$ is isomorphic to the homotopy colimit of the functor on $\mathbf{D}$ which sends every object to a one-point space. It is very useful to have a method for representing more complex homotopy colimits as nerves. The most general tool we know of for doing this is the Grothendieck Construction.

2.8 Definition. Suppose that $\mathbf{D}$ is a small category, and that $f : \mathbf{D} \to \mathbf{Cat}$ is a functor. The *Grothendieck Construction on* f, denoted $\mathbf{Gr}(f)$, is the category whose objects are the pairs (d, x) where d is an object of $\mathbf{D}$ and x is an object of $f(d)$. An arrow $(d, x) \to (d', x')$ in $\mathbf{Gr}(f)$ is a pair (u, v), where $u : d \to d'$ is a morphism in $\mathbf{D}$ and $v : (f(u))(x) \to x'$ is a morphism in $f(d')$. Arrows compose according to the rule $(u, v) \cdot (u', v') = (u'', v'')$, where u'' is the composite $u \cdot u'$ and v'' is the composite of v with the image of v' under the functor $f(u)$.

Thomason discovered the following remarkable property of this construction.

2.9 Theorem. [19, 1.2] *Suppose that* $\mathbf{D}$ *is a small category and* $f : \mathbf{D} \to \mathbf{Cat}$ *is a functor. Let* $\mathbf{Gr}(f)$ *be the Grothendieck Construction on* f*. Then there is a natural weak equivalence*

$$\operatorname{hocolim} |f| \xrightarrow{\simeq} |\mathbf{Gr}(f)| \ .$$

2.10 Variations. If $\mathbf{D}$ is a small category, then the nerve of its opposite category $\mathbf{D}^{\mathrm{op}}$ is weakly equivalent to $|\mathbf{D}|$ in a natural way. Even better, Quillen exhibits a category $\mathbf{D}'$, depending functorially on $\mathbf{D}$, together with functors $\mathbf{D}' \to \mathbf{D}$ and $\mathbf{D}' \to \mathbf{D}^{\mathrm{op}}$ which induce weak equivalences on nerves [18, p. 94]. Suppose that $f : \mathbf{D} \to \mathbf{Cat}$ is a functor. Let f^{op} denote the composite of f with the "opposite" construction $\mathbf{Cat} \to \mathbf{Cat}$; note that f^{op} is again a functor $\mathbf{D} \to \mathbf{Cat}$. It follows from the above remarks and the homotopy invariance of homotopy colimits [5, p. 335] that the four categories $\mathbf{Gr}(f)$, $\mathbf{Gr}(f)^{\mathrm{op}}$, $\mathbf{Gr}(f^{\mathrm{op}})$ and $\mathbf{Gr}(f^{\mathrm{op}})^{\mathrm{op}}$ *all* have nerves which are weakly equivalent in a natural way to $\operatorname{hocolim}|f|$.

2.11 Remark. Suppose that f is a functor from $\mathbf{D}$ to the category of sets. We can treat the values of f as *discrete* categories, i.e., categories with no nonidentity morphisms, and think of f as a special type of functor $\mathbf{D} \to \mathbf{Cat}$. For such an f it is easy to check that $\operatorname{hocolim} f$ is in fact isomorphic to $|\mathbf{Gr}(f)|$.

The following propositions illustrate how 2.9 is used (in the form 2.11). If G is a finite group and $\mathcal{C}$ is a collection of subgroups of G, let $\mathrm{E}_{\mathcal{C}}$ denote the homotopy colimit of the inclusion $\mathcal{J} : \mathbf{O}_{\mathcal{C}} \to \mathbf{Spaces}$ (1.7). The space $\mathrm{E}_{\mathcal{C}}$ obtains a G-action from the fact that $\mathcal{J}$ actually takes values in the category of G-spaces.

2.12 Proposition. *Suppose that* G *is a finite group. Then for any collection* $\mathcal{C}$ *of subgroups of* G *there is a* G*-map* $\mathrm{E}_{\mathcal{C}} \to |\mathbf{S}_{\mathcal{C}}|$ *(2.4) which is a homotopy equivalence of spaces.*

Proof. Consider $\mathcal{J}$ as a functor from $\mathbf{O}_{\mathcal{C}}$ to discrete categories (2.11), and let $\mathbf{U}$ be the corresponding Grothendieck construction. The objects of $\mathbf{U}$ are pairs (O, x), where O is a G-orbit of the form G/H, $H \in \mathcal{C}$, and $x \in O$. A morphism $(O, x) \to (O', x')$ is a G-map $f : O \to O'$ such that $f(x) = x'$. The nerve $|\mathbf{U}|$ is isomorphic in a natural way to $\mathrm{E}_{\mathcal{C}}$ (2.11). The group G acts on $\mathbf{U}$, with an element $g \in G$ giving the functor from $\mathbf{U}$ to itself which sends an object (O, x) to (O, gx). Upon passage to nerves this map induces the action of G on $\mathrm{E}_{\mathcal{C}}$.

Consider the functor $F : \mathbf{U} \to \mathbf{S}_{\mathcal{C}}$ which assigns to an object (O, x) the isotropy subgroup G_x of x. This functor commutes with the actions of G on the two categories, since translating $x \in O$ by $g \in G$ has the effect of conjugating the isotropy subgroup of x by g. It is also clear that F is an equivalence of categories; an inverse equivalence is obtained by sending a subgroup $H \in \mathcal{C}$ to the pair $(G/H, eH)$. By 2.2, applying the nerve construction to F gives the desired map $\mathrm{E}_{\mathcal{C}} \to |\mathbf{S}_{\mathcal{C}}|$. □

2.13 Remark. Essentially the same argument gives the following more elaborate result. Suppose that $\mathcal{C}$ is a collection of subgroups of G. If $H \subset G$ is a subgroup, not necessarily in $\mathcal{C}$, let $\mathbf{S}_{\mathcal{C}}(H)$ denote the poset consisting of all elements of $\mathcal{C}$ which contain H.

2.14 Proposition. *Let* G *be a finite group,* $\mathcal{C}$ *a collection of subgroups of* G*, and* $H \subset G$ *a subgroup. Then there is a map*

$$(\mathrm{E}_{\mathcal{C}})^H \to |\mathbf{S}_{\mathcal{C}}(H)|$$

which is a homotopy equivalence of spaces and is equivariant with respect to the natural actions of $N_G(H)/H$ on the spaces involved.

Proof. Observe that $(\mathrm{E}_{\mathcal{C}})^H = \operatorname{hocolim}(\mathcal{J}^H)$ and repeat the argument above. □

2.15 Corollary. *Let G be a finite group and $\mathcal{C}$ a collection of subgroups of G. Then for any $H \in \mathcal{C}$, $(\mathrm{E}_{\mathcal{C}})^H$ is contractible.*

Proof. By 2.14, $(\mathrm{E}_{\mathcal{C}})^H$ is homotopy equivalent to $|\mathbf{S}_{\mathcal{C}}(H)|$. This nerve is contractible by 2.6, since $\mathbf{S}_{\mathcal{C}}(H)$ is a poset with H itself as a minimal object. □

2.16 Homotopy orbit spaces as homotopy colimits. Let G be a group which acts on a space X, and $f_X : \mathbf{G} \to \mathbf{Spaces}$ the corresponding functor (2.3). It is easy to check from the definitions that $X_{\mathrm{h}G}$ (1.2) is isomorphic to $\operatorname{hocolim} f_X$. Suppose that $X = |\mathbf{D}|$ for some category $\mathbf{D}$, and that the action of G on X is induced by an action of G on $\mathbf{D}$. Let $f_{\mathbf{D}} : \mathbf{G} \to \mathbf{Cat}$ be the functor given by this categorical action. According to 2.9, $X_{\mathrm{h}G}$ is weakly equivalent to $|\mathbf{Gr}(f_{\mathbf{D}})|$. We will call $\mathbf{Gr}(f_{\mathbf{D}})$ the Grothendieck construction of the action of G on $\mathbf{D}$.

For example, if G is a finite group and $\mathcal{C}$ is a collection of subgroups of G, the action of G on $|\mathbf{S}_{\mathcal{C}}|$ arises from an action of G on the poset (2.4) $\mathbf{S}_{\mathcal{C}}$. By Proposition 2.5, the question of whether or not $\mathcal{C}$ is ample can be studied by looking at the nerve of the Grothendieck construction on the action of G on $\mathbf{S}_{\mathcal{C}}$.

§3. Three homology decompositions

In this section we will prove the three homology decomposition theorems described in §1. Throughout the section, G denotes some particular finite group.

Our approach is to use Grothendieck constructions to find explicit categorical models for the homotopy colimits which appear as the domains of the decomposition maps $a_{\mathcal{C}}$ (1.6), $b_{\mathcal{C}}$ (1.8) and $d_{\mathcal{C}}$ (1.10). Inspecting the categories involved then reveals that each of these domains is weakly equivalent to $|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G}$ in a way which respects the natural maps from these spaces to $\mathrm{B}G$.

3.1 The centralizer decomposition. Let $\mathcal{C}$ be a collection of subgroups of G, and $\mathbf{A}_{\mathcal{C}}$ the $\mathcal{C}$-conjugacy category described in 1.5. Associated to an object (H, Σ) of $\mathbf{A}_{\mathcal{C}}$ is the groupoid $\tilde{\alpha}_{\mathcal{C}}(H, \Sigma)$ whose objects consist of all homomorphisms $i : H \to G$ with $i \in \Sigma$; a morphism $i \to i'$ in this groupoid is an element $g \in G$ such that $gig^{-1} = i'$. This construction gives a functor from $\mathbf{A}_{\mathcal{C}}^{\mathrm{op}}$ to the category of groupoids. It is clear that $\tilde{\alpha}_{\mathcal{C}}(H, \Sigma)$ is a connected groupoid in which the automorphism group of an object i is the centralizer $C_G(i(H))$, so the nerve $|\tilde{\alpha}_{\mathcal{C}}(H, \Sigma)|$ is equivalent to $\mathrm{B}C_G(i(H))$ (see 2.3). We let $\alpha_{\mathcal{C}} = |\tilde{\alpha}_{\mathcal{C}}|$; this is a functor $\mathbf{A}_{\mathcal{C}}^{\mathrm{op}} \to \mathbf{Spaces}$ of the type promised in 1.5.

Let $\mathbf{G}$ be the category of the group G (2.3). For each object (H, Σ) of $\mathbf{A}_{\mathcal{C}}$ there is a functor $\tilde{\alpha}_{\mathcal{C}}(H, \Sigma) \to \mathbf{G}$ which assigns to a morphism $gig^{-1} = i'$ of $\tilde{\alpha}_{\mathcal{C}}(H, \Sigma)$ the element $g \in G$ which determines it. These functors combine to give a natural transformation from $\tilde{\alpha}_{\mathcal{C}}$ to the constant functor on $\mathbf{A}_{\mathcal{C}}^{\mathrm{op}}$ with value $\mathbf{G}$, and consequently a natural transformation from $\alpha_{\mathcal{C}}$ to the constant functor with value $|\mathbf{G}| = \mathrm{B}G$. This gives a map [5, p. 329]

$$a_{\mathcal{C}} : \operatorname{hocolim} \alpha_{\mathcal{C}} \to \mathrm{B}G \ .$$

In order to prove Theorem 1.6 it is enough to show that $a_{\mathcal{C}}$ can be identified up to weak equivalence with the map $q_{\mathcal{C}}$ of 2.7.

Let $\mathbf{U}$ be the category $\mathbf{Gr}((\tilde{\alpha}_{\mathcal{C}})^{\mathrm{op}})^{\mathrm{op}}$ (see 2.10). The objects of $\mathbf{U}$ are pairs (H, i) such that H is a group and $i : H \to G$ is a monomorphism with $i(H) \in \mathcal{C}$. A morphism $(H, i) \to (H', i')$ is a pair (j, g) where $j : H \to H'$ is a homomorphism and $g \in G$ is an element such that $g(i'j)g^{-1} = i$. By 2.10, $|\mathbf{U}|$ is weakly equivalent to $\operatorname{hocolim} \alpha_{\mathcal{C}}$. The functor $\mathbf{U} \to \mathbf{G}$ which sends a morphism (j, g) to g induces a map $|\mathbf{U}| \to |\mathbf{G}| = \mathrm{B}G$ which corresponds (by naturality) to $a_{\mathcal{C}}$.

Let $\mathbf{V}$ be the Grothendieck construction of the action of G on $\mathbf{S}_{\mathcal{C}}$ (2.16). The objects of $\mathbf{V}$ are the subgroups H of G. A morphism $H \to H'$ in this category consists of an element $g \in G$ such that $gHg^{-1} \subset H'$. By 2.9, $|\mathbf{V}|$ is weakly equivalent to $|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G}$. There is a functor $|\mathbf{V}| \to \mathbf{G}$ which sends a morphism $g : gHg^{-1} \subset H'$ to the element $g \in G$ which determines it; this induces a map $|\mathbf{V}| \to |\mathbf{G}| = \mathrm{B}G$ which corresponds to $q_{\mathcal{C}}$.

Consider the functor $F : \mathbf{U} \to \mathbf{V}$ which sends an object (H, i) to the subgroup $i(H) \subset G$. It is easy to see that F is an equivalence of categories; an inverse equivalence is given by the functor $\mathbf{V} \to \mathbf{U}$ which sends a subgroup H to the pair (H, ι) where $\iota : H \to G$ is the inclusion. It follows that F induces a weak equivalence on nerves. The proof of 1.6 is completed by observing that F commutes with the functors from $\mathbf{U}$ and $\mathbf{V}$ to $\mathbf{G}$. □

3.2 The subgroup decomposition. Let $\mathcal{J} : \mathbf{O}_{\mathcal{C}} \to G$–**Spaces** be the inclusion functor and $\mathrm{E}_{\mathcal{C}} = \operatorname{hocolim} \mathcal{J}$ the space of 2.12. Since homotopy colimits (cf. 2.16) commute with one another, there is a natural weak equivalence

$$\operatorname{hocolim} \beta_{\mathcal{C}} = \operatorname{hocolim}(\mathcal{J}_{\mathrm{h}G}) \simeq (\operatorname{hocolim} \mathcal{J})_{\mathrm{h}G} = (\mathrm{E}_{\mathcal{C}})_{\mathrm{h}G} \ .$$

Under this equivalence the map $b_{\mathcal{C}}$ corresponds to the map

$$b'_{\mathcal{C}} : (\mathrm{E}_{\mathcal{C}})_{\mathrm{h}G} \to \mathrm{B}G$$

induced by the G-map $\mathrm{E}_{\mathcal{C}} \to *$. By 2.12, there is a weak equivalence $\mathrm{E}_{\mathcal{C}} \to |\mathbf{S}_{\mathcal{C}}|$ which is G-equivariant; this map induces a weak equivalence $(\mathrm{E}_{\mathcal{C}})_{\mathrm{h}G} \to |\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G}$ which commutes with the respective maps $b'_{\mathcal{C}}$ and $q_{\mathcal{C}}$ from these spaces to $\mathrm{B}G$. This implies that $q_{\mathcal{C}}$ is an $\mathbf{F}_p$-equivalence if and only if $b'_{\mathcal{C}}$ or equivalently $b_{\mathcal{C}}$ is. By 2.7, this proves 1.8. □

3.3 The normalizer decomposition. The category $\mathrm{s}\mathbf{S}_{\mathcal{C}}$ of simplices in $K_{\mathcal{C}}$ has as objects the simplices σ of $K_{\mathcal{C}}$, that is, the finite subsets σ of $\mathcal{C}$ which are totally ordered by inclusion:

$$\sigma = \{H_i \,|\, 0 \le i \le n(\sigma),\, H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{n(\sigma)}\} \ . \tag{3.4}$$

There is exactly one morphism $\sigma \to \sigma'$ in $\mathrm{s}\mathbf{S}_{\mathcal{C}}$ if $\sigma' \subset \sigma$, and there are no other morphisms. It may be a little surprising that we have chosen to have morphisms correspond to *reverse* inclusions, but this simplifies a construction below. The conjugation action of G on $\mathcal{C}$ induces an action of G on $\mathrm{s}\mathbf{S}_{\mathcal{C}}$. The geometric realization of $|\mathrm{s}\mathbf{S}_{\mathcal{C}}|$ is the barycentric subdivision of $K_{\mathcal{C}}$ (cf. 2.5), and in fact $|\mathrm{s}\mathbf{S}_{\mathcal{C}}|$ is weakly equivalent to $|\mathbf{S}_\mathcal{C}|$ in a way which respects the actions of G on these spaces. For instance, such a weak equivalence is induced given by the functor $\mathrm{s}\mathbf{S}_\mathcal{C} \to \mathbf{S}_\mathcal{C}$ which sends the object 3.4 of $\mathrm{s}\mathbf{S}_\mathcal{C}$ to the object H_0 of $\mathbf{S}_\mathcal{C}$; see [9, §5] for more details.

Let $\mathbf{U}$ be the Grothendieck construction of the action of G on $\mathrm{s}\mathbf{S}_\mathcal{C}$. The objects of $\mathbf{U}$ are the simplices σ of $K_\mathcal{C}$; a morphism $\sigma \to \sigma'$ is an element $g \in G$ such that $g\sigma' g^{-1} \subset \sigma$. There is a functor $\mathrm{s}\tilde{q}_\mathcal{C} : \mathbf{U} \to \mathbf{G}$ which sends a morphism $g\sigma' g^{-1} \subset \sigma$ to the element g which determines it. By 2.7, 2.16, and the homotopy invariance of homotopy colimits, $|\mathbf{U}|$ is equivalent to $|\mathbf{S}_\mathcal{C}|_{\mathrm{h}G}$ in such a way that the map

$$\mathrm{s}q_\mathcal{C} = |\mathrm{s}\tilde{q}_\mathcal{C}| : |\mathbf{U}| \to |\mathbf{G}| = \mathrm{B}G \tag{3.5}$$

corresponds to $q_\mathcal{C}$ (2.7).

The category $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ of "orbit simplices" has as objects the equivalence classes $\bar{\sigma}$ of objects of $\mathrm{s}\mathbf{S}_\mathcal{C}$ under the conjugation action of G. More explicitly, an object of $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ is an equivalence class

$$\bar{\sigma} = \langle \{H_i \,|\, 0 \le i \le n(\bar{\sigma}),\, H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{n(\bar{\sigma})}\}\rangle$$

of totally ordered subsets of $\mathcal{C}$, where two subsets $\{H_i\}$ and $\{H_i'\}$ are considered equivalent if there is a single element $g \in G$ such that $gH_ig^{-1} = H_i'$, $0 \le i \le n(\bar{\sigma})$. Suppose that $\sigma \in \bar{\sigma}$ and $\sigma' \in \bar{\sigma}'$. Then there is exactly one morphism $\bar{\sigma} \to \bar{\sigma}'$ in $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ if there exists $g \in G$ such that $g\sigma' \subset \sigma$. There are no other morphisms.

For each object $\bar{\sigma}$ of $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$, let $\tilde{\delta}_\mathcal{C}(\bar{\sigma})$ denote the groupoid whose objects are the elements $\sigma \in \bar{\sigma}$. A morphism $\sigma \to \sigma'$ in $\tilde{\delta}_\mathcal{C}(\bar{\sigma})$ is an element $g \in G$ such that $g\sigma = \sigma'$. It is clear that this groupoid is connected (2.3), and that $|\tilde{\delta}_\mathcal{C}(\bar{\sigma})|$ is equivalent to $\mathrm{B}G_\sigma$, where G_σ is the isotropy subgroup in G of an element $\sigma = \{H_i\} \in \bar{\sigma}$. The group G_σ can be calculated by inspection:

$$G_\sigma = \bigcap_{i=1}^{n(\bar{\sigma})} N_G(H_i)\,.$$

We will now describe how the construction $\tilde{\delta}_\mathcal{C}$ gives a functor from $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ to groupoids. Suppose that $\bar{\sigma}$, $\bar{\sigma}'$ are objects of $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ and that $f : \bar{\sigma} \to \bar{\sigma}'$ is a map in $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$. Let $\{H_i \,|\, 0 \le i \le n(\bar{\sigma})\}$ be an object of $\tilde{\delta}_\mathcal{C}(\bar{\sigma})$. It is not hard to check that there is a *unique* list $(i_k \,|\, 0 \le k \le n(\bar{\sigma}'))$ of distinct integers, $0 \le i_k \le n(\bar{\sigma})$, such that the subset $\{H_{i_k} \,|\, 0 \le k \le n(\bar{\sigma}')\}$ belongs to $\bar{\sigma}'$. (Uniqueness follows from the fact that a subgroup of G cannot be conjugate to a proper subgroup of itself.) The functor $\tilde{\delta}_\mathcal{C}(f)$ then takes the object $\{H_i\}$ of $\tilde{\delta}_\mathcal{C}(\bar{\sigma})$ to the object $\{H_{i_k}\}$ of $\tilde{\delta}_\mathcal{C}(\bar{\sigma}')$. The effect of $\tilde{\delta}_\mathcal{C}(f)$ on morphisms of $\tilde{\delta}_\mathcal{C}(\bar{\sigma})$ is more or less evident.

Let $\delta_\mathcal{C}(\bar{\sigma}) = |\tilde{\delta}_\mathcal{C}(\bar{\sigma})|$, so that $\delta_\mathcal{C} : \bar{\mathrm{s}}\mathbf{S}_\mathcal{C} \to \mathbf{Spaces}$ is a functor of the type promised in 1.9. Let $\mathbf{G}$ be the category of the group G. For each object $\bar{\sigma}$ of $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ there is a a functor $\delta_\mathcal{C}(\bar{\sigma}) \to \mathbf{G}$ which sends a morphism determined by an element $g \in G$ to the morphism g of $\mathbf{G}$. These functors combine to give a natural transformation from $\tilde{\delta}_\mathcal{C}$ to the constant functor on $\bar{\mathrm{s}}\mathbf{S}_\mathcal{C}$ with value $\mathbf{G}$. Passing to nerves gives a natural transformation from $\delta_\mathcal{C}$ to the constant functor with value $|\mathbf{G}| = \mathrm{B}G$, and hence a map $d_\mathcal{C} : \operatorname{hocolim} \delta_\mathcal{C} \to \mathrm{B}G$, as required 1.9.

In order to prove Theorem 1.10, it is enough to show that $d_{\mathcal{C}}$ can be identified up to weak equivalence with the map $q_{\mathcal{C}}$ of 2.7, or even with the map $\mathrm{s}q_{\mathcal{C}}$ of 3.5. Let $\mathbf{V}$ be the Grothendieck construction of $\tilde{\delta}_{\mathcal{C}}$. An object of $\mathbf{V}$ is a simplex σ of $K_{\mathcal{C}}$, and a morphism $\sigma \to \sigma'$ is an element $g \in G$ such that $g\sigma' \subset \sigma$. By 2.9 and 2.2, it is enough to check that the category $\mathbf{V}$ is equivalent to the category $\mathbf{U}$ of 3.5 in a way which respects the relevant functors from these two categories to $\mathbf{G}$. This is clear. □

§4. G-spaces and fixed point sets

In this section we will recall some facts about the relationship between a G-space X and the fixed point sets X^H for various subgroups H of G. Throughout, G is a fixed finite group. By our conventions a G-space is a simplicial set with an action of G, but the results remain true for G-CW complexes.

A map $Y \to X$ of G-spaces is said to be a *weak G-equivalence* (resp. *a G-$\mathbf{F}_p$-equivalence*) if for every subgroup H of G the induced map $Y^H \to X^H$ is a weak equivalence (resp. an $\mathbf{F}_p$-equivalence). Let $\mathrm{Iso}_G(X)$ or $\mathrm{Iso}(X)$ denote the collection of subgroups of G consisting of the isotropy subgroups of the action of G on X. We will be interested in the following two fairly well-known results, which will be proved at the end of this section.

4.1 Proposition. *Suppose that $f : X \to Y$ is a map of G-spaces, and that $\mathcal{C} = \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$. If $f^H : X^H \to Y^H$ is a weak equivalence (resp. an $\mathbf{F}_p$-equivalence) for all $H \in \mathcal{C}$, then f is a weak G-equivalence (resp. a G-$\mathbf{F}_p$-equivalence).*

4.2 Proposition. *Suppose that X is a G-space and $\mathcal{C}$ is a collection of subgroups of G with $\mathrm{Iso}(X) \subset \mathcal{C}$. Then there exists a zigzag of G-maps*

$$X \xleftarrow{f} X' \xrightarrow{h} \mathrm{E}_{\mathcal{C}}$$

such that f is a weak G-equivalence. If X^H is weakly contractible (resp. $\mathbf{F}_p$-acyclic) for all $H \in \mathcal{C}$, then h is a weak G-equivalence (resp. a G-$\mathbf{F}_p$-equivalence).

4.3 Remark. These propositions will be applied mostly to spaces X which themselves are of the form $\mathrm{E}_{\mathcal{C}'}$. For this it is necessary to have information about $\mathrm{Iso}(\mathrm{E}_{\mathcal{C}})$. The following statements are clear from the construction of the homotopy colimit (see 4.7 or [5, p. 338]).

4.4 Lemma. *For any collection $\mathcal{C}$ of subgroups of G, $\mathrm{Iso}(\mathrm{E}_{\mathcal{C}}) = \mathcal{C}$.*

4.5 Lemma. *Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are collections of subgroups of G. Assume that $\mathcal{C} \subset \mathcal{C}'$ (so that $\mathrm{E}_{\mathcal{C}}$ is a subspace of $\mathrm{E}_{\mathcal{C}'}$) and that given $H \in \mathcal{C}$ and $K \in \mathcal{C}' \setminus \mathcal{C}$, H is not a subgroup of K. Then every simplex of $\mathrm{E}_{\mathcal{C}'} \setminus \mathrm{E}_{\mathcal{C}}$ has isotropy subgroup contained in $\mathcal{C}' \setminus \mathcal{C}$.*

Results similar to 4.1 and 4.2 are proved in [16, Appendix] by an induction on orbit types which uses a pushout formula like the one below in Lemma 8.6 in the inductive step. We will derive 4.1 and 4.2 by referring to a general method for building G-spaces from fixed point data.

4.6 Definition. Suppose that $\mathbf{D}$ is a small category, and that $f : \mathbf{D}^{\mathrm{op}} \to \mathbf{Spaces}$ and $g : \mathbf{D} \to \mathbf{Spaces}$ are functors. The *double bar construction* $B(f, \mathbf{D}, g)$ is the simplicial space which in dimension k consists of the following coproduct (indexed by strings of composable arrows in $\mathbf{D}$)

$$\coprod_{d_k \to \cdots \to d_0} g(d_k) \times f(d_0) ,$$

and which has the usual face and degeneracy operators. See [5, XI §2], [13, §3], or [8, §9] (which uses the notation $N_{**}(f, \mathbf{D}, g)$). The *homotopy coend* of f and g, denote $\mathrm{hocoend}(f, g)$, is the realization or diagonal of $B(f, \mathbf{D}, g)$ [5, XII, 3.4, 5.3]. See [14, §3] for the topological version of this construction.

4.7 Remark. Homotopy coends have many convenient properties. Sometimes they can be obtained up to homotopy as the nerves of generalized Grothendieck constructions [8, §9]. Let $*$ denote the constant one-point valued functor on $\mathbf{D}$, considered as necessary to be either covariant or contravariant. In the situation of 4.6 there are natural isomorphisms

$$\mathrm{hocolim}\, f \cong \mathrm{hocoend}(f, *) \quad \mathrm{hocolim}\, g \cong \mathrm{hocoend}(*, g) \quad |\mathbf{D}| \cong \mathrm{hocoend}(*, *) \ .$$

If $u : f \to f'$ and $v : g \to g'$ are natural transformations such that, for each object d of $\mathbf{D}$, $u(d)$ and $v(d)$ are weak equivalences (resp. $\mathbf{F}_p$-equivalences), then the map $\mathrm{hocoend}(u, v) : \mathrm{hocoend}(f, g) \to \mathrm{hocoend}(f', g')$ is a weak equivalence [5, XII, 4.3] (resp. $\mathbf{F}_p$-equivalence [5, XII, 5.7]).

Suppose that G is a finite group, that $\mathcal{C}$ is a collection of subgroups of G, and that X is a G-space. There is a *fixed point functor* $\Phi^X_{\mathcal{C}} : \mathbf{O}^{\mathrm{op}}_{\mathcal{C}} \to \mathbf{Spaces}$ given by

$$\Phi^X_{\mathcal{C}}(G/H) = \mathrm{Map}_G(G/H, X) \ (= X^H) \ .$$

as well as an inclusion functor $\mathcal{J} : \mathbf{O}_{\mathcal{C}} \to G\text{-}\mathbf{Spaces}$. Let $X_{\mathcal{C}} = \mathrm{hocoend}(\Phi^X_{\mathcal{C}}, \mathcal{J})$. The action of G on the values of $\mathcal{J}$ induces by naturality an action of G on $X_{\mathcal{C}}$.

4.8 Theorem. *Let G be a finite group, $\mathcal{C}$ a collection of subgroups of G, and X a G-space. Then there is a natural G-map $X_{\mathcal{C}} \to X$. If $\mathcal{C}$ contains* $\mathrm{Iso}(X)$*, this map is a weak G-equivalence.*

Results like this were first proved by Elmendorf [13] in the case in which $\mathcal{C}$ is the collection consisting of all subgroups of G.

Proof. According to the description of the homotopy coend in 4.6, $X_{\mathcal{C}}$ is the realization of a simplicial space which in dimension k is a disjoint union of spaces of the form $G/H_k \times \mathrm{Map}_G(G/H_0, X)$, the union indexed by chains $G/H_k \to \cdots \to G/H_0$ in $\mathbf{O}_{\mathcal{C}}$. Such a chain gives a composite map $G/H_k \to G/H_0$, which can be combined with the evaluation map $G/H_0 \times \mathrm{Map}_G(G/H_0, X) \to X$ to give a map $G/H_k \times \mathrm{Map}_G(G/H_0, X) \to X$. These maps are compatible with the simplicial operators and upon passage to the realization give a map $X_{\mathcal{C}} \to X$; for details consult [13, §3].

One can check directly that $X_{\mathcal{C}} \to X$ is a weak equivalence if $X = G/H$ for some $H \in \mathcal{C}$. This follows for instance from the Reduction Theorem [14, 4.4] and the fact that if $X = G/H$ then $\Phi^X_{\mathcal{C}}$ is the representable functor $\mathrm{Map}(-, G/H)$ on $\mathbf{O}_{\mathcal{C}}$. There is an explicit argument in [13, §3]. The functor $X \mapsto X_{\mathcal{C}}$ preserves weak G-equivalences (4.7). It also preserves coproducts, pushouts in which one of the maps is a monomorphism of simplicial sets, and sequential colimits. It follows that the map $X_{\mathcal{C}} \to X$ is a weak equivalence whenever X can be constructed from the collection of "G-cells" $\{G/H \times \Delta[k] \mid H \in \mathcal{C},\, k \geq 0\}$ by these three operations. This can be done if and only if $\mathrm{Iso}(X) \subset \mathcal{C}$. □

Proof of 4.1. By 4.8 and naturality there is a commutative diagram

$$\begin{array}{ccc} X_{\mathcal{C}} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y_{\mathcal{C}} & \longrightarrow & Y \end{array}$$

in which the the horizontal arrows are weak G-equivalences. By 4.7 the left vertical arrow is a weak G-equivalence (resp. a G-$\mathbf{F}_p$-equivalence). □

Proof of 4.2. Let $*$ be the contravariant functor on $\mathbf{O}_{\mathcal{C}}$ whose value is the one-point space. There is a unique natural transformation $\Phi^X_{\mathcal{C}} \to *$, which gives rise to a map

$$X_{\mathcal{C}} = \mathrm{hocoend}(\Phi^X_{\mathcal{C}}, \mathcal{J}) \to \mathrm{hocoend}(*, \mathcal{J}) = \mathrm{hocolim}\, \mathcal{J} = \mathrm{E}_{\mathcal{C}}\,.$$

The required zigzag is $X \leftarrow X_{\mathcal{C}} \to \mathrm{E}_{\mathcal{C}}$. It has the necessary properties by 4.8 and 4.7. □

§5. Finite p-groups

In this section we recall some properties of finite p-groups. The properties are elementary and very well-known, but since they are so important in what follows it seems worthwhile to state them explicitly.

5.1 Proposition. *Suppose that X is a finite set with an action of the finite p-group P. Then the cardinality of X^P is congruent mod p to the cardinality of X.*

Proof. All of the non-trivial orbits of the action of P have cardinality divisible by p. □

5.2 Proposition. *Let P and Q be finite p-groups, and suppose that P acts on Q via group automorphisms. Then there exists a nonidentity element x in the center of Q such that x is fixed by the action of P.*

Proof. Let G be the semidirect product of P with Q, so that the conjugation action of Q on itself combines with the given action of P on Q to give an action of G on Q. Since G fixes the identity element $e \in Q$, counting (5.1) shows that G must fix a nonidentity element x. □

5.3 Proposition. *Suppose that P is a finite p-group and that M is an $\mathbf{F}_p[P]$-module which is finite dimensional as a vector space over $\mathbf{F}_p$. Then M has a finite filtration by P-submodules with the property that P acts trivially on the filtration quotients.*

Proof. Use induction on the order of M. By 5.2 there exists a nonzero submodule $M' \subset M$ on which P acts trivially, and by induction the quotient module M/M' has a filtration of the required type. □

Remark. The finite-dimensionality hypothesis in 5.3 is not necessary. This follows from the fact that 5.3 applies to the universal case, i.e., the module given by the left action of P on $\mathbf{F}_p[P]$.

5.4 Proposition. *Let Q be a finite p-group and P a proper subgroup of Q. Then the normalizer $N_Q(P)$ is strictly larger than P.*

Proof. The quotient $N_Q(P)/P$ is the fixed point set of the action of P on Q/P. Since P is a proper subgroup of Q, the cardinality of Q/P is divisible by p. The identity coset eP is fixed by P, and so counting (5.1) shows that there are other fixed cosets. □

§6. The collection of nontrivial p-subgroups

Let G be a fixed finite group. In this section we prove that if p divides the order of G the collection of nontrivial p-subgroups of G is ample. We will actually be concerned with a something more delicate than this, which is described in the following definition.

6.1 Definition. Let $\mathcal{C}$ be a collection of subgroups of G and M a G-module. The collection $\mathcal{C}$ is said to be M-*ample* if the map $q_{\mathcal{C}}^K$ of 1.4 (equivalently, the map $q_{\mathcal{C}}$ of 2.7) induces an isomorphism on (twisted) homology with coefficients in M.

6.2 Remark. If $\mathbf{F}_p$ denotes the trivial G-module, then $\mathcal{C}$ is $\mathbf{F}_p$-ample in the sense of 6.1 if and only if $\mathcal{C}$ is ample. An easy calculation shows that $\mathcal{C}$ is $\mathbf{F}_p[G]$-ample if and only if the spaces $K_{\mathcal{C}}$ and $|\mathbf{S}_{\mathcal{C}}|$ are $\mathbf{F}_p$-acyclic; in this situation $\mathcal{C}$ is M-ample for *any* G-module M. The arguments of §3 show that $\mathcal{C}$ is M-ample if and only if the three decomposition maps $a_{\mathcal{C}}$ (1.6), $b_{\mathcal{C}}$ (1.8) and $d_{\mathcal{C}}$ (1.10) induce isomorphisms on homology with coefficients in M. If $|\mathbf{S}_{\mathcal{C}}|$ is weakly contractible, then all three decomposition maps are weak equivalences, and $\mathcal{C}$ is M-ample for *any* G-module M. For instance, this is the case if $\mathcal{C}$ is a trivial collection (1.12).

6.3 Theorem. *Let $\mathcal{C}$ be the collection of all nontrivial p-subgroups of G. Suppose that M is a module over $\mathbf{F}_p[G]$ such that p divides the order of the kernel of the action map $G \to \mathrm{Aut}(M)$. Then $\mathcal{C}$ is M-ample.*

Remark. Theorem 6.3 is a result of Jackowski-McClure-Oliver traveling in disguise. Under the hypotheses of 6.3 they prove that certain groups $\Lambda^*(G; M)$ vanish [16, 5.5], but it is not hard to see using the ideas in 3.2 that these groups are isomorphic to the relative groups $\mathrm{H}^*(\mathrm{B}G, |\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G}; M)$. We will give an independent proof of 6.3, since the argument in [16] is indirect and depends on results which we eventually want to prove ourselves, *6.4 Remark.* Theorem 6.3 implies that if p divides the order of G, then the collection of all nontrivial p-subgroups of G is ample. This was first proved by K. Brown (cf. [1, V.3.1]). There is a spectral sequence approach (which proves a lot more) due to P. Webb [1, V.3.2]. Finally, this can be proved by working backwards, using 1.6 and Quillen's theorem (1.14), from the theorem of Jackowski-McClure [15] that for the collection of nontrivial elementary abelian p-subgroups of G, the centralizer decomposition map (1.5) is an $\mathbf{F}_p$-equivalence.

The key ingredient in the proof of 6.3 is the following lemma.

6.5 Lemma. *Let M be as in 6.3, and let*

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

be a chain of nontrivial p-subgroups of G. Then there exists an element $g \in G$ of order p such that g normalizes each of the subgroups P_i and g acts trivially on M.

Proof. In fact we can find an element $g \in G$ of order p such that g centralizes P_n and acts trivially on M. Let P be a Sylow p-subgroup of G which contains P_n, and let H be the kernel of the action map $G \to \mathrm{Aut}(M)$. Since H is normal in G, $Q = H \cap P \neq \{e\}$ is a Sylow p-subgroup of H and is normal in P. Choose g to be some element of order p in Q which is fixed by the conjugation action of P on Q (5.2). □

Proof of 6.3. If C is a chain complex of $\mathbf{F}_p[G]$-modules, in practice of finite length, we will let $\mathrm{H}_*(G;C)$, $\mathrm{H}^*(G;C)$ and $\hat{\mathrm{H}}^*(G;C)$ denote respectively the hyperhomology, hypercohomology and Tate hypercohomology of G with coefficients in C [6, XVII] [1, P. 164] [20, p. 166]. By the way in which complete (Tate) resolutions are constructed [1, II.7] there is a doubly infinite exact sequence

$$\text{(6.6)} \qquad \cdots \to \mathrm{H}_i(G;C) \xrightarrow{\nu} \mathrm{H}^{-i}(G;C) \to \hat{\mathrm{H}}^{-i}(G;C) \to \mathrm{H}_{i-1}(G;C) \to \cdots .$$

Let C_G (resp. C^G) be the chain complex obtained by applying $\mathrm{H}_0(G;-)$ (resp. $\mathrm{H}^0(G;-)$) dimensionwise to C. Further inspection of the recipe for a complete resolution shows that the map ν in the above exact sequence can be factored as a composite

$$\mathrm{H}_i(G;C) \to \mathrm{H}_i(C_G) \xrightarrow{(\bar{\nu}_G)_*} \mathrm{H}^i(C^G) \to \mathrm{H}^{-i}(G;C)\,.$$

where $\bar{\nu}_G : C_G \to C^G$ is induced by the norm endomorphism $\nu_G = \sum_{g\in G} g$ of C. In particular, if ν_G is the trivial endomorphism of C, the map ν is trivial.

Let C be the normalized simplicial $\mathbf{F}_p$-chain complex of $|\mathbf{S}_{\mathcal{C}}|$, or equivalently (2.5) the cellular chain complex of $K_{\mathcal{C}}$, and $\tilde{C}$ the kernel of the map $C \to \mathbf{F}_p$ induced by the map from $|\mathbf{S}_{\mathcal{C}}|$ to a point. Denote by $C \otimes M$ and $\tilde{C} \otimes M$ the chain complexes obtained by taking tensor products over $\mathbf{F}_p$ and using diagonal G-actions. Recall that $\mathrm{H}_*(G;C)$ is naturally isomorphic to $\mathrm{H}_*(|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G};\mathbf{F}_p)$ (cf. [1, p. 184]); more generally, $\mathrm{H}_*(G;C\otimes M)$ is naturally isomorphic to $\mathrm{H}_*(|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G};M)$. Similarly, $\mathrm{H}_*(G;\mathbf{F}_p\otimes M) = \mathrm{H}_*(G;M)$ is naturally isomorphic to $\mathrm{H}_*((*)_{\mathrm{h}G};M) = \mathrm{H}_*(\mathrm{B}G;M)$. Thus it is enough to show that the map $C\otimes M \to \mathbf{F}_p \otimes M$ induces an isomorphism $\mathrm{H}_*(G;C\otimes M) \cong \mathrm{H}_*(G;\mathbf{F}_p\otimes M)$, or even, by a long exact sequence argument, enough to show that $\mathrm{H}_*(G;\tilde{C}\otimes M) = 0$.

Let $P \subset G$ be a Sylow p-subgroup. The argument in [1, proof of V.3.1] shows that $C \to \mathbf{F}_p$ induces an isomorphism $\hat{\mathrm{H}}^*(P; C) \cong \hat{\mathrm{H}}(P; \mathbf{F}_p)$. Essentially the same argument shows that $C \otimes M \to \mathbf{F}_p \otimes M = M$ induces an isomorphism $\hat{\mathrm{H}}^*(P; C \otimes M) \to \hat{\mathrm{H}}^*(P; M)$. It is only necessary to check that if F is a free module over $\mathbf{F}_p[P]$ then $F \otimes M$ (with the diagonal action) is also a free module, and this is true for any $\mathbf{F}_p[P]$-module M. It follows that $\hat{\mathrm{H}}^*(P; \tilde{C} \otimes M) = 0$, and from that by a transfer argument that $\hat{\mathrm{H}}^*(G; \tilde{C} \otimes M) = 0$.

Let $\{m_\alpha\}$ be an $\mathbf{F}_p$-basis of M. The chain complex $C \otimes M$ has an $\mathbf{F}_p$-basis in which each element is of the form $\sigma \otimes m_\alpha$ for some simplex $\sigma = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of $K_{\mathcal{C}}$. According to 6.5 each such basis element is fixed by an element in G of order p, and so each basis element maps to zero under the norm endomorphism ν_G. (Keep in mind that $C \otimes M$ is a chain complex of vector spaces over $\mathbf{F}_p$.) It follows that ν_G is also the trivial endomorphism of $\tilde{C} \otimes M$, and from the remarks above that for each i the map $\nu : \mathrm{H}_i(G; \tilde{C} \otimes M) \to \mathrm{H}^{-i}(G; \tilde{C} \otimes M)$ is zero. Clearly, then, the fact that $\hat{\mathrm{H}}^*(G; \tilde{C} \otimes M) = 0$ implies that $\mathrm{H}_*(G; \tilde{C} \otimes M) = 0$. □

We record some related results for future use.

6.7 Proposition. *Let Y be a G-space, M a local coefficient system of exponent p on $Y_{\mathrm{h}G}$, and $\mathcal{C}$ the collection of all nontrivial p-subgroups of G. Suppose that there is an element of order p in G which acts trivially on $\mathrm{H}_*(Y; M)$ (with respect to the Serre action 6.8). Then the map*

$$(Y \times |\mathbf{S}_{\mathcal{C}}|)_{\mathrm{h}G} \to Y_{\mathrm{h}G}$$

induced by the projection $Y \times |\mathbf{S}_{\mathcal{C}}| \to Y$ induces an isomorphism on $\mathrm{H}_*(-; M)$.

Remark. The G-action on $Y \times |\mathbf{S}_{\mathcal{C}}|$ implicit in the statement of the proposition is the diagonal one. The letter M is used to denote both the original local coefficient system on $Y_{\mathrm{h}G}$ and the systems on Y and on $(Y \times |\mathbf{S}_{\mathcal{C}}|)_{\mathrm{h}G}$ pulled back over the evident maps of these spaces to $Y_{\mathrm{h}G}$.

Proof of 6.7. There are two Serre spectral sequences

$$\begin{aligned} E^2_{*,*} &= \mathrm{H}_*(|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}G}; \mathrm{H}_*(Y; M)) \Rightarrow \mathrm{H}_*((Y \times |\mathbf{S}_{\mathcal{C}}|)_{\mathrm{h}G}; M) \\ E^2_{*,*} &= \mathrm{H}_*(\mathrm{B}G; \mathrm{H}_*(Y; M)) \Rightarrow \mathrm{H}_*(Y_{\mathrm{h}G}; M) \end{aligned} \tag{6.8}$$

and a map between them which on abutments gives the homology map we are interested in. By 6.3 the map on E^2-terms is an isomorphism. □

The following is a theorem of Quillen.

6.9 Theorem. *Suppose that G has a nontrivial normal p-subgroup, and let $\mathcal{C}$ be the collection of all nontrivial p-subgroups of G. Then $|\mathbf{S}_{\mathcal{C}}|$ is contractible, and so $\mathcal{C}$ is M-ample for any G-module M.*

Proof. Let P be a nontrivial normal p-subgroup of G. For each $Q \in \mathcal{C}$ there are maps (inclusions) $Q \to PQ \leftarrow P$; these give a zigzag of natural transformations between the identity functor of $\mathbf{S}_{\mathcal{C}}$ and a constant functor. The result follows from 2.1.

To help orient the reader, we will prove that $|\mathbf{S}_{\mathcal{C}}|$ is $\mathbf{F}_p$-acyclic in a way which ties this fact into 6.3. Let $P \subset G$ be a nontrivial normal p-subgroup. By 6.3 the collection $\mathcal{C}$ is M-ample for $M = \mathbf{F}_p[G/P]$. For any G-space X there is a natural isomorphism $\mathrm{H}_*(X_{\mathrm{h}G}; \mathbf{F}_p[G/P]) \cong \mathrm{H}_*(X_{\mathrm{h}P}; \mathbf{F}_p)$ (Shapiro's lemma), and so it follows that the map $|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}P} \to \mathrm{B}P$ is an $\mathbf{F}_p$-equivalence. But $\mathbf{F}_p[P]$ has a finite filtration by P-submodules such that P acts trivially on the associated graded groups (5.3), so an induction using long exact sequence comparisons and the five lemma shows that the map

$$\mathrm{H}_*(|\mathbf{S}_{\mathcal{C}}|_{\mathrm{h}P}; \mathbf{F}_p[P]) \to \mathrm{H}_*((*)_{\mathrm{h}P}; \mathbf{F}_p[P])$$

is an isomorphism. By Shapiro's lemma again, the groups on the left are the mod p homology groups of $|\mathbf{S}_{\mathcal{C}}|$, and those on the right are the mod p homology groups of a point. □

§7. p-STUBBORN COLLECTIONS

Let G be a finite group and let $\mathcal{C}$ be a collection of p-subgroups of G which contains all p-stubborn subgroups. In this section we will show that $\mathcal{C}$ is M-ample for all G-modules M (6.2). In fact, we will show something stronger. For the statement, recall the definition of $\mathbf{S}_{\mathcal{C}}(H)$ (H a subgroup of G) from 2.13.

7.1 Theorem. *Let $\mathcal{C}$ be any collection of p-subgroups of G which contains all p-stubborn subgroups. Then for any p-subgroup P of G the nerve $|\mathbf{S}_{\mathcal{C}}(P)|$ is weakly contractible.*

7.2 Remark. If H is the trivial subgroup of G then $|\mathbf{S}_{\mathcal{C}}(H)| = |\mathbf{S}_{\mathcal{C}}|$.

7.3 Remark. Let $\mathcal{C}$ be the collection of p-stubborn subgroups of G. The fact that $|\mathbf{S}_{\mathcal{C}}|$ is contractible is a simple consequence of Bouc's theorem (1.18). To see this, let P be the maximal normal p-subgroup of G. If $P = \{e\}$, then $\{e\} \subset G$ is p-stubborn, and $|\mathbf{S}_{\mathcal{C}}|$ is contractible by 1.12. If $P \neq \{e\}$, then $\mathcal{C}$ is the Bouc collection, and $|\mathbf{S}_{\mathcal{C}}|$ is contractible by a combination of Bouc's theorem (1.18) and Quillen's theorem (6.9). We include the argument below because it is short, it proves something a little more than the weak contractibility of $|\mathbf{S}_{\mathcal{C}}|$, it sets the stage for §8, and it suggests an approach which generalizes to the case of compact Lie groups.

Theorem 7.1 raises the disquieting suspicion that working with a p-stubborn collection $\mathcal{C}$ of subgroups does not in any sense involve focusing on the structure of the group G at p, since the decomposition maps associated with $\mathcal{C}$ are weak equivalences (6.2). Indeed, the reader who is familiar with [16] may wonder why Theorem 7.1 is so strong; the results of [16, §2] suggest that if $\mathcal{C}$ is the collection of all p-stubborn subgroups of G then $|\mathbf{S}_{\mathcal{C}}|$ should at best be $\mathbf{F}_p$-acyclic, not weakly contractible. The explanation for this is partly visible in the proof below. A key element in the analysis of $|\mathbf{S}_{\mathcal{C}}|$ is the study of $W(P) = N_G(P)/P$ for p-subgroups P which are *not* p-stubborn. In such a case $W(P)$ has a nontrivial normal p-subgroup, and so Theorem 6.9 provides a certain contractibility statement for $W(P)$. Suppose now that P is a p-toral subgroup of a compact Lie group G, and that, in the appropriate Lie group sense [16, 1.3], P is not p-stubborn. If $W(P)$ is finite, Theorem 6.9 applies to $W(P)$ and again provides a contractibility statement. If $W(P)$

is positive-dimensional, an analogous theorem applies (cf. [16, 2.11]), but provides only $\mathbf{F}_p$-acyclicity. This difference propagates through the theory and accounts for the fact that p-stubborn decomposition theorems for general compact Lie groups are slightly weaker than the corresponding ones for finite groups.

Proof of 7.1. The proof is by downward induction on the size of the subgroup P. We have to check that the result is true if P is as large as possible, i.e., if P is a Sylow p-subgroup of G. In this case, though, P is p-stubborn, the poset $\mathbf{S}_{\mathcal{C}}(P)$ has only one element, and $|\mathbf{S}_{\mathcal{C}}(P)|$ is a single point.

Let $P \subset G$ be a p-subgroup of G and assume that for all p-subgroups Q of larger order, $|\mathbf{S}_{\mathcal{C}}(Q)|$ is weakly contractible. We will show that $|\mathbf{S}_{\mathcal{C}}(P)|$ is weakly contractible. If $P \in \mathcal{C}$, then P itself is a minimal element of $\mathbf{S}_{\mathcal{C}}(P)$, so $|\mathbf{S}_{\mathcal{C}}(P)|$ is contractible (see 2.6) and we are done. Assume then that $P \notin \mathcal{C}$.

Let $X = \mathrm{E}_{\mathcal{C}}$. According to 2.14 it is enough to show that the space X^P is weakly contractible. Let N be the normalizer $N_G(P)$ and W the quotient group N/P. The group W acts on X^P in a natural way and the isotropy subgroups of this action are (4.4) of the form $(Q \cap N)/P$ for subgroups $Q \in \mathcal{C}$ such that $Q \supset P$. These isotropy subgroups are p-subgroups of W and in fact *nontrivial* p-subgroups of W: to see this recall that $P \notin \mathcal{C}$, so that if $Q \in \mathcal{C}$ and $Q \supset P$, then $Q \supsetneq P$ and, by 5.4, $Q \cap N \supsetneq P$. Let $\bar{Q}$ be a nontrivial p-subgroup of W and $Q \subset N$ its preimage. By the inductive hypothesis (and 2.14) the fixed point space $(X^P)^{\bar{Q}} = X^Q$ is weakly contractible. An application of 4.2 to the group W and the W-space X^P shows that X^P is weakly W-equivalent to $|\mathbf{S}_{\mathcal{C}'}|$, where $\mathcal{C}'$ is the collection of all nontrivial p-subgroups of W. Since P is not p-stubborn, W has a nontrivial normal p-subgroup and $|\mathbf{S}_{\mathcal{C}'}|$ is contractible (6.9); this proves that X^P is weakly contractible. □

Remark. The main idea above is lifted from an argument of Quillen [2, 6.9.2] which identifies up to homotopy the simplicial complex associated to the poset of all p-subgroups of G which properly contain a given p-subgroup. This argument was reproduced in a different context by Jackowski-McClure-Oliver [16, 5.4].

§8. p-CENTRIC COLLECTIONS

In this section we prove generalizations of the results described in 1.19 and 1.20. As usual, G is a fixed finite group. Suppose that $H \subset G$ is a subgroup and that M is a G-module. Let $C_G(H, M)$ denote the subgroup of G consisting of elements which both centralize H and act trivially on M.

8.1 Definition. Let M be a module over $\mathbf{F}_p[G]$. A p-subgroup P of G is said to be *M-centric* if $C_G(P, M)$ is the product of a subgroup of P and a finite group of order prime to p. Equivalently, P is M-centric if $C_G(P, M) \cap P$ is a Sylow p-subgroup of $C_G(P, M)$.

8.2 Definition. Let M be a module over $\mathbf{F}_p[G]$. A collection $\mathcal{C}$ of p-subgroups of G is said to be *M-admissible* if

(1) $\mathcal{C}$ contains all M-centric subgroups, and
(2) $\mathcal{C}$ is closed under the process of passing to p-supergroups: if $P \in \mathcal{C}$ and Q is a p-subgroup of G with $Q \supset P$ then $Q \in \mathcal{C}$.

8.3 Theorem. *Let M be a module over $\mathbf{F}_p[G]$ and $\mathcal{C}$ a collection of p-subgroups of G which is M-admissible. Then $\mathcal{C}$ is M-ample.*

Remark. We can recover 1.19 from 8.3 by observing that

(1) the collection of all M-centric subgroups of G is M-admissible, and
(2) a p-subgroup P is M-centric for $M = \mathbf{F}_p$ if and only if P is p-centric in the sense of 1.19.

The following properties of the homotopy orbit space construction are well known.

8.4 Lemma. *Suppose that that $K \subset G$ is a subgroup and that X is a G-space. Then there is a natural weak equivalence $X_{\mathrm{h}K} \simeq (G \times_K X)_{\mathrm{h}G}$.*

Proof. $(G \times_K X)_{\mathrm{h}G}$ can be identified with $(\mathrm{E}G \times X)/K$. The natural map $\mathrm{E}K \to \mathrm{E}G$ is a map between contractible spaces on which K acts freely, and so induces a weak equivalence $(\mathrm{E}K \times X)/K \to (\mathrm{E}G \times X)/G$. □

To avoid clutter in the next statement, we define a *proxy action* of a group W on a space X to be some associated space X', a weak equivalence $X \to X'$, and a (genuine) action of W on X'. Given such a proxy action, $X_{\mathrm{h}W}$ stands for $X'_{\mathrm{h}W}$.

8.5 Lemma. *Suppose that $K \subset G$ is a normal subgroup with quotient group $W = G/K$, and that X is a G-space. Then there is a natural proxy action of W on $X_{\mathrm{h}K}$ and a weak equivalence*

$$(X_{\mathrm{h}K})_{\mathrm{h}W} \simeq X_{\mathrm{h}G} \ .$$

If K acts trivially on X, then $X_{\mathrm{h}K}$ is $\mathrm{B}K \times X$ and the proxy action of W on $\mathrm{B}K \times X$ is a diagonal one.

Proof. The map

$$(\mathrm{E}W \times (\mathrm{E}G \times X)/K)/W \to ((\mathrm{E}G \times X)/K)/W = (\mathrm{E}G \times X)/G$$

is a weak equivalence because W acts freely on $(\mathrm{E}G \times X)/K$. This gives an equivalence

$$((\mathrm{E}G \times X)/K)_{\mathrm{h}W} \simeq X_{\mathrm{h}G} \, .$$

As in the proof above, $(\mathrm{E}G \times X)/K$ is weakly equivalent in a natural way to $X_{\mathrm{h}K}$. The last statement is clear. □

8.6 Lemma. *Let Y be a G-space, $X \subset Y$ a G-subspace, $H \subset G$ a subgroup, and $N = N_G(H)$ its normalizer. Suppose that all of the simplices of Y which are not in X have isotropy subgroup conjugate to H. Then there is a (homotopy) pushout diagram of G-spaces:*

$$\begin{array}{ccc} G \times_N X^H & \longrightarrow & X \\ \downarrow & & \downarrow \\ G \times_N Y^H & \longrightarrow & Y \end{array} \qquad (8.7)$$

Proof. This amounts to the observation that all of the simplices of Y which are not in X lie in the G-orbit of Y^H; the part of this orbit which lies in X is the orbit of X^H. The diagram is a homotopy pushout diagram because the left vertical arrow is the inclusion of a G-subcomplex. □

Proof of 8.3. The proof is by downward induction on the size of $\mathcal{C}$. The statement is clearly true (by 1.12) if $\mathcal{C}$ is the M-admissible collection of all p-subgroups of G. Suppose then that $\mathcal{C}$ is M-admissible, $\mathcal{C}$ does not contain all p-subgroups of G, and that $\mathcal{C}'$ is M-ample for all M-admissible collections of subgroups with $\mathcal{C}' \supset \mathcal{C}$. Let $P \subset G$ be a p-subgroup of G which is maximal with respect to the property that $P \notin \mathcal{C}$, and let $\mathcal{C}'$ be the union of $\mathcal{C}$ with the set of all conjugates of P. Clearly $\mathcal{C}'$ is M-admissible, and hence by induction M-ample. Since $\mathcal{C}'$ contains all M-centric subgroups of G, P is *not* M-centric.

Let $X = \mathrm{E}_{\mathcal{C}}$ and $Y = \mathrm{E}_{\mathcal{C}'}$ (see 2.12), so that X can be considered as a subspace of Y. By induction and 2.12 the natural map $Y_{\mathrm{h}G} \to \mathrm{B}G$ induces an isomorphism on $\mathrm{H}_*(\text{–}; M)$; we must show that the same holds when Y is replaced by X. This amounts to showing the map $X_{\mathrm{h}G} \to Y_{\mathrm{h}G}$ induces an isomorphism on $\mathrm{H}_*(\text{–}; M)$.

Let $N = N_G(P)$. The simplices of Y which do not lie in X all have isotropy subgroup conjugate to P (4.5), and so by 8.6 there is a homotopy pushout diagram:

$$\begin{array}{ccc} G \times_N X^P & \longrightarrow & X \\ \downarrow & & \downarrow \\ G \times_N Y^P & \longrightarrow & Y \end{array} .$$

We are interested in the spaces in the left hand column. By 2.15 the space Y^P is contractible. Let $W = N/P$. The group W acts on X^P and, as explained in the proof of 7.1, Proposition 5.4 guarantees that the isotropy subgroups of this action are nontrivial p-subgroups of W. Let $\bar{Q}$ be a nontrivial p-subgroup of W and $Q \subset N$ its preimage. Since $Q \in \mathcal{C}$, it follows from 2.15 that $(X^P)^{\bar{Q}} = X^Q$ is contractible. Let $|\mathbf{S}_{\mathcal{P}(W)}|$ denote as usual the nerve of the poset $\mathcal{P}(W)$ of nontrivial p-subgroups of W. By 4.2 and 2.12, X^P is weakly W-equivalent to $|\mathbf{S}_{\mathcal{P}(W)}|$.

In the light of 8.4 and 8.5, applying the homotopy orbit space construction to 8.7 gives a homotopy pushout diagram of the following form:

(8.8)
$$\begin{array}{ccc} (\mathrm{B}P \times |\mathbf{S}_{\mathcal{P}(W)}|)_{\mathrm{h}W} & \longrightarrow & X_{\mathrm{h}G} \\ f\downarrow & & \downarrow \\ (\mathrm{B}P)_{\mathrm{h}W} & \longrightarrow & Y_{\mathrm{h}G} \end{array} .$$

Here we have used $Y^P \simeq *$ to give $(Y^P)_{\mathrm{h}P} \simeq \mathrm{B}P$. The action of W on $\mathrm{B}P$ in this diagram is easily seen to correspond to the conjugation action of W on P (via outer automorphisms). Since P is not M-centric, there is an element of order p in W which acts trivially on $\mathrm{H}_*(\mathrm{B}P; M)$ (see 8.9). By 6.7 the left vertical arrow in 8.8 induces an isomorphism on $\mathrm{H}_*(\text{–}; M)$, and so by Mayer-Vietoris the right vertical arrow does too. □

8.9 Remark. We continue to use the notation in the proof above. The reader may be uneasy about the assertion that the fact that P is not M-centric implies that there is an element of order p in W which acts trivially on $\mathrm{H}_*(\mathrm{B}P;M)$. We will show how to check this. Let **Mod** be the category whose objects are pairs (K,A), where K is a group and A is a K-module. A map $(K,A)\to(K',A')$ is a pair (u,v), where $u:K\to K'$ is a group homomorphism and $v:A\to A'$ is an abelian group map such that $v(xa)=u(x)v(a)$, $x\in K$, $a\in A$. Homology gives a functor $(K,A)\mapsto \mathrm{H}_*(\mathrm{B}K;A)$ from **Mod** to graded abelian groups. For any element $y\in K$ let $(c_y,y)=(u,v)$ be the automorphism of (K,A) for which $u(x)=yxy^{-1}$ and $v(a)=ya$. The usual argument that inner automorphisms of K act trivially on $\mathrm{H}_*(\mathrm{B}K;\mathbf{F}_p)$ generalizes to show that the automorphism (c_y,y) of (K,A) acts trivially on $\mathrm{H}_*(K;A)$. In the situation of the proof above, N acts on the object (P,M) of **Mod** by similar automorphisms (c_y,y), $y\in N$, and it follows that the induced action of N on $\mathrm{H}_*(\mathrm{B}P;M)$ factors through an action of $W=N/P$ on these homology groups. We will show that it is this action of W on $\mathrm{H}_*(\mathrm{B}P;M)$ which enters into 8.8. It is then clear that the image of $C_G(P,M)$ in W has order divisible by p (because P is not M-centric), and acts trivially on $\mathrm{H}_*(\mathrm{B}P;M)$.

Write $\otimes_P$ for tensor product over $\mathbf{F}_p[P]$, and let C_*X denote the $\mathbf{F}_p$-chain complex of a simplicial set X. Consider the following diagram of chain complexes

$$C_*\,\mathrm{E}P\otimes_P M \leftarrow C_*\,\mathrm{E}P\otimes_P (C_*\,\mathrm{E}N\otimes_{\mathbf{F}_p} M)\to \mathbf{F}_p\otimes_P(C_*\,\mathrm{E}N\otimes_{\mathbf{F}_p} M)\ .$$

The action of P on $C_*\,\mathrm{E}N\otimes_{\mathbf{F}_p} M$ is the diagonal one. Since $C_*\,\mathrm{E}P$ and $C_*\,\mathrm{E}N$ are free resolutions over $\mathbf{F}_p[P]$ of the trivial module $\mathbf{F}_p$, the maps in this diagram, which are obtained from the augmentations $C_*\,\mathrm{E}P\to\mathbf{F}_p$ and $C_*\,\mathrm{E}N\to\mathbf{F}_p$, induce isomorphisms on homology. Pick $x\in N$. Let c denote simultaneously the automorphism of P given by conjugation with x and the automorphism of $C_*\,\mathrm{E}P$ induced by this conjugation. Let ℓ denote the automorphisms of M and of $C_*\,\mathrm{E}N$ given by left multiplication by x. The group N acts compatibly on the three chain complexes above, with x acting respectively by $c\otimes_c\ell$, $c\otimes_c(\ell\otimes_1\ell)$ and $1\otimes_c(\ell\otimes_1\ell)$. The proof of 8.5 shows that it is the action of N on the right hand chain complex which induces the action of W on $\mathrm{H}_*(\mathrm{B}P;M)$ figuring in 8.8. The action of N on the left hand chain complex is the well-behaved one discussed in the preceding paragraph. The diagram shows how to identify these two actions with one another.

We will finish the section by proving the statement in 1.20. We leave it to the reader to formulate and prove a generalization involving a G-module M and an appropriate notion of M-admissible collection.

8.10 Proof of 1.20. Let $\mathcal{C}'$ be the collection of all p-centric subgroups of G, and $\mathcal{C}\subset\mathcal{C}'$ the collection of subgroups which in addition are p-stubborn. By 2.12 and 8.3, it is enough to show that the natural map $\mathrm{E}_{\mathcal{C}}\to\mathrm{E}_{\mathcal{C}'}$ is a weak equivalence. By 4.4 and 4.1, this will follow if we can prove that for all $P\in\mathcal{C}'$ the map $(\mathrm{E}_{\mathcal{C}})^P\to(\mathrm{E}_{\mathcal{C}'})^P$ is a weak equivalence. The target of this map is weakly contractible (2.15), so this amounts to showing that for all $P\in\mathcal{C}'$, $(\mathrm{E}_{\mathcal{C}})^P$ is weakly contractible.

The proof is by downward induction on the size of P. (Note that $\mathcal{C}'$ is closed under passage to p-supergroups.) As in the proof of 7.1, the result is clear if P is of maximal size, i.e., if P is a Sylow p-subgroup of G. Suppose then that $P\in\mathcal{C}'$ and that $(\mathrm{E}_{\mathcal{C}})^Q$ is weakly contractible for all p-subgroups Q of G which properly contain P. If $P\in\mathcal{C}$ then $(\mathrm{E}_{\mathcal{C}})^Q$ is contractible by 2.15. Otherwise P is not p-stubborn, and the argument at the end of the proof of 7.1 shows that $(\mathrm{E}_{\mathcal{C}})^P$ is weakly contractible. □

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University of Notre Dame, Notre Dame, Indiana 46556

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E-mail address: dwyer.1@nd.edu