# SHARP HOMOLOGY DECOMPOSITIONS FOR CLASSIFYING SPACES OF FINITE GROUPS

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## §1. Introduction

Let $G$ be a finite group and $p$ a prime number. A *homology decomposition* for the classifying space $\mathrm{B}G$ is a way of building $\mathrm{B}G$ up to mod $p$ homology out of classifying spaces of subgroups of $G$. More precisely, such a decomposition consists of a mod $p$ homology isomorphism

$$\text{(1.1)} \qquad \operatorname{hocolim} F = \operatorname{hocolim}^{\mathbf{D}} F \xrightarrow[p]{\sim} \mathrm{B}G \ ,$$

where $\mathbf{D}$ is some small category, $F$ is a functor from $\mathbf{D}$ to the category of spaces, and, for each object $d \in \mathbf{D}$, $F(d)$ has the homotopy type of $\mathrm{B}H$ for some subgroup $H$ of $G$. The operator "hocolim" is the homotopy colimit in the sense of Bousfield and Kan [4], which amounts to a homotopy invariant method of gluing together the values of the functor $F$ according to the pattern of the maps in $\mathbf{D}$. There is a systematic study of some of the more common homology decompositions in [6] (see 1.3 below). Here we go further along the same lines. Bousfield and Kan associate to 1.1 a first quadrant homology spectral sequence [4, XII.5.7]

$$\text{(1.2)} \qquad E^2_{i,j} = \operatorname{colim}_i \mathrm{H}_j(F;\mathbf{F}_p) \Rightarrow \mathrm{H}_{i+j}(\mathrm{B}G;\mathbf{F}_p) \ .$$

In the description of this $E^2$-term, $\operatorname{colim}_i$ stands for the $i$'th left derived functor of the colimit construction on the category of functors from $\mathbf{D}$ to abelian groups. Call a homology decomposition *sharp* if its spectral sequence *collapses onto the vertical axis* in the sense that $E^2_{i,j} = 0$ for $i > 0$. The appeal of a sharp homology decomposition is that it induces an isomorphism

$$\operatorname{colim} \mathrm{H}_*(F;\mathbf{F}_p) = \operatorname{colim}_0 \mathrm{H}_*(F;\mathbf{F}_p) \xrightarrow{\cong} \mathrm{H}_*(\mathrm{B}G;\mathbf{F}_p)$$

and so gives a closed formula for the homology of $\mathrm{B}G$ in terms of the homology of classifying spaces of subgroups of $G$. In this paper we do our best to determine which of the decompositions from [6] are sharp.

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**1.3 Homology decompositions.** Recall from [6] that *a collection* $\mathcal{C}$ of subgroups of $G$ is by definition a set of subgroups of $G$ which is closed under conjugation, in the sense that if $H \in \mathcal{C}$ and $g \in G$ then $gHg^{-1} \in \mathcal{C}$. Such a collection $\mathcal{C}$ is a poset under the inclusion relation between subgroups. Let $K_{\mathcal{C}}$ denote the associated simplicial complex [2, 6.2]: the vertices of $K_{\mathcal{C}}$ are the elements of $\mathcal{C}$, and the $n$-simplices ($n \geq 1$) are the subsets of $\mathcal{C}$ of cardinality $(n+1)$ which are totally ordered by inclusion. The action of $G$ on $\mathcal{C}$ given by conjugating one subgroup to another induces an action of $G$ on $K_{\mathcal{C}}$.

Let $\mathrm{E}G$ be the universal cover of $\mathrm{B}G$; if $X$ is a $G$-space, the *Borel construction* or *homotopy orbit space* $X_{\mathrm{h}G}$ is defined to be the quotient $(\mathrm{E}G \times X)/G$. Let $*$ denote the one-point space with trivial $G$ action.

*1.4 Definition.* A collection $\mathcal{C}$ of subgroups of $G$ is said to be *ample* if the map $K_{\mathcal{C}} \to *$ induces a mod $p$ homology isomorphism $(K_{\mathcal{C}})_{\mathrm{h}G} \to (*)_{\mathrm{h}G} = \mathrm{B}G$.

Associated to any ample collection $\mathcal{C}$ are three homology decompositions of $G$, called the *centralizer decomposition*, the *subgroup* decomposition, and the *normalizer decomposition*. The names signify that the constituents of the three homology decomposition formulas are of the form $\mathrm{B}H$, where $H \subset G$ is respectively, the centralizer in $G$ of some element of $\mathcal{C}$, an element of $\mathcal{C}$ itself, or an intersection in $G$ of normalizers of elements of $\mathcal{C}$. We will describe these three decompositions in a general way; there are more details in [6] or at the end of §3.

*1.5 The centralizer decomposition.* The *$\mathcal{C}$-conjugacy category* $\mathbf{A}_{\mathcal{C}}$ is the category in which the objects are pairs $(H, \Sigma)$, where $H$ is a group and $\Sigma$ is a conjugacy class of monomorphisms $i : H \to G$ with $i(H) \in \mathcal{C}$. A morphism $(H, \Sigma) \to (H', \Sigma')$ is a group homomorphism $j : H \to H'$ which under composition carries $\Sigma'$ into $\Sigma$. If $H \subset G$ is a subgroup, let $C_G(H)$ denote the centralizer of $H$ in $G$. There is a natural functor

$$\alpha_{\mathcal{C}} : (\mathbf{A}_{\mathcal{C}})^{\mathrm{op}} \to \mathbf{Spaces}$$

which assigns to each object $(H, \Sigma)$ a space which has the homotopy type of $\mathrm{B}C_G(i(H))$ for any $i \in \Sigma$. There is also a map

$$a_{\mathcal{C}} : \operatorname{hocolim} \alpha_{\mathcal{C}} \to \mathrm{B}G$$

which is a mod $p$ homology isomorphism if and only if $\mathcal{C}$ is ample.

*1.6 The subgroup decomposition.* The *$\mathcal{C}$-orbit category* $\mathbf{O}_{\mathcal{C}}$ is the category whose objects are the $G$-sets $G/H$, $H \in \mathcal{C}$, and whose morphisms are $G$-maps. There is an inclusion functor $\mathcal{J}$ from $\mathbf{O}_{\mathcal{C}}$ to the category of $G$-spaces. Composing $\mathcal{J}$ with the Borel construction $(-)_{\mathrm{h}G}$ gives a functor

$$\beta_{\mathcal{C}} : \mathbf{O}_{\mathcal{C}} \to \mathbf{Spaces}$$

whose value $(G/H)_{\mathrm{h}G}$ at an object $G/H$ has the homotopy type of $\mathrm{B}H$. The natural maps $\beta_{\mathcal{C}}(G/H) \to \mathrm{B}G$ are compatible as $G/H$ varies and induce a map

$$b_{\mathcal{C}} : \operatorname{hocolim} \beta_{\mathcal{C}} \to \mathrm{B}G\ .$$

This map is a mod $p$ homology isomorphism if and only if $\mathcal{C}$ is ample.

*1.7 The normalizer decomposition.* Let $\bar{s}\mathbf{S}_{\mathcal{C}}$ be the category of "orbit simplices" for the action of $G$ on $K_{\mathcal{C}}$. The objects of $\bar{s}\mathbf{S}_{\mathcal{C}}$ are the orbits $\bar{\sigma}$ of the action of $G$ on the simplices of $K_{\mathcal{C}}$, and there is one morphism $\bar{\sigma} \to \bar{\sigma}'$ if for some simplices $\sigma \in \bar{\sigma}$ and $\sigma' \in \bar{\sigma}'$, $\sigma'$ is a face of $\sigma$. If $H$ is a subgroup of $G$, let $N_G(H)$ denote the normalizer of $H$ in $G$. There is a natural functor

$$\delta_{\mathcal{C}} : \bar{s}\mathbf{S}_{\mathcal{C}} \to \mathbf{Spaces}$$

which assigns to the orbit of a simplex $\sigma = \{H_i\}$ a space which has the homotopy type of $\mathrm{B}(\cap_i N_G(H_i))$ There is also a map

$$d_{\mathcal{C}} : \operatorname{hocolim} \delta_{\mathcal{C}} \to \mathrm{B}G$$

which is a mod $p$ homology isomorphism if and only if $\mathcal{C}$ is ample.

**Sharp homology decompositions.** Suppose that $\mathcal{C}$ is an ample collection of subgroups of $G$. Some of the homology decompositions associated above to $\mathcal{C}$ might be sharp, and others might not be. We need some terminology to describe this.

*1.8 Definition.* A collection $\mathcal{C}$ of subgroups of $G$ is said to be *centralizer-sharp* (resp. *subgroup-sharp*, resp. *normalizer-sharp*) if $\mathcal{C}$ is ample and the centralizer functor $\alpha_{\mathcal{C}}$ (resp. the subgroup functor $\beta_{\mathcal{C}}$, resp. the normalizer functor $\delta_{\mathcal{C}}$) gives a sharp homology decomposition of $\mathrm{B}G$.

In this paper we determine the sharpness properties of a number of common collections. Here are some examples.

*Trivial examples.* Suppose that $\mathcal{C}$ contains only the trivial subgroup of $G$. Then $\mathcal{C}$ is both centralizer-sharp and normalizer-sharp, since both the centralizer and normalizer decomposition diagrams reduce to the trivial diagram with $\mathrm{B}G$ as its only constituent. By inspection, $\mathcal{C}$ is subgroup-sharp if and only if the mod $p$ homology of $\mathrm{B}G$ is trivial, which is the case if and only if the order of $G$ is relatively prime to $p$.

Suppose that $\mathcal{C}$ contains only $G$ itself. Then $\mathcal{C}$ is both subgroup-sharp and normalizer-sharp, since both the subgroup and normalizer decomposition diagrams reduce to the trivial diagram with $\mathrm{B}G$ as its only constituent. Let $Z$ be the center of $G$. The category $\mathbf{A}_{\mathcal{C}}$ has one object whose group of self-maps is $G/Z$ and the functor $\alpha_{\mathcal{C}}$ assigns to this object the space $\mathrm{B}Z$. The Bousfield-Kan homology spectral sequence for the associated homotopy colimit is the Lyndon-Hochschild-Serre spectral sequence of the group extension $Z \to G \to G/Z$. From this it is not hard to see that $\mathcal{C}$ is centralizer-sharp if and only if $Z$ contains a Sylow $p$-subgroup of $G$; this is the case if and only if $G$ is the product of an abelian $p$-group and a group of order prime to $p$.

*1.9 Nontrivial p-subgroups.* Let $\mathcal{C}$ be the collection of all nontrivial $p$-subgroups of $G$. As long as $p$ divides the order of $G$, $\mathcal{C}$ is centralizer-sharp, subgroup-sharp, and normalizer-sharp. This is proved in §7.

*1.10 Elementary abelian subgroups.* Recall that a finite abelian group is said to be an *elementary abelian p-group* if it a module over $\mathbf{F}_p$. Let $\mathcal{C}$ be the collection of all nontrivial elementary abelian $p$-subgroups of $G$. If $p$ divides the order of $G$, then $\mathcal{C}$ is both centralizer-sharp and normalizer-sharp; it cannot possibly be subgroup-sharp unless the mod $p$ cohomology of $G$ is detected on elementary abelian $p$-subgroups. We will discuss this and related collections in §8.

*1.11 p-centric subgroups.* Recall that a $p$-subgroup $P$ of $G$ is said to be *p-centric* if the center of $P$ is a Sylow $p$-subgroup of $C_G(P)$. Let $\mathcal{C}$ be the collection of all $p$-centric subgroups of $G$. Then $\mathcal{C}$ is subgroup-sharp (10.3).

*1.12 Subgroups both p-centric and p-stubborn..* Recall that a $p$-subgroup $P$ of $G$ is said to be *p-stubborn* if $N_G(P)/P$ has no nontrivial normal $p$-subgroups. Let $\mathcal{C}$ be the collection of all subgroups of $G$ which are both $p$-centric and $p$-stubborn. Then $\mathcal{C}$ is subgroup-sharp (10.4).

*1.13 A slight extension.* Suppose that $M$ is a module over $\mathbf{F}_p[G]$, so that $M$ gives rise in the usual way to a local coefficient system on $\mathrm{B}G$. We will say a a collection $\mathcal{C}$ of subgroups of $G$ is *ample for M* if the map $(K_\mathcal{C})_{\mathrm{h}G} \to (*)_{\mathrm{h}G} = \mathrm{B}G$ induced by $K_\mathcal{C} \to *$ gives an isomorphism on the local coefficient homology groups $\mathrm{H}_*(–; M)$. For such a collection $\mathcal{C}$ there are local coefficient spectral sequences

$$E^2_{i,j} = \mathrm{colim}_i\, \mathrm{H}_j(F; M) \Rightarrow \mathrm{H}_{i+j}(\mathrm{B}G; M)\ .$$

where $F = \alpha_\mathcal{C}$, $\beta_\mathcal{C}$ or $\delta_\mathcal{C}$. We will say that $\mathcal{C}$ is centralizer-sharp (resp. subgroup-sharp, resp. normalizer-sharp) *for M* if the spectral sequence for $\alpha_\mathcal{C}$ (resp. $\beta_\mathcal{C}$, resp. $\delta_\mathcal{C}$) collapses onto the vertical axis. In some of the examples described above, we will look at this more general notion of sharpness.

*Organization of the paper.* Section 2 discusses the "isotropy spectral sequence" associated to a $G$-space, and §3 shows how to identify the Bousfield-Kan spectral sequences of the above homology decompositions as isotropy spectral sequences. Section 4 gives a homological interpretation of the $E^2$-term of the isotropy spectral sequence, and the next two sections develop techniques for showing that this $E^2$-term has appropriate vanishing properties. The most interesting technique is probably the method of discarded orbits (6.8). The final sections apply these techniques to study the homology decompositions associated to the collection of all non-trivial $p$-subgroups (§7), certain collections of elementary abelian $p$-subgroups (§8), and certain collections of subgroups which are $p$-centric and/or $p$-stubborn (§10). There is a pause in §9 to look at the effect on a collection of "pruning" (i.e. removing) a single conjugacy class of subgroups.

This paper is mostly independent of [6], although we sometimes refer to a result or argument which it did not seem worthwhile to reproduce.

*Related work.* There is a large literature about homology decompositions. For instance, see [1, V.3] or [15] for the normalizer decomposition associated to the collection of all nontrivial $p$-subgroups, [8] or [3] for centralizer decompositions associated to collections of nontrivial elementary abelian $p$-subgroups, and [9] for the subgroup decomposition associated to the collection of $p$-stubborn subgroups. There is significant overlap between this paper and [14], although [14] takes a different point of view. The reader can also find a great deal of interesting related material in [11], [12] and [13].

*Notation and terminology.* This paper is written with the convention that "space" means "simplicial set" [4, VIII]. A map between spaces is said to be an *equivalence* or a *weak equivalence* if its geometric realization gives a homotopy equivalence between topological spaces [10]. The letter $G$ always stands for some finite group. If $x$ belongs to a $G$-set, then $G_x$ is the isotropy subgroup of $x$. If $M$ is a $G$-module, then $\mathrm{H}_*(G;M) = \mathrm{H}_*(\mathrm{B}G;M)$ stands for the group homology of $G$ with coefficients in $M$, or equivalently for the homology of $\mathrm{B}G$ with coefficients in the local system determined by $M$.

Throughout the paper $p$ is a fixed prime number and $\mathbf{F}_p$ is the field with $p$ elements. The vector space over $\mathbf{F}_p$ generated by a set $X$ is denoted $\mathbf{F}_p[X]$. The *nerve* of a category $\mathbf{D}$ is written $|\mathbf{D}|$ (cf. 3.3, [4, XI.2]).

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## §2. The isotropy spectral sequence

In this section we discuss a well-known spectral sequence, called the *isotropy spectral sequence*, associated to an action of $G$ on a space. This spectral sequence will be linked to homology decompositions in §3. The isotropy spectral sequence is a special case of the Leray spectral sequence.

**2.1 The Leray spectral sequence.** Recall that the *n-skeleton* $\mathrm{sk}_n B$ ($n \geq 0$) of a simplicial set $B$ is the subobject of $B$ generated by all simplices of dimension $\leq n$. Let $f : X \to B$ be a map of simplicial sets, and let $X_n$ denote $f^{-1}(\mathrm{sk}_n B)$. The *Leray spectral sequence* of $f$ is the (mod $p$) homology spectral sequence associated to the filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$$

of $X$; it is usually indexed so that $E^1_{i,j} = \mathrm{H}_{i+j}(X_i, X_{i-1}; \mathbf{F}_p)$. Since $X_n$ contains $\mathrm{sk}_n X$, $E^1_{i,j} = 0$ for $j < 0$ and this is a first quadrant, strongly convergent homology spectral sequence. In particular, the differential $d^r$ has bidegree $(-r, r-1)$. If $M$ is a local coefficient system on $X$ there is also a Leray spectral sequence with coefficients in $M$. Here are a few examples.

*Collapse map.* If $Y$ is a subspace of $X$, the $E^2$-term of the Leray spectral sequence of $f : X \to X/Y$ vanishes except for the groups $E^2_{0,0} = \mathrm{H}_0(X; \mathbf{F}_p)$, $E^2_{0,j} = \mathrm{H}_j(Y; \mathbf{F}_p)$ ($j > 0$) and $E^2_{i,0} = \mathrm{H}_i(X/Y; \mathbf{F}_p))$ ($i > 0$). The various differentials running from the horizontal axis to the vertical axis give the connecting homomorphisms in the long exact homology sequence of the pair $(X, Y)$.

*Fibration.* If $f : X \to Y$ is a fibration, the Leray spectral sequence of $f$ can be identified with the Serre spectral sequence of $f$.

*2.2 Homotopy colimit.* Suppose that $\mathbf{D}$ is some small category and $F : \mathbf{D} \to \mathbf{Spaces}$ is a functor. The unique natural transformation from $F$ to the constant functor $*$ with value the one-point space induces a map

$$f : \mathrm{hocolim}\, F \to \mathrm{hocolim} * \cong |\mathbf{D}|$$

The Leray spectral sequence of $f$ can be identified from $E^2$ onwards with the Bousfield-Kan spectral sequence [4, XII.5.7]

$$E^2_{i,j} = \mathrm{colim}_i\, \mathrm{H}_j(F; \mathbf{F}_p) \Rightarrow \mathrm{H}_{i+j}(\mathrm{hocolim}\, F; \mathbf{F}_p)\ .$$

This can be seen by inspecting the definitions of the two spectral sequences.

**2.3 The isotropy spectral sequence.** We will be interested in a specific Leray spectral sequence associated to a $G$-space $X$.

*Definition.* The *isotropy spectral sequence* of a $G$-space $X$ is the Leray spectral sequence of the map

$$f : X_{\mathrm{h}G} = (\mathrm{E}G \times X)/G \to X/G$$

More generally, suppose that $M$ is a module over $\mathbf{F}_p[G]$. The isotropy spectral sequence of $X$ with coefficients in $M$ is the Leray spectral sequence of $f$ with coefficients in the local system on $X_{\mathrm{h}G}$ derived from $M$.

*2.4 Remark.* Consider the diagram

$$\begin{array}{ccccc} & & \{\mathrm{B}G_x\} & & \\ & & \downarrow & & \\ X & \longrightarrow & X_{\mathrm{h}G} & \xrightarrow{q} & (*)_{\mathrm{h}G} = \mathrm{B}G \\ & & f\downarrow & & \\ & & X/G & & \end{array} \tag{2.5}$$

Suppose that $M$ is a module over $\mathbf{F}_p[G]$. The horizontal lineup forms a fibration, and the Leray spectral sequence of $q$ is the usual Serre spectral sequence converging to $\mathrm{H}_*(X_{\mathrm{h}G}; M)$. The vertical lineup is meant to suggest that the geometric fibres of $f$ in general differ from point to point, and that the fibre over a simplex $\bar{x} \in X/G$ can be identified up to homotopy with $\mathrm{B}G_x$, where $G_x$ is the isotropy subgroup of a simplex $x \in X$ above $\bar{x}$. The Leray spectral sequence of $f$ with coefficients in $M$ is the isotropy spectral sequence of $X$ with coefficients in $M$, and it too converges to $\mathrm{H}_*(X_{\mathrm{h}G}; M)$. These spectral sequences are special cases of the hyperhomology spectral sequences from [5, XVII]. Let $C^{\mathrm{n}}_*Y$ denote the normalized $\mathbf{F}_p$-chain complex of a simplicial set $Y$. For each $i \geq 0$, $C^{\mathrm{n}}_iY$ is the quotient $\mathbf{F}_p[Y_i]/(s_0\mathbf{F}_p[Y_{i-1}] + \cdots s_{i-1}\mathbf{F}_p[Y_{i-1}])$; the boundary map $C^{\mathrm{n}}_iY \to C^{\mathrm{n}}_{i-1}Y$ is obtained by taking an alternating sum of the face maps $d_k : \mathbf{F}_p[Y_i] \to \mathbf{F}_p[Y_{i-1}]$, $0 \leq k \leq i$ and observing that this homomorphism passes to the necessary quotient groups. By the Künneth theorem, $C^{\mathrm{n}}_*(X_{\mathrm{h}G})$ is naturally chain homotopy equivalent to $C^{\mathrm{n}}_*(\mathrm{E}G) \otimes_{\mathbf{F}_p[G]} C^{\mathrm{n}}_*X$. This tensor product has two filtrations, $\{F_n\}$ and $\{F'_n\}$, given by

$$F_n = C^{\mathrm{n}}_*(\mathrm{sk}_n\,\mathrm{E}G) \otimes_{\mathbf{F}_p[G]} C^{\mathrm{n}}_*X \quad \text{and} \quad F'_n = C^{\mathrm{n}}_*(\mathrm{E}G) \otimes_{\mathbf{F}_p[G]} C^{\mathrm{n}}_*(\mathrm{sk}_n\,X)\,.$$

From $E^2$-onwards, the spectral sequence associated to $\{F_n\}$ is the Serre spectral sequence of $X_{\mathrm{h}G} \to \mathrm{B}G$, and the spectral sequence associated to $\{F'_n\}$ is the isotropy spectral sequence of $X$.

*2.6 Examples.* If $G$ acts freely on $X$ then the map $f$ of 2.5 is an equivalence with contractible fibres, and the isotropy spectral sequence of $X$ collapses onto the horizontal axis. More generally, suppose that $K$ is a normal subgroup of $G$ which acts trivially on $X$, and that the quotient group $W = G/K$ acts freely on $X$. Then the map $f$ of 2.5 is a fibration with fibre $\mathrm{B}K$, and the isotropy spectral sequence of $X$ can be identified with the Serre spectral sequence of the fibration $\mathrm{B}K \to X_{\mathrm{h}G} \to X_{\mathrm{h}W}$.

*2.7 The $E^1$-term of the isotropy spectral sequence.* The $E^1$-term of the isotropy spectral sequence has $E^1_{i,j} = \mathrm{H}_j(G; C^{\mathrm{n}}_i X)$. For each $j \geq 0$ one can form a chain complex $\{\mathrm{H}_j(G; C^{\mathrm{n}}_* X), d\}$ which in dimension $i \geq 0$ contains the group $\mathrm{H}_j(G; C^{\mathrm{n}}_i X)$ and has boundary map $d$ induced by the boundary maps $C^{\mathrm{n}}_i X \to C^{\mathrm{n}}_{i-1} X$. The group in position $E^2_{i,j}$-term of the isotropy spectral sequence is then the $i$'th homology group of $\{\mathrm{H}_j(G; C^{\mathrm{n}}_* X), d\}$.

*2.8 An unnormalized description of the $E^2$-term.* Let $C_* Y$ denote the *unnormalized* chain complex of a simplicial set $Y$. For each $i \geq 0$, $C_i(Y)$ is the vector space $\mathbf{F}_p[Y_i]$, and the boundary map $C_i(Y) \to C_{i-1}(Y)$ is obtained by taking an alternating sum of the face maps $d_k : \mathbf{F}_p[Y_i] \to \mathbf{F}_p[Y_{i-1}]$, $0 \leq k \leq i$. By [10, 22.2], the kernel of the epimorphism $C_* Y \to C^{\mathrm{n}}_* Y$ is a natural summand of $C_* Y$ and has a natural chain contraction. Let $X$ as above be a $G$-space. For each $j \geq 0$ one can form a chain complex $\{\mathrm{H}_j(G; C_* X), d\}$ which in dimension $i \geq 0$ contains the group $\mathrm{H}_j(G; C_i X)$ and has boundary map $d$ induced by the boundary maps $C_i X \to C_{i-1} X$. It follows from the above remarks that the maps $C_* X \to C^{\mathrm{n}}_* X$ and $\{\mathrm{H}_j(G; C_* X), d\} \to \{\mathrm{H}_j(G; C^{\mathrm{n}}_* X), d\}$ $(j \geq 0)$ induce isomorphisms on homology. In particular, the group in position $E^2_{i,j}$ of the isotropy spectral sequence for $X$ is isomorphic to the $i$'th homology group of $\{\mathrm{H}_j(G; C_* X), d\}$.

More generally, if $M$ is an $\mathbf{F}_p[G]$-module, it is possible to form chain complexes $\{\mathrm{H}_j(G; M \otimes C_* X), d\}$ $(j \geq 0)$; here the tensor product is taken over $\mathbf{F}_p$ and the action of $G$ on $M \otimes C_i X$ is diagonal. The homology groups of these chain complexes form the $E^2$ term of the isotropy spectral sequence of $X$ with coefficients in $M$.

## §3. Homology decompositions and the isotropy spectral sequence

The goal of this section is to show that each of the homology decompositions of §1 is tied to an associated $G$-space $X$, in such a way that the Bousfield-Kan spectral sequence of the decomposition (1.2) is the isotropy spectral sequence of $X$ (2.3). This will allow us to work with the Bousfield-Kan spectral sequence by manipulating $G$-spaces.

*3.1 A general construction.* Let $\mathbf{D}$ be a small category and $\tilde{F}$ a functor from $\mathbf{D}$ to the category of transitive $G$-sets. Let $X$ be the $G$-space $\operatorname{hocolim} \tilde{F}$. The assumption that $\tilde{F}(d)$ is a transitive $G$-set means that there is an natural isomorphism

$$X/G = (\operatorname{hocolim} \tilde{F})/G = \operatorname{hocolim}(\tilde{F}/G) \cong \operatorname{hocolim} * = |\mathbf{D}| . \tag{3.2}$$

Let $F = \tilde{F}_{\mathrm{h}G}$. The values of $F$ have the homotopy type of $\mathrm{B}H$ for various subgroups $H$ of $G$; in fact, $F(d)$ has the homotopy type of $\mathrm{B}G_x$, where $G_x$ is the isotropy subgroup in $G$ of any element $x \in \tilde{F}(d)$. The natural maps $\tilde{F}(d) \to *$ induce maps $F(d) \to (*)_{\mathrm{h}G} = \mathrm{B}G$ which are compatible as $d$ varies and give a map $\operatorname{hocolim} F \to \mathrm{B}G$. If this map induces an isomorphism on mod $p$ homology, then $F$ provides a homology decomposition of $\mathrm{B}G$.

Parallel to 3.2 is an isomorphism

$$X_{\mathrm{h}G} = (\operatorname{hocolim} \tilde{F})_{\mathrm{h}G} \cong \operatorname{hocolim}(\tilde{F}_{\mathrm{h}G}) = \operatorname{hocolim} F$$

The natural map $\operatorname{hocolim} F \to |\mathbf{D}|$ from 2.2 corresponds to the map $f : X_{\mathrm{h}G} \to X/G$ from 2.3. It follows from 2.2 that the Bousfield-Kan spectral sequence of $F$ can be identified with the isotropy spectral sequence associated to the action of $G$ on $X$; this is true either with trivial coefficients $\mathbf{F}_p$ or with coefficients in a general $\mathbf{F}_p[G]$-module $M$.

*3.3 Remark.* We briefly recall that if $\mathbf{D}$ is a small category, the nerve $|\mathbf{D}|$ is the simplicial set in which the $n$-simplices ($n \geq 0$) are the chains

$$d_0 \xrightarrow{f_1} d_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} d_n$$

of composable morphisms in $\mathbf{D}$. The face maps $d_i : |\mathbf{D}|_n \to |\mathbf{D}|_{n-1}$ act by dropping $f_1$ if $i = 0$, composing $f_{i+1}$ with $f_i$ if $0 < i < n$, and dropping $f_n$ if $i = n$. The degeneracy maps $s_i : |\mathbf{D}|_n \to |\mathbf{D}|_{n+1}$ act by inserting appropriate identity morphisms. A functor between two categories induces a map between their nerves; a natural transformation between functors gives a homotopy between maps. If the group $H$ acts on $\mathbf{D}$, then the fixed point set of the induced action on $|\mathbf{D}|$ is the nerve of the fixed subcategory: $|\mathbf{D}|^H = |\mathbf{D}^H|$. One explanation for the usefulness of the nerve construction is that it is sometimes easier to manipulate functors and natural transformations than maps and homotopies.

Note by [6, 2.11] that the space $X = \operatorname{hocolim} F$ above is isomorphic to the nerve of the category $\mathbf{X}$ whose objects consist of pairs $(d, a)$ where $d$ is an object of $\mathbf{D}$ and $a \in F(d)$; a map $(d, a) \to (d', a')$ in $\mathbf{X}$ is a map $f : d \to d'$ in $\mathbf{D}$ such that $F(f)(a) = a'$. The action of $G$ on $X$ is induced by the action on $\mathbf{X}$ (via functors) given by $g \cdot (d, a) = (d, ga)$,

We now describe how the homology decompositions of 1.3 can be constructed by the method of 3.1 above. (In fact, *any* homology decomposition can be constructed in this way.) Let $\mathcal{C}$ be a collection of subgroups of $G$.

*3.4 Building the centralizer decomposition.* Let $\mathbf{A}_{\mathcal{C}}$ be the $\mathcal{C}$-conjugacy category (1.5) associated to $\mathcal{C}$, and $\tilde{\alpha}_{\mathcal{C}}$ the contravariant functor on $\alpha_{\mathcal{C}}$ which assigns to a pair $(H, \Sigma)$ the set $\Sigma$ itself, i.e., the set of all monomorphisms $H \to G$ contained in the conjugacy class $\Sigma$. The group $G$ acts transitively on $\tilde{\alpha}_{\mathcal{C}}(H, \Sigma)$ by conjugation. If $\mathcal{C}$ is ample the functor $\alpha_{\mathcal{C}} = (\tilde{\alpha}_{\mathcal{C}})_{\mathrm{h}G}$ gives as above the centralizer decomposition of $\mathrm{B}G$ associated to $\mathcal{C}$ [6, 3.1]. Let $X^{\alpha}_{\mathcal{C}}$ denote $\operatorname{hocolim} \tilde{\alpha}_{\mathcal{C}}$. By 3.1, the Bousfield-Kan homology spectral sequence associated to the centralizer decomposition is the isotropy spectral sequence (2.3) of the action of $G$ on $X^{\alpha}_{\mathcal{C}}$.

The space $X^{\alpha}_{\mathcal{C}}$ is the nerve of the category $\mathbf{X}^{\alpha}_{\mathcal{C}}$, whose objects consist of pairs $(H, i)$ where $H$ is a group and $i : H \to G$ is a monomorphism with $i(H) \in \mathcal{C}$. A morphism $(H, i) \to (H, i')$ is a group homomorphism $j : H \to H'$ with $i'j = i$. In order to make this a small category, the groups $H$ should be restricted to lie in some set which is large enough to include up to isomorphism all of the elements of $\mathcal{C}$. The action of $G$ on $X^{\alpha}_{\mathcal{C}}$ is induced by the action of $G$ on $\mathbf{X}^{\alpha}_{\mathcal{C}}$ given by $g \cdot (H, i) = (H, gig^{-1})$. The $n$-simplices of $X^{\alpha}_{\mathcal{C}}$ correspond to diagrams

$$H_0 \to H_2 \to \cdots \to H_n \to G \tag{3.5}$$

in which all of the maps are monomorphisms and all of the composite maps $H_i \to G$ have image contained in $\mathcal{C}$.

*3.6 Building the subgroup decomposition.* Let $\mathbf{O}_{\mathcal{C}}$ be the orbit category (1.6) associated to $\mathcal{C}$, and $\tilde{\beta}_{\mathcal{C}}$ the inclusion functor $\mathbf{O}_{\mathcal{C}}$ to the category of $G$-spaces. By construction, $G$ acts transitively on $\tilde{\beta}_{\mathcal{C}}(G/H)$. According to [6, 3.2], if $\mathcal{C}$ is ample the functor $\beta_{\mathcal{C}} = (\tilde{\beta}_{\mathcal{C}})_{\mathrm{h}G}$ gives a homology decomposition of $\mathrm{B}G$; this is the subgroup decomposition provided by $\mathcal{C}$. Let $X^{\beta}_{\mathcal{C}}$ denote $\operatorname{hocolim}\tilde{\beta}_{\mathcal{C}}$. By 3.1, the Bousfield-Kan homology spectral sequence associated to the subgroup decomposition is the isotropy spectral sequence of the action of $G$ on $X^{\beta}_{\mathcal{C}}$. In [6] the space $X^{\beta}_{\mathcal{C}}$ was denoted $\mathrm{E}_{\mathcal{C}}$.

The space $X^{\beta}_{\mathcal{C}}$ is the nerve of the category $\mathbf{X}^{\beta}_{\mathcal{C}}$ whose objects consist of pairs $(x, G/H)$, where $H \in \mathcal{C}$ and $x \in G/H$. A morphism $(x, G/H) \to (x', G/H')$ is a $G$-map $f : G/H \to G/H'$ with $f(x) = x'$. The action of $G$ on $X^{\beta}_{\mathcal{C}}$ is induced by the action of $G$ on $\mathbf{X}^{\beta}_{\mathcal{C}}$ given by $g \cdot (x, G/H) = (gx, G/H)$.

*3.7 Building the normalizer decomposition.* Let $\bar{\mathrm{s}}\mathbf{S}_{\mathcal{C}}$ be the category of orbit simplices (1.7) for the action of $G$ on the simplicial complex $K_{\mathcal{C}}$, and $\tilde{\delta}_{\mathcal{C}}$ the functor which assigns to a particular object $\bar{\sigma}$ of $\bar{\mathrm{s}}\mathbf{S}_{\mathcal{C}}$ the set $\bar{\sigma}$ itself, considered as a set of simplices in $K_{\mathcal{C}}$. By construction, the group $G$ acts transitively on $\tilde{\delta}_{\mathcal{C}}(\bar{\sigma})$. According to [6, 3.3], if $\mathcal{C}$ is ample the functor $\delta_{\mathcal{C}} = (\tilde{\delta}_{\mathcal{C}})_{\mathrm{h}G}$ gives a homology decomposition of $\mathrm{B}G$; this is the normalizer decomposition provided by $\mathcal{C}$. Let $\mathrm{sd}X^{\delta}_{\mathcal{C}}$ denote $\operatorname{hocolim}\tilde{\delta}_{\mathcal{C}}$. By 3.1, the Bousfield-Kan spectral sequence associated to the normalizer decomposition is the isotropy spectral sequence of the action of $G$ on $\mathrm{sd}X^{\delta}_{\mathcal{C}}$.

Let $X^{\delta}_{\mathcal{C}}$ be the nerve of the poset category $\mathbf{X}^{\delta}_{\mathcal{C}}$ whose objects are the subgroups $H$ of $G$ with $H \in \mathcal{C}$. In this category there is one morphism $H \to H'$ if $H \subset H'$, and no other morphisms. There is an action of $G$ on $X^{\delta}_{\mathcal{C}}$ induced by the action of $G$ on $\mathbf{X}^{\delta}_{\mathcal{C}}$ given by $g \cdot H = gHg^{-1}$. Since $\mathrm{sd}X^{\delta}_{\mathcal{C}}$ is just the barycentric subdivision of $X^{\delta}_{\mathcal{C}}$, the Bousfield-Kan spectral sequence associated to the normalizer decomposition can be identified with the isotropy spectral sequence of the action of $G$ on $X^{\delta}_{\mathcal{C}}$. The space $X^{\delta}_{\mathcal{C}}$ is the simplicial set which corresponds to the (ordered) simplicial complex $K_{\mathcal{C}}$ from 1.3.

**3.8 Relationships among the three decompositions.** There are $G$-equivariant functors

$$\mathbf{X}^{\alpha}_{\mathcal{C}} \xrightarrow{u} \mathbf{X}^{\delta}_{\mathcal{C}} \xleftarrow{v} \mathbf{X}^{\beta}_{\mathcal{C}}$$

given by setting $u(H, i) = i(H) \subset G$ and $v(x, G/H) = G_x$. In light of 3.1, the following proposition explains why the centralizer, subgroup, and normalizer diagrams for a collection $\mathcal{C}$ act together when it comes to success or failure at being homology decompositions.

**3.9 Proposition.** *The $G$-maps $X^{\alpha}_{\mathcal{C}} \xrightarrow{|u|} X^{\delta}_{\mathcal{C}} \xleftarrow{|v|} X^{\beta}_{\mathcal{C}}$ are weak equivalences.*

*3.10 Remark.* The maps $|u|$ and $|v|$ are *not* necessarily weak $G$-equivalences (4.6). This explains (4.8) why the centralizer, subgroup and normalizer diagrams can have different sharpness properties.

*Proof of 3.9.* By the remarks in 3.3, it is enough to find functors $u^{-1} : \mathbf{X}^{\delta}_{\mathcal{C}} \to \mathbf{X}^{\alpha}_{\mathcal{C}}$ and $v^{-1} : \mathbf{X}^{\delta}_{\mathcal{C}} \to \mathbf{X}^{\beta}_{\mathcal{C}}$ such that the composites $uu^{-1}$, $u^{-1}u$, $vv^{-1}$ and $v^{-1}v$ are connected to the appropriated identity functors by natural transformations. To obtain these, set $u^{-1}(H) = (H, i)$, where $i : H \to G$ is the inclusion, and set $v^{-1}(H) = (eG/sg, G/H)$. Note (cf. 3.10) that $u^{-1}$ and $v^{-1}$ are *not* $G$-equivariant. □

## §4. Bredon Homology

The previous two sections identify the Bousfield-Kan spectral sequence derived from a homology decomposition of B$G$ as the isotropy spectral sequence of an associated $G$-space $X$. The $E^2$-term of this isotropy spectral sequence is a kind of homological functor of $X$, and in this section we describe some of its formal properties. It is convenient to work in a setting slightly more abstract than that of §2-3.

*4.1 Definition.* A *coefficient functor* for $G$ is a functor $\mathcal{H}$ from the category of $\mathbf{F}_p[G]$-modules to the category of $\mathbf{F}_p$ vector spaces which preserves arbitrary direct sums. If $K$ is a subgroup of $G$, $\mathcal{H}|_K$ denotes the coefficient functor for $K$ given by $\mathcal{H}|_K(A) = \mathcal{H}(\mathbf{F}_p[G] \otimes_{\mathbf{F}_p[K]} A)$.

*4.2 Example.* If $M$ is an $\mathbf{F}_p[G]$-module, there are associated coefficient functors $\mathcal{H}^M_j$ given by $\mathcal{H}^M_j(A) = \mathrm{H}_j(G; M \otimes A)$. Here the tensor product is over $\mathbf{F}_p$ and the $G$-action is diagonal. These are the coefficient functors we will interested in, particularly for the trivial module $M = \mathbf{F}_p$. Note (by Shapiro's lemma) that if $K$ is a subgroup of $G$ and $\mathcal{H} = \mathrm{H}_j(G; M \otimes -)$, then $\mathcal{H}|_K = \mathrm{H}_j(K; M \otimes -)$.

*4.3 Definition.* Suppose that $\mathcal{H}$ is a coefficient functor for $G$ and that $X$ is a $G$-space. Let $(C^G_*(X;\mathcal{H}), d)$ be the chain complex with $C^G_n(X;\mathcal{H}) = \mathcal{H}(\mathbf{F}_p[X_n])$ and with boundary $d$ induced by the alternating sum of the face maps in $X$. The *Bredon homology groups* of $X$ with coefficients in $\mathcal{H}$, denoted $\mathrm{H}^G_*(X;\mathcal{H})$, are defined to be the homology groups of $C^G_*(X;\mathcal{H})$.

*4.4 Example.* Suppose that $X$ is a $G$-space and that $M$ is a module over $\mathbf{F}_p[G]$, with associated coefficient functors $\mathcal{H}^M_j$ as in 4.2. The $E^2$ term of the isotropy spectral sequence for $X$ with coefficients in $M$ is then (2.8) given by

$$E^2_{i,j} = \mathrm{H}^G_i(X;\mathcal{H}^M_j) \ .$$

*Remark.* It is possible to use coefficients for Bredon homology more general than the functors we have allowed above; in fact, any functor from the orbit category of $G$ to abelian groups can be used to construct a Bredon homology theory. We will not use this greater generality.

**4.5 Properties of $\mathrm{H}^G_*(X;\mathcal{H})$.** The groups $\mathrm{H}^G_*(X;\mathcal{H})$ have a certain basic invariance property.

*4.6 Definition.* A map $f : X \to Y$ of $G$-spaces is said to be a *weak $G$-equivalence* if $f^H : X^H \to Y^H$ is a weak equivalence for every subgroup $H$ of $G$.

*4.7 Remark.* If $X$ is a $G$-space, let $\mathrm{Iso}(X)$ denote the set of subgroups of $G$ which appear as isotropy groups of simplices of $X$. By [6, 4.1], a map $f : X \to Y$ of $G$-spaces is a weak $G$-equivalence if and only if it induces a weak equivalence $f^H : X^H \to Y^H$ for all $H \in \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$.

**4.8 Proposition.** *Suppose that $f : X \to Y$ is a a weak $G$-equivalence and that $\mathcal{H}$ is as in 4.1. Then $f$ induces isomorphisms $\mathrm{H}^G_*(X;\mathcal{H}) \cong \mathrm{H}^G_*(Y;\mathcal{H})$.*

For completeness, we will sketch a proof. If $(X, A)$ is a pair of $G$-spaces (i.e., $A$ is a subspace of $X$), let $C^G_*(X, A;\mathcal{H})$ denote the quotient complex $C^G_*(X;\mathcal{H})/C^G_*(A;\mathcal{H})$ and $H^G_*(X, A;\mathcal{H})$ its homology. It is clear that there is a long exact sequence relating $H^G_*(A;\mathcal{H})$, $H^G_*(X;\mathcal{H})$, and $H^G_*(X, A;\mathcal{H})$.

Let $K$ be a normal subgroup of $G$. A pair $(X, A)$ is said to be *relatively free mod* $K$ if $K$ acts trivially on the simplices of $X$ not in $A$ and $G/K$ acts freely on these simplices. Let $R = \mathbf{F}_p[G/K]$. If $(X, A)$ is relatively free mod $K$, then the relative simplicial chain complex $C_*(X, A;\mathbf{F}_p)$ is a chain complex of free $R$-modules, and there is an evident isomorphism $C^G_*(X, A;\mathcal{H}) \cong \mathcal{H}(R) \otimes_R C_*(X, A)$. The next lemma follows from basic homological algebra.

**4.9 Lemma.** *Suppose that $f : (X, A) \to (Y, B)$ is a map between pairs of $G$-spaces which are relatively free mod $K$, and that $f$ induces an isomorphism $\mathrm{H}_*(X, A;\mathbf{F}_p) \cong \mathrm{H}_*(Y, B;\mathbf{F}_p)$. Then $f$ induces an isomorphism $H^G_*(X, A;\mathcal{H}) \cong H^G_*(Y, B;\mathcal{H})$.*

*Proof of 4.8.* Pick representatives $\{K_i\}_{i=0}^m$ for the conjugacy classes of subgroups of $G$ and label the representatives in such a way that if $K_i$ is conjugate to a subgroup of $K_j$ then $i \geq j$. If $Z$ is a $G$-space, write $Z^{(n)}$ for the subspace of $Z$ consisting of all $z \in Z$ such that $G_z$ is conjugate to one of the groups $K_i$ for $i \leq n$. Let the height of $Z$ be the least integer $n$ such that $Z = Z^{(n)}$. The proof of 4.8 will be by induction on the heights of the spaces $X$ and $Y$ involved. The result is easy to check if $G$ acts trivially on $X$ and $Y$, i.e., if the heights of these spaces are $\leq 0$.

Assume by induction that the statement of 4.8 is true if the $G$-spaces involved have height $\leq n-1$. Suppose that $X$ and $Y$ are $G$-spaces of height $\leq n$ and that $f : X \to Y$ is a map which induces weak equivalences $X^H \to Y^H$ for all subgroups $H$ of $G$. Let $A = X^{(n-1)}$, $B = Y^{(n-1)}$, $K = K_n$ and $N = N_G(K)$. We must prove that the map $H^G_*(X;\mathcal{H}) \to H^G_*(Y;\mathcal{H})$ is an isomorphism.

It is easy to check that there is a map of pushout squares

$$\begin{array}{ccc} G \times_N A^K & \longrightarrow & A \\ \downarrow & & \downarrow \\ G \times_N X^K & \longrightarrow & X \end{array} \quad \longrightarrow \quad \begin{array}{ccc} G \times_N B^K & \longrightarrow & B \\ \downarrow & & \downarrow \\ G \times_N Y^K & \longrightarrow & Y \end{array}$$

and that in these squares the vertical arrows are monic. By 4.7 the map $A \to B$ is a weak $G$-equivalence, so by induction and a long exact sequence argument it is enough to show that the map $H^G_*(X, A;\mathcal{H}) \to H^G_*(Y, B;\mathcal{H})$ is an isomorphism. Given the above diagram of squares, this is equivalent to showing that the map $H^N_*(X^K, A^K;\mathcal{H}|_N) \to H^N_*(Y^K, B^K;\mathcal{H}|_N)$ is an isomorphism. Since the maps $A^K \to B^K$ and $X^K \to Y^K$ are weak equivalences of spaces, the map $\mathrm{H}_*(X^K, A^K;\mathbf{F}_p) \to \mathrm{H}_*(Y^K, B^K;\mathbf{F}_p)$ is an isomorphism. The desired result now follows from 4.9, since the $N$-space pairs $(X^K, A^K)$ and $(Y^K, B^K)$ are relatively free mod $K$, because all of the simplices which are added in going from $A$ to $X$ or from $B$ to $Y$ have isotropy group conjugate to $K$. □

The Bredon homology groups also have a gluing property. Say that a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ Y' & \xrightarrow{v} & Y \end{array} \tag{4.10}$$

of $G$-spaces is a *homotopy pushout square* if for each subgroup $H$ of $G$ the induced diagram of $H$-fixed subspaces is a homotopy pushout square of spaces. If 4.10 is a pushout square of $G$-spaces and at least one of the maps $u$ or $f'$ is monic, then 4.10 is a homotopy pushout square. In general, let $W$ be the double mapping cone of $u$ and $f'$. Then 4.10 is a homotopy pushout square if and only if the natural map $W \to Y$ is a weak $G$-equivalence. Alternatively, 4.10 is a homotopy pushout square of $G$-spaces if and only if for each subgroup $H$ of $G$ the induced square of $H$-fixed-point sets is a homotopy pushout square of spaces.

**4.11 Lemma.** *Suppose that 4.10 is a homotopy pushout square of $G$-spaces, and that $\mathcal{H}$ is a coefficient functor for $G$. Then there is a long exact sequence*

$$\cdots \to \mathrm{H}_i^G(X';\mathcal{H}) \to \mathrm{H}_i^G(X;\mathcal{H}) \oplus \mathrm{H}_i^G(Y';\mathcal{H}) \to \mathrm{H}_i^G(Y;\mathcal{H}) \to \mathrm{H}_{i-1}^G(X';\mathcal{H}) \to \cdots$$

*Proof.* By 4.8 we can replace $Y$ by the double mapping cone of $u$ and $f'$ and thus assume, for instance, that $u$ is monic. The result then follows from the fact that there is a chain complex short exact sequence:

$$0 \to C_*^G(X';\mathcal{H}) \to C_*^G(X;\mathcal{H}) \oplus C_*^G(Y';\mathcal{H}) \to C_*^G(Y;\mathcal{H}) \to 0\,. \quad \square$$

## §5. The transfer

All of our techniques for dealing with the $G$-spaces from §3 are based in one way or another on the transfer. We assume in this section that $\mathcal{H}$ is a coefficient functor (4.1), in practice a functor given by some homology construction (4.2).

**5.1 The transfer.** Suppose that $f : S \to T$ is a map of $G$-sets. There is an induced map $\mathbf{F}_p[S] \to \mathbf{F}_p[T]$, also denoted $f$, as well as a map

$$f_* = \mathcal{H}(f) : \mathcal{H}(\mathbf{F}_p[S]) \to \mathcal{H}(\mathbf{F}_p[T])\,.$$

Say that $f$ is *finite-to-one* if for each $x \in T$ the set $f^{-1}(x)$ is finite. For such an $f$ there is a $G$-map $\tau(f) : \mathbf{F}_p[T] \to \mathbf{F}_p[S]$, called the *pretransfer*, which sends $x \in T$ to $\sum_{y\in f^{-1}(x)} y$. The induced map

$$\tau_*(f) : \mathcal{H}(\mathbf{F}_p[T]) \to \mathcal{H}(\mathbf{F}_p[S])$$

is the *transfer* associated to $f$.

*5.2 Example.* Suppose that $H \subset K$ are subgroups of $G$, let $M$ be an $\mathbf{F}_p[G]$-module and let $\mathcal{H}$ be the functor $\mathrm{H}_j(G; M \otimes -)$ (cf. 4.2). Suppose that $f : G/H \to G/K$ is the projection map. By Shapiro's lemma there are isomorphisms

$$\mathcal{H}(\mathbf{F}_p[G/H]) \cong \mathrm{H}_j(H; M) \qquad \mathcal{H}(\mathbf{F}_p[G/K]) \cong \mathrm{H}_j(K; M) \ .$$

Under these identifications, $f_* : \mathrm{H}_j(H; M) \to \mathrm{H}_j(K; M)$ is the map induced by the inclusion $H \subset K$ and $\tau_*(f) : \mathrm{H}_j(K; M) \to \mathrm{H}_j(H; M)$ is the associated group homology transfer map.

The transfer has the following basic properties, which are easy to verify by calculations with pretransfers. Recall that $\mathcal{H}$ is assumed to commute with direct sums.

**5.3 Lemma.** *Suppose that $f_1 : S_1 \to T_1$ and $f_2 : S_2 \to T_2$ are maps of $G$-sets. If $f_1$ and $f_2$ are finite-to-one, the so is $f_1 \amalg f_2 : S_1 \amalg S_2 \to T_1 \amalg T_2$, and $\tau_*(f_1 \amalg f_2) = \tau_*(f_1) \oplus \tau_*(f_2)$.*

**5.4 Lemma.** *Suppose that $f_1 : S_1 \to T$ and $f_2 : S_2 \to T$ are maps of $G$-sets. If $f_1$ and $f_2$ are finite-to-one, the so is $f_1 + f_2 : S_1 \amalg S_2 \to T$, and $\tau_*(f_1 + f_2) = (\tau_*(f_1), \tau_*(f_2))$*

*5.5 Remark.* It follows from 5.2, 5.3, and 5.4 that if $f : S \to T$ is a map of $G$-sets which is finite-to-one, then $\tau_*(f)$ can be computed in terms of a sum of transfers associated to the projections $G/G_x \subset G/G_{f(x)}$, $x \in S$. We will call such a transfer the transfer *associated to the inclusion* $G_x \to G_{f(x)}$.

**5.6 Lemma.** *Suppose that $f : S \to T$ and $g : T \to R$ are maps of $G$-sets. If $f$ and $g$ are finite-to-one then so is $g \cdot f$, and $\tau_*(g \cdot f) = \tau_*(f) \cdot \tau_*(g)$.*

**5.7 Lemma.** *Suppose that*

$$\begin{array}{ccc} S' & \xrightarrow{s} & S \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ T' & \xrightarrow{t} & T \end{array}$$

*is a pullback square of $G$-sets (i.e. a commutative diagram which induces an isomorphism from $S'$ to the pullback of $S$ and $T'$ over $T$). Then if $f$ is finite-to-one, so is $f'$, and $\tau_*(f) \cdot t_* = s_* \cdot \tau_*(f')$.*

*5.8 Definition.* A map $f : S \to T$ of $G$-sets is said to be *an even covering mod p* if it is finite-to-one and the cardinality mod $p$ of $f^{-1}(x)$ does not depend on the choice of $x \in T$. The common value mod $p$ of these inverse image cardinalities is called the *degree* of $f$ and denoted $\deg(f)$.

**5.9 Lemma.** *Suppose that $f : S \to T$ is a map of $G$-sets which is an even covering mod $p$. Then the composite $f_* \cdot \tau_*(f)$ is the endomorphism of $\mathcal{H}(\mathbf{F}_p[T])$ given by multiplication by* $\deg(f)$.

*5.10 Example.* Suppose that $H$ is a subgroup of $G$ and that $S$ is a $G$-set. The action map $a : G \times_H S \to S$ is finite-to-one and has degree given by the index of $H$ in $G$. Moreover, if $S' \to S$ is a map of $G$-sets, the diagram

$$\begin{array}{ccc} G \times_H S' & \longrightarrow & G \times_H S \\ {\scriptstyle a}\downarrow & & {\scriptstyle a}\downarrow \\ S' & \longrightarrow & S \end{array}$$

is a pullback square. It follows that the maps $\tau_*(a)$ give a natural map

$$\mathcal{H}(\mathbf{F}_p[S]) \xrightarrow{\tau_*(a)} \mathcal{H}(\mathbf{F}_p[G \otimes_H S])$$

on the category of $G$-sets. Moreover, the composite $a_* \cdot \tau_*(a)$ is the endomorphism of $\mathcal{H}(\mathbf{F}_p[S])$ given by multiplication by the index of $H$ in $G$.

## §6. Acyclicity for $G$-spaces

In this section we explicitly translate the question of whether a homology decomposition of B$G$ is sharp into a question about the Bredon homology the an associated $G$-space $X$. We then study this second question. The symbol $\mathcal{H}$ denotes a coefficient functor (4.1) for $G$.

*6.1 Definition.* A $G$-space $X$ is said to be *acyclic for* $\mathcal{H}$ if the map $X \to *$ induces an isomorphism $\mathrm{H}^G_*(X; \mathcal{H}) \to \mathrm{H}^G_*(*; \mathcal{H})$.

*6.2 Remark.* Note that $\mathrm{H}^G_i(*; \mathcal{H})$ vanishes for $i > 0$ and $\mathrm{H}^G_0(*; \mathcal{H}) = \mathcal{H}(\mathbf{F}_p)$. Let $M$ be a module over $\mathbf{F}_p[G]$ and $\mathcal{C}$ a collection of subgroups of $G$. By §3 and 4.4, the centralizer, subgroup and normalizer decompositions associated to $\mathcal{C}$ are sharp for $M$ (1.13) if and only if the $G$-spaces $X^\alpha_{\mathcal{C}}$, $X^\beta_{\mathcal{C}}$, and $X^\delta_{\mathcal{C}}$ (respectively) are acyclic for the functors $\mathrm{H}_i(G; M \otimes -)$, $i \geq 0$.

We have three ways to show that a $G$-space $X$ is acyclic for $\mathcal{H}$.

**6.3 The direct transfer method.** This uses the fact that if $K$ is a subgroup of $G$ of index prime to $p$ then the transfer exhibits $\mathrm{H}^G_*(X; \mathcal{H})$ as a retract of $\mathrm{H}^K_*(X; \mathcal{H}|_K)$.

**6.4 Theorem.** *Suppose that $X$ is a $G$-space, $\mathcal{H}$ is a coefficient functor for $G$, and $K$ is a subgroup of $G$ of index prime to $p$. If $X$ is acyclic as a $K$-space for $\mathcal{H}|_K$, then $X$ is acyclic as a $G$-space for $\mathcal{H}$.*

*Proof.* The transfers (5.1) associated to the maps $q : G \times_K X_n \to X_n$ provide a map $t : C^G_*(X; \mathcal{H}) \to C^K_*(X; \mathcal{H}|_K)$ (4.1). By 5.10 this map commutes with differentials, and the index assumption implies that the composite $C^G_*(X; \mathcal{H}) \xrightarrow{t} C^K_*(X; \mathcal{H}|_K) \xrightarrow{q} C^G_*(X; \mathcal{H})$ is an isomorphism. By naturality, then, the map $\mathrm{H}^G_*(X; \mathcal{H}) \to \mathrm{H}^G_*(*; \mathcal{H})$ is a retract of $\mathrm{H}^K_*(X; \mathcal{H}|_K) \to \mathrm{H}^K_*(*; \mathcal{H}|_K)$, and the theorem follows from the fact that a retract of an isomorphism is an isomorphism. □

**6.5 The Mayer-Vietoris method.** This gives a way to study $G$-spaces which are constructed by gluing.

**6.6 Theorem.** *Suppose that*

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Y' & \xrightarrow{v} & Y \end{array}$$

*is a homotopy pushout square of $G$-spaces (4.8) and that either the space $X$ or the space $Y$ is acyclic for $\mathcal{H}$. Then the other member of the pair $\{X, Y\}$ is acyclic for $\mathcal{H}$ if and only if the map $f'$ induces an isomorphism $\mathrm{H}^G_*(X';\mathcal{H}) \to \mathrm{H}^G_*(Y';\mathcal{H})$.*

*Proof.* This is clear from the exact sequence of 4.11. □

**6.7 The method of discarded orbits.** This a more sophisticated version of the direct transfer method which exploits the fact that $K$-orbits can be discarded if they do not contribute to the transfer.

**6.8 Theorem.** *Let $X$ be a $G$-space, $K$ a subgroup of $G$ of index prime to $p$, and $Y$ a subspace of $X$ which is closed under the action of $K$. Assume that $Y$ is acyclic for $\mathcal{H}|_K$, and that for each simplex $x \in X \setminus Y$ the transfer map $\mathcal{H}(\mathbf{F}_p[G/G_x]) \to \mathcal{H}(\mathbf{F}_p[G/K_x])$ is zero (cf. 5.5). Then $X$ is acyclic for $\mathcal{H}$.*

*Proof of 6.8.* The transfers associated to the maps $q : G \times_K X_n \to X_n$ provide a map $t : C^G_*(X;\mathcal{H}) \to C^K_*(X;\mathcal{H}|_K)$ (4.1). By 5.10 this map commutes with differentials, and the index assumption implies that the composite $C^G_*(X;\mathcal{H}) \xrightarrow{t} C^K_*(X;\mathcal{H}|_K) \xrightarrow{q} C^G_*(X;\mathcal{H})$ is an isomorphism. The transfer hypothesis shows that the image of $t$ is actually in the subcomplex $C^K_*(Y;\mathcal{H}|_K)$ of $C^K_*(X;\mathcal{H}|_K)$. By naturality, the homology map $\mathrm{H}^G_*(X;\mathcal{H}) \to \mathrm{H}^G_*(*;\mathcal{H})$ is a retract of $\mathrm{H}^K_*(Y;\mathcal{H}|_K) \to \mathrm{H}^K_*(*;\mathcal{H}|_K)$, and the theorem follows from the fact that a retract of an isomorphism is an isomorphism. □.

*6.9 Example.* Let $X$ be a $G$-space, $K$ a subgroup of $G$ of index prime to $p$, and $Y$ a subspace of $X$ which is closed under the action of $K$. Suppose that $M$ is a $G$-module, and that $Y$ is acyclic for the functors $\mathrm{H}_j(K, M\otimes -)$, $j \geq 0$. Assume finally that for each $x \in X \setminus Y$ the transfer map $\mathrm{H}_*(G_x; M) \to \mathrm{H}_*(K_x; M)$ is zero. In light of 4.2, Proposition 6.8 implies that $X$ is acyclic for the functors $\mathrm{H}_j(G; M\otimes -)$, $j \geq 0$.

A special case of the above is due to Webb [15] and Adem-Milgram [1, V.3].

**6.10 Corollary.** *Let $X$ be a $G$-space, $P$ a Sylow $p$-subgroup of $G$, and $M$ a module over $\mathbf{F}_p[G]$. Suppose that for every nonidentity subgroup $Q$ of $P$ the fixed point space $X^Q$ is contractible, and that for any $x \in X$ there is an element of order $p$ in $G_x$ which acts trivially on $M$. Then $X$ is acyclic for the functors $\mathrm{H}_j(G; M\otimes -)$, $j \geq 0$.*

*Proof.* Let $Y$ be the $P$-subspace of $X$ consisting of simplices which are fixed by a nonidentity element of $P$. By 4.7 the map $Y \to *$ is a weak $P$-equivalence, and so by 4.8 the space $Y$ is acyclic for $\mathrm{H}_j(P; M \otimes -)$. Moreover, for any $j \geq 0$ and $x \in X \setminus Y$ the transfer map

$$\mathrm{H}_j(G_x; M) \to \mathrm{H}_j(P_x; M) = \mathrm{H}_j(\{e\}; M)$$

is trivial. This is true for $j > 0$ because the target group is zero, and true for $j = 0$ because by assumption the norm map $\sum_{g\in G_x} g : M \to M$ is trivial. The result follows from 6.8 (cf. 6.9). □

## §7. Nontrivial $p$-subgroups

Let $\mathcal{C}$ be the collection of all nontrivial $p$-subgroups of $G$. Assume that $\mathcal{C}$ is nonempty, i.e., that the order of $G$ is divisible by $p$. In this section we will show that the three homology decompositions derived from $\mathcal{C}$ are sharp for suitable $\mathbf{F}_p[G]$-modules $M$. Let $P$ denote a Sylow $p$-subgroup of $G$. In all three cases we use the method of Webb and Adem-Milgram (6.10) to show that the spaces $X^\delta_{\mathcal{C}}$, $X^\beta_{\mathcal{C}}$, and $X^\alpha_{\mathcal{C}}$ are acyclic for the coefficient functors $\mathrm{H}_j(G; M\otimes -)$ (see 6.2).

The following fact is from [6, pf. of 6.5].

**7.1 Lemma.** *Suppose $Q$ is a $p$-subgroup of $G$ and that $M$ is an $\mathbf{F}_p[G]$-module with the property that the kernel of the action map $G \to \mathrm{Aut}(M)$ has order divisible by p. Then there exists an element $x$ in $G$ of order p such that $x$ commutes with $Q$ and $x$ acts trivially on $M$.*

*7.2 The normalizer decomposition.* Suppose that $M$ is as in 7.1. We will show that $X^\delta_{\mathcal{C}}$ is acyclic for the coefficient functors $\mathrm{H}_j(G; M\otimes -)$. The first step is to show that for each nonidentity subgroup $Q$ of $P$ the space $(X^\delta_{\mathcal{C}})^Q$ is contractible. By 3.7 and 3.3, $(X^\delta_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\delta_{\mathcal{C}}$ generated by the objects $H$ of $\mathbf{X}^\delta_{\mathcal{C}}$ (equivalently, elements $H \in \mathcal{C}$) such that $Q \subset N_G(H)$. The inclusions

$$H \subset H \cdot Q \supset Q$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value $Q$. The existence of this zigzag implies that $|\mathbf{D}|$ is contractible (3.3).

A typical $n$-simplex $Q_0 \subset \cdots \subset Q_n$ ($Q_i \in \mathcal{C}$) of $X^\delta_{\mathcal{C}}$ has isotropy subgroup $\cap_i N_G(Q_i)$. The fact that there is an element of order $p$ in this isotropy subgroup which acts trivially on $M$ follows from 7.1. Now use 6.10. □

*7.3 The centralizer decomposition.* Suppose that $M$ is as in 7.1. We again show that $X^\alpha_{\mathcal{C}}$ is acyclic for the coefficient functors $\mathrm{H}_j(G; M\otimes -)$. We first prove that for any nonidentity subgroup $Q$ of $P$ the space $(X^\alpha_{\mathcal{C}})^Q$ is contractible. By 3.4 and 3.3, $(X^\alpha_{\mathcal{C}})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\alpha_{\mathcal{C}}$ generated by the objects $(H, i)$ with the property that $Q \subset C_G(i(H))$. Let $Z$ be the center of $Q$ and $j : Z \to G$ the inclusion. For an object $(H, i)$ of $\mathbf{D}$, let $H'$ denote the image of the product map $H \times Z \to G$ and $i' : H' \to G$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j)$$

give a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value $(Z, j)$. As above, then, $|\mathbf{D}|$ is contractible.

The isotropy subgroup of a typical simplex (3.5) of $X^\alpha_{\mathcal{C}}$ has the form $C_G(Q)$ for some $Q \in \mathcal{C}$. It follows from 7.1 that such an isotropy subgroup contains an element of order $p$ which acts trivially on $M$. Now use 6.10. □

*7.4 The subgroup decomposition.* In this case we deal only with the trivial module $M = \mathbf{F}_p$, and show that $X^\beta_\mathcal{C}$ is acyclic for the coefficient functors $\mathrm{H}_j(G;-)$. As above the first problem is to show that for any nontrivial subgroup $Q$ of $P$ the space $(X^\beta_\mathcal{C})^Q$ is contractible. By 3.6 and inspection $(X^\beta_\mathcal{C})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\beta_\mathcal{C}$ generated by the pairs $(x, G/H)$ with $Q \subset G_x$. The category $\mathbf{D}$ has $(eQ, G/Q)$ as an initial element; in other words, for any object $(x, G/H)$ of $\mathbf{D}$ there is a unique map $(eQ, G/Q) \to (x, G/H)$. These maps give a natural transformation between the identity functor of $\mathbf{D}$ and the constant functor with value $(eQ, G/Q)$, which as above implies that $|\mathbf{D}|$ is contractible.

A typical simplex of $X^\beta_\mathcal{C}$ has as its isotropy subgroup a group of the form $Q$ for some $Q \in \mathcal{C}$. Any such isotropy subgroup contains an element of order $p$ (which of acts trivially on $\mathbf{F}_p$). Now, again, use 6.10. □

## §8. Elementary abelian $p$-subgroups

In this section we prove several sharpness statements about collections of elementary abelian $p$-subgroups of $G$.

**Nontrivial elementary abelian $p$-subgroups.** Let $\mathcal{C}$ be the collection of all nontrivial elementary abelian $p$-subgroups of $G$. We will show that $\mathcal{C}$ is both both centralizer-sharp and normalizer-sharp for any $\mathbf{F}_p[G]$-module $M$ such the the kernel of the action map $G \to \mathrm{Aut}(M)$ has order divisible by $p$. The arguments mimic the ones in §7. Let $P$ be a Sylow $p$-subgroup of $G$. In each case we use the method of Webb and Adem-Milgram (6.10) to show that the spaces $X^\delta_\mathcal{C}$ and $X^\alpha_\mathcal{C}$ are acyclic for the functors $\mathrm{H}_j(G; M \otimes -)$.

*The normalizer decomposition.* We follow 7.2. The first step is to show that for any nontrivial subgroup $Q$ of $P$ the space $(X^\delta_\mathcal{C})^Q$ is contractible. By 3.7 and 3.3, $(X^\delta_\mathcal{C})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\delta_\mathcal{C}$ determined by the objects $H$ of $\mathbf{X}^\delta_\mathcal{C}$ (equivalently, elements $H \in \mathcal{C}$) such that $Q \subset N_G(H)$. Let $Z$ be the group of elements of exponent $p$ in the center of $Q$, and given an object $H$ of $\mathbf{D}$, let $H'$ be the group of elements of exponent $p$ in the center of $QH$. The inclusions

$$H \supset H \cap H' \subset H'Z \supset Z$$

give a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value $Z$. (See [6, 5.2] for the fact that $H \cap H'$ is nontrivial.) By 3.3, $|\mathbf{D}|$ is contractible.

As in 7.2, the isotropy subgroup of any simplex of $X^\delta_\mathcal{C}$ contains an element of order $p$ which acts trivially on $M$. □

*8.1 The centralizer decomposition.* [8] We follow 7.3. The first step is to show that if $Q$ is a nontrivial subgroup of $P$, then $(X^\alpha_\mathcal{C})^Q$ is contractible. By 3.4 and 3.3, $(X^\alpha_\mathcal{C})^Q$ is the nerve of the full subcategory $\mathbf{D}$ of $\mathbf{X}^\alpha_\mathcal{C}$ generated by objects $(H, i)$ such that $Q \subset C_G(i(H))$. Let $Z$ denote group of elements of exponent $p$ in the center of $Q$ and $j : Z \to G$ the inclusion. For an object $(H, i)$ of $\mathbf{D}$, let $H'$ denote the image of the product map $H \times Z \to G$ and $i' : H' \to G$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j) \tag{8.2}$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{D}$ and the constant functor with value $(Z, j)$. As above, $|\mathbf{D}|$ is contractible.

As in 7.3, the isotropy subgroup of any simplex of $X^\alpha_{\mathcal{C}}$ contains an element of order $p$ which acts trivially on $M$. $\square$

**Smaller collections.** If $\mathcal{C}$ is a collection of subgroups of $G$ and $K$ is a subgroup of $G$, let $\mathcal{C} \cap 2^K$ denote the set of all elements of $\mathcal{C}$ which are subgroups of $K$. Clearly $\mathcal{C} \cap 2^K$ is a collection of subgroups of $K$. We are aiming at the following theorem.

**8.3 Theorem.** *Let $K$ be a subgroup of $G$ of index prime to $p$, and $\mathcal{C}$ a collection of elementary abelian $p$-subgroups of $G$. If $\mathcal{C} \cap 2^K$ is centralizer-sharp (as a $K$-collection) then $\mathcal{C}$ is centralizer-sharp (as a $G$-collection).*

*8.4 Example.* Theorem 8.3 can be used as a substitute for the argument of 8.1 in showing that the collection $\mathcal{C}$ of all nontrivial elementary abelian $p$-subgroups of $G$ is centralizer-sharp (for the trivial module $\mathbf{F}_p$). To see this, let $P$ be a Sylow $p$-subgroup of $G$. It is enough to prove that the collection $\mathcal{C}' = \mathcal{C} \cap 2^P$ of all nontrivial elementary abelian $p$-subgroups of $P$ is centralizer-sharp as a $P$-collection. We can derive this from 4.8 by showing that $X^\alpha_{\mathcal{C}'}$ is $P$-equivariantly contractible. Let $j : Z \to P$ be the inclusion of the group of elements of exponent $p$ in the center of $P$. If $(H, i)$ is an object of $\mathbf{X}^\alpha_{\mathcal{C}'}$, let $H'$ denote the image of the product map $H \times Z \to P$ and $i' : H' \to P$ the inclusion. The maps

$$(H, i) \to (H', i') \leftarrow (Z, j) \tag{8.5}$$

provide a zigzag of natural transformations between the identity functor of $\mathbf{X}^\alpha_{\mathcal{C}'}$ and the constant functor with value $(Z, j)$. This zigzag respects the action of $P$ on $\mathbf{X}^\alpha_{\mathcal{C}'}$, and so gives an equivariant contraction of $X^\alpha_{\mathcal{C}'}$.

*8.6 Example.* [3] It is possible to do better than the above. Let $P$ be a Sylow $p$-subgroup of $G$, and let $Z$ be any nontrivial central elementary abelian $p$-subgroup of $P$. Let $\mathcal{C}$ be the smallest collection of subgroups of $G$ which contains $Z$ and has the property that if $V \in \mathcal{C}$ and $V$ commutes with $Z$ then $\langle Z, V \rangle \in \mathcal{C}$. An argument virtually identical to the one in 8.4 shows that if $\mathcal{C}' = \mathcal{C} \cap 2^P$, then $X^\alpha_{\mathcal{C}'}$ is $P$-equivariantly contractible. It follows from 8.3 and 4.8 that $\mathcal{C}$ is centralizer-sharp.

The proof of 8.3 depends on the following observation.

**8.7 Lemma.** *Suppose that $K$ is a subgroup of $G$ and that $V$ is an elementary abelian subgroup of $G$ not entirely contained in $K$. Then the transfer map*

$$\mathrm{H}_*(C_G(V); \mathbf{F}_p) \to \mathrm{H}_*(C_G(V) \cap K; \mathbf{F}_p)$$

*associated to $C_G(V) \cap K \to C_G(V)$ (cf. 5.5) is zero.*

*Proof.* Let $C_1 = C_G(V) \cap K$ and $C_2 = C_G(V)$. Choose $v \in V$ with $v \notin K$, and let $C_1' \cong C_1 \times \langle v \rangle$ be the subgroup of $C_2$ generated by $C_1$ and $v$. The inclusion $C_1 \to C_2$ factors as the composite of $f' : C_1' \to C_2$ with $f : C_1 \to C_1'$, so the transfer in question factors (5.6) as a parallel composite $\tau_*(f)\tau_*(f')$. However, the map $\tau_*(f)$ is zero; this follows from the fact that the map

$$f_* : \mathrm{H}_*(C_1; \mathbf{F}_p) \to \mathrm{H}_*(C_1'; \mathbf{F}_p)$$

is a monomorphism ($f$ has a left inverse) and the fact that the composite $\tau_*(f) \cdot f_*$ is multiplication by $p$ (5.9). □

*Proof of 8.3.* Let $X$ be the $G$-space $X^\alpha_\mathcal{C} = |\mathbf{X}^\alpha_\mathcal{C}|$; we have to show that $X$ is acyclic for the functors $\mathrm{H}_i(G;-)$. The strategy is to use the method of discarded orbits (6.7). Let $\mathbf{Y}$ be the full subcategory of $\mathbf{X}^\alpha_\mathcal{C}$ determined by the objects $(H, i)$ such that $i(H)$ is a subgroup of $K$, and let $Y = |\mathbf{Y}|$, so that $Y$ is a subspace of $X^\alpha_\mathcal{C}$. The action of $G$ on $X^\alpha_\mathcal{C}$ restricts to an action of $K$ on $Y$, and it is clear that $Y$ is equivalent as a $K$-space to $X^\alpha_{\mathcal{C}'}$, where $\mathcal{C}' = \mathcal{C} \cap 2^K$. In particular, $Y$ is by hypothesis acyclic for the functors $\mathrm{H}_i(K;-)$, $i \geq 0$. Let $x$ as in 3.5 be a simplex of $X \setminus Y$, and let $V$ be the image of $H_n$ in $G$. Since $V$ is not contained in $K$, Lemma 8.7 guarantees that the homology transfer map associated to the inclusion

$$K_x = C_G(V) \cap K \to G_x = C_G(V)$$

is zero. The desired result follows from 6.9. □

## §9. Pruning a collection

Let $\mathcal{C}'$ be a collection of subgroups of $G$, $K$ an element of $\mathcal{C}'$, and $\mathcal{C}$ the collection obtained by deleting from $\mathcal{C}'$ all conjugates of $K$. We say that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by "pruning" the subgroup $K$. Our goal in this section is to obtain relationships between the sharpness properties of $\mathcal{C}'$ and those of $\mathcal{C}$.

*9.1 Definition.* If $H$ and $K$ are two subgroups of $G$, then $H$ is said to be *comparable* to $K$ if $H \subset K$ or $H \supset K$.

**9.2 Pruning the subgroup decomposition.** Let $\mathbf{link}^\beta_{\mathcal{C}'}(K)$ denote the full subcategory of $X^\beta_{\mathcal{C}'}$ given by the objects $(x, G/H)$ such that $G_x$ is comparable to $K$ but $G_x \neq K$. Denote the nerve of this category by $\mathrm{link}^\beta_{\mathcal{C}'}(K)$. The action of $G$ on $X^\beta_{\mathcal{C}'}$ induces an action of $N_G(K)$ on $\mathrm{link}^\beta_{\mathcal{C}'}(K)$.

Recall that if $H$ is a group, $\mathrm{E}H$ denotes the universal cover of $\mathrm{B}H$; in particular, $\mathrm{E}H$ is a contractible space on which $H$ acts freely. We say that a map between spectral sequences is *an $E^2$-isomorphism* if it induces an isomorphism between $E^2$-terms.

**9.3 Proposition.** *Suppose that $\mathcal{C}'$ is a collection of subgroups of $G$, and that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by pruning $K$. Let $N = N_G(K)$ and $W = N/K$. Suppose that either $\mathcal{C}$ or $\mathcal{C}'$ is subgroup-sharp. Then the other collection of the pair $\{\mathcal{C}, \mathcal{C}'\}$ is subgroup-sharp if and only if the projection $\mathrm{E}W \times \mathrm{link}^\beta_{\mathcal{C}'}(K) \to \mathrm{E}W$ (which is a map of $N$-spaces) induces an $E^2$-isomorphism of isotropy spectral sequences.*

The proof depends on a lemma. Let $\mathbf{star}^\beta_{\mathcal{C}'}(K)$ denote the full subcategory of $\mathbf{X}^\beta_{\mathcal{C}'}$ given by the objects $(x, G/H)$ with $G_x$ comparable to $K$, and $\mathrm{star}^\beta_{\mathcal{C}'}(K)$ its nerve. There is a natural action of $N_G(K)$ on $\mathrm{star}^\beta_{\mathcal{C}'}(K)$, and an equivariant inclusion $\mathrm{link}^\beta_{\mathcal{C}'}(K) \subset \mathrm{star}^\beta_{\mathcal{C}'}(K)$.

**9.4 Lemma.** *In the situation of 9.3 there is a homotopy pushout square (4.10) of G-spaces*

$$\begin{array}{ccc} G\times_N(\mathrm{E}W\times \mathrm{link}^{\beta}_{\mathcal{C}'}(K)) & \longrightarrow & X^{\beta}_{\mathcal{C}} \\ \downarrow & & \downarrow \\ G\times_N(\mathrm{E}W\times \mathrm{star}^{\beta}_{\mathcal{C}'}(K)) & \longrightarrow & X^{\beta}_{\mathcal{C}'} \end{array} \tag{9.5}$$

*Proof.* By examining simplices it is easy to check that there is a pushout square of $G$-spaces

$$\begin{array}{ccc} G\times_N \mathrm{link}^{\beta}_{\mathcal{C}'}(K) & \longrightarrow & X^{\beta}_{\mathcal{C}} \\ \downarrow & & \downarrow \\ G\times_N \mathrm{star}^{\beta}_{\mathcal{C}'}(K) & \longrightarrow & X^{\beta}_{\mathcal{C}'} \end{array}$$

which is in fact a homotopy pushout square because the left vertical arrow is monic. To complete the proof of the proposition it is enough to show that the square

$$\begin{array}{ccc} G\times_N(\mathrm{E}W\times \mathrm{link}^{\beta}_{\mathcal{C}'}(K)) & \longrightarrow & G\times_N \mathrm{link}^{\beta}_{\mathcal{C}'}(K) \\ \downarrow & & \downarrow \\ G\times_N(\mathrm{E}W\times \mathrm{star}^{\beta}_{\mathcal{C}'}(K)) & \longrightarrow & G\times_N \mathrm{star}^{\beta}_{\mathcal{C}'}(K) \end{array}$$

is also a homotopy pushout square. Here the horizontal arrows are induced by projections. This is equivalent to the assertion that the square

$$\begin{array}{ccc} \mathrm{E}W\times \mathrm{link}^{\beta}_{\mathcal{C}'}(K) & \longrightarrow & \mathrm{link}^{\beta}_{\mathcal{C}'}(K) \\ \downarrow & & \downarrow \\ \mathrm{E}W\times \mathrm{star}^{\beta}_{\mathcal{C}'}(K) & \longrightarrow & \mathrm{star}^{\beta}_{\mathcal{C}'}(K) \end{array}$$

is a homotopy pushout square of $N$-spaces. Let $H$ be a subgroup of $N$ and consider the square of fixed point sets

$$\begin{array}{ccc} (\mathrm{E}W\times \mathrm{link}^{\beta}_{\mathcal{C}'}(K))^H & \longrightarrow & \mathrm{link}^{\beta}_{\mathcal{C}'}(K)^H \\ \downarrow & & \downarrow \\ (\mathrm{E}W\times \mathrm{star}^{\beta}_{\mathcal{C}'}(K))^H & \longrightarrow & \mathrm{star}^{\beta}_{\mathcal{C}'}(K)^H \end{array} \tag{9.6}$$

If $H$ is not contained in $K$, the spaces in the right hand column of 9.6 are empty, and the left vertical map $\mathrm{link}^{\beta}_{\mathcal{C}'}(K)^H \to \mathrm{star}^{\beta}_{\mathcal{C}'}(K)^H$ is by inspection an isomorphism. If $H\subset K$, then $H$ acts trivially on $\mathrm{E}W$ and the horizontal maps in 9.6 are equivalences. In either case, the square 9.6 is a homotopy pushout square of spaces. □

*Proof of 9.3.* Suppose for concreteness that $\mathcal{C}$ is subgroup-sharp. By 6.2 and 6.6, $\mathcal{C}'$ is subgroup-sharp if and only if the left vertical map in 9.5 gives an $E^2$-isomorphism of isotropy spectral sequences. By Shapiro's lemma, though, the isotropy spectral sequence of an $N$-space $Y$ is naturally isomorphic to the isotropy spectral sequence of the associated $G$-space $G \times_N Y$. It follows that $\mathcal{C}'$ is subgroup-sharp if and only if the map of $N$-spaces $\mathrm{E}W \times \mathrm{link}^{\beta}_{\mathcal{C}'}(K) \to \mathrm{E}W \times \mathrm{star}^{\beta}_{\mathcal{C}'}(K)$ induces an $E^2$-isomorphism of isotropy spectral sequences. To complete the proof it is enough to show that the map $f : \mathrm{E}W \times \mathrm{star}^{\beta}_{\mathcal{C}'}(K) \to \mathrm{E}W$ gives an $E^2$-isomorphism of isotropy spectral sequences. We do this by proving that $f$ is in fact a weak $N$-equivalence (cf. 4.8). Let $J$ be a subgroup of $N$. If $J$ is not contained in $K$, then both the domain and range of $f^J$ are empty. If $J \subset K$, then the range of $f^J$ is the contractible space $\mathrm{E}W$, and the domain is $\mathrm{E}W \times \mathrm{star}^{\beta}_{\mathcal{C}'}(K)^J$, so that it is enough to prove that $\mathrm{star}^{\beta}_{\mathcal{C}'}(K)^J$ is contractible. The space $\mathrm{star}^{\beta}_{\mathcal{C}'}(K)^J$ is the nerve of the subcategory $\mathbf{D}$ of $\mathbf{star}^{\beta}_{\mathcal{C}'}(K)$ consisting of pairs $(x, G/H)$ such that $J \subset G_x$. Let $F : \mathbf{D} \to \mathbf{D}$ be the functor given by the $F(x, G/H) = (eK, G/K)$ if $G_x \subset K$ and $F(x, G/H) = (x, G/H)$ otherwise. The unique maps

$$(x, G/H) \to F(x, G/H) \leftarrow (eK, G/K)$$

provide a zigzag of natural transformations connecting the identity functor of $\mathbf{D}$ to the constant functor with value $(eK, G/K)$. By 3.3, $|\mathbf{D}|$ is contractible. □

**9.7 Pruning the centralizer and normalizer decompositions.** Suppose as above that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by pruning the subgroup $K$. Let $\mathbf{link}^{\alpha}_{\mathcal{C}'}(K)$ denote the full subcategory of $\mathbf{X}^{\alpha}_{\mathcal{C}}$ given by the objects $(H, i)$ such that $i(H)$ is comparable (9.1) to $K$ but $i(H) \neq K$. Let $\mathbf{link}^{\delta}_{\mathcal{C}'}(K)$ denote the full subcategory of $X^{\delta}_{\mathcal{C}}$ given by the objects $H$ of $X^{\delta}_{\mathcal{C}}$ such that $H$ is comparable to $K$ but $H \neq K$. Denote the nerves of these categories by $\mathrm{link}^{\alpha}_{\mathcal{C}'}(K)$ and $\mathrm{link}^{\delta}_{\mathcal{C}'}(K)$ respectively; the categories and their nerves are naturally furnished with actions of $N_G(K)$. We will not use the following results and we omit the proofs; they are similar to the proof of 9.3.

**9.8 Proposition.** *Suppose that $\mathcal{C}'$ is a collection of subgroups of $G$, and that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by pruning $K$. Let $N = N_G(K)$, $C = C_G(K)$, and $W' = N/C$. Suppose that either $\mathcal{C}$ or $\mathcal{C}'$ is centralizer-sharp. Then the other collection of the pair $\{\mathcal{C}, \mathcal{C}'\}$ is centralizer-sharp if and only if the projection $\mathrm{E}W' \times \mathrm{link}^{\alpha}_{\mathcal{C}'}(K) \to \mathrm{E}W'$ (which is a map of $N$-spaces) induces an $E^2$-isomorphism of isotropy spectral sequences.*

**9.9 Proposition.** *Suppose that $\mathcal{C}'$ is a collection of subgroups of $G$, and that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by pruning $K$. Let $N = N_G(K)$. Suppose that either $\mathcal{C}$ or $\mathcal{C}'$ is normalizer-sharp. Then the other collection of the pair $\{\mathcal{C}, \mathcal{C}'\}$ is normalizer-sharp if and only if the map $\mathrm{link}^{\delta}_{\mathcal{C}'}(K) \to *$ (which is a map of $N$-spaces) induces an $E^2$-isomorphism of isotropy spectral sequences.*

*Remark.* The map $\mathrm{link}^{\delta}_{\mathcal{C}'}(K) \to *$ of 9.9 induces an $E^2$-isomorphism of isotropy spectral sequences if and only if the $N$-space $\mathrm{link}^{\delta}_{\mathcal{C}'}(K)$ is acyclic for the coefficient functors $\mathrm{H}_j(N; –)$, $j \geq 0$.

**9.10 Fixed point sets of normalizers acting on links.** Propositions 9.3, 9.8 and 9.9 motivate studying the action of $N = N_G(K)$ on the spaces $\mathrm{link}^{\beta}_{\mathcal{C}'}(K)$, $\mathrm{link}^{\alpha}_{\mathcal{C}'}(K)$ and $\mathrm{link}^{\delta}_{\mathcal{C}'}(K)$. Let $J$ be a subgroup of $N_G(K)$; we indicate how to identify up to homotopy the fixed point sets of the action of $J$ on these links. The space $\mathrm{link}^{\beta}_{\mathcal{C}'}(K)^J$ is the nerve of the full subcategory of $\mathbf{link}^{\beta}_{\mathcal{C}'}(K)$ given by the objects $(x, G/H)$ with $G_x \supset J$; the functor $v$ from 3.8 gives an equivalence between this nerve and the nerve of the poset of all subgroups $H$ of $G$ such that $H$ is comparable (9.1) to $K$, $H \in \mathcal{C}$, and $H \supset J$. The space $\mathrm{link}^{\alpha}_{\mathcal{C}'}(K)^J$ is the nerve of the full subcategory of $\mathbf{link}^{\beta}_{\mathcal{C}'}(K)$ given by the objects $(H, i)$ such that $i(H)$ is centralized by $J$; the functor $u$ from 3.8 gives an equivalence between this nerve and the nerve of the poset of all subgroups $H$ of $G$ such that $H$ is comparable to $K$, $H \in \mathcal{C}$, and $H$ is centralized by $J$. (The proofs of these statements are similar to the proof of 3.9.) The space $\mathrm{link}^{\delta}_{\mathcal{C}'}(K)^J$ is the nerve of the poset of all subgroups $H$ of $G$ such that $H$ is comparable to $K$, $H \in \mathcal{C}$, and $H$ is normalized by $J$.

## §10. $p$-CENTRIC COLLECTIONS

In this section we show that various collections involving $p$-centric subgroups of $G$ are subgroup-sharp. We leave it to the reader to formulate a more elaborate result involving "$M$-centric" collections as in [6, §8]. It takes us two stages to reach the collection of all $p$-subgroups $P$ of $G$ such that $P$ is both $p$-centric and $p$-stubborn. We use an upward induction on the size of the omitted group to discard all subgroups which are not $p$-centric, and then a downward induction to eliminate all subgroups which are not in addition $p$-stubborn. This parallels the argument in [6, §8]. The starting point is this.

**10.1 Proposition.** *The collection $\mathcal{C}$ of all $p$-subgroups of $G$ is centralizer-sharp, subgroup-sharp and normalizer-sharp.*

*Proof.* In fact, these collections are sharp for any $\mathbf{F}_p[G]$-module $M$. This is proved as in §7, but substituting the direct transfer method (6.4) for the method of Webb and Adem-Milgram. □

**10.2 Elimination of non-$p$-centric subgroups.** The main result here is the following. Note that it is a consequence of the definition of "$p$-centric subgroup" that if $P$ is a $p$-centric subgroup of $G$ and $Q$ is a $p$-subgroup of $G$ which contains $P$, then $Q$ is also $p$-centric. In other words, the collection of all $p$-centric subgroups of $G$ is closed under passage to $p$-supergroups.

**10.3 Theorem.** *Let $\mathcal{C}$ be a collection of $p$-subgroups of $G$ which contains all $p$-centric subgroups and is closed under passage to $p$-supergroups. Then $\mathcal{C}$ is subgroup-sharp.*

*Proof.* The proof is by downward induction on the size of $\mathcal{C}$ or equivalently upward induction on the size of the $p$-subgroups of $G$ omitted from $\mathcal{C}$. The case in which $\mathcal{C}$ is the collection of all $p$-subgroups of $G$ is covered by 10.1. Suppose then that $\mathcal{C}$ is as described in the statement of the theorem and $\mathcal{C}$ does not contain all $p$-subgroups of $G$. Let $P \subset G$ be a $p$-subgroup of $G$ which is maximal (under the inclusion ordering) with respect to the property that $P \notin \mathcal{C}$, and let $\mathcal{C}'$ be the union of $\mathcal{C}$ with the set of all conjugates of $P$, so that $\mathcal{C}$ is obtained from $\mathcal{C}'$ by pruning $P$. The group $P$ is not $p$-centric, and we can assume by induction that $\mathcal{C}'$ is subgroup-sharp.

In order to prove that $\mathcal{C}$ is sharp, we will use 9.3 with $K = P$. Let $N = N_G(P)$, $W = N/P$, and $L = \mathrm{link}^{\beta}_{\mathcal{C}'}(P)$. Since $\mathcal{C}'$ contains no proper subgroups of $P$, the group $P$ itself acts trivially on $L$ and the action of $N$ on $L$ factors through an action of $W$. Let $\mathcal{P}(W)$ be the collection of all nonidentity $p$-subgroups of $W$. It follows from the argument in [6, pf. of 8.3] that the space $L$ is weakly $W$-equivalent to the space $X^{\beta}_{\mathcal{P}(W)}$. Briefly, it is clear by inspection and [6, 5.4] that the set of isotropy subgroups of $L$ is the set of nontrivial $p$-subgroups of $W$, and by [6, 2.15] or a calculation with 9.10 that $L^Q$ is contractible for each nontrivial $p$-subgroup $Q$ of $W$. The fact that $L$ is weakly $W$-equivalent to $X^{\beta}_{\mathcal{P}(W)}$ is then a consequence of [6, 4.3]. By 4.8, we can replace $L = \operatorname{link}^{\beta}_{\mathcal{C}'}(P)$ by $X^{\beta}_{\mathcal{P}(W)}$ in checking the isotropy spectral sequence condition from 9.3.

As in 2.6, comparing the isotropy spectral sequence of $\mathrm{E}W \times X^{\beta}_{\mathcal{P}(W)}$ to that of $\mathrm{E}W$ amounts to looking at the pullback square

$$\begin{array}{ccc} (\mathrm{B}P \times X^{\beta}_{\mathcal{P}(W)})_{\mathrm{h}W} & \longrightarrow & \mathrm{B}P_{\mathrm{h}W} = \mathrm{B}N \\ \downarrow & & \downarrow \\ (X^{\beta}_{\mathcal{P}(W)})_{\mathrm{h}W} & \longrightarrow & (*)_{\mathrm{h}W} = \mathrm{B}W \end{array}$$

and comparing the Serre spectral sequences of the two vertical fibrations. In fact, these two spectral sequences are isomorphic at $E^2$. Let $M$ be one of the $W$-modules $\mathrm{H}_j(\mathrm{B}P; \mathbf{F}_p)$, $j \geq 0$. Since $P$ is not $p$-centric, there is an element of order $p$ in $W$ which acts trivially on $M$ (cf. [6, 8.9]). By §7, $\mathcal{P}(W)$ is normalizer-sharp for $M$; in particular, $\mathcal{P}(W)$ is ample for $M$, so that the map $(X^{\alpha}_{\mathcal{P}(W)})_{\mathrm{h}W} \to \mathrm{B}W$ induces an isomorphism

$$\mathrm{H}_*((X^{\alpha}_{\mathcal{P}(W)})_{\mathrm{h}W}; M) \xrightarrow{\cong} \mathrm{H}_*(\mathrm{B}W; M) .$$

As $M$ varies through all of the groups $\mathrm{H}_j(\mathrm{B}P; \mathbf{F}_p)$, these isomorphisms provide the required isomorphism of spectral sequence $E^2$-terms. □

**10.4 Elimination of all non-$p$-stubborn subgroups.** Let $\mathcal{C}'$ be the collection of all $p$-centric subgroups of $G$, and $\mathcal{C} \subset \mathcal{C}'$ the collection of all subgroups which are in addition $p$-stubborn. By 10.3 the collection $\mathcal{C}'$ is subgroup-sharp, and we wish to show that $\mathcal{C}$ is subgroup-sharp too. By 4.8 and 4.4, it is enough to show that the map $X^{\beta}_{\mathcal{C}} \to X^{\beta}_{\mathcal{C}'}$ is a weak $G$-equivalence. Since the isotropy subgroups of these two spaces are contained in $\mathcal{C}'$, it is enough to check that the fixed point map $(X^{\beta}_{\mathcal{C}})^P \to (X^{\beta}_{\mathcal{C}'})^P$ is a weak equivalence for each $P \in \mathcal{C}'$. This is done in [6, 8.10] by descending induction on the size of $P$.

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