MODEL STRUCTURES ON PRO-CATEGORIES

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ABSTRACT. We introduce a notion of a filtered model structure and use this notion to produce various model structures on pro-categories. This framework generalizes the examples of [13], [15], and [16]. We give several examples, including a homotopy theory for G-spaces, where G is a pro-finite group. The class of weak equivalences is an approximation to the class of underlying weak equivalences.

1. Introduction

The goal of this paper is to give a general framework for constructing model structures on pro-categories. Given a proper model structure on $\mathcal{C}$, there is a strict model structure on pro-$\mathcal{C}$ [15]. However, for most purposes, the class of strict weak equivalences on pro-categories is too small. For example, generalized cohomology theories in the strict model structure on pro-spaces do not typically have good computational properties. A more useful class of maps are those that induce pro-isomorphisms on pro-homotopy groups [13]. It turns out that this class of equivalences is exactly the class of maps that are isomorphic to strict n-equivalences for all integers n. In this paper we axiomatize this situation so that it includes many other interesting model structures on pro-categories. Also, we desire to streamline the technical arguments that are usually required in establishing a model structure on a pro-category. In the example described above, the approach in this paper avoids many technical issues from [13] involving basepoints.

The key idea is the notion of a *filtered* model structure. Our main theorem (see Section 5) states that every proper filtered model structure on $\mathcal{C}$ gives rise to a model structure on pro-$\mathcal{C}$.

Because the list of axioms for a filtered model structure is complicated (see Section 4), for now we will give an example that gives a feeling for filtered model structures. Let $\mathcal{C}$ be a category, and let A be a directed set such that (W_a, C_a, F_a) is a model structure on $\mathcal{C}$ for every a in A. Moreover, assume that W_a and C_a are contained in W_b and C_b respectively for $a \geq b$ (i.e., the classes W_a and C_a are "decreasing", so the classes F_a are automatically "increasing"). This is a particular example of a filtered model structure, so there results an associated model structure on pro-$\mathcal{C}$. In pro-$\mathcal{C}$, a pro-map f is a weak equivalence if for all a in A, f is isomorphic to a pro-map that belongs to W_a levelwise. The cofibrations are defined analogously. The fibrations, as usual, can be defined via a lifting property, but we will give a more concrete description of them.

A filtered model structure is a generalization of the situation in the previous paragraph. We still have a directed set A and classes W_a, C_a, and F_a of maps for

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each a in A. However, we do not assume that (W_a, C_a, F_a) is necessarily a model structure. For example, instead of requiring the two-out-of-three property for each class W_a, we only require an "up-to-refinement" property. Namely, for every a in A, there must exist a b in A such that if two out of the three maps f, g, and gf are in W_b, then the third is in W_a. One concrete example of this kind of phenomenon occurs when A is the set of natural numbers and W_n is the class of n-equivalences of spaces. This may seem like an unnatural generalization of the more natural situation from the previous paragraph, but it is important in our main examples. The very nature of pro-categories suggests that up-to-refinement definitions are a sensible approach.

We suspect that this axiomatization is more complicated than it has to be, but we do not know any way to simplify it so that it still includes all the examples of interest.

A t-model structure is a stable model structure $\mathcal{C}$ with a t-structure on its (triangulated) homotopy category, together with a lift of the t-structure to $\mathcal{C}$. This notion is studied in detail in [8], where it is shown that a particularly well-behaved filtered model structure on $\mathcal{C}$ (and thus a model structure on pro-$\mathcal{C}$) can be associated to any t-model structure.

This paper grew out of an attempt to find useful model structures on the category of pro-G-spaces and pro-G-spectra when G is a pro-finite group. Section 8 contains a detailed description of a model structure on G-spaces as an illustration of our general theory. Analogous results for pro-G-spectra are presented in detail in [8] and [9].

We now summarize our interest in pro-G-spaces when G is a pro-finite group.

Let G be a finite group. There is an obvious generalization to pro-G-spaces of the model structure for pro-non-equivariant spaces in the first paragraph. Now the weak equivalences are maps f such that for every n, f is equivalent to a level map g with the property that g^H is a levelwise n-equivalence for every subgroup H of G. One can make similar model structures for arbitrary topological groups.

When G is a pro-finite group, the model structure on pro-G-spaces described above is probably not the right construction. For a pro-finite group, it is the continuous cohomology $\operatorname{colim}_U H^*(G/U; M)$ rather than the group cohomology $H^*(G^\delta; M)$ that is of interest. Here U ranges over the finite-index normal subgroups of G, M is a discrete continous G-module, and G^δ is the group G considered as a discrete group. For finite groups, the group cohomology of G is equal to the Borel cohomology of a point. In other words, the homotopy-orbit space $*_{hG}$ of a point is BG. In model-theoretic terms, what is happening is that one takes a cofibrant replacement EG for $*$ and then takes actual G-orbits to obtain BG.

We desire a model structure on pro-G-spaces such that the system $\{E(G/U)\}$ plays the role of the free contractible space EG. In other words, $\{E(G/U)\}$ should be a cofibrant replacement for $*$. Then $*_{hG}$ is equal to $\{B(G/U)\}$, and the cohomology of $*_{hG}$ is the continuous cohomology of G. Such a model structure is obtained using our machinery by defining a weak equivalence of pro-G-spaces to be a map such that for every integer n, the map is isomorphic to a levelwise map f with f^U an n-equivalence for some finite-index subgroup U of G. The details of this particular example are given in Section 8.

In fact, the n-equivalences in the previous paragraph are irrelevant for the purposes of ordinary continuous cohomology. We could just as well consider a weak equivalence to be a map that is isomorphic to a levelwise map f with f^U a weak equivalence for some finite-index subgroup U of G. The resulting model structure still behaves well with respect to continuous cohomology, but it does not behave well with respect to generalized continuous cohomology theories. The non-equivariant analogue of this phenomenon is explained in detail in [16].

In the context of equivariant model categories that behave well with respect to continuous cohomology, the paper [10] should also be mentioned.

1.1. **Organization.** We begin with a review of pro-categories, including a technical discussion of essentially levelwise properties. Afterwards, we define filtered model structures and prove our main result in Theorem 5.15, which establishes the existence of model structures on pro-categories. The next section considers Quillen functors in this context.

Then we proceed to examples. We first give a few examples of our general theory. Then we focus on constructing G-equivariant homotopy theories when G is a pro-finite group.

1.2. **Background.** We assume that the reader is familiar with model categories, especially in the context of equivariant homotopy theory. The original reference is [18], but we will refer to more modern treatments [11] [12]. This paper is a generalization of [15], and we use specific pro-model category techniques from it.

2. Pro-categories

We begin with a review of the necessary background on pro-categories. This material can be found in [1], [2], [5], [6], and [15].

2.1. Pro-Categories.

Definition 2.1. For a category $\mathcal{C}$, the category **pro-$\mathcal{C}$** has objects all cofiltering diagrams in $\mathcal{C}$, and

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}}(X,Y) = \lim_s \operatorname*{colim}_t \mathrm{Hom}_{\mathcal{C}}(X_t, Y_s).$$

Composition is defined in the natural way.

A category I is **cofiltering** if the following conditions hold: it is non-empty and small; for every pair of objects s and t in I, there exists an object u together with maps $u \to s$ and $u \to t$; and for every pair of morphisms f and g with the same source and target, there exists a morphism h such that fh equals gh. Recall that a category is **small** if it has only a set of objects and a set of morphisms. A diagram is said to be **cofiltering** if its indexing category is so. Beware that some papers on pro-categories, such as [2] and [17], consider cofiltering categories that are not small. All of our pro-objects will be indexed by small categories.

Objects of pro-$\mathcal{C}$ are functors from cofiltering categories to $\mathcal{C}$. We use both set-theoretic and categorical language to discuss indexing categories; hence "$t \geq s$" and "$t \to s$" mean the same thing when the indexing category is actually a cofiltering partially ordered set.

The word *pro-object* refers to an object of a pro-category. A **constant** pro-object is one indexed by the category with one object and one (identity) map. Let $\mathbf{c} : \mathcal{C} \to \text{pro-}\mathcal{C}$ be the functor taking an object X to the constant pro-object with value X. Note that this functor makes $\mathcal{C}$ into a full subcategory of pro-$\mathcal{C}$.

2.2. **Level Maps.** A **level map** $X \to Y$ is a pro-map that is given by a natural transformation (so X and Y must have the same indexing category); this is a very special kind of pro-map. Up to pro-isomorphism, every map is a level map [2, App. 3.2].

Let M be a collection of maps in a category $\mathcal{C}$. A level map g in pro-$\mathcal{C}$ is a **levelwise M-map** if each g_s belongs to M. A pro-map is an **essentially levelwise M-map** if it is pro-isomorphic, in the category of arrows in pro-$\mathcal{C}$, to a levelwise M-map.

We will return to level maps in more detail in Section 3.

2.3. **Cofiniteness.** A partially ordered set $(I, \leq)$ is **directed** if for every s and t in I, there exists u such that $u \geq s$ and $u \geq t$. A directed set $(I, \leq)$ is **cofinite** if for every t, the set of elements s of I such that $s \leq t$ is finite. A pro-object or level map is **cofinite directed** if it is indexed by a cofinite directed set.

Every pro-object is isomorphic to a cofinite directed pro-object [6, Th. 2.1.6] (or [1, Exposé 1, 8.1.6]). Similarly, up to isomorphism, every map is a cofinite directed level map. Cofiniteness is critical for inductive arguments.

Let $f : X \to Y$ be a cofinite directed level map. For every index t, the **relative matching map $M_t f$** is the map

$$X_t \to \lim_{s<t} X_s \times_{\lim_{s<t} Y_s} Y_t.$$

The terminology is motivated by the fact that these maps appear in Reedy model structures [11, Defn. 16.3.2].

Definition 2.2. Let M be any class of maps in $\mathcal{C}$. A map in pro-$\mathcal{C}$ is a **special M-map** if it is isomorphic to a cofinite directed level map f with the property that for each t, the relative matching map $M_t f$ belongs to M.

We will need the following lemma in several places later. Its proof is contained in the proof of [15, Lem. 4.4].

Lemma 2.3. *Let $f : X \to Y$ be a cofinite directed level map. For every s, the map $\lim_{t<s} X_t \to \lim_{t<s} Y_t$ is a finite composition of base changes of the relative matching maps $M_t f$ for $t < s$.*

2.4. **Simplicial structures on pro-categories.** Recall that a category $\mathcal{C}$ is simplicial if for every object X of $\mathcal{C}$ and every simplicial set K, there exist objects $X \otimes K$ (called a tensor) and X^K (called a cotensor) of $\mathcal{C}$ satisfying certain adjointness properties [11, Sec. 9.1]. Moreover, for every X and Y in $\mathcal{C}$, there is a simplicial set $\mathrm{Map}(X, Y)$ that interacts appropriately with the tensor and cotensor.

If $\mathcal{C}$ is a simplicial category, then pro-$\mathcal{C}$ is again a simplicial category. Tensors and cotensors with finite simplicial sets are defined levelwise, while tensors and cotensors with arbitrary simplicial sets are defined via limits and colimits. If X and Y belong to pro-$\mathcal{C}$, then $\mathrm{Map}(X, Y)$ is defined to be $\lim_s \mathrm{colim}_t \mathrm{Map}(X_t, Y_s)$. See [13, Sec. 16] for more details.

3. Essentially levelwise properties

Later we will frequently encounter situations where a single pro-map is an essentially levelwise M_a-map for all a, where $\{M_a\}$ is a collection of classes of maps (see especially Definitions 5.1, 5.2, and 8.9). This situation has some subtleties that are worth exploring.

If f is an essentially levelwise M_a-map for all a, it does not follow that f is an essentially levelwise $(\cap M_a)$-map. The problem is that different values of a might require different level maps, even though they are all pro-isomorphic to f. However, we will show below in Corollary 3.3 that f does have a slightly more complicated but still concrete property.

We first need the following technical lemma for constructing isomorphisms in pro-categories.

Lemma 3.1. *Let Y be a pro-object. Suppose that for some of the maps $t \to s$ in the indexing diagram for Y, there exists an object Z_{ts} and a factorization $Y_t \to Z_{ts} \to Y_s$ of the structure map $Y_t \to Y_s$. Also suppose that for every s, there exists at least one $t \to s$ with this property. The objects Z_{ts} assemble into a pro-object Z that is isomorphic to Y.*

Proof. We may assume that Y is indexed by a directed set I because every pro-object is isomorphic to a pro-object indexed by a directed set [6, Thm. 2.1.6]. Define a new directed set K as follows. The elements of K consist of pairs (t, s) of elements of I such that $t \geq s$ and a factorization $Y_t \to Z_{ts} \to Y_s$ exists. If (t, s) and (t', s') are two elements of K, we say that $(t', s') \geq (t, s)$ if $s' \geq t$. It can easily be checked that this makes K into a directed set.

Note that the function $K \to I : (t, s) \mapsto s$ is cofinal in the sense of [2, App. 1]. This means that we may reindex Y along this functor and assume that Y is indexed by K; thus we write $Y_{(t,s)} = Y_s$.

We define the pro-object Z to be indexed by K by setting $Z_{(t,s)} = Z_{ts}$. If $(t', s') \geq (t, s)$, then the structure map $Z_{(t',s')} \to Z_{(t,s)}$ is the composition

$$Z_{t's'} \to Y_{s'} \to Y_t \to Z_{ts},$$

It can easily be checked that this gives a functor defined on K; here is where we use that the composition $Y_t \to Z_{ts} \to Y_s$ equals $Y_t \to Y_s$.

Finally, we must show that Z is isomorphic to Y. We use the criterion from [13, Lem. 2.3] for detecting pro-isomorphisms. Given any (t, s) in K, choose u such that (u, t) is in K. Then there exists a diagram

$$\begin{array}{ccc} Z_{(u,t)} & \longrightarrow & Y_{(u,t)} = Y_t \\ \downarrow & \swarrow & \downarrow \\ Z_{(t,s)} & \longrightarrow & Y_{(t,s)} = Y_s. \end{array}$$

□

In the rest of this section, we will frequently consider a collection of classes C_a indexed by a directed set such that C_b is contained in C_a whenever $b \geq a$. We say that the classes C_a are a **decreasing collection** if they satisfy this property.

Lemma 3.2. *Let $\{C_a\}$ be a decreasing collection of classes of objects indexed by a directed set A. For each a in A, let C_a be a class of objects such that C_b is contained in C_a when $b \geq a$. An object in pro-$\mathcal{C}$ belongs to C_a essentially levelwise for all a if and only if it is isomorphic to a pro-object X indexed by a directed set I such that for all a in A, there exists an element s in I (depending on a) with the property that X_t belongs to C_a for all $t \geq s$.*

The idea is that for every a, X "eventually" belongs to C_a levelwise. However, the height at which X belongs to C_a may depend on a.

Proof. Suppose that X is a pro-object indexed by a directed set I such that for all a in A, there exists an element s in I (depending on a) with the property that X_t belongs to C_a for all $t \geq s$. Given a, choose s such that X_t belongs to C_a for all $t \geq s$. Let I' be the subset of I consisting of all $t \geq s$. By restricting to I', we obtain a new pro-object X' such that X' is isomorphic to X. By construction, X' belongs to C_a levelwise. Since the above argument works for all a in A, this shows that X belongs to C_a essentially levelwise for every a.

Now we consider the other direction. Let X be a pro-object such that X belongs to C_a essentially levelwise for all a. We may assume that X is indexed by a directed set I because every pro-object is isomorphic to a pro-object indexed by a directed set [6, Thm. 2.1.6]. Reindex X so that its indexing set is $I \times A$, where $X_{s,a}$ equals X_s. This only changes X up to isomorphism.

For every element a of A, choose an isomorphism $X \to Y^a$ where Y^a is a pro-object that belongs to C_a levelwise. The existence of this isomorphism implies that for any s in I, there exists $s' \geq s$ such that the structure map $X_{s',a} \to X_{s,a}$ factors through $Y^a_{\phi(s,a)}$ for some $\phi(s,a)$ belonging to the indexing category of Y^a (see [13, Lem. 2.3] for a similar situation). By Lemma 3.1, the objects $Y^a_{\phi(s,a)}$ assemble into a pro-object Z indexed by $I \times A$ that is isomorphic to X. By construction, Z has the property that $Z_{s,a'}$ belongs to C_a for all s in I and all $a' \geq a$. □

Corollary 3.3. *Let $\{M_a\}$ be a decreasing collection of classes of maps indexed by a directed set A. A map in pro-$\mathcal{C}$ is an essentially levelwise M_a-map for all a if and only if it has a level representation f indexed by a directed set I such that for all a in A, there exists an element s in I (depending on a) with the property that f_t belongs to M_a for all $t \geq s$.*

Proof. This follows immediately from Lemma 3.2, using that the category Ar(pro-$\mathcal{C}$) of arrows in pro-$\mathcal{C}$ is equivalent to the category pro-Ar($\mathcal{C}$) [17]. □

We now discuss how functors preserve essentially levelwise properties. This issue will be particularly relevant later in Section 6 when we consider Quillen functors between pro-categories.

If $F : \mathcal{C} \to \mathcal{C}'$ is a functor, then F induces another functor pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ in a natural way; we abuse notation and use the same symbol F for this functor. If $X = \{X_s\}$ is a filtered diagram, then FX is defined to be the filtered diagram $\{FX_s\}$.

Lemma 3.4. *Let $\{C_a\}$ and $\{C'_{a'}\}$ be decreasing collections of classes of objects in $\mathcal{C}$ and $\mathcal{C}'$ respectively indexed on directed sets A and A' respectively. Suppose for every $a' \in A'$ that there is an $a \in A$ such that $F(C_a)$ is contained in $C_{a'}$. Consider the class of objects in pro-$\mathcal{C}$ that belong to C_a essentially levelwise for all a in A. Then $F :$ pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ takes this class to the class of objects in pro-$\mathcal{C}'$ that belong to $C'_{a'}$ essentially levelwise for all a' in A'.*

Proof. For every a', there is an a such that F takes pro-objects that belong levelwise to C_a to pro-objects that belong levelwise to $C'_{a'}$. Finally, just recall that the functor F preserves isomorphisms. □

Corollary 3.5. *Let $\{M_a\}$ and $\{M'_{a'}\}$ be decreasing collections of classes of maps in $\mathcal{C}$ and $\mathcal{C}'$ respectively indexed on directed sets A and A' respectively. Suppose for every $a' \in A'$ that there is an $a \in A$ such that $F(M_a)$ is contained in $M'_{a'}$. Consider the class of maps in pro-$\mathcal{C}$ that are essentially levelwise M_a-maps for all a in A. Then F : pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ takes this class to the class of maps in pro-$\mathcal{C}'$ that are essentially levelwise $M'_{a'}$-maps for all a' in A'.*

Proof. This follows immediately from Lemma 3.4, again using that the categories Ar(pro-$\mathcal{C}$) and pro-Ar($\mathcal{C}$) are equivalent. □

With some assumptions, the sufficient condition given in Lemma 3.4 is in fact necessary. Before we can prove this, we need another technical lemma.

Lemma 3.6. *Let $\mathcal{C}$ be a category and let C be a class of objects in $\mathcal{C}$ closed under retract. Let X be a pro-object indexed by a directed set I such that for all $t \geq s$, the structure map $X_t \to X_s$ has a left inverse (so X_t is a retract of X_s). If X belongs to C essentially levelwise, then there is an element $s \in I$ such that X_t is in C for all $t \geq s$.*

Proof. Let Y belong to C levelwise such that X is isomorphic to Y. The existence of this isomorphism implies that for every u, there exists $s \geq u$ and v such that the structure map $X_s \to X_u$ factors through Y_v. Since X_s is a retract of X_u, it follows that X_s is also a retract of Y_v. But Y_u belongs to C, so X_s also belongs to C.

For any $t \geq s$, X_t is a retract of X_s, so X_t also belongs to C. □

Lemma 3.7. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories such that $\mathcal{C}$ is cocomplete possessing an initial and terminal object $*$. Let $\{C_a\}$ and $\{C'_{a'}\}$ be decreasing collections of classes of objects in $\mathcal{C}$ and $\mathcal{C}'$ respectively that are indexed by directed sets A and A'. Suppose that each C_a is closed under small coproducts and that each $C'_{a'}$ is closed under retracts. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor such that the associated functor F : pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ has the property that if X belongs to C_a essentially levelwise for all a in A, then $F(X)$ belongs to $C'_{a'}$ essentially levelwise for all a' in A'. Then for every a' in A', there exists an element a of A such that $F(C_a)$ is contained in $C'_{a'}$.*

Proof. Let a' be an element in A'. Suppose for contradiction that for every a in A, $F(C_a)$ is not contained in $C'_{a'}$. For each b in A, we can choose an object X_b in C_b such that FX_b is not in $C'_{a'}$. We construct a pro-object Y indexed on A by letting Y_a be

$$\amalg_{c \geq a} X_c.$$

The structure maps are given by the canonical inclusions whenever $b \geq a$.

Since the classes C_a are decreasing and closed under coproducts, each Y_b belongs to C_a for $b \geq a$. Lemma 3.2 implies that Y belongs to C_a essentially levelwise for all a. Therefore, FY belongs to $C'_{a'}$ essentially levelwise for all a'.

Since $\mathcal{C}$ is pointed, each structure map $Y_b \to Y_a$ has a left inverse given by projecting some of the factors to $*$. Therefore each structure map $FY_b \to FY_a$ of FY has a left inverse. Lemma 3.6 applies, so for every a', there exists b such that FY_c belongs to $C'_{a'}$ whenever $c \geq b$.

Note that X_a is a retract of Y_a for all a, again using that $\mathcal{C}$ is pointed. Therefore FX_a is a retract of FY_a for all a. We have shown in the previous paragraph that FY_c belongs to $C'_{a'}$ for some value of c. Since $C'_{a'}$ is closed under retract, FX_c also belongs to $C'_{a'}$. This contradicts the way in which we chose X_c. □

Corollary 3.8. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories such that $\mathcal{C}$ is cocomplete possessing an initial and terminal object $*$. Let $\{M_a\}$ and $\{M'_{a'}\}$ be decreasing collections of classes of maps in $\mathcal{C}$ and $\mathcal{C}'$ respectively that are indexed by directed sets A and A'. Suppose that each M_a is closed under small coproducts and that each $M'_{a'}$ is closed under retracts. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor such that the associated functor F : pro-$\mathcal{C}$ $\to$ pro-$\mathcal{C}'$ has the property that if f is an essentially levelwise M_a-map for all a in A, then Ff is an essentially levelwise $M'_{a'}$-map for all a' in A'. Then for every a' in A', there exists an element a of A such that $F(M_a)$ is contained in $M'_{a'}$.*

Proof. This follows immediately from the previous lemma, again using that the categories Ar(pro-$\mathcal{C}$) and pro-Ar($\mathcal{C}$) are equivalent. □

4. Filtered model structures

We now describe the axiomatic setup that we use to produce model structures on pro-categories.

Definition 4.1. A **filtered model structure** consists of a complete and cocomplete category $\mathcal{C}$ equipped with:

(1) a directed set A,
(2) for each a in A, a class C_a of maps in $\mathcal{C}$ such that C_b is contained in C_a whenever $b \geq a$,
(3) for each a in A, a class W_a of maps in $\mathcal{C}$ such that W_b is contained in W_a whenever $b \geq a$,
(4) for each a in A, a class F_a of maps in $\mathcal{C}$ such that F_a is contained in F_b whenever $b \geq a$.

These classes must satisfy Axioms 4.2 through 4.6, which are described below.

Axiom 4.2. For all a in A, there exists b in A such that if f and g are two composable morphisms of $\mathcal{C}$ with any two of the three maps f, g, and gf in W_b, then the third is in W_a.

Axiom 4.3. For all a in A, the classes W_a, C_a, and F_a are closed under retract. The classes C_a and F_a are closed under composition. The class C_a is closed under arbitrary cobase change, and the class F_a is closed under arbitrary base change.

Recall that if M is any class of maps, then **inj-M** is the class of maps that have the right lifting property with respect to every map in M. Similarly, **proj-M** is the class of maps that have the left lifting property with respect to every map in M.

Axiom 4.4. For every a in A, the class inj-C_a is contained in $W_a \cap F_a$, and the class proj-F_a is contained in $W_a \cap C_a$.

Axiom 4.5. For every map f in $\mathcal{C}$ and every a in A, f can be factored as pi, where i belongs to C_a and p belongs to inj-C_a, or as qj, where j belongs to proj-F_a and q belongs to F_a.

Axiom 4.6. If f belongs to W_a for some a in A, then there exists $b \geq a$ such that f factors as pi, where i belongs to proj-F_b and p belongs to inj-C_b.

Definition 4.7. Let $\boldsymbol{F}$ be the union of the classes F_a for all a. Let **inj-C** be the union of the classes inj-C_a for all a.

We think of $\{F_a\}$ as an increasing filtration on F. Since the classes C_a are decreasing, the classes inj-C_a are increasing. Thus, $\{\text{inj-}C_a\}$ is an increasing filtration on inj-C.

Definition 4.8. A filtered model structure is **proper** if it satisfies the following two additional axioms.

Axiom 4.9. For every a in A, cobase changes of maps in W_a along maps in C_a are in W_a.

Axiom 4.10. For every a in A, base changes of maps in W_a along maps in F are in W_a.

Axioms 4.9 and 4.10 are a kind of properness. They turn out to be necessary in various technical arguments concerning pro-objects. It may appear at first glance that Axiom 4.10 is significantly stronger than Axiom 4.9. However, this is not the case. Recall that the classes C_b are decreasing.

Definition 4.11. A filtered model structure $\mathcal{C}$ is **simplicial** if $\mathcal{C}$ is a simplicial category satisifying the following additional axiom.

Axiom 4.12.

(1) If $j : K \to L$ is a cofibration of finite simplicial sets and $i : A \to B$ belongs to C_a, then the pushout-product map
$$f : A \otimes L \amalg_{A\otimes K} B \otimes K \to B \otimes L$$
belongs to C_a.

(2) If in addition j is a weak equivalence or i belongs to proj-F_a, then f belongs to proj-F_a.

Axiom 4.12 can be reformulated in the following two equivalent ways. The usual arguments with adjoints establish the equivalence.

Reformulation 4.13.

(1) If $j : K \to L$ is a cofibration of finite simplicial sets and $p : X \to Y$ belongs to F_a, then the map
$$f : X^L \to Y^L \times_{Y^K} X^K$$
belongs to F_a.

(2) If in addition j is a weak equivalence or p belongs to inj-C_a, then f belongs to inj-C_a.

Reformulation 4.14.

(1) If $i : A \to B$ belongs to C_a and $p : X \to Y$ belongs to F_a, then the map
$$f : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$
is a fibration of simplicial sets.

(2) If in addition i belongs to proj-F_a or p belongs to inj-C_a, then f is an acyclic fibration.

Remark 4.15. The axioms for a filtered model structure are almost but not quite symmetric; see for example the inclusion relations for the classes C_a and F_a. The reason for this asymmetry is that the construction of pro-$\mathcal{C}$ from $\mathcal{C}$ is not symmetric. If we were interested in producing model structures on the ind-category ind-$\mathcal{C}$, then we would need to dualize the notion of a filtered model category.

At this point, we prove one simple lemma about filtered model structures that we will need later.

Lemma 4.16. *For any two elements a and b of A, the class inj-C_a is contained in the class W_b.*

Proof. Since A is directed, we may choose c such that $c \geq a$ and $c \geq b$. Since C_c is contained in C_a, it follows formally that inj-C_a is contained in inj-C_c. Now Axiom 4.4 implies that inj-C_c is contained in W_c, which is contained in W_b. □

As an illustration of the definitions in this section, we provide several general situations that produce filtered model structures.

Proposition 4.17. *Suppose that A consists of only a single element. A filtered model structure (resp., proper filtered model structure, simplicial filtered model structure) indexed by A is the same as an ordinary model structure (resp., proper model structure, simplicial model structure) on $\mathcal{C}$.*

Proof. It is easy to verify that an ordinary model structure (resp., ordinary proper model structure, ordinary simplicial model structure) gives a filtered model structure where A has only one element.

For the converse, suppose given a filtered model structure where A has only one element. We write C, W, and F for the three classes given in the definition. The two-out-of-three and retract axioms are immediate from Axioms 4.2 and 4.3. The factorization axiom follows from Axioms 4.4 and 4.5.

The lifting axiom requires more explanation. Given a map p in $W \cap F$, use Axiom 4.6 to factor it as qj, where j belongs to proj-F and q belongs to inj-C. By the retract argument, p is a retract of q. Since inj-C is formally closed under retracts, it follows that p also belongs to inj-C. This shows that maps in C lift with respect to maps in $W \cap F$. The proof of the other half of the lifting axiom is identical. This finishes the proof of the first claim.

For the second claim, if the filtered model structure is proper, then properness for the ordinary model structure is given by Axioms 4.9 and 4.10.

Finally, for the third claim, if the filtered model structure is simplicial, then Axiom 4.12 implies that the ordinary model structure is simplicial. Note that for formal reasons (see, for example, [13, Prop. 16.1]), it is enough to check the axioms for a simplicial model structure only for finite simplicial sets. □

Proposition 4.18. *Let A be a directed set. For each a in A, let (C_a, W_a, F_a) be a proper model structure on $\mathcal{C}$ such that C_a and W_a are contained in C_b and W_b respectively when $a \geq b$. Then (A, C, W, F) is a proper filtered model structure on $\mathcal{C}$.*

Proof. Using that inj-C_a equals $W_a \cap F_a$ and proj-F_a equals $W_a \cap C_a$, the verification of Axioms 4.2 through 4.10 follow immediately from basic properties of model structures. □

Proposition 4.19. *Let $\mathcal{C}$ be a proper cofibrantly generated model category with a set I of generating cofibrations and a set J of generating acyclic cofibrations. Let A be a directed set, and for each a in A, let I_a be a subset of I such that I_a is contained in I_b if $a \geq b$. For each a in A, define C_a to be the class of all cofibrations, F_a to be inj-$(I_a \cup J)$, and W_a to be the maps that are the composition of a map in proj-F_a*

followed by an acyclic fibration. If each W_a is closed under retract and Axiom 4.2 is satisfied, then (A, C, W, F) is a filtered model structure.

Later in Section 7 we describe several concrete examples of this situation. Beware that the filtered model structures arising from this proposition are not necessarily proper.

Proof. We need to show that Axioms 4.2 through 4.6 are satisfied. We have assumed that Axiom 4.2 holds.

For Axiom 4.3, we have assumed that each W_a is closed under retract. The class C_a is closed under retract because the class of cofibrations in $\mathcal{C}$ is closed under retract. The class F_a is closed under retract because it is defined by a right lifting property.

The first part of Axiom 4.4 is satisfied because W_a and F_a both contain the acyclic fibrations of $\mathcal{C}$. For the second part, every map in proj-F_a is a cofibration because F_a contains the acyclic fibrations of $\mathcal{C}$. Therefore, proj-F_a is contained in C_a. It remains to explain why proj-F_a is contained in W_a, but this follows immediately from the definition of W_a.

The first factorizations required by Axiom 4.5 are just the usual factorizations in $\mathcal{C}$ of maps into cofibrations followed by acyclic fibrations. The second factorizations can be produced by applying the small object argument to the set $I_a \cup J$.

Axiom 4.6 is satisfied by definition of W_a. □

4.1. **A possible weakening of the axioms.** For expository reasons, we have chosen a form of Axiom 4.3 that is unnecessarily strong. To develop most of our theory, it suffices to assume only that the classes C_a and W_a are closed under retract. In any case, we can always replace C_a and F_a by the smallest classes of maps containing C_a and F_a respectively and satisfying the conditions in 4.3. These new classes (with W_a unchanged) will still satisfy all of the rest of the axioms for a filtered model structure. However, Axioms 4.9, 4.10, and 4.12 might no longer be satisfied.

Axioms 4.9, 4.10, and 4.12 are not up-to-refinement properties, unlike Axiom 4.2. In fact, there is an obvious way to weaken these three axioms to make them up-to-refinement. We have chosen to not formalize this in our definitions because we know of no examples in which the added generality is necessary.

The more general form of Axiom 4.9 states that for every a in A, there exists b and c in A such that cobase changes of maps in W_b along maps in C_c are in W_a.

The more general form of Axiom 4.10 states that for every a in A, there exists b in A such that base changes of maps in W_b along maps in F are in W_a.

The more general form of Axiom 4.12 states that for every a in A, there exists b in A such that if $j : K \to L$ is a cofibration of finite simplicial sets and $i : A \to B$ belongs to C_b, then the pushout-product map

$$f : A \otimes L \amalg_{A \otimes K} B \otimes K \to B \otimes L$$

belongs to C_a; if in addition j is a weak equivalence or i belongs to proj-F_b, then f belongs to proj-F_a.

5. Model structures on pro-$\mathcal{C}$

Throughout this section, let $\mathcal{C}$ be a category equipped with a proper filtered model structure (A, C, W, F) as in the previous section.

Definition 5.1. A map in pro-$\mathcal{C}$ is a **cofibration** if it is an essentially levelwise C_a-map for every a in A.

Definition 5.2. A map in pro-$\mathcal{C}$ is a **weak equivalence** if it is an essentially levelwise W_a-map for all a in A.

If i is a cofibration, there is no guarantee that i has one level replacement that is a level C_a-map for every a. Typically, the choice of level replacement depends on a. The same warning applies to weak equivalences.

Corollary 3.3 can be used to give a slightly more concrete description of the cofibrations and weak equivalences. For understanding specific examples, this more concrete description is often helpful. However, for proving general results, we prefer to work with the more abstract definition.

Definition 5.3. A map in pro-$\mathcal{C}$ is a **fibration** if it is a retract of a special F-map.

Recall that an **acyclic cofibration** is a map that is both a cofibration and a weak equivalence. Similarly, an **acyclic fibration** is a map that is both a fibration and a weak equivalence.

We will eventually prove that these definitions yield a model structure on pro-$\mathcal{C}$. First we need a series of lemmas that will lead to the proof. Our approach follows [15].

Lemma 5.4. *Suppose that f and g are two composable morphisms in pro-$\mathcal{C}$. If any two of f, g, and gf are weak equivalences, then so is the third.*

Proof. Suppose that two of f, g, and gf are weak equivalences. We need to show that the third is an essentially levelwise W_a-map for every a. Choose b such that if any two of ϕ, ψ, and $\psi\phi$ are in W_b, then the third is in W_a; this is possible by Axiom 4.2.

If we assume that any two of f, g, and gf are essentially levelwise W_b-maps, then the proofs of [15, Lem. 3.5] and [15, Lem. 3.6] can be applied to conclude that the third is an essentially levelwise W_a-map. Note that [15, Lem. 3.2] works for C_b and inj-C_b (or for proj-F_b and F_b) because of Axiom 4.5 (see [15, Rem. 3.3]). To make these proofs work, we need Axioms 4.9 and 4.10.

To illustrate the point, we describe in detail how to adapt the proof of [15, Lem. 3.5] to our situation. Suppose that f and g are essentially levelwise W_b-maps. We wish to show that gf is an essentially levelwise W_a-map.

We may assume that f and g are levelwise W_b-maps, but their index categories are not necessarily the same. However, we can obtain a levelwise diagram

$$X \xrightarrow{f} Y \xleftarrow[\cong]{h} Z \xrightarrow{g} W$$

in which f and g are levelwise W_b-maps while h is a pro-isomorphism (but not a levelwise isomorphism). We must construct a levelwise W_a-map isomorphic to the composition $gh^{-1}f$.

By [15, Lem. 3.2], after reindexing we can factor $h : Z \to Y$ into a levelwise C_b-map $Z \to A$ followed by a levelwise F_b-map $A \to Y$ such that both maps are pro-isomorphisms. Here we are using Axiom 4.5 to provide the necessary factorizations in $\mathcal{C}$ (and also Axiom 4.4 to identify that a map in inj-C_a is necessarily in F_a). We now have a diagram

$$X \longrightarrow Y \xleftarrow{\cong} A \xleftarrow{\cong} Z \longrightarrow W$$

in which the first and fourth maps are levelwise W_b-maps, and the second and third are pro-isomorphisms.

Let B be the pullback $X \times_Y A$, and let C be the pushout $A \amalg_Z W$, which we may construct levelwise. The map $B \to A$ is levelwise a base change of a map in W_b along a map in F_b. By Axiom 4.10, $B \to A$ is a levelwise W_b-map. Similarly, the map $A \to C$ is levelwise a cobase change of a map in W_b along a map in C_b. By Axiom 4.9, $A \to C$ is a levelwise W_b-map.

The maps $B \to X$ and $W \to C$ are pro-isomorphisms since base and cobase changes preserve isomorphisms. Hence the composition $B \to C$ is isomorphic to $gh^{-1}f$ as desired. Moreover, $B \to C$ is levelwise a composition of two maps in W_b, which means that it is a levelwise W_a-map because of the way in which b was chosen. □

Recall the following lemma from [14, Thm. 5.5].

Lemma 5.5. *Let M be any class of maps in $\mathcal{C}$. Then the class of essentially levelwise M-maps in pro-$\mathcal{C}$ is closed under retracts.*

Corollary 5.6. *The class of cofibrations and the class of weak equivalences in pro-$\mathcal{C}$ are closed under retract.*

Proof. The class of cofibrations is the intersection of a set of classes, each of which is closed under retract by Lemma 5.5. The same argument applies to the weak equivalences. □

Lemma 5.7. *Every map $f : X \to Y$ in pro-$\mathcal{C}$ factors as a cofibration $i : X \to Z$ followed by a special inj-C-map $p : Z \to Y$.*

Proof. We may suppose that f is a level map indexed by a cofinite directed set I. Moreover, by adding isomorphisms to cofiltered diagrams, we may assume that I has cardinality larger than A. Choose an arbitrary function $\phi : I \to A$ such that $\phi(s) \geq \phi(t)$ if $s \geq t$ and such that for all a in A, there exists s in I such that $\phi(s) \geq a$; this function can be constructed inductively because I is cofinite and because the cardinality of I is larger than the cardinality of A.

Suppose for induction that the maps $i_t : X_t \to Z_t$ and $p_t : Z_t \to Y_t$ have already been defined for $t < s$. Consider the map

$$X_s \to Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} Z_t.$$

Use Axiom 4.5 to factor it into a map $i_s : X_s \to Z_s$ belonging to $C_{\phi(s)}$ followed by a map

$$q_s : Z_s \to Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} Z_t$$

belonging to inj-$C_{\phi(s)}$. Let p_s be the map $Z_s \to Y_s$ induced by q_s. This extends the factorization to level s.

It follows immediately from its construction that p is a special inj-C-map. To show that i is a cofibration, just apply Lemma 3.2. □

Lemma 5.8. *Every map $f : X \to Y$ in pro-$\mathcal{C}$ factors into a map $i : X \to Z$ followed by a special F-map $p : Z \to Y$, where i is an essentially levelwise proj-F_a-map for every a.*

Proof. The proof is identical to the proof of Lemma 5.7, except that we factor the map

$$X_s \to Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} Z_t$$

into a map belonging to proj-$F_{\phi(s)}$ followed by a map belonging to $F_{\phi(s)}$. □

Lemma 5.9. *A map in pro-$\mathcal{C}$ is a cofibration if and only if it has the left lifting property with respect to all retracts of special inj-C-maps. Also, a map in pro-$\mathcal{C}$ is a retract of a special inj-C-map if and only if it has the right lifting property with respect to all cofibrations.*

Proof. First we will show that cofibrations have the left lifting property with respect to retracts of special inj-C-maps. Let $i : A \to B$ be a cofibration, and let p be a retract of a special inj-C-map. Since retracts preserve lifting properties, it suffices to assume that p is a special inj-C-map. Moreover, as shown in [15, Prop. 5.2], a special inj-C-map is a composition along a transfinite tower, each of whose maps is a base change of a map of the form $cX \to cY$, where $X \to Y$ belongs to inj-C. Since base changes and transfinite compositions preserve lifting properties, it suffices to assume that p is the map $cX \to cY$, where $X \to Y$ belongs to inj-C.

The map $X \to Y$ belongs to inj-C_a for some a. Since i is a cofibration, it is an essentially levelwise C_a-map. Therefore, we may assume that i is a levelwise C_a-map.

Suppose given a square

$$\begin{array}{ccc} A & \longrightarrow & cX \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & cY \end{array}$$

in pro-$\mathcal{C}$. This square is represented by a square

$$\begin{array}{ccc} A_s & \longrightarrow & X \\ {\scriptstyle i_s}\downarrow & & \downarrow \\ B_s & \longrightarrow & Y \end{array}$$

in $\mathcal{C}$. This square has a lift because $X \to Y$ has the right lifting property with respect to all maps in C_a. Finally, this lift represents a map $B \to cX$ that is our desired lift.

Now suppose that a map $i : A \to B$ has the left lifting property with respect to all special inj-C-maps. Use Lemma 5.7 to factor i as a cofibration $i' : A \to B'$ followed by a special inj-C-map $p : B' \to B$. Since i has the left lifting property with respect to p by assumption, the retract argument implies that i is a retract of i'. But retracts preserve cofibrations by Corollary 5.6, so i is again a cofibration.

Finally, suppose that $p : X \to Y$ has the right lifting property with respect to all cofibrations. Use Lemma 5.7 to factor p as a cofibration $i : X \to X'$ followed by a special inj-C-map $p' : X' \to Y$. Similarly to the previous paragraph, p is a retract of p', so p is a retract of a special inj-C-map. □

Lemma 5.10. *A map in pro-$\mathcal{C}$ is an essentially levelwise proj-F_a-map for all a if and only if it has the left lifting property with respect to all fibrations. Also, a map in pro-$\mathcal{C}$ is a fibration if and only if it has the right lifting property with respect to all essentially levelwise proj-F_a-maps for all a.*

Proof. The proof is the same as the proof of Lemma 5.9, except that the role of cofibrations is replaced by the maps that are essentially levelwise proj-F_a-maps for every a and special inj-C-maps are replaced by special F-maps. Lemma 5.8 is relevant instead of Lemma 5.7. Note also that retracts preserve the maps that are essentially levelwise proj-F_a-maps for every a; the proof is like the proof of Corollary 5.6. □

Proposition 5.11. *A map in pro-$\mathcal{C}$ is an acyclic cofibration if and only if it is an essentially levelwise proj-F_a-map for every a.*

Proof. One implication follows from the definitions and the fact that proj-F_a is contained in $W_a \cap C_a$ by Axiom 4.4.

For the other implication, let $i : A \to B$ be a weak equivalence and cofibration. Fix an arbitrary a; we will show that i is an essentially levelwise proj-F_a-map.

Given a, begin by choosing b as in Axiom 4.2. We may assume that i is a level map such that each i_s belongs to W_b. As in the proof of Lemma 5.7, we may use Axiom 4.6 to factor i into a map $i' : A \to B'$ followed by a map $p : B' \to B$ such that for each s, i'_s belongs to proj-F_c and the relative matching map

$$M_s p : B'_s \to B_s \times_{\lim_{t<s} B_t} \lim_{t<s} B'_t$$

belongs to inj-C_c for some $c \geq a$.

Actually, in order to apply Axiom 4.6, we need to prove inductively that the map $A_s \to B'_s \times_{\lim_{t<s} B'_t} \lim_{t<s} B_t$ belongs to W_a. To do this, consider the diagram

$$\begin{array}{ccccc} A_s & \longrightarrow & B_s \times_{\lim_{t<s} B_t} \lim_{t<s} B'_t & \longrightarrow & \lim_{t<s} B'_t \\ & \searrow & \downarrow & & \downarrow \\ & & B_s & \longrightarrow & \lim_{t<s} B_t. \end{array}$$

Using Axiom 4.2, we just need to show that the diagonal map and the left vertical map belong to W_b. The diagonal map belongs to W_b by the assumption on i. On the other hand, the right vertical map belongs to inj-C by Lemma 2.3 and the induction assumption. Therefore, the left vertical map is also in inj-C since inj-C is closed under base changes for formal reasons. Now Lemma 4.16 implies that it belongs to W_b.

At this point, we have factored i as pi', where p is a special inj-C-map. Since i is a cofibration, it has the left lifting property with respect to p by Lemma 5.9. The retract argument now implies that i is a retract of i'. Because essentially levelwise proj-F_a-maps are closed under retract by Lemma 5.5, it suffices to show that i' is a levelwise proj-F_a-map. Recall that each i'_s belongs to proj-F_c for some $c \geq a$. Since F_a is contained in F_c, it follows that proj-F_c is contained in proj-F_a; therefore, each i'_s belongs to proj-F_a. □

The following proposition, although not actually necessary to establish the existence of the model structure on pro-$\mathcal{C}$, is a useful detection principle for acyclic cofibrations. It says that the fibrations are "generated" by a certain very simple class of fibrations.

Proposition 5.12. *A map $i : A \to B$ is an acyclic cofibration if and only if it has the left lifting property with respect to all constant pro-maps $cX \to cY$ in which $X \to Y$ belongs to F.*

Proof. Lemma 5.10 and Proposition 5.11 imply that i is an acyclic cofibration if and only if it has the left lifting property with respect to all special F-maps. By [15, Prop. 5.2], every special F-map is a transfinite composition of a tower of maps, each of whose maps is a base change of a map of the form $cX \to cY$ with $X \to Y$ in F. By a formal argument with lifting properties, a map has the left lifting property with respect to all special F-maps if and only if it has the left lifting property with respect to maps of the form $cX \to cY$ with $X \to Y$ in F. □

Proposition 5.13. *A map p is an acyclic fibration if and only if it is a retract of a special inj-C-map.*

Proof. First suppose that p is a retract of a special inj-C-map. The class of acyclic fibrations is closed under retract by Corollary 5.6 and by the definition of fibrations, so it suffices to assume that p is a special inj-C-map. Since each class inj-C_a is contained in F_a by Axiom 4.4, it follows that the union inj-C is contained in the union F. Therefore, every special inj-C-map is a special F-map. This shows that p is a fibration.

It remains to show that p is a weak equivalence. Given a fixed s, we will show that $p_s : X_s \to Y_s$ belongs to W_a for all a. Then p is a level W_a-map for every a and thus a weak equivalence. In order to do this, Lemma 4.16 tells us that we only have to show that p_s belongs to inj-C.

We may choose an element a of A such that $M_t p$ belongs to inj-C_a for every $t \leq s$. This follows from cofiniteness and the fact that the classes inj-C_b are increasing.

The map $p_s : X_s \to Y_s$ factors as

$$X_s \xrightarrow{M_s p} Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} X_t \xrightarrow{q_s} Y_s.$$

Our goal is to show that p_s belongs to inj-C_a. Since compositions and base changes preserve inj-C_a for formal reasons, it suffices to show that $\lim_{t<s} p_t : \lim_{t<s} X_t \to \lim_{t<s} Y_t$ belongs to inj-C_a. This last map is a finite composition of maps that are base changes of the maps $M_t p$ for $t \leq s$ (see Lemma 2.3). Finally, recall that each $M_t p$ belongs to inj-C_a. This finishes one implication.

Now suppose that $p : X \to Y$ is an acyclic fibration. Use Lemma 5.7 to factor p into a cofibration $i : X \to X'$ followed by a special inj-C-map $p' : X' \to Y$. By the two-out-of-three axiom (see Lemma 5.4), we know that i is in fact an acyclic cofibration. Then Proposition 5.11 says that i is an essentially levelwise proj-F_a-map for every a, so Lemma 5.10 implies that p has the right lifting property with respect to i. The retract argument then gives that p is a retract of p', as desired. □

The following lemma will be needed to show that the model structure on pro-$\mathcal{C}$ is right proper.

Lemma 5.14. *Any special F-map is a levelwise F-map.*

Proof. Suppose given a cofinite directed level map $p : X \to Y$ for which each relative matching map $M_s p$ belongs to F. The map p_s is the composition of $M_s p$ followed by the projection $Y_s \times_{\lim_{t<s} Y_t} \lim_{t<s} X_t \to Y_s$. This projection is a base change of the map $\lim_{t<s} X_t \to \lim_{t<s} Y_t$, which is a finite composition of base changes of the maps $M_t p$ for $t < s$ by Lemma 2.3. So Axiom 4.3 implies that p_s belongs to F. □

Theorem 5.15. *Let (A, C, W, F) be a proper filtered model structure on $\mathcal{C}$. Then pro-$\mathcal{C}$ has a proper model structure given by Definitions 5.1, 5.2, and 5.3.*

Proof. The category pro-$\mathcal{C}$ is complete and cocomplete because $\mathcal{C}$ is complete and cocomplete [13, Prop. 11.1]. The two-out-of-three axiom for weak equivalences is not automatic; we proved this in Lemma 5.4. Corollary 5.6 shows that cofibrations and weak equivalences are closed under retract. Fibrations are closed under retract by definition.

Factorizations into cofibrations followed by acyclic fibrations are given in Lemma 5.7. Here we have to use Proposition 5.13 to identify the second map as an acyclic fibration. Factorizations into acyclic cofibrations followed by fibrations are given in Lemma 5.8. Now we have to use Proposition 5.11 to identify the first map as an acyclic cofibration.

Cofibrations lift with respect to acyclic fibrations by Lemma 5.9; we use Proposition 5.13 to identify the acyclic fibrations. Acyclic cofibrations lift with respect to fibrations by Lemma 5.10; now we use Proposition 5.11 to identify the acyclic cofibrations.

We now show that the model structure is proper. For right properness, consider a pullback square

$$\begin{array}{ccc} W & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[p]{} & Z \end{array}$$

in which f is a weak equivalence and p is a fibration. We want to show that g is also a weak equivalence; that is, we want to show that g is an essentially levelwise W_a-map for all a.

Lemma 5.14 says that p is an essentially levelwise F-map. Therefore, the proof of [15, Thm. 4.13] can be applied to show that base changes of essentially levelwise W_a-equivalences along fibrations are essentially levelwise W_a-equivalences. We need that F is closed under arbitrary base changes and Axiom 4.10 for the proof to work. Since f is an essentially levelwise W_a-equivalence, we conclude that g is an essentially levelwise W_a-equivalence.

The proof of left properness is dual but easier because we know from the definition that cofibrations are essentially levelwise C_a-maps. Now we need to use that C_a is closed under arbitrary cobase changes and Axiom 4.9. □

Theorem 5.16. *Let (A, C, W, F) be a simplicial proper filtered model structure on $\mathcal{C}$. Then the model structure of Theorem 5.15 is also simplicial.*

Proof. As explained in Section 2.4, pro-$\mathcal{C}$ is a simplicial category. Let $j : K \to L$ be a cofibration of simplicial sets, let $i : X \to Y$ be a cofibration in pro-$\mathcal{C}$, and let f be the map

$$f : X \otimes L \amalg_{X \otimes K} Y \otimes K \to Y \otimes L.$$

As explained in [13, Prop. 16.1], it suffices to assume that K and L are finite simplicial sets.

Let a be any element of A. We may assume that i is a levelwise C_a-map. Because K and L are finite, the map f may be constructed levelwise; that is, f_s equals

$$X_s \otimes L \amalg_{X_s \otimes K} Y_s \otimes K \to Y_s \otimes L.$$

Because of part (1) of Axiom 4.12, f is a levelwise C_a-map. This shows that f is a cofibration in pro-$\mathcal{C}$.

Next, assume that j is an acyclic cofibration. As in the previous paragraph but using part (2) of Axiom 4.12, f is an essentially levelwise proj-F_a-map for every a. By Proposition 5.11, it follows that f is an acyclic cofibration.

Finally, assume that i is an acyclic cofibration. By Proposition 5.11, i is an essentially levelwise proj-F_a-map for every a. As before but using the other part of part (2) of Axiom 4.12, f is an essentially levelwise proj-F_a-map for every a, so it is an acyclic cofibration by Proposition 5.11. □

The following result shows that the model structures produced by Theorem 5.15 are a generalization of the strict model structures of [6] and [15].

Proposition 5.17. *Let $\mathcal{C}$ be a proper model category considered as a proper filtered model category indexed on a set A with only one element. The associated model structure on pro-$\mathcal{C}$ is the strict model structure on pro-$\mathcal{C}$.*

Proof. This follows from the definitions, Theorem 5.15, and Proposition 4.17. □

Recall from [15] that if $\mathcal{C}$ is simplicial, then the strict model structure on pro-$\mathcal{C}$ is also simplicial. This result now is an immediate corollary of Theorem 5.16.

We include one more minor technical lemma in this section that will be needed later.

Lemma 5.18. *Suppose that Y is a fibrant object in pro-$\mathcal{C}$. Then Y is isomorphic to an object Y' such that each map $Y'_s \to *$ belongs to F.*

Proof. Let D be the class of objects X in $\mathcal{C}$ such that $X \to *$ belongs to F. We want to show that Y belongs to D essentially levelwise.

We may assume that Y is cofinite directed. If we use Lemma 5.8 to factor $Y \to *$ into an acyclic cofibration followed by a fibration and then apply the retract argument, we see that $Y \to *$ is a retract of a special F-map $Z \to *$. Lemma 5.14 implies that Z belongs to D levelwise, and [14, Thm. 5.5] implies that Y belongs to D essentially levelwise. □

6. Quillen functors

In this section, we consider two proper filtered model structures (A, C, W, F) and (A', C', W', F') on the categories $\mathcal{C}$ and $\mathcal{C}'$ respectively. We will compare the associated model structures on pro-$\mathcal{C}$ and pro-$\mathcal{C}'$.

Recall that if if $L : \mathcal{C} \to \mathcal{C}'$ is a functor, then L induces another functor pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ by applying L levelwise. Moreover, if $R : \mathcal{C}' \to \mathcal{C}$ is the right adjoint of L, then $R :$ pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ is the right adjoint of $L :$ pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$.

We will give precise conditions telling us when the induced functors $L :$ pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ and $R :$ pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ are a Quillen adjoint pair. Then we will give some conditions that imply that L and R induce a Quillen equivalence between pro-$\mathcal{C}$ and pro-$\mathcal{C}'$.

Theorem 6.1. *Let (A, C, W, F) and (A', C', W', F') be proper filtered model structures on the categories $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint of $R : \mathcal{C}' \to \mathcal{C}$. The induced functors $L :$ pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ and $R :$ pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ are a Quillen adjoint pair when pro-$\mathcal{C}$ and pro-$\mathcal{C}'$ are equipped with the model structures of Theorem 5.15 if and only if $R(F')$ is contained in F and $R($inj-$C')$ is contained in inj-C.*

The properness assumption on the filtered model structures is only to guarantee that the model structures on pro-$\mathcal{C}$ and pro-$\mathcal{C}'$ exist.

Proof. First suppose that R takes F' into F and takes inj-C' into inj-C. Since R commutes with finite limits, it takes special F'-maps to special F-maps. This means that R takes retracts of special F'-maps to retracts of special F-maps, which means that R preserves fibrations.

To show that R preserves acyclic fibrations, recall that an acyclic fibration in pro-$\mathcal{C}'$ is a retract of a special inj-C'-map (see Proposition 5.13). We can use the same argument as in the previous paragraph, using that R takes inj-C' to inj-C.

Now suppose that L and R are a Quillen adjoint pair. Let $p : X \to Y$ belong to F'. We want to show that Rp belongs to F. The constant pro-map $cp : cX \to cY$ is a special F'-map, so it is a fibration in pro-$\mathcal{C}'$. Since R is a right Quillen functor, $R(cp)$ must be a fibration in pro-$\mathcal{C}$, so it is a retract of a special F-map $q : Z \to W$. Note that $R(cp)$ equals $c(Rp)$.

We have a diagram

$$\begin{array}{ccccc} cRX & \longrightarrow & Z & \longrightarrow & cRX \\ \downarrow & & \downarrow & & \downarrow \\ cRY & \longrightarrow & W & \longrightarrow & cRY \end{array}$$

in pro-$\mathcal{C}$, which is represented by a diagram

$$\begin{array}{ccccc} RX & \longrightarrow & Z_s & \longrightarrow & RX \\ \downarrow & & \downarrow & & \downarrow \\ RY & \longrightarrow & W_s & \longrightarrow & RY \end{array}$$

in $\mathcal{C}$. By Lemma 5.14 each $Z_s \to W_s$ belongs to F. Now F is closed under retract by Axiom 4.3, so $RX \to RY$ belongs to F. This is what we wanted to prove.

Using Proposition 5.13, the proof that $R(\text{inj-}C')$ is contained in $R(\text{inj-}C)$ is identical. Note that inj-C is closed under finite compositions and arbitrary base changes for formal reasons. □

Although Theorem 6.1 gives elegant necessary and sufficient conditions for a Quillen adjunction, it can sometimes be hard to verify these conditions in practice. We give other conditions that are sometimes easier to check.

Proposition 6.2. *Let (A, C, W, F) and (A', C', W', F') be proper filtered model structures on the categories $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint of $R : \mathcal{C}' \to \mathcal{C}$. Suppose that for every b in A', there exists a in A such that $L(C_a)$ is contained in C'_b and $L(W_a \cap C_a)$ is contained in $W'_b \cap C'_b$. Then the induced functors L : pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ and R : pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ are a Quillen adjoint pair when pro-$\mathcal{C}$ and pro-$\mathcal{C}'$ are equipped with the model structures of Theorem 5.15.*

Proof. To see that L preserves cofibrations, use Corollary 3.5 and the definition of cofibrations.

Now consider an acyclic cofibration i. By Proposition 5.11, we know that i is an essentially levelwise proj-F_a-map for every a. Using Axiom 4.4, it follows that i is an essentially levelwise $(W_a \cap C_a)$-map for every a. Now apply Corollary 3.5. □

We now consider Quillen equivalences.

Theorem 6.3. *Let (A, C, W, F) and (A', C', W', F') be proper filtered model structures on the categories $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint of $R : \mathcal{C}' \to \mathcal{C}$ such that the induced functors L : pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ and R : pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ are a Quillen adjoint pair when pro-$\mathcal{C}$ and pro-$\mathcal{C}'$ are equipped with the model structures of Theorem 5.15. Suppose also that:*

(1) *For every b in A', there exists a in A such that if $X \to RY$ is a map in $\mathcal{C}$ belonging to W_a, $* \to X$ belongs to some C_c, and $Y \to *$ belongs to F'; then the adjoint map $LX \to Y$ belongs to W'_b.*
(2) *For every b in A, there exists a in A' such that if $LX \to Y$ is a map in $\mathcal{C}'$ belonging to W'_a, $* \to X$ belongs to some C_c, and $Y \to *$ belongs to F'; then the adjoint map $X \to RY$ belongs to W_b.*

Then L and R are a Quillen equivalence between pro-$\mathcal{C}$ and pro-$\mathcal{C}'$.

Proof. Suppose that X is a cofibrant object of pro-$\mathcal{C}$ and Y is a fibrant object of pro-$\mathcal{C}'$. Since X is cofibrant, we may assume that each map $* \to X_s$ belongs to some C_c. Lemma 5.18 says that we may assume that each map $Y_t \to *$ belongs to F'.

Now suppose that $f : LX \to Y$ is a weak equivalence in pro-$\mathcal{C}'$. Our goal is to show that the adjoint map $g : X \to RY$ is a weak equivalence in pro-$\mathcal{C}$. Using the level replacement of [2, App. 3.2], we may reindex X and Y in such a way that f is a level map, each $* \to X_s$ still belongs to some C_c, and each $Y_t \to *$ still belongs to F'.

Factor f into an acyclic cofibration $LX \to Z$ followed by a fibration $Z \to Y$. If we use the method of Lemma 5.8 to produce this factorization, we find that for all a in A', there exists an $s(a)$ such that the map $LX_t \to Z_t$ belongs to W_a for all $t \geq s(a)$. Moreover, $Z \to Y$ is a levelwise F'-map by Lemma 5.14, so $Z_t \to *$ belongs to F' for all t.

Now condition (2) implies that for every b in A, there exists a in A' such that the map $X_t \to RZ_t$ belongs to W_b for all $t \geq s(a)$. In particular, the map $X \to RZ$ is an essentially levelwise W_b-map for every b. Thus $X \to RZ$ is a weak equivalence.

The map g factors as $X \to RZ \to RY$. We have just observed that the first map is a weak equivalence. For the second map, note that the two-out-of-three axiom implies that $Z \to Y$ is an acyclic fibration and that the right Quillen functor R preserves acyclic fibrations. This shows that g is a weak equivalence and finishes one half of the proof.

Now suppose that $g : X \to RY$ is a weak equivalence in pro-$\mathcal{C}$. Our goal is to show that the adjoint map $f : LX \to Y$ is a weak equivalence in pro-$\mathcal{C}'$. Using the level replacement of [2, App. 3.2], we may reindex X and Y in such a way that f is a level map, each $* \to X_s$ still belongs to some C_c, and each $Y_t \to *$ still belongs to F'.

Use the method of Lemma 5.8 to factor g into a cofibration $X \to Z$ followed by an acyclic fibration $Z \to RY$. In the notation of the proof of that lemma, we may choose $\phi(t)$ sufficiently large such that $X_t \to Z_t$ and $* \to X_t$ both belong to C_c for some value of c. Then the composition $* \to Z_t$ also belongs to C_c.

Also, $Z \to RY$ is a special inj-C-map. The proof of Lemma 5.14 can be adapted line by line to show that $Z \to RY$ is a levelwise inj-C-map. Lemma 4.16 implies that $Z \to RY$ is a levelwise W_a-map for every a. Now condition (1) of the theorem implies that the adjoint map $LZ \to Y$ is a levelwise W'_b-equivalence for all b in A'.

The map f factors as

$$LX \longrightarrow LZ \longrightarrow Y.$$

We have just observed that the second map is a weak equivalence. The two-out-of-three axiom implies that $X \to Z$ is an acyclic cofibration. Since the left Quillen functor L preserves acyclic cofibrations, it follows that $LX \to LZ$ is also a weak equivalence. Thus f is a weak equivalence. □

We can apply our general results above to the specific case of strict model structures. It was an oversight that this result did not appear in [15].

Theorem 6.4. *Let $L : \mathcal{C} \to \mathcal{C}'$ and $R : \mathcal{C}' \to \mathcal{C}$ be a Quillen pair between two proper model categories $\mathcal{C}$ and $\mathcal{C}'$. The induced functors L : pro-$\mathcal{C} \to$ pro-$\mathcal{C}'$ and R : pro-$\mathcal{C}' \to$ pro-$\mathcal{C}$ are a Quillen adjoint pair when pro-$\mathcal{C}$ and pro-$\mathcal{C}'$ are equipped with strict model structures. If L and R are a Quillen equivalence, then L and R induce a Quillen equivalence between the strict model structures on pro-$\mathcal{C}$ and pro-$\mathcal{C}'$.*

Proof. Recall from Proposition 5.17 that a strict model structure is the model structure associated to a filtered model structure indexed by a set with only one element. In the case when A and A' have only one element, the conditions of Theorem 6.1 reduce to the definition of a Quillen pair. Similarly, the conditions of Theorem 6.3 reduce to the definition of a Quillen equivalence. □

7. Examples

In this section, we describe several examples of model structures on pro-categories that can be established with Theorem 5.15.

Example 7.1. We give a concrete example that is an application of Proposition 4.18. Let $\mathcal{C}$ be the category of S-modules [7], and let $A = \mathbb{N}$. For each n, let E_n be a generalized homology theory such that if a map of spectra is an E_n-homology equivalence, then it is also an E_m-homology equivalence for all $m \leq n$. Let W_n be the class of E_n-homology equivalences. For each n there is a proper model structure on $\mathcal{C}$ whose weak equivalences are the class W_n and whose cofibrations are the class C of all cofibrations of spectra [7, VIII.1]. Let $C_n = C$ for all n. Now the classes C_n satisfy the necessary inclusion relationships trivially, and the classes W_n satisfy the necessary inclusion relationships by assumption on the homology theories E_n. By Proposition 4.18, there is a model structure on pro-$\mathcal{C}$ such that the weak equivalences are maps that are essentially levelwise E_n-homology equivalences for all n. If L_n is the E_n-localization functor, then the map $cX \to \{L_n X\}$ is a weak equivalence for all spectra X. Note that there are natural transformations $L_n \to L_{n-1}$ because of the assumption on the homology theories E_n.

The role of S-modules in this example is not central. Any of the standard model structures for stable homotopy theory will work just as well.

Example 7.2. We give a concrete illustration of Proposition 4.19. Let $\mathcal{C}$ be the category of simplicial sets. Let A be $\mathbb{N}$, and define I_n to be the set of generating cofibrations $\partial\Delta^k \to \Delta^k$ with $k > n$. The results of [13, Sec. 3] imply that W_n is the class of n-equivalences, and F_n is the class of co-n-fibrations, i.e., the class of maps that are both fibrations and co-n-equivalences. The hypotheses of Proposition 4.19 are satisfied because the class of n-equivalences is closed under retract and because if any two of f, g, and gf are n-equivalences, then the third is an $(n-1)$-equivalence.

Before we can use Theorem 5.15 to obtain a model structure on pro-simplicial sets, we must also observe that the filtered model structure of the previous paragraph is proper. This follows from the results of [13, Sec. 3]; see the end of the proof of Proposition 8.12 and Lemma 8.13 for the topological analogue.

The resulting model structure on the category of pro-simplicial sets is the same as the model structure of [13, Thm. 6.4]. The weak equivalences are the pro-maps f such that for every $n \geq 0$, f is isomorphic to a levelwise n-equivalence. This fact is not clearly stated in [13], but see [13, Prop. 6.8] for a "morally equivalent" claim. Compared to the technical methods of [13] involving basepoints, this approach is much simpler.

Example 7.3. Similarly to Example 7.2, there is a filtered model structure on the category of topological spaces. The class C_n is the class of Serre cofibrations, i.e., retracts of relative cell complexes; the class W_n is the class of n-equivalences; and the class F_n is the class of co-n-fibrations, i.e., Serre fibrations that are also co-n-equivalences. In this context, I_n is the set of generating cofibrations $S^{k-1} \to D^k$ with $k > n$. In the same way as the previous example, one can show that the hypotheses of Proposition 4.19 are satisfied and that the associated filtered model structure is proper. The resulting model structure on pro-spaces is Quillen equivalent to the model structure on pro-simplicial sets from the previous example.

Example 7.4. The following example is a stable version of Example 7.2. Let $\mathcal{C}$ be the category of spectra. Let A be $\mathbb{Z}$, and define I_n to be the set of generating cofibrations whose cofibers are spheres of dimension greater than n. The results of [16, Sec. 4] imply that W_n is the class of n-equivalences, and F_n is again the class of co-n-fibrations. As before, the hypotheses of Proposition 4.19 are satisfied, and the associated filtered model structure is proper. The resulting model structure on the category of pro-spectra is the same as the model structure of [16]. The weak equivalences can be described in terms of pro-homotopy groups, but the reformulation is not quite as obvious as one might expect. See [16] for details.

Example 7.5. Let $\mathcal{C}$ be the category of spaces, and let A equal $\mathbb{N}$. Let h_* be a homology theory on $\mathcal{C}$ that satisfies the colimit axiom. Let W_n be the class of maps f such that $h_i(f)$ is an isomorphism for $i < n$ and $h_n(f)$ is a surjection. In order to obtain a *proper* filtered model structure, we must assume that W_n is preserved by base changes along fibrations. If we let C_n be the class of cofibrations and F_n be the class inj-$(W_n \cap C_n)$, then we get a proper filtered model structure on spaces.

We outline the verification of the axioms for a filtered model structure in this case. Axioms 4.2 and 4.3 are obvious, as is the first half of Axiom 4.4. We defer the second half of Axiom 4.4 until later.

The first factorization of Axiom 4.5 is given by factorizations into cofibrations followed by acyclic fibrations. For the second part of this axiom, an adaptation of the small object argument in [3, Sec. 11] gives the desired factorization when the source is cofibrant. The basic idea is to replace all statements of the form "$h_*(K, L) = 0$" with statements of the form "$h_i(K, L) = 0$ for $i \leq n$". The factorization in general follows from standard arguments with model categories, together with properness for the category of spaces.

Now we return to the second half of Axiom 4.4. Using the factorization of the previous paragraph, a retract argument shows that proj-F_n equals $W_n \cap C_n$.

Having identified proj-F_n, the factorizations required by Axiom 4.6 are provided by factorizations into cofibrations followed by acyclic fibrations.

Axiom 4.9 follows from consideration of the long exact sequence in homology associated to a cofiber sequence. Axiom 4.10 is satisfied by our assumption on W_n.

If h_* is a periodic cohomology theory, then the model structure of this example is just the strict model structure associated to the h_*-local model structure on spaces [3].

8. The underlying model structure for pro-finite groups

Recall our goal of finding a model structure for pro-G-spaces in which $\{E(G/U)\}$ is a cofibrant replacement for $*$, where G is a pro-finite group and U ranges over all finite-index normal subgroups of G. We begin by describing the equivariant analogue of Example 7.2, but this model structure turns out not to have the desired property.

Let G be a (topological) group, and let $\mathcal{C}$ be the category of G-spaces. Let C be the class of retracts of relative G-cell complexes; this is the class of cofibrations in one of the usual model structures on the category of G-spaces. Now let A be $\mathbb{N}$, and let I_n be the set of generating cofibrations of the form $S^{k-1} \times G/H \to D^k \times G/H$, where $k > n$ and H is a (closed) subgroup of G. The framework of Proposition 4.19 applies; the hypotheses of this result can be verified just as in [13, Sec. 3]. The class W_n turns out to be the class of G-equivariant n-equivalences, i.e., the maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is an n-equivalence for all (closed) subgroups H of G. The class F_n is the class of equivariant co-n-fibrations, i.e., the maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a co-n-fibration for all (closed) subgroups H of G.

The resulting model structure is a G-equivariant analogue of the model structure of [13, Thm. 6.4]. It can be shown that a map $f : X \to Y$ of pro-G-spaces is a weak equivalence if and only if $\underline{\pi}_n f : \underline{\pi}_n X \to \underline{\pi}_n Y$ is a pro-isomorphism of pro-coefficient systems for all $n \geq 0$.

The map $\{E(G/U)\} \to c(*)$ induces a pro-isomorphism after applying π_n^V for any fixed V. However, it does *not* induce a pro-isomorphism after applying $\underline{\pi}_n$. The problem is that in choosing refinements, one must make different choices for different values of V. There is no one choice that works for all V.

The point of the previous paragraphs is that we must work harder to obtain our desired model structure. The rest of this section provides the details.

Let G be a pro-finite group. This means that G is a topological group such that $G \to \lim_U G/U$ is an isomorphism, where U ranges over all normal subgroups of G such that G/U is discrete and finite. Usually, a pro-finite group is viewed as a topological group, where the topology is totally disconnected, compact, and Hausdorff. It is also possible to think of G as a pro-object in the category of finite groups. We will use both viewpoints.

Let $\mathcal{C}$ be the category of compactly generated weak Hausdorff spaces equipped with a continuous G-action. First, recall that $\mathcal{C}$ is a simplicial category. If X is a G-space and K is a simplicial set, then $X \otimes K$ is defined to be $X \times |K|$, where the realization $|K|$ has a trivial G-action. Also X^K is the topological mapping space $F(|K|, X)$ of non-equivariant continuous maps $|K| \to X$. If X and Y are both G-spaces, then $\mathrm{Map}(X, Y)$ is the simplicial set whose n-simplices are equivariant maps $X \times |\Delta^n| \to Y$.

Definition 8.1. Let U be a finite-index subgroup of G. A G-equivariant map $f : X \to Y$ is a ***U*-weak equivalence** if the map f^V on V-fixed points is a weak equivalence for all finite-index subgroups V of U, and it is a ***U*-fibration** if f^V is a fibration for all finite-index subgroups V of U. It is a ***U*-cofibration** if it is a retract of a relative cell complex built from cells of the form $G/V \times S^n \to G/V \times D^{n+1}$, where V is a finite-index subgroup of U.

Recall that an n-cofibration [13, Defn. 3.2] is a cofibration that is also an n-equivalence (i.e., an isomorphism on homotopy groups up to dimension $n-1$ and a surjection on nth homotopy groups). Dually, a co-n-fibration is a fibration that is also a co-n-equivalence (i.e., an isomorphism on homotopy groups above dimension n and an injection on nth homotopy groups). We will now make similar definitions in the equivariant situation.

Definition 8.2. A map f is a ***U*-*n*-equivalence** (resp., ***U*-co-*n*-equivalence** if the map f^V is an n-equivalence (resp., co-n-equivalence) for all finite-index subgroups V of U. A map f is a ***U*-co-*n*-fibration** if f^V is a co-n-fibration for all finite-index subgroups V of U. Finally, a map is a ***U*-*n*-cofibration** if it is a U-cofibration such that f^V is an n-equivalence for all finite-index subgroups V of U.

The following two lemmas are proved in exactly the same way as their non-equivariant analogues [13, Sec. 3].

Lemma 8.3. *Any G-equivariant map can be factored into a U-n-cofibration followed by a U-co-n-fibration.*

Proof. Use the small object argument applied to the set of maps of the form $G/V \times S^k \to G/V \times D^{k+1}$, where V is a finite-index subgroup of U and $k \geq n$, together with all maps of the form $G/V \times I^m \to G/V \times I^{m+1}$, where V is a finite-index subgroup of U and m is arbitrary. Here I^m is the m-cube, and the map $I^m \to I^{m+1}$ is the inclusion of a face. □

Lemma 8.4. *The classes of U-n-cofibrations and U-co-n-fibrations are determined by lifting properties with respect to each other.*

Proof. This can be proved with an obstruction theory argument. See [13, Lem. 3.4] and [13, Lem. 3.6] for more details. □

Let $\boldsymbol{A}$ be the set consisting of all pairs (U, n), where U is a finite-index subgroup of G and n is a non-negative integer. We write $(U, n) \geq (V, m)$ if U is contained in V and if $n \geq m$. This makes A into a directed set. In other words, given (U_1, n_1) and (U_2, n_2), there exists (V, m) such that $(V, m) \geq (U_1, n_1)$ and $(V, m) \geq (U_2, n_2)$. To see why this is true, just observe that $U_1 \cap U_2$ is a finite-index subgroup of G whenever U_1 and U_2 are finite-index subgroups.

For each (U, n) in A, we will define three classes $C_{U,n}$, $W_{U,n}$, and $F_{U,n}$ of G-equivariant maps.

Definition 8.5. The class $\boldsymbol{C_{U,n}}$ is the class of all U-cofibrations. The class $\boldsymbol{W_{U,n}}$ is the class of maps that are V-n-equivalences for some V. The class $\boldsymbol{F_{U,n}}$ is the class of all U-co-n-fibrations.

Note that $C_{U,n}$ does not actually depend on n, and $W_{U,n}$ does not actually depend on U. The point of this seemingly confusing notation is that there is just one indexing set A for all three families of classes.

As in Definition 4.7, we write $\boldsymbol{F}$ for the union of the classes $F_{U,n}$.

Example 8.6. The following example emphasizes a subtlety in the definition of $W_{U,n}$, Consider the map

$$\amalg_U E(G/U) \to \amalg_U *,$$

where U ranges over the finite-index normal subgroups of G. This map is an underlying weak equivalence. However, it is not a V-equivalence for any finite-index subgroup V of G and thus does not belong to $W_{V,n}$ for any (V, n).

Lemma 8.7. *If $(V, m) \geq (U, n)$, then $C_{V,m}$ is contained in $C_{U,n}$, $W_{V,m}$ is contained in $W_{U,n}$, and $F_{U,n}$ is contained in $F_{V,m}$.*

Proof. All three claims follow immediately from the definitions. If V is contained in U, then the set of generating V-cofibrations is a subset of the set generating U-cofibrations. This shows that $C_{V,m}$ is contained in $C_{U,n}$.

If $m \geq n$, then an m-equivalence is automatically an n-equivalence; this shows that $W_{V,m}$ is contained in $W_{U,n}$.

If $m \geq n$, then a co-n-fibration is automatically a co-m-fibration. If V is contained in U, then a U-fibration is automatically a V-fibration. This shows that $F_{U,n}$ is contained in $F_{V,m}$. □

Lemma 8.8. *The class inj-$C_{U,n}$ (i.e., the class of maps that have the right lifting property with respect to all U-cofibrations equals the class of U-acyclic fibrations (i.e., maps that are both U-weak equivalences and U-fibrations). The class proj-$F_{U,n}$ (i.e., the class of maps that have the left lifting property with respect to all U-co-n-fibrations) equals the class of U-n-cofibrations.*

Proof. The first claim follows from standard equivariant homotopy theory. The second claim is immediate from Lemma 8.4. □

Recall that $\mathcal{C}$ is the category of compactly generated weak Hausdorff spaces with continuous G-actions and G-equivariant maps. The following definition is a special case of Definitions 5.1, 5.2, and 5.3.

Definition 8.9. A map in pro-$\mathcal{C}$ is a **cofibration** if it is an essentially levelwise $C_{U,n}$-map for every (U, n) in A. A map in pro-$\mathcal{C}$ is a **weak equivalence** if it is an essentially levelwise $W_{U,n}$-map for every (U, n) in A. A map in pro-$\mathcal{C}$ is a **fibration** if it is a retract of a special F-map.

Theorem 8.10. *Definition 8.9 is a proper simplicial model structure on the category pro-$\mathcal{C}$.*

Proof. Using Theorems 5.15 and 5.16, we just need to verify that Definition 8.5 is a proper simplicial filtered model structure. This is provided below in Propositions 8.11, 8.12, and 8.15. □

Proposition 8.11. *Definition 8.5 is a filtered model structure.*

Proof. We have to verify Axioms 4.2 through 4.6.

We have already observed that A is a directed set. Lemma 8.7 says that the containments given in Definition 4.1 are satisfied. Also, the category of G-spaces is complete and cocomplete; limits and colimits are constructed in the underlying category of topological spaces.

For Axiom 4.2, first observe that if any two of the maps f, g, and gf are V-$(n+1)$-equivalences, then a simple diagram chase shows that the third is a V-n-equivalence. If any two of f, g, and gf belong to $W_{U,n+1}$, then there exists a finite-index subgroup V of G such that the two maps are V-$(n+1)$-equivalences. The third map is a V-n-equivalence, which means that it belongs to $W_{U,n}$.

We now consider Axiom 4.3. The V-n-equivalences are closed under retract since non-equivariant n-equivalences are closed under retract, so $W_{U,n}$ is closed under retract. The class $C_{U,n}$ of U-cofibrations is closed under retract, finite compositions, and arbitrary cobase changes because it is defined in terms of retracts of relative cell complexes. The class $F_{U,n}$ is defined by a right lifting property (see Lemma 8.4), so it is closed under retract, finite compositions, and arbitrary base changes.

Axiom 4.4 is immediate from Lemma 8.8.

The first half of Axiom 4.5 is given by factorizations into U-cofibrations followed by U-acyclic fibrations; these factorizations are supplied by standard equivariant homotopy theory. The second half of Axiom 4.5 is given by Lemma 8.3.

For Axiom 4.6, let f belong to $W_{U,n}$. This means that f is a V-n-equivalence for some V. Now factor f into a V-cofibration i followed by a V-acyclic fibration p. By Lemma 8.8, this means that p belongs to inj-$C_{V,n}$. Because f is a V-n-equivalence, i is also a V-n-equivalence and hence a V-n-cofibration. By Lemma 8.8 again, this means that i belongs to proj-$F_{V,n}$. □

Proposition 8.12. *Definition 8.5 is a proper filtered model structure.*

Proof. We just need to verify Axioms 4.9 and 4.10.

For Axiom 4.9, suppose that $f : A \to B$ is a U-cofibration and that $g : A \to C$ is a V-n-equivalence for some finite-index subgroups U and V of G. We may replace V by $V \cap U$ to assume that V is a finite-index subgroup of U; this is allowed because g is a still a V-n-equivalence. We will show that the map $h : B \to B \amalg_A C$ is also a V-n-equivalence. If W is a finite-index subgroup of V, then $(B \amalg_A C)^W$ is equal to $B^W \amalg_{A^W} C^W$; this uses the fact that f is injective. Now W is a finite-index subgroup of U, so f^W is a non-equivariant cofibration. Since g^W is a non-equivariant n-equivalence, we need only show that cobase changes along non-equivariant cofibrations preserve non-equivariant n-equivalences. This is proved in Lemma 8.13 below.

For Axiom 4.10, suppose that $f : X \to Y$ is a U-fibration and that $g : Z \to Y$ is a V-n-equivalence for some finite-index subgroups U and V of G. Choose a finite-index subgroup W contained in both V and U. Then f is a W-fibration and g is a W-n-equivalence. Taking fixed points commutes with fiber products. Therefore, in order to show that $X \times_Y Z \to X$ is a W-n-equivalence, we only need prove that base changes of non-equivariant n-equivalences along non-equivariant fibrations are n-equivalences. This last fact follows from the five lemma and the long exact sequence of homotopy groups for a fibration. □

Lemma 8.13. *Let $f : A \to B$ be a cofibration of topological spaces, and let $g : A \to C$ be an n-equivalence. Then the map $h : B \to B \amalg_A C$ is also an n-equivalence.*

Proof. Since the usual model category on topological spaces is left proper, the pushout $B \amalg_A C$ is in fact a homotopy pushout. Therefore, we may replace g by a weakly equivalent cofibration; this will not change the weak homotopy type of h.

Now we have that g is an n-cofibration (i.e., a cofibration and an n-equivalence). The class of n-cofibrations is determined by a left lifting property; this is the non-equivariant version of Lemma 8.4. Therefore, the class of n-cofibrations is closed under arbitrary cobase changes, so h is also an n-cofibration. □

Remark 8.14. The reader may feel that it is not possible to prove Lemma 8.13 without using van Kampen's theorem. Van Kampen's theorem is necessary to prove that the model category of topological spaces is left proper, so we are in fact using it in a disguised way.

Proposition 8.15. *Definition 8.5 is a simplicial filtered model structure.*

Proof. We have already observed that $\mathcal{C}$ is a simplicial category, so we just need to prove that Axiom 4.12 holds.

Let $j : K \to L$ be a cofibration of finite simplicial sets, and let $i : A \to B$ be a U-cofibration. Standard equivariant homotopy theory implies that the map

$$f : A \otimes L \amalg_{A \otimes K} B \otimes K \to B \otimes L$$

is also a U-cofibration.

Similarly, if j is an acyclic cofibration, then standard equivariant homotopy theory implies that f is a U-acyclic cofibration. This implies that f is a U-n-cofibration, which means that it belongs to proj-$F_{U,n}$ by Lemma 8.8.

Next, suppose that j is a cofibration and that i belongs to proj-$F_{U,n}$. By Lemma 8.8, this means that i is a U-n-cofibration. We want to conclude that f is also a U-n-cofibration. We have already shown that f is a U-cofibration, so we just need to show that f^V is an n-equivalence for every finite-index subgroup of V of U.

The map f^V is equal to the map

$$A^V \otimes L \amalg_{A^V \otimes K} B^V \otimes K \to B^V \otimes L.$$

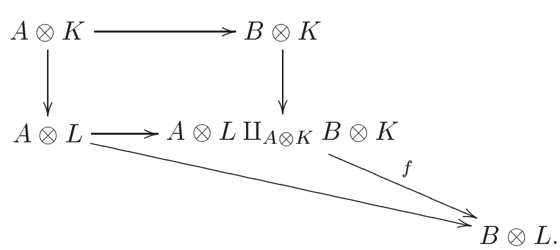
This follows from the fact that the G-actions on K and L are trivial and that taking V-fixed points commutes with the pushout because $A \otimes K \to A \otimes L$ is injective. Now the map $i^V : A^V \to B^V$ is an n-equivalence because i is a U-n-cofibration, so the desired conclusion follows from Lemma 8.16 below. □

Lemma 8.16. *Suppose that $j : K \to L$ is a cofibration of simplicial sets, and suppose that $i : A \to B$ is an n-cofibration of non-equivariant topological spaces. Then the map*

$$f : A \otimes L \amalg_{A \otimes K} B \otimes K \to B \otimes L$$

is an n-cofibration.

Proof. Consider the diagram



We will show that $A \otimes K \to B \otimes K$ and $A \otimes L \to B \otimes L$ are n-cofibrations. Then, since n-cofibrations are preserved by cobase changes, it follows that the map

$A \otimes L \to A \otimes L \amalg_{A \otimes K} B \otimes K$ is an n-cofibration. A small diagram chase proves that f is an n-equivalence.

It remains only to prove that $A \otimes K \to B \otimes K$ is an n-cofibration; the proof for $A \otimes L \to B \otimes L$ is identical. First, standard homotopy theory of topological spaces says that $A \times |K| \to B \times |K|$ is a cofibration. Second, since homotopy groups commute with products, the map $A \times |K| \to B \times |K|$ is an n-equivalence. □

Remark 8.17. The basic ideas of this section can be implemented in exactly the same way for naive G-spectra. One small difference is that the indexing set A consists of pairs (U, n) where U is a finite-index subgroup of G as before but n is an arbitrary integer, possibly negative. See [16] for details concerning n-cofibrations and co-n-fibrations of spectra.

Finally, we can establish our main motivation for producing the model structure of Theorem 8.10.

Proposition 8.18. *The pro-G-space $\{E(G/U)\}$ is a cofibrant replacement for the constant trivial pro-space $c(*)$ in the model structure of Theorem 8.10.*

Proof. Note first that for each finite-index subgroup U, $E(G/U)$ can be built from cells of the form $S^{k-1} \times G/U \to D^k \times G/U$. This means that the map from the empty set to $E(G/V)$ is a U-cofibration whenever V is contained in U. It follows that $\phi \to \{E(G/U)\}$ is a cofibration of pro-G-spaces.

Now we will show that the map $E(G/U) \to *$ is a U-weak equivalence for each U; this will imply that the map $\{E(G/U)\} \to c(*)$ is a weak equivalence of pro-G-spaces. If V is a finite-index subgroup of U then the V-fixed points of $E(G/U)$ equals $E(G/U)$. Since $E(G/U)$ is contractible, it follows that the map

$$E(G/U)^V \to *^V$$

is a weak equivalence. □

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