STRONGLY CLOSED SUBGROUPS AND THE CELLULAR STRUCTURE OF CLASSIFYING SPACES

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Abstract. In this paper we give a complete classification of the finite groups that contain a strongly closed p-subgroup, generalizing previous work of the second author to the case of an odd prime. We use this result to also obtain a description of the $B\mathbb{Z}/p$-cellularization (in the sense of Dror-Farjoun) of all the classifying spaces of finite groups.

1. Introduction

This paper brings to fruition previous endeavors of the two authors working independently in two areas of Mathematics — Finite Group Theory and Topology — by completing both the classification of finite groups containing a strongly closed p-subgroup for every prime p, and the characterization of cellularization of classifying spaces of all finite groups. These two classifications, the latter relying on the former, epitomize the rich interplay between their subject areas that has historically been evident and is currently even more vibrant. The results also mirror the striking advances in topics such as fusion systems, that have recently captured the interest of both topologists and group theorists. It is interesting that in the precursor to this paper, [FS07], the germs of the ideas for characterizing classifying spaces were already present. However, the list of simple groups containing strongly closed p-subgroups for p odd, characterized herein, is much more diverse; and indeed it uncovered new "obstructions" that had to be dealt with. In particular, one infinite family exhibits a unique fusion behavior that persists even in under more stringent (generation) conditions than just strong closure. Thus we see the necessity of having the full group-theoretic classification in order to effect the complete topological solution. Finally, although the techniques used in the two classifications tend to be quite different, the underlying fusion arguments that permeate the group theory sections

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provide deeper insight into, in essence, the fusion that may be "swept under the rug" for our topological considerations (in a sense to be made precise shortly). Indeed, the marriage of these elements is seen in high relief in Section 5 where we explore more explicit configurations that give rise to interesting — what might be called exotic — classifying spaces.

We now give the necessary terminology and some very brief historical contextualization before stating the main results. Recall that for any finite group G and subgroup S we say $x, y \in S$ are *fused* if they are conjugate in G (but not necessarily in S). This concept has played a central role in group theory and representation theory, particularly in the case when S is a Sylow p-subgroup of G. Of particular relevance to our work are the celebrated Glauberman Z^*-Theorem, [Gla66], and the Goldschmidt Theorem on strongly closed abelian 2-subgroups, [Gol74]. A subgroup A of S is called *strongly closed* in S with respect to G if for every $a \in A$, every element of S that is fused in G to a lies in A; in other words, $a^G \cap S \subseteq A$, where a^G denotes the G-conjugacy class of a. It is easy to verify that if A is a p-subgroup, then A is strongly closed in a Sylow p-subgroup if and only if it is strongly closed in $N_G(A)$, so the notion of strong closure for a p-subgroup does not depend on the Sylow subgroup containing it. For a p-group A we therefore simply say A is strongly closed. The Z^*-Theorem proved that if A is strongly closed and of order 2, then $\overline{A} \leq Z(\overline{G})$, where the overbars denote passage to $G/O_{2'}(G)$. Goldschmidt extended this by showing that if A is a strongly closed abelian 2-subgroup, then $\langle \overline{A}^{\overline{G}} \rangle$ is a central product of an abelian 2-group and quasisimple groups that have either BN-rank 1 or abelian Sylow 2-subgroups. These two theorems, in particular, played fundamental roles in the study of finite groups, especially in the Classification of the Finite Simple Groups.

The concept of strong closure has had ramifications even beyond finite group theory, as our work will illustrate (see also [Foo97a], for example). Furthermore, observe that if $A \leq S$ is strongly closed, $N_G(S)$ also normalizes A; in other words, every fusion automorphism of S restricts to an automorphism of A. This important property has been used to extend the notion of strong closure to more general frameworks. For example, in the context of p-local finite groups and linking systems ([BLO03]) the adequate concept of strong closure has been crucial in the study of extensions ([BCGLO07]), and in the definition and structure theory of saturated fusion systems ([Lin06] and [Asc07]).

In this paper we are essentially concerned with the relationship between the (group-theoretic) notion of strong closure and the concept of cellular classes in Homotopy Theory. Recall, following Dror-Farjoun [Far95], that given pointed spaces A and X, X is said to be A-*cellular* if it can be built from A by iterating pointed homotopy colimits. In a natural way, one may define an A-cellularization functor $\mathbf{CW}_A : \mathbf{Spaces}_* \to \mathbf{Spaces}_*$, that is idempotent and augmented, with the property that for every X the space $\mathbf{CW}_A X$ is A-cellular, and the augmentation $\mathbf{CW}_A X \to X$ induces a weak homotopy equivalence: $\mathrm{map}_*(A, \mathbf{CW}_A X) \simeq \mathrm{map}_*(A, X)$. Moreover, the *cellularity class* of A is defined as the minimal class of spaces that contains all the A-cellular spaces. In this way the category of pointed spaces is divided into cellularity classes. All these concepts were formalized by Chachólski [Cha96], and they have been recently applied in very different contexts such as the theory of algebraic varieties, commutative rings, stable homotopy and, more generally, Duality Theory ([DGI01]).

In this paper we classify, for every prime p, the finite groups possessing a strongly closed p-subgroup. This enables us to give a complete description of the $B\mathbb{Z}/p$-cellularization of classifying spaces of all finite groups. The philosophy behind our work is the following: whenever one has a space X with a notion of p-fusion (and then strong closure), the knowledge of the strongly closed subobjects of X is deeply related (and in some cases, almost equivalent) to the A-cellular structure of X, for a certain p-torsion space A. This strategy opens up new perspectives to analyze (from the point of view of (Co)localization Theory) the p-primary structure of a wide class of homotopy meaningful spaces, such as p-local finite groups, classifying spaces of compact Lie groups, p-compact groups or, more generally p-local compact groups ([BLO07]).

To describe the main results we introduce some new notation. Henceforth p is any prime, S is a Sylow p-subgroup of the finite group G and A is a subgroup of S. It is completely elementary that there is a unique normal subgroup of G, denoted by $\mathcal{O}_A(G)$, that is maximal with respect to the property that A contains one of its Sylow p-subgroups, i.e., $A \cap \mathcal{O}_A(G) \in Syl_p(\mathcal{O}_A(G))$. Note that $A \leq \mathcal{O}_A(G)$ if and only if A is a Sylow p-subgroup of its normal closure $\langle A^G \rangle$ in G. In the latter circumstance A is strongly closed, i.e., Sylow p-subgroups of normal subgroups of G are the "generic" instances of strongly closed p-subgroups. One may therefore view our classification as a determination of the "obstructions" to this generic reason that strongly closed

p-subgroups arise. In what follows let overbars denote passage to $G/\mathcal{O}_A(G)$, so that $\overline{A}$ does not contain a Sylow p-subgroup of any nontrivial normal subgroup of $\overline{G}$. The complete description of groups possessing a strongly closed p-subgroup, Theorems 2.1 and 2.2, is too lengthy and technical to warrant interrupting the flow of this introduction, so — in the spirit of our cursory statement of Goldschmidt's Theorem — we present only an abridged version at this point:

Theorem A. *Let p be any prime, let G be a finite group possessing a strongly closed p-subgroup A and assume A is not a Sylow p-subgroup of $\langle A^G \rangle$. Then $\overline{A} \neq 1$ and*

$$\langle \overline{A}^{\overline{G}} \rangle = (L_1 \times L_2 \times \cdots \times L_r)\Delta$$

where $r \geq 1$, each L_i is a simple group belonging to an explicitly listed family, $A_i = \overline{A} \cap L_i$ is a homocyclic abelian group, and Δ is a (possibly trivial) group acting as automorphisms on each L_i.

This statement combines the $p = 2$ case (Theorem 2.1) and the p odd case (Theorem 2.2). In addition to the families of simple groups being explicitly listed, all possibilities for each $\overline{A}_i$ are given, as is the precise action of Δ on each L_i. A crucial consequence of this theorem is the following:

Corollary B. *Assume the hypothesis of Theorem A, let $A \leq S \in Syl_p(G)$, and assume that G is generated by conjugates of A. Then $N_{\overline{G}}(\overline{A})$ controls strong fusion in $\overline{S}$.*

This classification was the main ingredient we lacked in order to finish the characterization of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for all finite groups G, solving a problem that was posed by Dror-Farjoun [Far95, 3.C] in the case $G = \mathbb{Z}/p^r$, and partially solved in [Flo07] and [FS07] (see Section 4.1 below for an analysis of the previous cases). The latter paper showed the importance of a specific strongly closed subgroup: for S a Sylow p-subgroup of G let $\mathfrak{A}_1(S)$ denote the unique minimal strongly closed subgroup of S that contains all elements of order p in S (i.e., contains $\Omega_1(S)$, the group generated by elements of order p in S — sometimes called the *p-socle* of S). The first main result on classifying spaces is the following.

Theorem C. *Let G be a finite group generated by its elements of order p, let $A = \mathfrak{A}_1(S)$, and assume $A \neq S$. Let overbars denote passage from G to the quotient group*

$G/\mathcal{O}_A(G)$. Then there exists a fibration

$$\mathbf{CW}_{B\mathbb{Z}/p}(BG_p^\wedge) \longrightarrow BG_p^\wedge \longrightarrow B(N_{\overline{G}}(\overline{A})/\overline{A})_p^\wedge.$$

This is Theorem 4.2. It makes precise what we glibly called "sweeping under the rug" earlier. It is important to note that although it is relatively elementary to prove that for any strongly closed subgroup A the subgroup $N_G(A)$ controls fusion of subgroups containing A, it does not generally control fusion inside A (or for subsets that intersect A but do not contain it). Thus the precise "shape" of the classification — factoring out the "correct" subgroup $\mathcal{O}_A(G)$ combined with an explicit knowledge of the structure of the quotient $G/\mathcal{O}_A(G)$ — is crucial. Furthermore, when $G/\mathcal{O}_A(G)$ is simple and equal to one of the "obstruction groups" L_i of Theorem A, in all but one family the "smaller" group $N_{\overline{G}}(\overline{S})$ controls strong fusion in S (and this was always the situation for $p = 2$); however, in that one exceptional family this replacement is not possible, as we explicate in Section 5.

To describe the culminating main result let $\Omega_1(G)$ denote the subgroup of G generated by the elements of order p in G (where p is always clear from the context), and let $BG_p^\wedge$ denote Bousfield-Kan p-completion of BG. For the definition and properties of this functor, we refer the reader to [BK72] for a thorough account and to [Flo07, Section 2] for a brief survey. By 4.1 and 4.3 of the latter, the inclusion $\Omega_1(G) \hookrightarrow G$ induces a homotopy equivalence $\mathbf{CW}_{B\mathbb{Z}/p}B\Omega_1(G) \simeq \mathbf{CW}_{B\mathbb{Z}/p}BG$. Thus the following result (which is Theorem 4.3 below and combines the information obtained in [Flo07], [FS07] and the present article), gives all the possible homotopy structures for $\mathbf{CW}_{B\mathbb{Z}/p}BG$:

Theorem D. *Let G be a finite group generated by its elements of order p, let $S \in Syl_p(G)$, and let $A = \mathfrak{A}_1(S)$ be the minimal strongly closed subgroup of S containing $\Omega_1(S)$. Then the $B\mathbb{Z}/p$-cellularization of BG has one the following shapes:*

1. *If $G = S$ is a p-group then BG is $B\mathbb{Z}/p$-cellular.*
2. *If G is not a p-group and $A = S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q \neq p} BG_q^\wedge$.*
3. *If G is not a p-group and $A \neq S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the map $BG \to B(N_{\overline{G}}(\overline{A})/\overline{A})_p^\wedge \times \prod_{q \neq p} BG_q^\wedge$.*

The classification of groups containing a strongly closed p-subgroup gives a very precise description of the fiber of the augmentation $\mathbf{CW}_{B\mathbb{Z}/p}BG \to BG$ in terms of normalizers of strongly closed p-subgroups in the simple components of $G/\mathcal{O}_A(G)$. In this sense the results here further improve those of [FS07], where this degree of sharpness was only obtained in the description of some concrete examples.

The overall organization of the paper is as follows: Section 2 begins by recapitulating the basic terminology and previous results in group theory; it then states the main group-theoretic classification in detail. After some preliminary results the Sylow structure and Sylow normalizers of simple groups containing strongly closed p-subgroups are elucidated. The main group-theoretic theorem is derived at the end of this section as a consequence of an inductive special case describing the "minimal configuration" on groups containing a minimal strongly closed p-subgroup, p odd (Theorem 2.3). Section 3 consists of the proof for this minimal configuration. In Section 4 we provide additional background and more precise statements of previous results of a topological nature, and then the main results on cellularization are established. Section 5 illustrates the efficacy of our methods by describing $N_G(A)$ and $N_G(S)$ as well as computing $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for specific cases of G. More explicitly, we describe first $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for G simple, and then for split extensions, and finally for certain nonsplit extensions of simple groups. The latter are very illuminating in the sense that they give an alluring glimpse of what "should be" the $B\mathbb{Z}/p$-cellularization of more general objects.

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2. Strongly closed p-subgroups

Throughout this section G is a finite group, p is a prime and A is a p-subgroup of G. Following the definitions and description in the Introduction, this section and the next complete the classification of groups possessing a strongly closed p-subgroup by carrying out for all odd p the classification scheme that was done in [Foo97] for $p = 2$ (see Theorem 2.1 below).

In general let R be any p-subgroup of G. If N_1 and N_2 are normal subgroups of G with $R \cap N_i \in Syl_p(N_i)$ for both $i = 1, 2$, then $R \cap N_1 N_2$ is a Sylow p-subgroup of $N_1 N_2$. Thus there is a unique largest normal subgroup N of G for which $R \cap N \in Syl_p(N)$; denote this subgroup by $\mathcal{O}_R(G)$. Thus

$$R \text{ is a Sylow } p\text{-subgroup of } \langle R^G \rangle \text{ if and only if } R \leq \mathcal{O}_R(G).$$

Note that $O_{p'}(G/\mathcal{O}_R(G)) = 1$; in particular, if $R = 1$ is the identity subgroup then $\mathcal{O}_1(G) = O_{p'}(G)$. In general, $R\mathcal{O}_R(G)/\mathcal{O}_R(G)$ does not contain the Sylow p-subgroup of any nontrivial normal subgroup of $G/\mathcal{O}_R(G)$; in other words, $\mathcal{O}_{\overline{R}}(\overline{G}) = 1$, where overbars denote passage to $G/\mathcal{O}_R(G)$. Throughout this section we freely use the observation that strong closure passes to quotient groups (cf. Lemma 2.8), so when analyzing groups where $R \not\leq \mathcal{O}_R(G)$ we may factor out $\mathcal{O}_R(G)$. With this in mind, the classification for strongly closed 2-subgroups from [Foo97] is as follows:

Theorem 2.1. *Let G be a finite group that possesses a strongly closed 2-subgroup A. Assume A is not a Sylow 2-subgroup of $\langle A^G \rangle$, and let $\overline{G} = G/\mathcal{O}_A(G)$. Then $\overline{A} \neq 1$ and $\langle \overline{A}^{\overline{G}} \rangle = L_1 \times L_2 \times \cdots \times L_r$, where each L_i is isomorphic to $U_3(2^{n_i})$ or $Sz(2^{n_i})$ for some n_i, and $\overline{A} \cap L_i$ is the center of a Sylow 2-subgroup of L_i.*

The classification for p odd, which is the principal objective of this section, yields a more diverse set of "obstructions" with added "decorations" as well.

Theorem 2.2. *Let p be an odd prime and let G be a finite group that possesses a strongly closed p-subgroup A. Assume A is not a Sylow p-subgroup of $\langle A^G \rangle$, and let $\overline{G} = G/\mathcal{O}_A(G)$. Then $\overline{A} \neq 1$ and*

$$\langle \overline{A}^{\overline{G}} \rangle = (L_1 \times L_2 \times \cdots \times L_r)(D \cdot A_F) \tag{2.1}$$

where $r \geq 1$, each L_i is a simple group, and $A_i = \overline{A} \cap L_i$ is a homocyclic abelian group. Furthermore, $D = [D, A_F]$ is a (possibly trivial) p'-group normalizing each L_i, and A_F is a (possibly trivial) abelian subgroup of $\overline{A}$ of rank at most r normalizing D and each L_i and inducing outer automorphisms on each L_i, and the extension $(A_1 \cdots A_r) : A_F$ splits. Each L_i belongs to one of the following families:

(i) *L_i is a group of Lie type in characteristic $\neq p$ whose Sylow p-subgroup is abelian but not elementary abelian; in this case the Sylow p-subgroup of L_i is homocyclic of the same rank as A_i but larger exponent than A_i; here $D/(D \cap$*

$L_iC_{\overline{G}}(L_i))$ is a cyclic p'-subgroup of the outer diagonal automorphism group of L_i, and $A_F/C_{A_F}(L_i)$ acts as a cyclic group of field automorphisms on L_i.

(ii) *$L_i \cong U_3(p^n)$ or $Re(3^n)$ is a group of BN-rank 1 ($p = 3$ with n odd and ≥ 2 in the latter family); in the unitary case A_i is the center of a Sylow p-subgroup of L_i (elementary abelian of order p^n), and in the Ree group case A_i is either the center or the commutator subgroup of a Sylow 3-subgroup (elementary abelian of order 3^n or 3^{2n} respectively); in both families D and A_F act trivially on L_i.*

(iii) *$L_i \cong G_2(q)$ with $(q, 3) = 1$; here $|A_i| = 3$ and both D and A_F act trivially on L_i.*

(iv) *L_i is one of the following sporadic groups, where in each case A_i has prime order, and both D and A_F act trivially on L_i:*

$(p = 3)$ **:** J_2,

$(p = 5)$ **:** Co_3, Co_2, HS, Mc,

$(p = 11)$**:** J_4.

(v) *$L_i \cong J_3$, $p = 3$, and A_i is either the center or the commutator subgroup of a Sylow 3-subgroup (elementary abelian of order 9 or 27 respectively); here D and A_F act trivially on L_i.*

Remark. After factoring out $\mathcal{O}_A(G)$ — so that overbars may be omitted — the proof of Theorem 2.2 shows that $F^*(G) = L_1 \times \cdots \times L_r$, and (2.1) may also be written as

$$\langle A^G \rangle \cong ((L_1 \times \cdots \times L_i)D \times L_{i+1} \times \cdots \times L_j)A_F \times (L_{j+1} \times \cdots \times L_r)$$

where $L_1, \ldots, L_i$ are the components of type PSL or PSU over fields of characteristic $\neq p$, $L_{i+1}, \ldots, L_j$ are other groups listed in conclusion (i) (but not linear or unitary), and $L_{j+1}, \ldots, L_r$ are the components of types listed in (ii) to (v). Furthermore, assume $G = \langle A^G \rangle$ and let $A \leq S \in Syl_p(G)$ and $S^* = S \cap F^*(G)$. Then we may choose D generically as $[O_{p'}(C_G(S^*)), S]$, which is a p'-group normalized by S and centralized by the Sylow p-subgroup S^* of $L_1 \cdots L_r$.

Conversely, observe that any finite group that has a composition factor of one of the above types for L_i possesses a strongly closed p-subgroup that is not a Sylow p-subgroup of its normal closure in G. More detailed information about the structure of the Sylow p-subgroups and their normalizers for the simple groups L_i appearing in the conclusion to this theorem is given from Proposition 2.9 through Corollary 2.13 following.

Theorem 2.2 is derived at the end of this section as a consequence of the next result, which is the minimal configuration whose proof appears in the next section.

Theorem 2.3. *Assume the hypotheses of Theorem 2.2. Assume also that A is a minimal strongly closed subgroup of G, i.e., no proper, nontrivial subgroup of A is also strongly closed. Then the conclusion of Theorem 2.2 holds with the additional results that A is elementary abelian, $D = 1$, $A_F = 1$, and G permutes $L_1, \dots, L_r$ transitively (hence they are all isomorphic).*

Although these group-theoretic results are of independent interest, the important consequences we need for our main theorems on cellularization are the following — their proofs appears at the end of this section.

Corollary 2.4. *Let p be any prime, let G be a finite group containing a strongly closed p-subgroup A, let S be a Sylow p-subgroup of G containing A, and let $\overline{G} = G/\mathcal{O}_A(G)$. Assume that G is generated by the conjugates of A. Then $N_{\overline{G}}(\overline{A})$ controls strong $\overline{G}$-fusion in $\overline{S}$. Furthermore, if $p \neq 3$ or if $\overline{G}$ does not have a component of type $G_2(q)$ with $9 \mid q^2 - 1$, then $N_{\overline{G}}(\overline{S})$ controls strong $\overline{G}$-fusion in $\overline{S}$.*

In Section 5.3 we demonstrate that the exceptional case to the stronger conclusion in the last sentence of Corollary 2.4 is unavoidable, even if we impose the condition that $\Omega_1(S) \leq A$: we construct examples of groups G generated by conjugates of a strongly closed subgroup A containing $\Omega_1(S)$ and $G/\mathcal{O}_A(G) \cong G_2(q)$ where $N_{\overline{G}}(\overline{S})$ does not control fusion in $\overline{S}$.

The next result facilitates computation of $N_G(A)$ in groups satisfying the conclusion to the preceding corollary.

Corollary 2.5. *Assume the hypotheses of preceding corollary and the notation of Theorem 2.2. For each i let $C_i = C_{\overline{G}}(A_F) \cap N_{L_i}(A_i)$ and $S_i = \overline{S} \cap L_i$. Then*

$$N_{\overline{G}}(\overline{A})/\,\overline{A} = (S_1C_1/A_1) \times (S_2C_2/A_2) \times \cdots \times (S_rC_r/A_r).$$

In particular, if L_i is a component on which A_F acts trivially — which is the case for all components in conclusions (ii) to (v) of Theorem 2.2 — the i^{th} direct factor above may be replaced by just $N_{L_i}(A_i)/A_i$ (and this applies to all factors if $A_F = 1$).

Example. An example where both D and A_F are nontrivial is $G = PGL_{11}(q)\langle\, f\,\rangle$ with $p = 5$ and $q = 3^5$: Here the simple group $L = PSL_{11}(q)$ has an abelian Sylow

5-subgroup of type (25,25), $PGL_{11}(q)/L$ is the cyclic outer diagonal automorphism group of L of order 11 (this is DL/L), and $\langle f \rangle = A_F$ induces a group of order 5 of field automorphisms on $PGL_{11}(q)$; in particular, G/L is the non-abelian group of order 55. If $f \in S \in Syl_5(G)$, then $A = \Omega_1(S) = \langle f, \Omega_1(S \cap L) \rangle$ is elementary abelian of order 5^3 and strongly closed in S with respect to G, and $A^* = \Omega_1(S \cap L)$ is a minimal strongly closed subgroup of G.

In this example, to compute the normalizers of A and A^* it is easier to work in the universal group $GL_{11}(q)\langle f \rangle$ — also denoted by G — via its action on an 11-dimensional $\mathbb{F}_q$-vector space V (since the center of $GL_{11}(q)$ has order prime to 5) — see the proof of Lemma 3.4 for some general methodology. Let $G^* = GL_{11}(q)$ and $S^* = S \cap G^*$. Then one sees that $N_G(A^*) = N_G(S^*)$ is contained in a subgroup

$$H = ((G_1 \times G_2)\langle t \rangle \times C)\langle f \rangle$$

where $G_i \cong GL_4(q)$, $C \cong GL_3(q)$, t interchanges the two factors and f induces field automorphisms on all three factors and commutes with t (here $G_1 \times G_2 \times C$ acts naturally on a direct sum decomposition of V). Let $S_i = S \cap G_i$, so S_i is cyclic of order 25 and acts $\mathbb{F}_q$-irreducibly on the 4-dimensional submodule for G_i. By basic representation theory, $C_{G_i}(S_i)$ is cyclic of order $q^4 - 1$, and $N_{G_i}(S_i)/C_{G_i}(S_i)$ is cyclic of order 4. Thus

$$N_G(A^*) = N_G(S^*) \cong ((q^4 - 1) \cdot 4 \ \times \ (q^4 - 1) \cdot 4)\langle t, f \rangle \times GL_3(3^5).$$

Since $A_F = \langle f \rangle$ acts as a field automorphisms, similar considerations show that

$$N_G(A) = S(N_G(S^*) \cap C_{G^*}(f)) \cong (400 \cdot 4 \ \times \ 400 \cdot 4)\langle t, f \rangle \times GL_3(3).$$

The G-fusion in S is effected by the group $S(4 \times 4)\langle t \rangle$, which is the same for both normalizers. In this example we may choose $D = [C_G(S^*), f]$, which is of type $B \times B \times (SL_3(q) \cdot 121)$ where B is cyclic of order $(q^4 - 1)/5(3^4 - 1)$; a (smaller) group of diagonal automorphisms for D could be chosen inside the abelian factor $B \times B$.

The proof of Theorem 2.3 relies on the Classification of Finite Simple Groups. We reduce to the case where a minimal counterexample, G, is a simple group having a strongly closed p-subgroup A that is properly contained in a non-abelian Sylow p-subgroup S of G. The remainder of the proof involves careful investigation of the families of simple groups to determine precisely when this happens.

We note that "most" simple groups do possess a strongly closed p-subgroup that is proper in a Sylow p-subgroup, that is, conclusion (i) of Theorem 2.2 is the "generic obstruction" in the following sense. Let $\mathcal{L}_n(q)$ denote a simple group of Lie type and BN-rank n over the finite field $\mathbb{F}_q$ with $(q, p) = 1$. As we shall see in Section 2.1, for all but the finitely many primes dividing the order of the Weyl group of the untwisted version of $\mathcal{L}_n(q)$ the Sylow p-subgroups of $\mathcal{L}_n(q)$ are homocyclic abelian. Furthermore, the order of $\mathcal{L}_n(q)$ can be expressed as a power of q times factors of the form $\Phi_m(q)^{r_m}$ for various $m, r_m \in \mathbb{N}$, where $\Phi_m(x)$ is the m^{th} cyclotomic polynomial. Then by Proposition 2.9 below, if m_0 is the multiplicative order of $q \pmod p$, then p divides $\Phi_{m_0}(q)$ and the abelian Sylow p-subgroup of $\mathcal{L}_n(q)$ is homocyclic of rank r_{m_0} and exponent $|\Phi_{m_0}(q)|_p$. In particular it is not elementary abelian whenever $p^2 \bigm| \Phi_{m_0}(q)$. For example, this is the case in the groups $PSL_{n+1}(q)$ whenever $p > n+1$ and p^2 divides $q^m - 1$ for some $m \leq n+1$. Thus for fixed n and all but finitely many p, this can always be arranged by taking q suitably large.

2.1. **Preliminary Results.** The special case when A has order p has already been treated in [GLSv3, Proposition 7.8.2]. It is convenient to quote this special case, although with extra effort our arguments could be reworded to independently subsume it.

Proposition 2.6. *If K is simple and $G = AK$ is a subgroup of $Aut(K)$ such that A is strongly closed and $|A| = p$, then $A \leq K = G$ and either the Sylow p-subgroups of G are cyclic, or G is isomorphic to $U_3(p)$ or one of the simple groups listed in conclusions (iii) and (iv) of Theorem 2.2.*

The authors of this result remark that an immediate consequence of this is the odd-prime version of Glauberman's celebrated Z^*-Theorem.

Proposition 2.7. *If an element of odd prime order p in any finite group X does not commute with any of its distinct conjugates then it lies in $Z(X/O_{p'}(X))$.*

We record some basic facts about strongly closed subgroups (the second of which relies on the odd-prime Z^*-Theorem).

Lemma 2.8. *For p any prime let A be a strongly closed p-subgroup of G.*

(1) *If N is any normal subgroup of G then AN/N is a strongly closed p-subgroup of G/N.*

(2) *If A normalizes a subgroup H of G with $O_{p'}(H) = 1$ and $A \cap H = 1$ then A centralizes H.*

Proof. In part (1) let $A \leq S \in Syl_p(G)$. This result follows immediately from the definition of strongly closed applied in the Sylow p-subgroup SN/N of G/N together with Sylow's Theorem. The proof of (2) is the same as for $p = 2$ since, as noted earlier, the Z^*-Theorem holds also for odd primes: by induction reduce to the case where $G = AH$ and $C_A(H) = 1$. Then any element of order p in A is isolated, hence lies in the center. □

The next few results gather facts about the simple groups appearing in the conclusions to Theorems 2.1 and 2.2.

The cross-characteristic Sylow structures of the simple groups of Lie type groups are beautifully described in [GL83, Section 10] and reprised in [GLSv3, Section 4.10]. Let $\mathcal{L}(q)$ denote a universal Chevalley group or twisted variation over the field $\mathbb{F}_q$. (In the notation of [GLSv3], $\mathcal{L}(q) = {}^dL(q)$, where $d = 1, 2, 3$ corresponds to the untwisted, Steinberg twisted, or Suzuki-Ree twisted variations respectively). Let W denote the Weyl group of the untwisted group corresponding to $\mathcal{L}(q)$. Except for some small order exceptions, $\mathcal{L}(q)$ is a quasisimple group; for example $\mathcal{A}_\ell(q) \cong SL_{\ell+1}(q)$ and ${}^2A_\ell(q) \cong SU_{\ell+1}(q)$. There is a set $\mathcal{O}(\mathcal{L}(q))$ of positive integers, and "multiplicities" r_m for each $m \in O(\mathcal{L}(q))$, such that

$$|\mathcal{L}(q)| = q^N \prod_{m \in \mathcal{O}(\mathcal{L}(q))} (\Phi_m(q))^{r_m}$$

where $\Phi_m(x)$ is the cyclotomic polynomial for the m^{th} roots of unity.

Let p be an odd prime not dividing q and assume S is a nontrivial Sylow p-subgroup of $\mathcal{L}(q)$. Let m_0 be the smallest element of $\mathcal{O}(\mathcal{L}(q))$ such that $p \bigm| \Phi_{m_0}(q)$. Let

$$\mathcal{W} = \{m \in \mathcal{O}(\mathcal{L}(q)) \mid m = p^a m_0,\ a \geq 1\} \qquad \text{and} \qquad b = \sum_{m \in \mathcal{W}} r_m \tag{2.2}$$

where $b = 0$ if $\mathcal{W} = \emptyset$. The main structure theorem is as follows.

Proposition 2.9. *Under the above notation the following hold:*

(1) *m_0 is the multiplicative order of $q \pmod p$.*
(2) *Except in the case where $\mathcal{L}(q) = {}^3D_4(q)$ with $p = 3$, S has a nontrivial normal homocyclic subgroup, S_T, of rank r_{m_0} and exponent $|\Phi_{m_0}(q)|_p$.*

(3) *With the same exception as in (2), S is a split extension of S_T by a (possibly trivial) subgroup S_W of order p^b (where b is defined in (2.2)), and S_W is isomorphic to a subgroup of W. In particular, if $p \nmid |W|$ or if $pm_0 \nmid m$ for all $m \in \mathcal{O}(\mathcal{L}(q))$, then $S = S_T$ is homocyclic abelian.*

(4) *If $\mathcal{L}(q) = {}^3D_4(q)$ with $p = 3$ and $|q^2 - 1|_3 = 3^a$, then S is a split extension of an abelian group of type $(3^{a+1}, 3^a)$ by a group of order 3, and S has rank 2.*

(5) *If $\mathcal{L}(q)$ is a classical group (linear, unitary, symplectic or orthogonal) then every element of order p is conjugate to some element of S_T.*

(6) *Except in ${}^3D_4(q)$ (where S_W is not defined), S_W acts faithfully on S_T; and in the simple group $\mathcal{L}(q)/Z(\mathcal{L}(q)) = \overline{\mathcal{L}(q)}$ we have $\overline{S_W} \cong S_W$ acts faithfully on $\overline{S_T}$ except when $p = 3$ with $\mathcal{L}(q) \cong SL_3(q)$ (with $3 \mid q-1$ but $9 \nmid q-1$) or $SU_3(q)$ (with $3 \mid q+1$ but $9 \nmid q+1$).*

(7) *If a Sylow p-subgroup of the simple group $\mathcal{L}(q)/Z(\mathcal{L}(q))$ is abelian but not elementary abelian then p does not divide the order of the Schur multiplier of $\mathcal{L}(q)$.*

Proof. For parts (1) to (6) see [GL83, 10-1, 10-2] or [GLSv3, Theorems 4.10.2, 4.10.3]. If the odd prime p divides the order of the Schur multiplier of $\mathcal{L}(q)$ then by [GLSv3, Table 6.12] we must have $\mathcal{L}(q)$ of type $SL_n(q)$, $SU_n(q)$, $E_6(q)$ or ${}^2E_6(q)$ with p dividing $(n, q-1)$, $(n, q+1)$, $(3, q-1)$ or $(3, q+1)$ respectively. It follows easily from (6) that in each of the corresponding simple groups a Sylow p-subgroup cannot be abelian of exponent $\geq p^2$. □

We shall frequently adopt the efficient shorthand from the sources just cited for the latter families.

Notation. Denote $SL_n(q)$ by $SL_n^+(q)$ and $SU_n(q)$ by $SL_n^-(q)$ (likewise for the general linear and projective groups); and say a group is of type $SL_n^\epsilon(q)$ according to whether $p \mid q - \epsilon$ for $\epsilon = +1, -1$ respectively (dropping the "1" from ± 1). The analogous convention is adopted for $E_6(q) = E_6^+(q)$ and ${}^2E_6(q) = E_6^-(q)$.

The following general result is especially important for the groups of Lie type.

Proposition 2.10. *If G is any simple group with an abelian Sylow p-subgroup S for any prime p, then $N_G(S)$ acts irreducibly and nontrivially on $\Omega_1(S)$, and so S is homocyclic. In particular, a nontrivial subgroup of S is strongly closed if and only if it is homocyclic of the same rank as S.*

Proof. See [GLSv3, Proposition 7.8.1] and [GL83, 12-1]. □

Proposition 2.11. *Let G be a simple group of Lie type over $\mathbb{F}_q$ and let p be an odd prime not dividing q. Assume a Sylow p-subgroup S of G is abelian and let $A = \Omega_1(S)$. Then $N_G(A) = N_G(S)$.*

Proof. The result is trivial if $S = A$ so assume this is not the case; in particular the exponent of S is at least p^2. By part (7) of Proposition 2.9, p does not divide the order of the Schur multiplier of G, so we may assume G is the (quasisimple) universal cover of the simple group. Clearly $N_G(S) \leq N_G(A)$. Moreover, since $S \in Syl_p(C_G(A))$, by Frattini's Argument $N_G(A) = C_G(A)N_G(S)$. Thus it suffices to show $C_G(A) = C_G(S)$. Since $C_G(A)$ has an abelian Sylow p-subgroup and since any nontrivial p'-automorphism of S must act nontrivially on A, by Burnside's Theorem $C_G(A)$ has a normal p-complement. Let $\Delta = [O_{p'}(C_G(A)), S]$. It suffices to prove S centralizes Δ.

Let $\overline{G}$ be the simply connected universal algebraic group over the algebraic closure of $\mathbb{F}_q$, and let σ be a Steinberg endomorphism whose fixed points equal G. In the notation of Proposition 2.9, since $S = S_T$, by the proof of [GLSv3, Theorem 4.10.2] there is a σ-stable maximal torus $\overline{T}$ of $\overline{G}$ containing S. Let $\overline{C}$ denote the connected component of $C_{\overline{G}}(\overline{A})$, so $\overline{C}$ is also σ-stable. Note that $\overline{T} \leq \overline{C}$ and since Δ is generated by conjugates of S, so too $\Delta \leq \overline{C}$. We may now follow the basic ideas in the proof of [GLSv3, Theorem 7.7.1(d)(2)], where more background is provided. By [SS70, 4.1(b)], $\overline{C}$ is reductive, so by the general theory of connected reductive groups

$$\overline{C} = \overline{Z}\,\overline{L}$$

where $\overline{Z}$ is the connected component of the center of $\overline{C}$, $\overline{L}$ is the semisimple component (possibly trivial), and $\overline{Z} \cap \overline{L}$ is a finite group. Since $\Delta \leq \overline{C}'$ we have $\Delta \leq \overline{L}$. The group of fixed points of σ on $\overline{L}$ is a commuting product $L_1 \cdots L_n$ of (possibly solvable) groups of Lie type over the same characteristic as G and smaller rank, and S induces inner or diagonal automorphisms on each L_i. Since $\Delta \leq O_{p'}(C_G(A))$ we have

$$\Delta \leq O_{p'}(L_1 \cdots L_n) = O_{p'}(L_1) \cdots O_{p'}(L_n).$$

If L_i is a p'-group, then $\mathrm{Inndiag}(L_i)$ is also a p'-group and so S centralizes L_i. On the other hand, if p divides the order of L_i, then $O_{p'}(L_i) \leq Z(L_i)$; in this case $\mathrm{Inndiag}(L_i)$ centralizes $Z(L_i)$. In all cases S centralizes $O_{p'}(L_i)$, as needed. □

Proposition 2.12. *Let p be any prime, let G be a simple group containing a strongly closed p-subgroup, let $S \in Syl_p(G)$ and let $Z = Z(S)$.*

(1) *Assume $G \cong U_3(q)$ with $q = p^n$, or $G \cong Sz(q)$ with $p = 2$ and $q = 2^n$. Then S is a special group of type q^{1+2} or q^{1+1} respectively, and $N_G(S) = N_G(Z) = SH$, where the Cartan subgroup H is cyclic of order $(q^2-1)/(3, q+1)$ or $q-1$ respectively. In both families H acts irreducibly on both Z and S/Z, and $Z = \Omega_1(S)$ is the unique nontrivial, proper strongly closed subgroup of S.*

(2) *Assume $G \cong Re(q)$ with $p = 3$ and $q = 3^n$, $n > 1$. Then S is of class 3, $Z \cong E_q$ and $S' = \Phi(S) = \Omega_1(S) \cong E_{q^2}$. Furthermore, $N_G(S) = N_G(Z) = SH$, where the Cartan subgroup H is cyclic of order $q-1$ and acts irreducibly on all three central series factors: Z and S'/Z and S/S'. Thus Z and $\Omega_1(S)$ are the only nontrivial proper strongly closed subgroups of S.*

(3) *Assume $G \cong G_2(q)$ for some q with $(q, 3) = 1$ and $p = 3$. Then $Z \cong Z_3$ is the only nontrivial proper strongly closed subgroup of S. Furthermore, $N_G(Z) \cong SL_3^\epsilon(q) \cdot 2$ according to whether $3 \mid q - \epsilon$. An element of order 2 in $N_G(Z) - C_G(Z)$ inverts Z, and $N_G(S)/S \cong QD_{16}$ or E_4 according as $|S| = 3^3$ or $|S| > 3^3$ respectively. No automorphism of G of order 3 normalizes S and centralizes both S/Z and a $3'$-Hall subgroup of $N_G(S)$.*

(4) *Assume G is isomorphic to one of the sporadic groups: J_2 (with $p = 3$); Co_2, Co_3, HS, Mc (with $p = 5$); or J_4 (with $p = 11$). In each case S is non-abelian of order p^3 and exponent p, and Z is the only nontrivial proper strongly closed subgroup of S. The normalizer of Z [in G] is: $3PGL_2(9)$ [in J_2], $5^{1+2}((4 * SL_3(3)) \cdot 2)$ [in Co_2], $5^{1+2}((4YS_3) \cdot 4)$ [in Co_3], $5^{1+2}(8 \cdot 2)$ [in HS], $(5^{1+2} \cdot 3) \cdot 8$ [in Mc], or $(11^{1+2} \cdot SL_2(3)) \cdot 10$ [in J_4]. In $G = J_2$ we have $N_G(S)/S \cong Z_8$; and in all other cases $N_G(S) = N_G(A)$.*

(5) *Assume $G \cong J_3$ with $p = 3$. Then $Z \cong E_9$ and $\Omega_1(S) \cong E_{27}$ are the only nontrivial proper strongly closed subgroups of S. Furthermore, $N_G(Z) = N_G(S) = SH$ where $H \cong Z_8$ acts fixed point freely on $\Omega_1(S)$ and irreducibly on Z.*

Proof. Part (1) may be found in [HKS72] and [Suz62]. Part (2) appears in [Wa66]. All parts of (4) and (5) appear in [GLSv3, Chapter 5] with references therein.

In part (3), by [GL83, 14-7] the center of S has order 3 and $C = C_G(Z) \cong SL_3^\epsilon(q)$ according to the condition $3 \mid q - \epsilon$. The same reference shows G has two conjugacy classes of elements of order 3: the two nontrivial elements of Z are in one class, and all elements of order 3 in $S - Z$ lie in the other. Now $S \leq SL_3^\epsilon(q)$ acts absolutely irreducibly on its natural 3-dimensional module over $\mathbb{F}_q$ (or $\mathbb{F}_{q^2}$ in the unitary case), hence by Schur's Lemma the centralizer of S in C consists of scalar matrices. Thus $Z = C_C(S) = C_G(S)$. Since the two nontrivial elements of Z are conjugate in G, $N_G(Z) = C\langle t \rangle$ where an involution t may be chosen to normalize S and induce a graph (transpose-inverse) automorphism on C. By canonical forms, all non-central elements of order 3 in $SL_3^\epsilon(q)$ are conjugate in $GL_3^\epsilon(q)$ to the same diagonal matrix $u = \text{diag}(\lambda, \lambda^{-1}, 1)$, where λ is a primitive cube root of unity, but are also conjugate in $SL_3^\epsilon(q)$ to u because the outer (diagonal) automorphism group induced by GL_3^ϵ may be represented by diagonal matrices that commute with u. Thus all elements of order 3 in $S - Z$ are conjugate in C.

If $|S| = 27$, then since S/Z is abelian of type (3,3), all elements of order 3 in S/Z are conjugate under the action of $N_C(S/Z) = N_C(S)/Z$; hence they are conjugate under the faithful action of a $3'$-Hall subgroup, H_0, of $N_C(S)$ on S/Z. This shows $|H_0| \geq 8$. Since a $3'$-Hall subgroup H of $N_G(S)$ acts faithfully on S/Z and has order $2|H_0|$, it must be isomorphic to a Sylow 2-subgroup, QD_{16}, of $GL_2(3)$ as claimed.

If $|S| = 3^{2a+1} > 27$ then we may describe S as the group, S_T, of diagonal matrices of 3-power order acted upon by a permutation matrix w of order 3 (where $\langle w \rangle = S_W$). Then $S_T \cong Z_{3^a} \times Z_{3^a}$ is the unique abelian subgroup of S of index 3 (as $|Z| = 3$), so $N_C(S)$ normalizes S_T. Let H_0 be a $3'$-Hall subgroup of $N_C(S)$. One easily sees that H_0 must act faithfully on $\Omega_1(S_T)$ (and centralize Z), hence $|H_0| \leq 2$. Since there is a permutation matrix of order 2 in C normalizing S, $|H_0| = 2$. Thus $N_G(S)/S$ has order 4, and is seen to be a fourgroup by its action on $\Omega_1(S_T)$.

To see that Z is the unique nontrivial strongly closed subgroup that is proper in S suppose B is another, so that $Z < B$. If B contains an element of order 9 — hence an element of order 9 represented by a diagonal matrix in C — then by conjugating in C one easily computes that $B - Z$ contains an element of order 3. Since all such are conjugate in C this shows $\Omega_1(S) \leq B$. It is an exercise that $\Omega_1(S) = S$ (the details appear at the end of the proof of Lemma 3.4), a contradiction.

Finally, suppose f is an automorphism of G of order 3 that normalizes S and centralizes S/Z. Then $|S : C_S(f)| \leq 3$ so f cannot be a field automorphism as $|G_2(r^3) : G_2(r)|_3 \geq 3^2$ for all r prime to 3. Thus f must induce an inner automorphism on G, hence act as an element of order 3 in S_T. We have already seen that no such element centralizes a $3'$-Hall subgroup of $N_G(S)$, a contradiction. This completes all parts of the proof. □

Corollary 2.13. *Let p be any prime, let L be a finite simple group possessing a strongly closed p-subgroup A that is properly contained in the Sylow p-subgroup S of L. Assume further that L is isomorphic to one of the groups L_i in the conclusion of Theorem 2.1 or Theorem 2.2. Then one of the following holds:*

(1) $N_L(S) = N_L(A)$,
(2) $|A| = 3$ *and* $L \cong G_2(q)$ *for some* q *with* $(q, 3) = 1$, *or*
(3) $|A| = 3$, $L \cong J_2$ *and* $N_L(A) \cong 3PGL_2(9)$.

Proof. This is immediate from Propositions 2.11 and 2.12. □

2.2. **The Proof of Theorem 2.2.**

This subsection derives Theorem 2.2 as a consequence of Theorem 2.3, which is proved in the next section. Throughout this subsection G is a minimal counterexample to Theorem 2.2.

Since strong closure inherits to quotient groups, if $\mathcal{O}_A(G) \neq 1$ we may apply induction to $G/\mathcal{O}_A(G)$ and see that the asserted conclusion holds. Thus we may assume $\mathcal{O}_A(G) = 1$, and consequently

$$A \cap N \text{ is not a Sylow } p\text{-subgroup of } N \text{ for any nontrivial } N \trianglelefteq G \\ \text{and } O_{p'}(G) = 1. \tag{2.3}$$

Likewise if $G_0 = \langle A^G \rangle$ then by Frattini's Argument, $G = G_0 N_G(A)$, whence $\langle A^G \rangle = \langle A^{G_0} \rangle$. Thus we may replace G by G_0 to obtain

$$G \text{ is generated by the conjugates of } A. \tag{2.4}$$

By strong closure $A \cap O_p(G) \trianglelefteq G$, whence by (2.3), $A \cap O_p(G) = 1$. Since $[A, O_p(G)] \leq A \cap O_p(G) = 1$, by (2.4) we have

$$O_p(G) \leq Z(G). \tag{2.5}$$

Consequently $F^*(G)$ is a product of subnormal quasisimple components $L_1, \ldots, L_r$ with $O_{p'}(L_i) = 1$ for all i. Moreover $S_i = S \cap L_i$ is a Sylow p-subgroup of L_i and $S_i \neq 1$ by (2.3).

We argue that each component of G is normal in G. By way of contradiction assume $\{L_1, \ldots, L_s\}$ is an orbit of size ≥ 2 for the action of G on its components.

Let $Z = A \cap Z(S)$, so that Z normalizes each L_i. Thus $N = \cap_{i=1}^{s} N_G(L_i)$ is a proper normal subgroup of G possessing a nontrivial strongly closed p-subgroup, $B = A \cap N$ that is not a Sylow subgroup of N. By induction — keeping in mind that components of N are necessarily components of G and $\mathcal{O}_B(N) = 1$ — and after possible renumbering, there are simple components $L_1, \ldots, L_t$ of N that satisfy the conclusion of Theorem 2.2 with $B \cap L_i \neq 1$, these are all the components of N satisfying the latter condition, and $t \geq 1$. By Frattini's Argument $G = N_G(B)N$ from which it follows that $L_1 \cdots L_t \trianglelefteq G$. The transitive action of G in turn forces $t = s$. Thus A permutes $\{L_1, \ldots, L_s\}$ and $1 \neq A \cap L_i < S_i$. If A does not normalize one of these components, say $L_i^a = L_j$ for some $i \neq j$ and $a \in A$, then $S_i S_i^a = S_i \times S_j$. But then $[S_i, a] \not\leq (A \cap L_i) \times (A \cap L_j)$, contrary to $A \trianglelefteq S$. Thus A must normalize L_i for $1 \leq i \leq s$. Since $A \leq N \trianglelefteq G$, (2.4) gives $N = G$, a contradiction. This proves

$$\text{every component of } G \text{ is normal in } G. \tag{2.6}$$

The preceding results also show that A acts nontrivially on each L_i. By Lemma 2.8, $A_i = A \cap L_i \neq 1$ and A_i is not Sylow in L_i for every i. By Theorem 2.3 applied to each L_i using a minimal strongly closed subgroup of A_i we obtain

$$F^*(G) = L_1 \times L_2 \times \cdots \times L_r \tag{2.7}$$

and each L_i is one of the simple groups described in the conclusion of Theorem 2.2. Moreover, in each of conclusions (i) to (v), by Propositions 2.10 and 2.12, A_i is a subgroup of L_i described in the respective conclusion.

It remains to verify that the action of A is as claimed when $A \not\leq F^*(G)$. The automorphism group of each L_i is described in detail in [GLSv3, Theorem 2.5.12 and Section 5.3] — these results are used without further citation.

Let $S^* = S \cap F^*(G) = S_1 \times \cdots \times S_r$, let $H^* = H_1 \times \cdots \times H_r$, where H_i is a p'-Hall complement to S_i in $N_{L_i}(S_i)$, and let $N^* = AS^*H^*$. Note that $O_{p'}(N^*) = C_{H^*}(S^*)$ is A-invariant. Now in all cases $[A, S_i] \leq A \cap S_i \leq \Phi(S_i)$, that is, A commutes with the action of H^* on $S^*/\Phi(S^*)$. This forces $A \leq O_{p',p}(N^*)$. By strong closure of A we get that $AO_{p'}(N^*) \trianglelefteq N^*$. Thus $N_{N^*}(A)$ covers $H^*/C_{H^*}(S^*)$. Let H be a p'-Hall complement to AS^* in $N_{N^*}(A)$; we may assume $H^* = HC_{H^*}(S^*)$. We have a Fitting decomposition

$$A = [A, H]A_F \quad \text{where} \quad A_F = C_A(H). \tag{2.8}$$

By Propositions 2.10 and 2.12 each A_i is abelian and H_i, hence also H, acts without fixed points on each A_i. Since $[A, H] \le A \cap F^*(G)$ we therefore obtain

$$[A, H] = A_1 \times \cdots \times A_r \qquad \text{and} \qquad A_F \cap [A, H] = 1. \tag{2.9}$$

We now determine the action of A_F on L_i for each isomorphism type in conclusions (i) to (v).

First suppose A_F acts trivially on some L_i, say for $i = 1$. In this situation $A = A_1 \times B$ where $B = (A_2 \times \cdots \times A_r)A_F = A \cap C_G(L_1)$. Then $\langle A^G \rangle = L_1 \times \langle B^G \rangle$, and so we may proceed inductively to identify $\langle B^G \rangle$ and conclude that Theorem 2.2 is valid. We now observe that A_F acts trivially on all components listed in conclusions (ii) to (v) as follows: If L_1 is one of these cases, it follows from Proposition 2.12 that $C_{H_1}(S_1) = 1$ and so A_F centralizes a p'-Hall subgroup of $N_{L_1}(S_1)$. In case (ii) of the conclusions, if L_1 is a Lie-type simple group in characteristic p and BN-rank 1, by [GL83, 9-1] no automorphism of order p centralizes a Cartan subgroup of L_1, so A_F acts trivially on L_1. If $L_1 \cong G_2(q)$ is described by case (iii) of the conclusion, then since $[S_1, A_F] \le A_1$, the last assertion of Proposition 2.12(3) shows that A_F acts trivially on L_1. And in cases (iv) and (v) of the conclusions, when L_1 is a sporadic group, none of the target groups admits an outer automorphism of order p, and no inner automorphism that normalizes a Sylow p-subgroup also commutes with a p'-Hall subgroup of its normalizer. Thus A_F acts trivially in these instances too.

It remains to consider when every L_i is described by conclusion (i): $L = L_i$ is a group of Lie type over the field $\mathbb{F}_{q_i}$ where $p \nmid q_i$ and the Sylow p-subgroups are abelian but not elementary abelian. Since A_F commutes with the action of a p'-Hall subgroup of $N_L(S_i)$, it follows from Proposition 2.10 that A_F induces outer automorphisms on L. The outer diagonal automorphism group of L has order dividing the order of the Schur multiplier of L, so by Proposition 2.9(7) no element of G induces a nontrivial outer diagonal automorphism of p-power order on L. Since Sylow 3-subgroups of $D_4(q)$ and ${}^3D_4(q)$ are non-abelian, L does not admit a nontrivial graph or graph-field automorphism when $p = 3$. This shows A_F must act as field automorphisms on L, and hence $A_F/C_{A_F}(L)$ is cyclic.

Now G is generated by the conjugates of A, hence the group $\widetilde{G} = G/LC_G(L)$ of outer automorphims on L is generated by conjugates of $\widetilde{A}_F$. This implies via [GLSv3, Theorem 2.5.12] that

$$\widetilde{G} = \widetilde{D}\widetilde{A}_F \qquad \text{and} \qquad \widetilde{D} = [\widetilde{D}, \widetilde{A}_F] \tag{2.10}$$

where $\widetilde{D}$ is a cyclic p'-subgroup of the outer-diagonal automorphism group of L normalized by the cyclic p-group $\widetilde{A}_F$ of field automorphisms. Moreover, since $p > 3$ when L is of type $E_6(q)$, ${}^2E_6(q)$ or $D_{2m}(q)$, the action of $\widetilde{A}_F$ on $\widetilde{D}$ in (2.10) implies that $\widetilde{D}$ is trivial except in the cases where L is a linear or unitary group.

A p'-order subgroup D that covers the section $\widetilde{D}$ for every L_i may be defined as follows (even in the presence of L_i that are not of type (i)): We have now established that $S = S^*A_F$, and that S^* is a Sylow p-subgroup of the (normal) subgroup G_D of G inducing only inner and diagonal automorphisms on $F^*(G)$. Thus $N_{G_D}(S^*)$ has a p'-Hall complement, which is then a complement to $S = S^*A_F$ in $N_G(S^*)$. Since $[S^*, A_F] \leq \Phi(S^*)$, A_F commutes with the action on S^* of this p'-Hall subgroup. As $\widetilde{D} = [\widetilde{D}, A_F]$, any choice for D must lie in $C_G(S^*)$. However, $C_G(S^*)$ has a normal p-complement, so any D must lie in $O_{p'}(C_G(S^*))$. Thus $[O_{p'}(C_G(S^*)), A_F] = [O_{p'}(C_G(S^*)), S]$ covers $\widetilde{D}$ for every component L_i (and centralizes all components that are not of type PSL or PSU).

Finally note that in every case A_F' centralizes L_i for every i. Since then A_F' centralizes $F^*(G)$, it must be trivial, that is, A_F is abelian. Since $A_F/C_{A_F}(L_i)$ is cyclic for all i, it follows that $A_F = A_F/\cap_{i=1}^r C_{A_F}(L_i)$ has rank at most r, as asserted. This completes the proof of Theorem 2.2. □

2.3. **The Proofs of Corollaries 2.4 and 2.5.**

Considering both corollaries at once, assume the hypotheses of Corollary 2.4 hold. The result is trivial if either $A = S$ (in which case $\mathcal{O}_A(G) = G$) or $A = 1$ (in which case $G = 1$). By passing to $G/\mathcal{O}_A(G)$ we may assume $\mathcal{O}_A(G) = 1$. Since G is generated by conjugates of A, Theorems 2.1 and 2.2 imply that

$$G = (L_1 \times \cdots \times L_r)(D \cdot A_F), \tag{2.11}$$

where the L_i, D and A_F are described in their conclusions (with both D and A_F trivial when $p = 2$). Let $S_i = S \cap L_i$ and $A_i = A \cap L_i$.

For each i let Z_i be a minimal nontrivial strongly closed subgroup of $A \cap L_i$, and let $Z = Z_1 \times \cdots \times Z_r$. Then Z is strongly closed in G, and by Propositions 2.10 and 2.12, Z is contained in the center of S. It is immediate from Sylow's Theorem and the weak closure of Z that $N_G(Z)$ controls strong G-fusion in S. Now

$$N_G(Z) = (N_{L_1}(Z_1) \times \cdots \times N_{L_r}(Z_r))(D \cdot A_F)$$

where by the proof of Theorem 2.2, $D = [D, A_F]$ may be chosen to be an S-invariant p'-subgroup centralizing each S_i. This implies

$$M = (N_{L_1}(Z_1) \times \cdots \times N_{L_r}(Z_r))A_F \;\text{ controls strong } G\text{-fusion in } S. \tag{2.12}$$

It suffices therefore to show that $N_M(A)$ controls strong M-fusion in S. Furthermore, $N_M(A)$ controls strong M-fusion in S if and only if the corresponding fact holds in $M/O_{p'}(M)$; so we may pass to this quotient and therefore assume $O_{p'}(M) = 1$ (without encumbering the proof with overbar notation, since all normalizers considered are for p-groups).

If L_i is a Lie-type component with S_i abelian then, as noted in the proof of Theorem 2.2, $N_{L_i}(Z_i) = N_{L_i}(S_i)$ and A_F commutes with the action on S_i of an A_F-stable p'-Hall subgroup H_i of this normalizer. Since $O_{p'}(M) = 1$ it follows that H_i acts faithfully on S_i, and so $[A_F, H_i] = 1$. On the other hand, if L_i is not of this type, $[A_F, L_i] = 1$. Thus (reading modulo $O_{p'}(M)$) we have

$$M = SC_M(A_F) \tag{2.13}$$

and so $N_M(A) = N_M(A^*)$, where $A^* = A_1 \cdots A_r$.

For every component L_i that is not of type $G_2(q)$ or J_2, by Corollary 2.13, $N_{L_i}(Z_i) = N_{L_i}(S_i)$; and therefore in these components $N_{L_i}(Z_i) = N_{L_i}(A_i)$ too. However, for a component L_i of type $G_2(q)$ or J_2 (with $p = 3$), by Proposition 2.12 we must have $Z_i = A_i$. In all cases we have $N_{L_i}(Z_i) = N_{L_i}(A_i)$. Hence $N_M(A^*) = N_M(Z) = M$ and the first assertion of Corollary 2.4 holds by (2.12). This also establishes the second assertion unless $p = 3$ and some components L_i are of type $G_2(q)$ or J_2, where the possibility that $|S_i| > 3^3$ in these exceptions is excluded by the hypotheses of Corollary 2.4.

In the remaining case let $S^* = S_1 \times \cdots \times S_r$, where $S_1, \ldots, S_k$ are the Sylow 3-subgroups of the components of type $G_2(q)$ or J_2, and $S_{k+1}, \ldots, S_r$ are the remaining ones. Again by (2.13), $N_M(S) = N_M(S^*)$ so we must prove the latter normalizer controls strong M-fusion in S; indeed, it suffices to prove control of fusion in S^*. Now $N_M(S^*)$ controls strong M-fusion in S^* if and only if the corresponding result holds in each direct factor. This is trivial for $i > k$ as S_i is normal in that factor. For $1 \le i \le k$ the result is true since $S_i = 3^{1+2}$, i.e., S_i has a central series $1 < Z_i < S_i$ whose terms are all weakly closed in S_i with respect to $N_{L_i}(Z_i)$ (see, for example, [GiSe85]). This establishes the final assertion of Corollary 2.4.

In Corollary 2.5 observe that by Theorem 2.2, once $\mathcal{O}_A(G)$ is factored out we have equation (2.11) holding, and since A_F acts without fixed points on the cyclic quotient $D/(D \cap L_1 \cdots L_r)$, we must have $N_G(A) \leq (L_1 \cdots L_r)A_F$. Thus by (2.13) we have

$$N_G(A) = N_M(A) \leq SC_M(A_F)O_{p'}(M).$$

Since $N_M(A) \cap O_{p'}(M)$ centralizes A we have $N_G(A) \leq SC_M(A_F)$, and hence $N_G(A) = SC_M(A_F)$. All parts of Corollary 2.5 now follow.

□

3. The Proof of Theorem 2.3

We now prove Theorem 2.3. Throughout this section p is an odd prime, G is a minimal counterexample, and A is a nontrivial strongly closed subgroup of G that is a proper subgroup of the Sylow p-subgroup S of G. The minimality implies that if H is any proper section of G containing a nontrivial minimal strongly closed (with respect to H) p-subgroup A_0, then either A_0 is a Sylow subgroup of its normal closure in H or the normal closure of $\overline{A_0}$ in $\overline{H}$ is a direct product of isomorphic simple groups, as described in the conclusion of Theorem 2.2, where overbars denote passage to $H/\mathcal{O}_{A_0}(H)$. In particular, A_0 does not even have to be a subgroup of A, although for the most part we will be applying this inductive assumption to subgroups $A_0 \leq A \cap H$ (which we often show is nontrivial by invoking part (2) of Lemma 2.8).

Familiar facts about the families of simple groups, including the sporadic groups, are often stated without reference. All of these can be found in the excellent, encyclopedic source [GLSv3]. Specific references are cited for less familiar results that are crucial to our arguments.

Lemma 3.1. *G is a simple group.*

Proof. Since strong closure inherits to quotient groups, if $\mathcal{O}_A(G) \neq 1$ we may apply induction to $G/\mathcal{O}_A(G)$ and see that the asserted conclusion holds. Thus we may assume $\mathcal{O}_A(G) = 1$, i.e.,

(3.1) $A \cap N$ is not a Sylow p-subgroup of N for any nontrivial $N \trianglelefteq G$.

In particular,

$$O_{p'}(G) = 1. \tag{3.2}$$

Let $G_0 = \langle A^G \rangle$ and assume $G_0 \neq G$. By (3.1), A is not a Sylow p-subgroup of G_0. Let $1 \neq A_0 \leq A$ be a minimal strongly closed subgroup of G_0. By the inductive hypothesis A_0 is contained in a semisimple normal subgroup N of G_0 satisfying the conclusions of the theorem. Since $N \trianglelefteq G$ it follows that $M = \langle N^G \rangle$ is a semisimple normal subgroup of G whose simple components are described by Theorem 2.2. Since A is minimal strongly closed in G and $1 \neq A_0 \leq A \cap M$, $A \leq M$ and the conclusion of Theorem 2.3 is seen to hold. Thus

$$G \text{ is generated by the conjugates of } A. \tag{3.3}$$

By strong closure $A \cap O_p(G) \trianglelefteq G$, hence by (3.1), $A \cap O_p(G) = 1$. Thus $[A, O_p(G)] \leq A \cap O_p(G) = 1$, i.e., A centralizes $O_p(G)$. Since G is generated by conjugates of A,

$$O_p(G) \leq Z(G). \tag{3.4}$$

By (3.2) and (3.4), $F^*(G)$ is a product of commuting quasisimple components, $L_1, \dots, L_r$, each of which has a nontrivial Sylow p-subgroup. Since A acts faithfully on $F^*(G)$, by Lemma 2.8 $A \cap F^*(G) \neq 1$. The minimality of A then forces $A \leq F^*(G)$. Thus A normalizes each L_i, whence so does G by (3.3). Now A acts nontrivially on one component, say L_1, so again by Lemma 2.8, $A \cap L_1 \neq 1$. By minimality of A we obtain $A \leq L_1 \trianglelefteq G$, so by (3.3)

$$G = L_1 \text{ is quasisimple (with center of order a power of } p).$$

Finally, assume $Z(G) \neq 1$ and let $\widetilde{G} = G/Z(G)$. Since $A \neq S$ but $A \cap Z(G) = 1$, by Gaschütz's Theorem we must have that $S \neq AZ(G)$ and so $\widetilde{A}$ is strongly closed but not Sylow in the simple group $\widetilde{G}$. Since $|\widetilde{G}| < |G|$, the pair $(\widetilde{G}, \widetilde{A})$ satisfy the conclusions of Theorem 2.3; in particular, $\widetilde{A} = \Omega_1(Z(\widetilde{S}))$ in all cases. If $\widetilde{G}$ is a group of Lie type in conclusion (i), again by Gaschütz's Theorem together with the irreducible action of $N_{\widetilde{G}}(\widetilde{S})$ on $\Omega_1(\widetilde{S})$, $\widetilde{A}$ must lift to a non-abelian group in G. In this situation $Z(G) \leq A'$, contrary to $A \cap Z(G) = 1$. In conclusions (ii), (iii) and (iv) the p-part of the multipliers of the simple groups are all trivial, so $Z(G) = 1$ in these cases. In case (v) when $\widetilde{G} \cong J_3$ and $\widetilde{A} = Z(\widetilde{S})$ by the fixed point free action of an element of order 8 in $N_G(S)$ on S it again follows easily that $\tilde{A}$ must lift to the non-abelian group of order 27 and exponent 3 in G, contrary to $A \cap Z(G) = 1$. This shows $Z(G) = 1$ and so G is simple. The proof is complete. □

Lemma 3.2. *A is not cyclic and S is non-abelian.*

Proof. If A is cyclic then since $\Omega_1(A)$ is also strongly closed, the minimality of A gives that $|A| = p$. Then G is not a counterexample by Proposition 2.6. Likewise if S is abelian, by Proposition 2.10 it is homocyclic with $N_G(S)$ acting irreducibly and nontrivially on $\Omega_1(S)$. By minimality of A we must then have $A = \Omega_1(S)$ and the exponent of S is greater than p. None of the sporadic or alternating groups or groups of Lie type in characteristic p contain such Sylow p-subgroups, so G must be a group of Lie type in characteristic $\neq p$. Again, G is not a counterexample, a contradiction. □

Note that because A is a noncyclic normal subgroup of S and p is odd, A contains an abelian subgroup U of type (p, p) with $U \trianglelefteq S$. Furthermore, $|S : C_S(U)| \leq p$ so U is contained in an elementary abelian subgroup of S of maximal rank.

Lemmas 3.3 to 3.7 now successively eliminate the families of simple groups as possibilities for the minimal counterexample. The argument used to eliminate the alternating groups is a prototype for the more complicated situation of Lie type groups, so slightly more expository detail is included.

Lemma 3.3. *G is not an alternating group.*

Proof. Assume $G \cong A_n$ for some n. Since S is non-abelian, $n \geq p^2$. If $p \nmid n$ then S is contained in a subgroup isomorphic to A_{n-1}, which contradicts the minimality of G (no alternating group satisfies the conclusions in Theorem 2.2). Thus $n = ps$ for some $s \in \mathbb{N}$ with $s \geq p$.

Let E be a subgroup of S be generated by s commuting p-cycles. Since E contains a conjugate of every element of order p in G, $A \cap E \neq 1$. We claim $E \leq A$. Let $z = z_1 \cdots z_r \in A \cap E$ be a product of commuting p-cycles z_i in E with r minimal. If $r \geq 3$ there is an element $\sigma \in A_n$ that inverts both z_r and z_{r-1} and centralizes all other z_i; and if $r = 2$, since $n \geq 3r$ there is an element $\sigma \in A_n$ that inverts z_2 and centralizes z_1. In either case, by strong closure $z^\sigma \in A \cap E$ and $zz^\sigma = z_1^2 \cdots z_{r-2}^2$ or z_1^2 respectively. Hence zz^σ is an element of $A \cap E$ that is a product of fewer commuting p-cycles, a contradiction. This shows A contains a p-cycle, hence by strong closure

$E \leq A$. Now A_n contains a subgroup H with

$$S \leq H = N_{A_n}(E) \qquad \text{and} \qquad H \cong Z_p \wr A_s. \tag{3.5}$$

By our inductive assumption H contains a normal subgroup $N = \mathcal{O}_A(H)$ with $E \leq N$ such that $A \cap N$ is a Sylow p-subgroup of N and H/N a product of simple components described in Theorem 2.2. Since H is a split extension over E and every element of H of order p is conjugate to an element of E, by strong closure $A \neq E$. Since $H/E \cong A_s$ is not one of the simple groups in Theorem 2.2 it follows that $N = H$ (in the cases where $s = 3$ or 4 as well), contrary to $A \neq S$. This contradiction establishes the lemma.

Alternatively, one could argue from (3.5) and induction that $S = \Omega_1(S)$, and so again $S = A$ by strong closure, a contradiction. □

Lemma 3.4. *G is not a classical group (linear, unitary, symplectic, orthogonal) over $\mathbb{F}_q$, where q is a prime power not divisible by p.*

Proof. Assume G is a classical simple group. Following the notation in [GLSv3, Theorem 4.10.2], let V be the classical vector space associated to G and let $X = \mathrm{Isom}(V)$. We may assume dim $V \geq 7$ in the orthogonal case because of isomorphisms of lower dimensional orthogonal groups with other classical groups (the dimension is over $\mathbb{F}_{q^2}$ in the unitary case). The tables in [KL90, Chapter 4] are helpful references in this proof.

First consider when G is neither a linear group with p dividing $q-1$ nor a unitary group with p dividing $q+1$. This restriction implies that $p \nmid |X : X'|$ and there is a surjective homomorphism $X' \to G$ whose kernel is a p'-group. Thus we may do calculations in X in place of G (taking care that conjugations are done in X'). Proposition 2.9 is realized explicitly in this case as follows: There is a decomposition

$$V = V_0 \perp V_1 \perp \cdots \perp V_s$$

of V ($\perp$ denotes direct sum in the linear case), where $\mathrm{Isom}(V_0)$ is a p'-group, the cyclic group of order p has an orthogonally indecomposable representation on each other V_i, the V_i are all isometric, and a Sylow p-subgroup of $\mathrm{Isom}(V_i)$ is cyclic. Furthermore, X' contains a subgroup isomorphic to A_s permuting $V_1, \ldots, V_s$ and the stabilizer in X' of the set $\{V_1, \ldots, V_s\}$ contains a Sylow p-subgroup of X. In other words, we may assume

$$S \le H \cong \mathrm{Isom}(V_1) \wr A_s. \tag{3.6}$$

In the notation of Proposition 2.9, let $S \cap \mathrm{Isom}(V_i) = \langle u_i \rangle$, where u_i acts trivially on V_j for all $j \neq i$. Then $S_T = \langle u_1, \dots, u_s \rangle$ and S_W is a Sylow p-subgroup of A_s. Since S is non-abelian, $S_W \neq 1$ and so $s \ge p \ge 3$. Let z_i be an element of order p in $\langle u_i \rangle$, and let

$$E = \langle z_1, \dots, z_s \rangle = \Omega_1(S_T) \cong E_{p^s}.$$

The faithful action of S_W on S_T forces $Z(S) \le S_T$, so $A \cap E \neq 1$.

We claim $E \le A$. As in the alternating group case, let z be a nontrivial element in $A \cap E$ belonging to the span of r of the basis elements z_i in E with r minimal. After renumbering and replacing each z_i by another generator for $\langle z_i \rangle$ if necessary, we may assume $z = z_1 \cdots z_r$. If $r \ge 3$ there is an element $\sigma \in G$ that acts trivially on $z_1, \dots, z_{r-2}$ and normalizes but does not centralize $\langle z_{r-1}, z_r \rangle$; and if $r = 2$, since $s \ge 3$ there is an element $\sigma \in G$ that centralizes z_1 and normalizes but does not centralize $\langle z_2 \rangle$. In both cases $z^\sigma z^{-1}$ is a nontrivial element of $A \cap E$ that is a product of fewer basis elements. This shows $z_i \in A$ for some i and so $E \le A$ since all z_j are conjugate in G.

By Proposition 2.9(5) in this setting, every element of order p in G is conjugate to an element of E. Since the extension in (3.6) is split, $A \not\le S_T$. By the overall induction hypothesis applied in H (or because a Sylow p-subgroup of A_s is generated by elements of order p), it follows that A covers S/S_T. We may therefore choose a numbering so that for some $x \in A$, $u_1^x = u_2$. Thus

$$u = u_1 u_2^{-1} = [u_2, x] \in A \cap \mathrm{Isom}(V_1 \perp \cdots \perp V_{s-1}).$$

Let $Y = G \cap \mathrm{Isom}(V_1 \perp \cdots \perp V_{s-1})$ so that Y is also a classical group of the same type as G over $\mathbb{F}_q$. Note that the dimension of the underlying space on which Y acts is at least $2(s-1)$ by our initial restrictions on q. Since Y is proper in G, by induction applied using a minimal strongly closed subgroup A_0 of $A \cap Y$ in Y we obtain the following: either A_0 (hence also A) contains a Sylow p-subgroup of Y, or the Sylow p-subgroups of Y are homocyclic abelian with $A_0 \cap Y$ elementary abelian of the same rank as a Sylow p-subgroup of Y. Furthermore, in the latter case a Sylow p-normalizer acts irreducibly on A_0, and hence the strongly closed subgroup $A \cap Y$ is also homocyclic abelian. Since $A \cap Y$ contains the element u of order d, where $d = |u_1|$, in either case $A \cap Y$ contains all elements of order d in $S \cap Y$. Since $u_1 \in S \cap Y$ this proves $u_1 \in A$. By (3.6) all u_i are conjugate in G to u_1, hence $S_T \leq A$ and so $A = S$ a contradiction.

It remains to consider the cases where V is of linear or unitary type and p divides $q-1$ or $q+1$ respectively (denoted as usual by $p \bigm| q-\epsilon$). Now replace the simple group G by its universal quasisimple covering $SL^\epsilon(V)$. Likewise replace A by the p-part of its preimage. Thus A is a noncyclic (hence noncentral) strongly closed p-subgroup of $SL^\epsilon(V)$. In this situation $S = S_T S_W$ where we may assume S_T is the group of p-power order diagonal matrices of determinant 1 (over $\mathbb{F}_{q^2}$ in the unitary case), and S_W is a Sylow p-subgroup of the Weyl group W of permutation matrices permuting the diagonal entries. Furthermore, S_T is homocyclic of exponent d, where $d = |q-\epsilon|_p$, and is a trace 0 submodule of the natural permutation module for W of exponent d and rank $m = \dim V$. Since A is noncyclic, it contains a noncentral element z of order p; and by Proposition 2.9, z is conjugate to an element of S_T, i.e., is diagonalizable. Arguing as above with $E = \Omega_1(S_T)$ we reduce to the case where z is represented by the matrix $\mathrm{diag}(\zeta, \zeta^{-1}, 1, \ldots, 1)$ for some primitive p^{th} root of unity ζ. The action of W again forces $E \leq A$. Again, every element of order p in S is conjugate in G to an element of E, so by strong closure

$$\Omega_1(S) \leq A. \tag{3.7}$$

Consider first when $m \geq 5$. Then $C_G(z)$ contains a quasisimple component $L \cong SL^\epsilon_{m-2}(q)'$. Since L contains a conjugate of z, the inductive argument used in the general case shows that $A \cap L$ contains a diagonal matrix element of order d, hence contains such an element centralizing an $n-2$ dimensional subspace. The strong closure of A then again yields $S_T \leq A$; and as before by induction or because $S = S_T \Omega_1(S)$ we get $A = S$, a contradiction.

Thus $\dim V \leq 4$, and since $S_W \neq 1$ we must have $p = 3$. If $G \cong SL^\epsilon_4(q)$ then let z be represented by the diagonal matrix $\mathrm{diag}(\zeta, \zeta, \zeta, 1)$, where ζ is a primitive 3^{rd} root of unity. Then $C_G(z)$ contains a Sylow 3-subgroup of G and a component of type $SL^\epsilon_3(q)$, so the preceding argument leads to a contradiction.

Finally, consider when $G \cong SL^\epsilon_3(q)$. The Sylow 3-subgroups of $SL^\epsilon_3(q)$ are described in the proof of Proposition 2.12. In both instances S_T is homocyclic of rank 2 and exponent d with generators u_1, u_2, and with $S_W = \langle\, w \,\rangle \cong Z_3$ acting by

$$u_1^w = u_2 \quad \text{and} \quad u_2^w = u_1^{-1} u_2^{-1}.$$

Thus $u_1 w$ has order 3, and so $u_1 = (u_1 w)w^{-1} \in \Omega_1(S)$. By (3.7), this again forces $A = S$, which gives the final contradiction. □

Lemma 3.5. *G is not an exceptional group of Lie type (twisted or untwisted) over $\mathbb{F}_q$, where q is a prime power not divisible by p.*

Proof. Assume $G = \mathcal{L}(q)$ is an exceptional group of Lie type over $\mathbb{F}_q$ with $p \nmid q$. Throughout this proof we rely on the Sylow structure for G as described in Proposition 2.9. It shows, in particular, that we need only consider when the odd prime p divides both order of the Weyl group of the untwisted group corresponding to G and $pm_0 \mid m$ for some $m \in \mathcal{O}(G)$; in all other cases the proposition gives that the Sylow p-subgroup is homocyclic abelian. The cyclotomic factors $\Phi_m(q)$ and their "multiplicities" r_m for each of the exceptional groups are listed explicitly in [GL83, Table 10:2]. Note that $3 \mid q^2 - 1$, so in this case m_0 is 1 or 2; also, $5 \mid q^4 - 1$, so in this case m_0 is 1, 2, or 4; finally, $7 \mid q^6 - 1$, so in this case m_0 is 1, 2, 3, or 6. In the notation of Proposition 2.9, except in the case ${}^3D_4(q)$ we have $S = S_T S_W$ (split extension) where S_T is a normal homocyclic abelian subgroup of exponent $|\Phi_{m_0}(q)|_p$ and rank r_{m_0}, and $|S_W| = p^b$, where b is defined in (2.2).

The exceptional groups are listed in Table 3A along with p dividing the order of the Weyl group, permissible m_0 such that $m = p^a m_0$ for some $m \in \mathcal{O}(G)$ with $a \geq 1$, and the corresponding r_{m_0} and p^b for each of these (in the case of ${}^3D_4(q)$ we define 3^b so that $|S| = (|\Phi_{m_0}(q)|_p)^{r_{m_0}} 3^b)$.

We consider all these cases, working from largest to smallest — the latter requiring more delicate examination. Table 4-1 in [GL83] is used frequently without specific citation: it lists all the "large" subgroups of various families of Lie type groups that we shall employ. It is helpful to keep in mind the description of the order of a Sylow p-subgroup in Proposition 2.9 when comparing the p-part of $|G|$ to that of its Lie-type subgroups.

Case $\boldsymbol{p = 7}$: $E_8(q)$ contains both $A_8(q)$ and ${}^2A_8(q)$ and so, by inspection of orders, shares a Sylow 7-subgroup with it in the cases (1,8,7) and (2,8,7) respectively (the Sylow 7-subgroup order is seen to be $7 \cdot |q - \epsilon|_7^8$ for each group). Likewise $E_7(q)$ contains both $A_7(q)$ and ${}^2A_7(q)$ and so shares a Sylow 7-subgroup with it in the cases (1,7,7) and (2,7,7) respectively. By minimality of G all the $p = 7$ cases are eliminated.

Table 3A

Group	Prime p	Permissible (m_0, r_{m_0}, p^b)
${}^3D_4(q)$	3	$(1, 2, 3^2)$, $(2, 2, 3^2)$
$G_2(q)$	3	$(1, 2, 3)$, $(2, 2, 3)$
$F_4(q)$	3	$(1, 4, 3^2)$, $(2, 4, 3^2)$
${}^2F_4(2^n)'$	3	$(2, 2, 3)$
$E_6(q)$	3	$(1, 6, 3^4)$, $(2, 4, 3^2)$
	5	$(1, 6, 5)$
${}^2E_6(q)$	3	$(1, 4, 3^2)$, $(2, 6, 3^4)$
	5	$(2, 6, 5)$
$E_7(q)$	3	$(1, 7, 3^4)$, $(2, 7, 3^4)$
	5	$(1, 7, 5)$, $(2, 7, 5)$
	7	$(1, 7, 7)$, $(2, 7, 7)$
$E_8(q)$	3	$(1, 8, 3^5)$, $(2, 8, 3^5)$
	5	$(1, 8, 5^2)$, $(2, 8, 5^2)$, $(4, 4, 5)$
	7	$(1, 8, 7)$, $(2, 8, 7)$

Case $p = 5$: The same containments in the preceding paragraph for $E_7(q)$ show these groups share a Sylow 5-subgroup in cases (1,7,5) and (2,7,5). Similarly, $E_8(q)$ contains $SU_5(q^2)$ and shares a Sylow 5-subgroup with it in the case (4,4,5). By minimality these $p = 5$ cases are eliminated.

Assume $G \cong E_8(q)$. Using the same large subgroups as in the $p = 7$ case, the Sylow 5-subgroup S has a subgroup S_0 of index 5 that lies in a subgroup G_0 of G of type $A_8(q)$ or ${}^2A_8(q)$ according to whether we are in cases $(1, 8, 5^2)$ or $(2, 8, 5^2)$ respectively. By Proposition 2.9 applied to G_0 it follows that S_0 is non-abelian; and since $|A| > 5$, $A \cap S_0 \neq 1$. Thus by induction applied to a minimal strongly closed subgroup $A_0 \leq A \cap S_0$ in G_0 we obtain $S_0 \leq A$. Moreover, by Proposition 2.9 it follows that $S_T \leq S_0$. Since A is non-abelian and since the normalizer of a Sylow 5-subgroup of the Weyl group of E_8 acts irreducibly on the Sylow 5-subgroup of W (which is abelian of type (5,5)), the strongly closed subgroup A containing S_T cannot have index 5 in S, a contradiction. This eliminates all $E_8(q)$ cases for $p = 5$.

Adopting the notation following Proposition 2.9, assume $G \cong E_6^\epsilon(q)$, where $5 \mid q-\epsilon$ and S_T has rank 6 and index 5 in S. Then G shares the Sylow 5-subgroup S with $G_0 = L_1 * L_2$, where L_1 and L_2 are central quotients of $SL_2^\epsilon(q)$ and $SL_6^\epsilon(q)$ respectively (both of whose centers have order prime to 5). Since A is not cyclic, it does not centralize L_2; hence it follows from Lemma 2.8 that $A \cap L_2 \neq 1$. Since $S \cap L_2$ is non-abelian, by induction $S \cap L_2 \leq A$. In particular, A contains a homocyclic abelian subgroup of rank 5 and exponent $|q-\epsilon|_5$, and S/A is cyclic. Now G also contains a subgroup $G_1 = K_1 * K_2 * K_3$ with each $K_i \cong SL_3^\epsilon(q)$, where we may assume $S \cap G_1 \in Syl_5(G_1)$. Each K_i contains a homocyclic abelian subgroup B_i of rank 2 and exponent $|q-\epsilon|_5$ with $N_{K_i}(B_i)$ acting irreducibly on $\Omega_1(B_i)$. Because S/A is cyclic it follows that $B_1 \times B_2 \times B_3 = S_T \leq A$; and since A is non-abelian, $A = S$. This completes the elimination of all $p = 5$ cases.

We next consider the various $p = 3$ cases, leaving the nettlesome groups of type $G_2(q)$ and ${}^3D_4(q)$ until the very end.

Case $p = 3$ and $m_0 = 1$: Here $3 \mid q-1$. If $G \cong F_4(q)$ then it contains the universal group $G_0 = B_4(q)^u$. By inspection of the order formulas, G_0 may be chosen to contain a subgroup S_0 of index 3 in S which, by Proposition 2.9, is non-abelian. Since $|A| > 3$ we have $S_0 \cap A \neq 1$ so, as usual, the minimality of G forces $S_0 \leq A$. Thus $S_0 = A$ has index 3 in S. Furthermore, since a Sylow 3-subgroup of the Weyl group of B_4 has order 3, we get that A has an abelian subgroup of index 3. But now by [GLSv3, Table 4.7.3A] there is an element t of order 3 in G such that $C = O^{3'}(C_G(t)) = L_1 * L_2$ where $L_i \cong SL_3(q)$ for $i = 1, 2$. Choose a suitable representative of this class so that $C_S(t) \in Syl_3(C)$. Then $A \cap L_i \not\leq Z(L_i)$, so because each Sylow subgroup $S \cap L_i$ is non-abelian, by induction $S \cap L_i \leq A$ for $i = 1, 2$. This gives a contradiction because $S \cap L_1L_2$ clearly does not have an abelian subgroup of index 3.

Since ${}^2E_6(q)$ shares a Sylow 3-subgroup with a subgroup of type $F_4(q)$ this family is eliminated by minimality of G.

Consider when G is one of $E_6(q)$, $E_7(q)$ or $E_8(q)$. In these cases S_T is homocyclic of the same rank as G and S_T lies in a maximal split torus T of G with $W = N_G(T)/C_G(T)$ isomorphic to the Weyl group of G. Note that W acts on the Sylow 3-subgroup S_T of T; moreover, in each case W acts irreducibly on $\Omega_1(S_T)$, and $Z(S) \leq S_T$. By strong closure of A we obtain

$$\Omega_1(S_T) \leq A. \tag{3.8}$$

There are containments: $F_4(q) \leq E_6(q) \leq E_7(q) \leq E_8(q)$, with corresponding containments of their maximal split tori. Thus by (3.8), in each exceptional family A nontrivially intersects a subgroup, G_0, of G of smaller rank in this chain. Since the Sylow 3-subgroups of each G_0 are non-abelian, by minimality of G and the preceding results we get that A contains a Sylow 3-subgroup of the respective subgroup G_0. Since then A is non-abelian, it is not contained in S_T. Now the Weyl group of G is of type $U_4(2) \cdot 2$, $Z_2 \times S_6(2)$, or $2 \cdot O_8^+(2) \cdot 2$, so by induction applied in $N_G(T)$ it follows that A covers a Sylow 3-subgroup of W. Finally, the irreducible action of W on $S_T/\Phi(S_T)$ forces $S_T \leq A$, and so $A = S$, a contradiction.

Case $\boldsymbol{p = 3}$ and $\boldsymbol{m_0 = 2}$: Here $3 \bigm| q+1$. The argument employed when $3 \bigm| q-1$ mutatis mutandis eliminates $F_4(q)$ as a possibility (using $L_i \cong SU_3(q)$ in this case). The groups ${}^2F_4(2^n)'$ — including the Tits simple group — share a Sylow 3-subgroup with their subgroups $SU_3(2^n)$, and so are eliminated by induction. Also, $E_6(q)$ shares a Sylow 3-subgroup with its subgroup $F_4(q)$, hence it is eliminated. To eliminate $E_8(q)$, $E_7(q)$ and ${}^2E_6(q)$ we refer to the table of centralizers of elements of order 3 in these groups: [GLSv3, Table 4.7.3A].

First assume $G \cong E_8(q)$. By [GLSv3, Table 4.7.3B], G contains a subgroup $X \cong L_1 \times L_2$, where the two components are conjugate and of type $U_5(q)$. We may assume $S \cap X \in Syl_3(X)$. Since $\Omega_1(S_T)$ is the unique elementary abelian subgroup of S of rank 8, $\Omega_1(S_T) \leq X$; in particular, $A \cap X \neq 1$. As usual, by minimality of G we obtain $S \cap X \leq A$, and the "toral subgroup" for $S \cap X$ lies in S_T. Order considerations then give $S_T \leq A$ and $|S : A| \leq 3^3$. Now the centralizer of an element of order 3 in $Z(S)$ is of type $({}^2E_6(q) * SU_3(q))3$, where the two factors share a common center of order 3. Since $S_T \leq A$ it follows that A acts nontrivially on, hence contains a Sylow 3-subgroup of, each component (or of $SU_3(2)$ when $q = 2$). This implies A covers $S/S_T \cong S_W$, as needed to give the contradiction $A = S$.

Let $G \cong E_7(q)$. Then G contains a subgroup $X \cong SU_8(q)$ with $S \cap X \in Syl_3(X)$. Since $S \cap X$ has the same "toral subgroup" as S, as usual we obtain $S \cap X \leq A$, $S_T \leq A$ and $|S : A| \leq 3^2$. Now S also contains an element of order 3 whose centralizer has a component of type ${}^2E_6(q)$ (universal version). Since as usual A contains a Sylow 3-subgroup of this component it follows that A covers S/S_T and so $A = S$, a contradiction.

Finally, assume $G \cong {}^2E_6(q)$. Since by [CCNPW] ${}^2E_6(2)$ shares a Sylow 3-subgroup with a subgroup of type Fi_{22}, by minimality of G we may assume $q > 2$. Let X be the centralizer of an element of order 3 in $Z(S)$, so $X \cong (L_1 * L_2 * L_3)(3 \times 3)$, where each $L_i \cong SU_3(q)$, the central product $L_1L_2L_3$ has a center of order 3, an element of S cycles the three components, and another element of S induces outer diagonal automorphisms on each L_i. As usual, it follows easily that A contains a Sylow 3-subgroup of $S \cap X$. By order considerations

$$|S_T : S_T \cap A| \le 3 \qquad \text{and} \qquad |S : A| \le 9.$$

Now there is an element t of order 3 in S such that

$$C = C_G(t) = D * T_1, \qquad \text{where } D \cong D_5^-(q) \quad \text{and} \quad T_1 \cong Z_{q+1},$$

and we may choose t so that $S_0 = C_S(t) \in Syl_3(C)$. Let $S_1 = S \cap D$ and $S_2 = S \cap T_1$, and note that $\langle\, t \,\rangle = \Omega_1(T_1)$. Since the Schur multiplier of D has order prime to 3, $S_0 = S_1 \times S_2$. It follows as usual that $S_1 \le A$.

Now let $w \in S - S_0$ and let $t_1 = t^w$. Then $t_1 \ne t$ and $S_0 \in Syl_3(C_G(t_1))$. By symmetry, the strongly closed subgroup A contains the Sylow 3-subgroup S_1^w of the component D^w of $C_G(t_1)$. Since t_1 acts faithfully on D, so too S_2^w acts faithfully on D, from which it follows that

$$S_2 \le S_1 S_1^w \le A.$$

Moreover, A contains the "toral subgroup" of C of type $(q+1)^6$ (in the universal version of G), so $S_T \le A$ and hence A is the subgroup of S that normalizes each component L_i of X. Since S_W is generated by elements of order 3 (in the universal version of G), $S = A\langle\, x \,\rangle$ for some element x of order 3. Since no conjugate of x lies in A we may further assume $C_S(x) \in Syl_3(C_G(x))$. Since $\langle\, x \,\rangle$ cycles L_1, L_2, L_2 it follows that the 3-rank of $C_G(x)$ is at most 5: this restricts the possibilities for the type of x in [GLSv3, Table 4.7.3A]. In all possible cases $C_G(x)$ contains a product, L, of one or two components with $C(L)$ cyclic. The same argument that showed $S_2 \le A$ may now be applied to show $x \in A$, a contradiction. This completes the proof for these families.

Case $G_2(q)$ and ${}^3D_4(q)$ where $q \equiv \epsilon \pmod 3$: If $G \cong G_2(q)$ then by Proposition 2.12 $Z(S) \cong Z_3$ is the unique candidate for A, contrary to Lemma 3.2. Thus the minimal counterexample is not of type $G_2(q)$.

Assume $G \cong {}^3D_4(q)$. Then G contains a subgroup G_0 isomorphic to $G_2(q)$ (the fixed points of a graph automorphism of order 3), and by order considerations we may assume $S_0 = S \cap G_0$ is Sylow in G_0 and so has index 3 in S. As noted above, $\langle z \rangle = Z(S_0)$ is of order 3 and is the unique nontrivial strongly closed (in G_0) proper subgroup of S_0. Consider first when $|A \cap S_0| > 3$. Then since S_0 is non-abelian, induction applied to G_0 gives $S_0 \leq A$, and so $A = S_0$. Since by Proposition 2.6, $z^{G_0} \cap S_0 = \{z^{\pm 1}\}$, whereas $\langle z \rangle$ is not strongly closed in G, there must be G-conjugates of z in $S - S_0$, contrary to A being strongly closed (one can see this fusion in a subgroup of ${}^3D_4(q)$ of type $PGL_3^{\epsilon}(q)$).

Thus $A \cap S_0 = \langle z \rangle$ and so by Lemma 3.2, $A = \langle z \rangle \times \langle y \rangle$ with $z \sim y$ in G. Since $[S, y] \leq \langle z \rangle$, y centralizes $\Phi(S)$. Since ${}^3D_4(q)$ has 3-rank 2 and $y \notin \Phi(S)$, by Proposition 2.9(4) we must have $|S| = 3^4$. But then S_0 is the non-abelian group of order 27 and exponent 3, and y centralizes a subgroup of index 3 in it, contrary to the 3-rank of ${}^3D_4(q)$ being 2. This eliminates the possibility that $G \cong {}^3D_4(q)$ and so completes the consideration of all cases. □

Lemma 3.6. *G is not a group of Lie type (untwisted or twisted) in characteristic p.*

Proof. Assume G is of Lie type (untwisted or twisted) over $\mathbb{F}_q$ where $q = p^n$. Since G is a counterexample, it follows from Proposition 2.10 that G has BN-rank ≥ 2. An end-node maximal parabolic subgroup P_1 for each of the Chevalley groups (untwisted or twisted) containing the Borel subgroup S is described in detail in [CKS76] and [GLS93] (for the classical groups these parabolics are the stabilizers in G of a totally isotropic one-dimensional subspace of the natural module.) For the groups of BN-rank 2 the other maximal parabolic, P_2, is also described in [GLS93]. In each group $P_i = Q_i L_i H$, where $Q_i = O_p(P_i)$, L_i is the component of a Levi factor of P_i and H is a p'-order Cartan subgroup.

Except for the 5-dimensional unitary groups and some groups over $\mathbb{F}_3$ (which will be dealt with separately), for some $i \in \{1, 2\}$ the group $M = O^{p'}(P_i)$ satisfies the following conditions:

Properties 3A.

(1) $S \leq M$,
(2) $F^*(M) = O_p(M)$,
(3) $\overline{M} = M/O_p(M)$ is a quasisimple group of Lie type in characteristic p,
(4) $\overline{M}$ is not isomorphic to $U_3(p^n)$ or $Re(3^n)$ (when $p = 3$), for any $n \geq 2$,

(5) $[O_p(M), \overline{M}] = O_p(M)$, and
(6) if $Q = O_p(M)$ and $Z = \Omega_1(Z(S))$, then one of the following holds:
(i): Q is elementary abelian of order q^k for some k, or
(ii): Q is special of type q^{1+k} for some k, all subgroups of order p in Z are conjugate in G, and $z^g \in S - Q$ for some $z \in Z$, $g \in G$.

Basic information about this parabolic is listed in Table 3B. The last column of Table 3B indicates which of the two alternatives in Properties 3A(6) holds. The proofs that the fusion in Properties 3A(6ii) holds in each case may be found in [CKS76].

Table 3B

Group	Parabolic	Q	$L/Z(L)$	3A(6)
$L_k(q)$, $k \geq 3$	P_1	q^{k-1}	$L_{k-1}(q)$	(i)
$O_k^{\pm}(q)$, $k \geq 7$	P_1	q^{k-2}	$O_{k-2}^{\pm}(q)$	(i)
$S_{2k}(q)$, $k \geq 2$	P_1	$q^{1+2(k-1)}$	$S_{2k-2}(q)$	(ii)
$U_k(q)$, $k \geq 4$, $k \neq 5$	P_1	$q^{1+2(k-2)}$	$U_{k-2}(q)$	(ii)
$E_6(q)$	P_1	q^{1+20}	$L_6(q)$	(ii)
$E_7(q)$	P_1	q^{1+32}	$O_{12}^{+}(q)$	(ii)
$E_8(q)$	P_1	q^{1+56}	$E_7(q)$	(ii)
${}^2E_6(q)$	P_1	q^{1+20}	$U_6(q)$	(ii)
$G_2(q)$, $q > 3$	P_2	q^{1+4}	$L_2(q)$	(ii)
$F_4(q)$	P_1	q^{1+14}	$S_6(q)$	(ii)
${}^3D_4(q)$	P_2	q^{1+8}	$L_2(q^3)$	(ii)
$U_5(q)$	P_1	q^{1+6}	$U_3(q)$	(ii)

Putting aside the last row for the moment, let $M = O^{p'}(P_i)$ be chosen according to Table 3B. Since M does not have any composition factors isomorphic to $U_3(p^n)$ or $Re(3^n)$, the minimality of G gives inductively that $A \in Syl_p(\langle A^M \rangle)$. If $A \not\leq Q$, then by the structure of M in Properties 3A(3) and (5), $M \leq \langle A^M \rangle$. But then $A = S$ by (1), a contradiction. Thus

$$A \leq Q \quad \text{and} \quad A \trianglelefteq M. \tag{3.9}$$

Assume first that Properties 3A(6ii) holds. Then since $A \trianglelefteq S$, $Z \cap A \neq 1$. The strong closure of A together with (6ii) forces $Z \leq A$, contrary to the existence of some $z^g \in S - Q$. This contradiction shows that G can only be among the families in the first two rows or the last row of Table 3B.

Assume now that Q is abelian, i.e., G is a linear or orthogonal group. In these cases Q is elementary abelian and is the natural module for $\overline{M}$; in particular, $\overline{M}$ acts irreducibly on Q. By (3.9) we obtain $A = Q$. However, in these cases when G is viewed as acting on its natural module, Q is a subgroup of G that stabilizes the one-dimensional subspace generated by an isotropic vector and acts trivially on the quotient space. Since the dimension of the space is at least 3, one easily exhibits noncommuting transvections that stabilize a common maximal flag; hence there are conjugates of elements of Q in S that lie outside of Q, a contradiction.

In $U_5(q)$ for $q \geq 3$ the unipotent radical of the parabolic P_1 is special of type q^{1+6} with $Z = Z(S) = Z(Q_1)$ and all subgroups of order p in Z conjugate in P_1 (so $Z \leq A$). As in the other unitary groups, there exist $z \in Z$ and $g \in G$ such that $z^g \in S - Q_1$. Now $L_1 \cong U_3(q)$ acts irreducibly on Q_1/Z and, by the strong closure of A, $A \cap Q$ is normal in P_1. Since $z^g \in A$ and $[Q_1, z^g] \leq A \cap Q_1$, the irreducible action of L_1 forces $Q_1 \leq A$. But now there is a root group U of type $U_3(q)$ with U contained in Q_1 such that $S = Q_1 U^x$, for some $x \in G$. Since $U \leq A$, this forces $A = S$, a contradiction.

It remains to treat the special cases when the Levi factors in Table 3B are not quasisimple: $G \cong L_2(q)$, $L_3(3)$, $G_2(3)$, $S_4(3)$, or $U_4(q)$ (in line 3 of Table 3B, $S_2(q) = L_2(q)$). Properties of small order groups may be found in [CCNPW]. The groups $L_2(q)$ have elementary abelian Sylow p-subgroups so G is not a counterexample in this instance. In $L_3(3)$ we have $S \cong 3^{1+2}$ and the action of the two maximal parabolic subgroups (stabilizers of one- and two-dimensional subspaces) easily show that the strong closure $Z(S)$ in S is all of S, contrary to $A \neq S$.

If $G \cong G_2(3)$ then since G has two (isomorphic) maximal parabolics containing S, A is not normal in one of them, say P_1. By [CCNPW], $P_1 = (W \times U) : L$ where $W \cong 3^{1+2}$, $U \cong Z_3 \times Z_3$, $O_3(P_1) = WU$, and $L \cong GL_2(3)$ acts naturally on both U and W/W'. Since A projects onto a subgroup of order 3 in $P_1/O_3(P_1) \cong L$, we see that $[A, W] \not\leq W'$ and $[A, U] \neq 1$. Both these commutators lie in the strongly closed subgroup A, so the action of L forces $O_3(P) \leq A$. Thus $A = S$, a contradiction.

If $G \cong S_4(3)$ there are maximal parabolics of type $P_1 = 3^{1+2} : SL_2(3)$ and $P_2 = 3^3(S_4 \times Z_2)$. Since $P_1 = N_G(Z(S))$ it follows that the S_4 Levi factor in P_2 acts irreducibly on $O_3(P_2)$. Now $A \cap O_3(P_2) \neq 1$ so $O_3(P_2) \leq A$. Likewise since A is a noncyclic strongly closed subgroup, it follows easily from the action of the Levi factor in P_1 that $O_3(P_1) \leq A$. These together give $A = S$, a contradiction.

Finally, assume $G \cong U_4(q)$. From the isomorphism $U_4(q) \cong O_6^+(q)$ we see that G contains a maximal parabolic $P_2 = q^4 O_4^+(q) \cong q^4 L_2(q^2)$, where the Levi factor is irreducible on the (elementary abelian) unipotent radical. This case has been eliminated by previous considerations. This final contradiction completes the proof of the lemma. □

Lemma 3.7. *G is not one of the sporadic simple groups.*

Proof. The requisite properties of the sporadic groups for this proof are nicely documented in [CCNPW], [GL83, Section 5], or [GLSv3, Section 5.3]; many of their proofs may be found in [Asc94]. Facts from these sources are quoted without further attribution. Verification that the sporadic groups in conclusions (iv) and (v) of Theorem 2.2 indeed have strongly closed subgroups as asserted may also be found in these references. We clearly only need to consider groups where p^2 divides the order; indeed, when the Sylow p-subgroup has order exactly p^2 it is elementary abelian and G is not a counterexample in these cases.

If $|S| = p^3$, then in all cases the Sylow p-subgroup is non-abelian of exponent p and, with the exception of M_{12}, $N_G(S)$ acts irreducibly on $S/Z(S)$. In M_{12} with $p = 3$: S contains distinct subgroups U_1 and U_2, each of order 9, such that $N_G(U_i)$ acts irreducibly on U_i for each i. Since A is noncyclic and strongly closed, in all cases these conditions force $A = S$, a contradiction. Thus we are reduced to considering when $|S| \geq p^4$.

We first argue that the following general configuration cannot occur in G:

Properties 3B.

(1) $Z(S) = Z \cong Z_p$,
(2) $N = N_G(Z)$ has $Q = O_p(N)$ extraspecial of exponent p and width $w > 1$ (denoted $Q \cong p^{1+2w}$),
(3) N acts irreducibly on Q/Q', and
(4) N/Q does not have a nontrivial strongly closed p-subgroup that is proper in a Sylow p-subgroup of N/Q.

By way of contradiction assume these conditions are satisfied in G. If $A \not\leq Q$ then by (4) we obtain that A covers a Sylow p-subgroup of N/Q. In this case, the irreducible action of N on Q/Q' then forces $Q \leq A$ and so $A = S$, a contradiction. Thus $A \leq Q$. Now $Z \leq A$ but $|A| > p$ so the irreducible action of N forces $A = Q$. Since A is minimal strongly closed, whence Z is not strongly closed, there is some $x \in Q - Z$ such that $x \sim z$ for $z \in Z$. Thus by Sylow's Theorem there is some $g \in G$ such that

$$C_Q(x)^g \leq S \qquad \text{and} \qquad x^g = z.$$

By strong closure, $C_Q(x)^g \leq Q$. But since Q has width > 1 we obtain $Z^g \leq (C_Q(x)^g)' \leq Q' = Z$ and so g normalizes Z. This contradicts the fact that $z^{g^{-1}} \notin Z$ and so proves these properties cannot hold in G.

Most sporadic groups are eliminated because they satisfy Properties 3B, or because they share a Sylow p-subgroup with a group that is eliminated inductively. All cases where $|S| \geq p^4$ are listed in Table 3C along with the isomorphism type of the corresponding normalizer of a p-central subgroup (or another "large" subgroup, or reason for elimination). Some additional arguments must be made in a few cases.

When $p = 5$ and $G \cong Co_1$ the extraspecial $Q = O_5(N)$ listed in the table has width 1. As before, if $A \not\leq Q$ then the irreducible action of N on Q/Q' forces $A = S$, a contradiction. Thus $A \leq Q$ and again the irreducible action yields $A = Q$. However G contains a subgroup $G_0 \cong Co_2$ whose Sylow 5-subgroup S_0 is isomorphic to Q and has index 5 in S. Since $|A \cap S_0| \geq 25$, the irreducible action of $N_{G_0}(S_0)$ on S_0/S_0' forces $S_0 \leq A$, and hence $S_0 = A$. But by Proposition 2.6, $Z(S_0)$ is strongly closed in G_0 but not strongly closed in G. Thus there is some $g \in G$ such that $Z(S_0)^g \leq S$ but $Z(S_0)^g \not\leq S_0$. This contradicts the fact that $A = S_0$ is strongly closed in G, and so $G \not\cong Co_1$.

When $p = 3$ and $G \cong Fi_{23}$ it contains a subgroup H of type $O_8^+(3) : S_3$ that may therefore be chosen to contain S. Let $H_0 = H'' \cong O_8^+(3)$. By Lemma 2.8, $A \cap H_0 \neq 1$; and so by induction A contains the non-abelian Sylow 3-subgroup $S_0 = S \cap H_0$ of H_0. Thus $|S : A| = 3$. Now H is generated by 3-transpositions in G, and so there are 3-transpositions t, t_1 such that

$$D_1 = \langle t, t_1 \rangle \cong S_3 \qquad \text{and} \qquad H = H_0 : D_1.$$

Likewise t inverts some element of order 3 in H_0, i.e., there is some $t_2 \in H_0\langle t \rangle$ such that $D_2 = \langle t, t_2 \rangle \cong S_3$. By the rank 3 action of G on its 3-transpositions, D_1 and D_2

Table 3C

Group	**Z(S) normalizer (or other reason)**
	$\boldsymbol{p=7}$
M	$7^{1+4}(3\times 2S_7)$
	$\boldsymbol{p=5}$
Ly	$5^{1+4}((4*SL_2(9)).2)$
Co_1	$5^{1+2}GL_2(5)$
HN	$5^{1+4}(2^{1+4}(5\cdot 4))$
B	$5^{1+4}(((Q_8*D_8)A_5)\cdot 4)$
M	$5^{1+6}((4*2J_2)\cdot 2)$
	$\boldsymbol{p=3}$
McL	$3^{1+4}(2S_5)$
Suz	$3U_4(3)2$
Ly	$3McL2$
O'N	(one class of Z_3 and $S=\Omega_1(S)$)
Co_1	$3^{1+4}GSp_4(3)$
Co_2	$3^{1+4}((D_8*Q_8)\cdot S_5)$
Co_3	$3^{1+4}((4*SL_2(9))\cdot 2)$
Fi_{22}	$(S\leq O_7(3))$
Fi_{23}	$(S\leq O_8^+(3):S_3)$
Fi_{24}'	$3^{1+10}(U_5(2)\cdot 2)$
HN	$3^{1+4}(4*SL_2(5))$
Th	(see separate argument)
B	$3^{1+8}(2^{1+6}O_6^-(2))$
M	$3^{1+12}(2Suz)\cdot 2$

are conjugate in G. Thus D_1' is conjugate to the subgroup D_2' of H_0, contrary to A being strongly closed. This proves $G\ncong Fi_{23}$.

Finally, assume $p=3$ and $G\cong Th$. Following the Atlas notation and the computations in [Wi98], the centralizer of an element of type $3A$ in S has isomorphism type

$$N = N_G(\langle 3A \rangle) \cong (Z_3 \times H).2 \qquad \text{where} \qquad H \cong G_2(3).$$

Since an element of type $3B$ in $Z(S) \cap A$ commutes with $3A$ and therefore acts nontrivially on H, by induction A contains a Sylow 3-subgroup of H. In the Atlas notation for characters of $G_2(3)$, the character χ of degree 248 of Th restricts to $Z_3 \times H$ as

$$\chi|_{Z_3 \times H} = 1 \otimes (\chi_1 + \chi_6) + (\omega + \overline{\omega}) \otimes \chi_5$$

where the characters of the Z_3 factor are denoted by their values on a generator. By comparison of the values of these on the $G_2(3)$-classes it follows that H contains a representative of every class of elements of order 3 in Th. The calculations in [Wi98] show that $S = \Omega_1(S)$, which leads to $A = S$, a contradiction.

This eliminates all sporadic simple groups as potential counterexamples, and so completes the proof of Theorem 2.3.

□

4. The $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups

Throughout this section p is an arbitrary prime, G is a finite group and $S \in Syl_p(G)$. Now that we have described precisely the structure of the finite groups possessing a strongly closed p-subgroup, we are prepared to analyze in detail how this is related with the $B\mathbb{Z}/p$-cellular structure of the classifying spaces of the groups. Before undertaking the complete description of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ we describe what it is known so far about this problem.

4.1. **Previous results.** As we said in the introduction, the starting point was the computation done by Dror-Farjoun in [Far95, 3.C], where he establishes that the $B\mathbb{Z}/p$-cellularization of the classifying space of a finite cyclic p-group has the homotopy type of $B\mathbb{Z}/p$.

Subsequently Rodríguez-Scherer investigated in [RS01] the $M(\mathbb{Z}/p, 1)$-cellularization, where $M(\mathbb{Z}/p, 1)$ denotes the corresponding Moore space for $\mathbb{Z}/p$. When the target is BG, this can be considered a precursor to our study because $M(\mathbb{Z}/p, 1)$ can be described as the 2-skeleton of $B\mathbb{Z}/p$. In their description the authors use the concept of cellularization in the category of groups (developed afterwards in [FGS07]). Their work in this subject allows one to prove, in particular, that the $B\mathbb{Z}/p$-cellularization of the classifying space of a p-group is the same as that of its p-socle; as the latter is

$B\mathbb{Z}/p$-cellular in this case ([Flo07, Proposition 4.14]), one obtains that $\mathbf{CW}_{B\mathbb{Z}/p}BG \simeq B\Omega_1(G)$ if G is a finite p-group.

The aforementioned is proved using a characterization of the cellularization discovered by Chachólski, that is perhaps the most useful tool available to attack these kind of problems. Because of its importance and ubiquity in our context we reproduce it here:

Theorem ([Cha96, 20.3]). *Let A and X be pointed spaces, and let C be the homotopy cofiber of the map $\bigvee_{[A,X]_*} A \to X$, defined as evaluation over all the homotopy classes of maps $A \to X$. Then $\mathbf{CW}_{B\mathbb{Z}/p}X$ has the homotopy type of the homotopy fiber of the composition $X \to C \to \mathbf{P}_{\Sigma A}C$.*

Here $\mathbf{P}$ denotes the nullification functor, first defined by A.K. Bousfield in [Bou94]. Recall that given spaces A and X, X is called A-*null* if the natural inclusion $X \hookrightarrow \mathrm{map}(A, X)$ is a weak equivalence. In this way one defines a functor $\mathbf{P}_A : \mathbf{Spaces} \to \mathbf{Spaces}$, coaugmented and idempotent, such that $\mathbf{P}_A X$ is A-null for every X, and such that for every A-null space Y the coaugmentation induces a weak equivalence $\mathrm{map}(\mathbf{P}_A X, Y) \simeq \mathrm{map}(X, Y)$. This functor can also be defined in the pointed category, and its main properties can be found in [Far95] and [Cha96].

In our case the role of A and X in the Chachólski result will be played by $B\mathbb{Z}/p$ and BG, respectively. If C is the corresponding cofiber, from now on we shall denote the $\Sigma B\mathbb{Z}/p$-nullification of C by P.

As a consequence of previous results, describing $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is equivalent to describing P, which is in general a more accessible problem. In our particular situation it is convenient to assume G generated by order p elements because this implies automatically (using Whitehead's Theorem) that P is a simply-connected space. Thus we can use fracture lemmas [BK72, VI.8.1]; and as P is rationally trivial, we see that $P \simeq \prod P^\wedge_p$, the product of the q-completions for all primes q. On the other hand, note that the additional hypothesis on the generation of G causes no restriction, as for every finite G there is a homotopy equivalence $\mathbf{CW}_{B\mathbb{Z}/p}B\Omega_1(G) \simeq \mathbf{CW}_{B\mathbb{Z}/p}BG$ induced by the natural inclusion ([Flo07, Proposition 4.1]).

It is easily proved that the q-torsion part of P (for every prime $q \neq p$) has the homotopy type of $\prod_{q\neq p} BG^\wedge_q$. Hence the difficult part is to compute $P^\wedge_p$. According to [FS07, 3.2], $P^\wedge_p$ coincides with the base of the Chachólski fibration when the total space is $BG^\wedge_p$, so we are actually studying the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$. More precisely, if G is generated by order p elements, the equivalence $\mathbf{CW}_{B\mathbb{Z}/p}(BG^\wedge_p) \simeq (\mathbf{CW}_{B\mathbb{Z}/p}BG)^\wedge_p$ is proved in [FS07, Proposition 3.2]. This is a very peculiar property for spaces which, in general, cannot be decomposed via an arithmetic square.

In the philosophy of [BLO03], the homotopy theory of $BG^\wedge_p$ is codified in the p-fusion data of G. From this point of view it can be observed that the structure of $P^\wedge_p$ strongly depends on the minimal strongly closed p-subgroup $\mathfrak{A}_1(S)$ of S that contains the p-socle of S (called $Cl\ S$ in [FS07]). In particular, it is a consequence of the Puppe sequence and the definition of nullification that if $\mathfrak{A}_1(S) = S$ then $P^\wedge_p$ is trivial. This shows one should consider the case in which $\mathfrak{A}_1(S)$ is strictly contained in S.

In [FS07] the latter case is studied under the additional assumption that $N_G(S)$ controls (strong) G-fusion in S. Recall that the normalizer *controls (strong) fusion* if for every subgroup $P \leq S$ and $g \in G$ such that $gPg^{-1} \leq S$, there exist $h \in N_G(S)$ and $c \in C_G(P)$, such that $g = hc$. Since $\mathfrak{A}_1(S)$ is normal in $N_G(S)$, [FS07] shows that $P^\wedge_p$ is homotopy equivalent to the p-completion of $B(N_G(S)/\mathfrak{A}_1(S))$; this also shows, roughly speaking, that the structure of the mapping space $\mathrm{map}_*(B\mathbb{Z}/p, BG^\wedge_p)$ depends heavily on $\mathfrak{A}_1(S)$. This result is used, in particular, to compute the $B\mathbb{Z}/2$-cellularization of classifying spaces of simple groups (relying on Theorem 2.1 there).

In the next subsection we describe the remaining cases, giving very explicit descriptions of $P^\wedge_p$ in all the situations in which this space is not trivial.

4.2. **The description of $CW_{B\mathbb{Z}/p}BG$.** In [FS07] it was already anticipated that a complete description of the $B\mathbb{Z}/p$-cellularization of classifying spaces of finite groups would depend on a structure theorem for groups that contain a non-trivial strongly closed p-subgroup that is not a Sylow p-subgroup; in that time such a classification was only known for $p = 2$ (Theorem 2.1). But even in this case there were examples of groups such that $\mathfrak{A}_1(S) \neq S$ and $N_G(S)$ does not control fusion in S — in other words, groups that were beyond the scope of [FS07].

The key step missing in that paper was the role of the subgroup $\mathcal{O}_A(G)$, whose importance was already evident — in an independent group-theoretic context — in [Foo97]. First, $\mathcal{O}_A(G)$ is by definition a subgroup of A, so in the case $A = \mathfrak{A}_1(S)$ it is likely that it controls a "part" of the structure of $\mathbf{CW}_{B\mathbb{Z}/p}BG$. Moreover, $\mathcal{O}_A(G)$ is normal in G, so one might expect a strong relationship between the cofiber of Chachólski fibration for $BG^\wedge_p$ and for $B(G/\mathcal{O}_A(G))^\wedge_p$. The significance of $\mathcal{O}_A(G)$ is also suggested by the fact that passing to the quotient $G/\mathcal{O}_A(G)$ gives a considerable simplification in the p-fusion structure of this quotient group in the sense that — as we shall see in Section 5 — there are normalizers (of subgroups/elements) that do not control fusion in G with images that do so in the quotient. These ideas, combined with the explication of the role of $\mathcal{O}_A(G)$ in the classification result Theorem 2.2, allow us to prove the main theorem, Theorem 4.2, which covers all extant cases for $\mathbf{CW}_{B\mathbb{Z}/p}BG$, subsuming all by a uniform treatment.

For the remainder of this subsection $A = \mathfrak{A}_1(S)$, and overbars will denote passage to the quotient $G \to G/\mathcal{O}_A(G)$. The normalizer of $\overline{A}$ in $\overline{G}$ will be denoted by $\overline{N}$, and $\overline{N}/\overline{A}$ will be called Γ.

We begin with a technical lemma that we will need in the proof of the main theorem.

Lemma 4.1. *The group Γ is p-perfect, i.e., has no normal subgroup of index p.*

Proof. By contradiction, assume that there is a nontrivial homomorphism $\varphi : \Gamma \to \mathbb{Z}/p$. Precomposing with the map $B\overline{N} \to B\Gamma$ induced by the canonical projection yields an essential map $B\overline{N} \to B\mathbb{Z}/p$, which is trivial when restricted to $B\overline{A}$. Moreover, as $\overline{N}$ controls p-fusion in $\overline{G}$, we have another nontrivial map $BG^\wedge_p \to B\overline{N}^\wedge_p \to B\mathbb{Z}/p$ that is induced by a (also nontrivial) homomorphism $\psi : G \to \mathbb{Z}/p$. As the image of A under the natural projection $G \to \overline{G}$ is $\overline{A}$, the definition of α implies that the composition $BA \to BG \to BG^\wedge_p \to B\mathbb{Z}/p$ is homotopically trivial. In particular, every element of order p of G should go to zero under the map ψ. But G is generated by order p elements, so ψ should be nontrivial. This is a contradiction, and we are done. □

Alternatively, Theorem B of [Gol75] says that for any strongly closed A we have $(G' \cap S)A = (N_G(A)' \cap S)A$. Thus if G has no normal subgroup of index p, neither does $N_G(A)/A$. Corollary 2.5 therefore also shows Γ is p-perfect since each L_i is simple.

Now we are in a position to prove the principal result of this section.

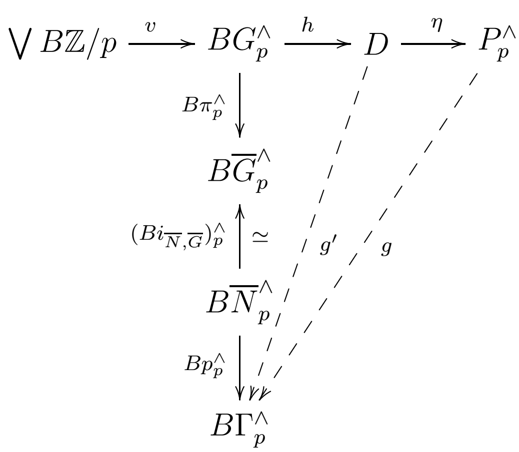
Theorem 4.2. *In the previous notation, $P^\wedge_p$ is homotopy equivalent to the p-completion of the classifying space of $\overline{N}/\overline{A}$.*

Proof. Let D be the cofiber of the Chachólski map $\bigvee B\mathbb{Z}/p \to BG^\wedge_p$ (extended to all the homotopy classes of maps $B\mathbb{Z}/p \xrightarrow{v} BG^\wedge_p$), whose $\Sigma B\mathbb{Z}/p$-nullification is $P^\wedge_p$. We denote by $h : BG^\wedge_p \to D$ the natural map, and by $\eta : D \to P^\wedge_p$ the canonical coaugmentation. Moreover, if $A_1 < A_2$, we will call i_{A_1,A_2} the group inclusion $A_1 \hookrightarrow A_2$.

We claim there are maps $B\Gamma_p^\wedge \to P_p^\wedge$ and $P_p^\wedge \to B\Gamma_p^\wedge$ that are homotopy inverses to one another.

First we define $P_p^\wedge \xrightarrow{g} B\Gamma_p^\wedge$. Recall that, as $\overline{N}$ controls $\overline{G}$-fusion in $\overline{S}$, the inclusion $\overline{N} \hookrightarrow \overline{G}$ induces a homotopy equivalence $B\overline{N}_p^\wedge \overset{(Bi_{\overline{N},\overline{G}})_p^\wedge}{\simeq} B\overline{G}_p^\wedge$ (see for example [MP98, Proposition 2.1]) . Now consider the diagram

(4.1)



and call α the composition of all vertical maps.

According to the definition of $\overline{G}$ and A, the composition $B\mathbb{Z}/p \to BG_p^\wedge \xrightarrow{\alpha} B\Gamma_p^\wedge$ is inessential for every map $B\mathbb{Z}/p \to BG_p^\wedge$. This implies that the composition $\alpha \circ v$ is so, and hence there exists the lifting g'. As $B\Gamma_p^\wedge$ is $\Sigma B\mathbb{Z}/p$-null, g' also lifts to g, and that is the map we were looking for.

To construct $f : B\Gamma_p^\wedge \to P_p^\wedge$ consider now the composition $BA \xrightarrow{(Bi_{A,G})_p^\wedge} BG_p^\wedge \xrightarrow{\eta\circ h} P_p^\wedge$. As the induced homomorphism of fundamental groups is trivial when restricted to every generator of A (by construction), the composition must be null-homotopic on BA. In particular, it is inessential when precomposing with the map $B(\mathcal{O}_A(G) \cap A) \xrightarrow{Bi_{\mathcal{O}_A(G)\cap A,A}} BA$. As $P_p^\wedge$ is p-complete (by [FS07, 3.2]), and $\mathcal{O}_A(G) \cap A$ is p-Sylow in $\mathcal{O}_A(G)$ by definition of $\mathcal{O}_A(G)$, we can apply [Dwy96, Theorem 1.4] to obtain that the composition $B\mathcal{O}_A(G) \xrightarrow{(Bi_{\mathcal{O}_A(G),G})_p^\wedge} BG_p^\wedge \longrightarrow P_p^\wedge$ is again homotopically trivial. Then, by Zabrodsky's Lemma [Dwy96, 3.4], there exists a lifting f'

$$\begin{array}{ccc} B\mathcal{O}_A(G) & & \\ \downarrow{\scriptstyle Bi_{\mathcal{O}_A(G),G}} & & \\ BG & \xrightarrow{\eta\circ h\circ(-)^\wedge_p} & P^\wedge_p \\ \downarrow{\scriptstyle B\pi} & \nearrow{\scriptstyle f'} & \\ B\overline{G} & & \end{array}$$

where $(-)^\wedge_p$ denotes the p-completion $BG \to BG^\wedge_p$. Now we have another map $B\overline{N} \to P^\wedge_p$ given by the composition

$$B\overline{N} \longrightarrow B\overline{N}^\wedge_p \stackrel{(Bi_{\overline{N},\overline{G}})^\wedge_p}{\longrightarrow} BG^\wedge_p \stackrel{(f')^\wedge_p}{\longrightarrow} P^\wedge_p,$$

where we have used the fact that $P^\wedge_p$ is p-complete.

The next diagram is clearly commutative by construction:

$$\begin{array}{ccccccc} BA & & \xrightarrow{Bi_{A,G}} & & BG & & \\ \downarrow{\scriptstyle B\pi_{|A}} & & & & \downarrow{\scriptstyle B\pi} & \searrow & \\ B\overline{A} & \xrightarrow{Bi_{\overline{A},\overline{N}}} & B\overline{N} & \xrightarrow{Bi_{\overline{N},\overline{G}}} & B\overline{G} & \longrightarrow & P^\wedge_p. \end{array}$$

Note that the composition $BA \stackrel{Bi_{A,G}}{\longrightarrow} BG \stackrel{\eta\circ h\circ(-)^\wedge_p}{\longrightarrow} P^\wedge_p$ is homotopically trivial, and hence the composition $BA \longrightarrow B\overline{A} \stackrel{Bi_{\overline{A},\overline{N}}}{\longrightarrow} B\overline{G} \stackrel{f'}{\longrightarrow} P^\wedge_p$ is so. As $\overline{A}$ is by definition a quotient of A, every generator of the former comes from a generator from the latter. This implies that the composition $B\overline{A} \stackrel{Bi_{\overline{A},\overline{N}}}{\longrightarrow} B\overline{G} \stackrel{f'}{\longrightarrow} P^\wedge_p$ is also homotopically trivial, and thus again by [Dwy96, 3.4], the map $(f')^\wedge_p$ lifts to a map f:

$$\begin{array}{ccc} B\overline{N}^\wedge_p & \xrightarrow{(f')^\wedge_p} & P^\wedge_p \\ \downarrow & \nearrow{\scriptstyle f} & \\ B\Gamma^\wedge_p. & & \end{array}$$

This is the map f that we wanted; and we have another commutative diagram:

$$\begin{array}{ccc} B\overline{G}^\wedge_p & \xrightarrow{\eta\circ h} & P^\wedge_p \\ \downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle f} & \\ B\Gamma^\wedge_p. & & \end{array} \tag{4.2}$$

It remains to prove that $f \circ g \simeq \mathrm{Id}_{P_p^\wedge}$ and $g \circ f \simeq \mathrm{Id}_{B\Gamma_p^\wedge}$. In the first case, the universal property of the nullification functor implies that it is enough to prove that $f \circ g'$ is homotopic to the coaugmentation η. As $P_p^\wedge$ is simply connected, we only need to prove that $f \circ g' \simeq \eta$ in the unpointed category. Hence we can use the long exact sequence of the cofibration $BG_p^\wedge \xrightarrow{h} D \to \vee\Sigma B\mathbb{Z}/p$ to establish that $f \circ g' \simeq \eta$ if and only if $f \circ g' \circ h \simeq \eta \circ h$. According to diagram 4.1, $g' \circ h$ is homotopic to α, and by diagram 4.2, $f \circ \alpha \simeq \eta \circ h$, so we are done.

To see that $g \circ f$ is homotopic to the identity of $B\Gamma_p^\wedge$, a repeated application of the universal property of the quotient shows that it is enough to prove that $g \circ f \circ \alpha \simeq \alpha$. By Lemma 4.1 and [BK72, II.5], $B\Gamma_p^\wedge$ is simply connected, and we can apply the previous arguments to ensure that it is enough to find the homotopy in the unpointed category. Again by diagram 4.2 the latter is homotopic to $g \circ \eta \circ h$, and has the same homotopy class as α by diagram 4.1. So the statement is proved. □

When we combine the previous statement with [Flo07, Proposition 4.14] and [FS07, Theorem 2.5] we obtain a complete description of $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for every p and every finite group G.

Theorem 4.3. *Let G be a finite group generated by its elements of order p, let $S \in Syl_p(G)$, and let $A = \mathfrak{A}_1(S)$ be the minimal strongly closed subgroup of S containing $\Omega_1(S)$. Then the $B\mathbb{Z}/p$-cellularization of BG has one the following shapes:*

1. *If $G = S$ is a p-group then BG is $B\mathbb{Z}/p$-cellular.*
2. *If G is not a p-group and $A = S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q\neq p} BG_q^\wedge$.*
3. *If G is not a p-group and $A \neq S$ then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the map $BG \to B(N_{\overline{G}}(\overline{A})/\overline{A})_p^\wedge \times \prod_{q\neq p} BG_q^\wedge$.*

Theorems 2.1 and 2.2 and Corollary 2.5 determine $N_G(\overline{A})/\overline{A}$, whose structure is very rigid and depends on a restricted set of well-known simple groups. It is very likely that an analogous classification can be obtained exactly in the same way for $\mathbf{CW}_{B\mathbb{Z}/p^r}BG$, $r > 1$, but we have restricted ourselves to the case $r = 1$ for the sake of simplicity (cf. also [CCS, Theorem 3.6]).

In the cases where $\mathcal{O}_A(G) = 1$ and $A \trianglelefteq G$ — which are implicit in the computations — the $B\mathbb{Z}/p$-cellularization of $BG_p^\wedge$ is the homotopy fiber of the natural map $BG_p^\wedge \to B(G/A)_p^\wedge$. It is then tempting to identify $\mathbf{CW}_{B\mathbb{Z}/p}BG$ with BA. But this would mean, in particular, that $\mathrm{map}_*(BZ/p, BG^\wedge_p)$ would be discrete. However, an analysis of the fibration

$$\mathrm{map}_*(B\mathbb{Z}/p, BG^\wedge_p) \to \mathrm{map}(B\mathbb{Z}/p, BG^\wedge_p) \to BG^\wedge_p$$

together with the description of its total space — which is given, for example, in [BK02, Appendix] — shows that $\mathrm{map}_*(B\mathbb{Z}/p, BG^\wedge_p)$ is non-discrete in general, and then usually $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is not an aspherical space.

It is conceivable that our results can also have interesting consequences from the point of view of homotopical representations of groups. In [FS07, Section 6] the results on cellularization gave rise to specific examples of nontrivial maps $BG \to BU(n)^\wedge_p$ that enjoyed two particular properties: they did not come from group homomorphisms $G \to U(n)$, and they were trivial when precomposing with any map $B\mathbb{Z}/p \to BG$. While there are a number of examples in the literature with the first feature (see for example [BW95] or [MT89]), no representations were known at this point for which the second property holds. The classification results of this paper give hope of finding a systematic and complete treatment of all these kinds of representations. We plan to undertake this task in a separate paper.

In the next section we show the applicability of our results by computing the $B\mathbb{Z}/p$-cellularization of various specific families of classifying spaces. We have chosen the simple groups (as they have shown their cornerstone role in the computation of $\mathbf{CW}_{B\mathbb{Z}/p}BG$), certain split extensions that signaled there was something beyond the results of [FS07], and certain nonsplit extensions of $G_2(q)$ that illuminate the roles of the normalizers of A and S in the $B\mathbb{Z}/p$-cellular context.

5. Examples

Throughout this section p is any prime, G is a finite group possessing a nontrivial Sylow p-subgroup S and $\mathfrak{A}_1(S)$, as before, denotes the unique smallest strongly closed (with respect to G) subgroup of S that contains $\Omega_1(S)$. In Theorem 4.3 a description of the $B\mathbb{Z}/p$-cellularization of BG for every group G and every prime p is given. In this section we describe some families of concrete examples for which $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is interesting. We begin with the case of simple groups.

5.1. **Simple groups.** In [FS07, Corollary 5.7] the $B\mathbb{Z}/2$-cellularization of the classifying spaces of all the simple groups was computed. In this section we use the classification, Theorem 2.2, to show that for every prime p and every simple group

G, $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is included in cases (ii) and (iii) of Theorem 4.3. The key result is the following immediate consequence of Theorems 2.1 and 2.2 (where $\mathcal{O}_A(G) = 1$ by the simplicity of G):

Corollary 5.1. *Let G be a simple group in which $\mathfrak{A}_1(S) \neq S$. Then G is isomorphic to one of the groups L_i that appear in the conclusions of Theorems 2.1 and 2.2.*

Next, recall that if G is the simple group $G_2(q)$ for some q with $(q, 3) = 1$, then we showed in the proof of Proposition 2.12 (and at the end of the proof of Lemma 3.4) that $S = \Omega_1(S)$. Then by Corollary 2.4, in all cases in Corollary 5.1 the normalizer of S controls strong fusion in S. Thus Theorem 4.3 yields the following characterization:

Proposition 5.2. *Let G be a simple group, let p a prime and let S be a Sylow p-subgroup of G. Then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ has one of the following two structures:*

(1) *If $\mathfrak{A}_1(S) = S$, then $\mathbf{CW}_{B\mathbb{Z}/p}BG$ is the homotopy fiber of the natural map $BG \to \prod_{q\neq p} BG^\wedge_q$.*

(2) *If $\mathfrak{A}_1(S) \neq S$, then we have a fibration*

$$\mathbf{CW}_{B\mathbb{Z}/p}BG \to BG \to B(N_G(S)/\mathfrak{A}_1(S))^\wedge_p \times \prod_{q\neq p} BG^\wedge_q.$$

Note that the inclusion $N_G(S) \hookrightarrow N_G(\mathfrak{A}_1(S))$ induces a homotopy equivalence $BN_G(S)^\wedge_p \simeq BN_G(\mathfrak{A}_1(S))^\wedge_p$ when $N_G(S)$ (and then $N_G(\mathfrak{A}_1(S))$) controls p-fusion in S. This happens for every simple group in the second case of the previous proposition, and in particular a comparison of Chachólski fibrations for $BN_G(S)^\wedge_p$ and $BN_G(\mathfrak{A}_1(S))^\wedge_p$ (which are homotopy equivalent) gives that the induced map $B(N_G(S)/\mathfrak{A}_1(S))^\wedge_p \simeq B(N_G(\mathfrak{A}_1(S))/\mathfrak{A}_1(S))^\wedge_p$ is also a homotopy equivalence.

In the following we use Theorem 4.3 to describe explicitly the $B\mathbb{Z}/p$-cellularization of the classifying spaces of the groups of the second kind in the previous statement, which turn out to be some of the ones that appear in the classification. With the exception of the groups of Lie type in characteristic $\neq p$, the Sylow-p normalizers of the simple groups appearing in the conclusions to Theorems 2.1 and 2.2 are described explicitly in Proposition 2.12. We therefore add here only some observations on the structure of the normalizers in the remaining case.

Let G be a group of Lie type over a field of characteristic $r \neq p$ and suppose the Sylow p-subgroup S of G is abelian but not elementary abelian (here p is odd). The overall structure of $N_G(S)$ is governed by the theory of algebraic groups, as invoked in the proof of Proposition 2.11. Recapping from that argument: since the Schur multiplier of G is prime to p we may work in the universal version of G to describe $N_G(S)$. Let $\overline{G}$ be the simply connected universal simple algebraic group over the algebraic closure of $\mathbb{F}_r$, and let σ be a Steinberg endomorphism whose fixed points equal G. In the notation of [SS70], p is not a torsion prime for $\overline{G}$, so by 5.8 therein $C_{\overline{G}}(S)$ is a connected, reductive group whose semisimple component is simply connected. The general theory of connected, reductive algebraic groups gives that $C_{\overline{G}}(S) = \overline{Z}\,\overline{L}$, where $\overline{Z}$ is the connected component of the center of $C_{\overline{G}}(S)$, $\overline{L}$ is the semisimple component (possibly trivial), and $\overline{Z} \cap \overline{L}$ is a finite group. Furthermore, $\overline{L}$ is a product of groups of Lie type over the algebraic closure of $\mathbb{F}_r$ of smaller rank than $\overline{G}$. It follows that $C_{\overline{G}}(S)$ is a commuting product of the fixed points of σ on $\overline{Z}$ and $\overline{L}$, i.e.,

$$C_{\overline{G}}(S) = C_{\overline{Z}}(\sigma)C_{\overline{L}}(\sigma)$$

where $S \leq C_{\overline{Z}}(\sigma)$ is an abelian group (a finite torus) and $C_{\overline{L}}(\sigma)$ is either solvable or a product of finite Lie type groups in characteristic r.

To complete the generic description of $N_G(S)$ we invoke additional facts from [SS70] and [GLSv3, Section 4.10]. As above, S is contained in a σ-stable maximal torus $\overline{T}_1$, where $\overline{T}_1$ is obtained from a σ-stable split maximal torus $\overline{T}$ by twisting by some element w of the Weyl group $W = N_{\overline{G}}(\overline{T})/\overline{T}$ of $\overline{G}$. Since S is characteristic in the finite torus $T_1 = \overline{(T_1)}_\sigma$ it follows that $N_G(S)/C_G(S) \cong N_G(T_1)/T_1$. In most cases, by 1.8 of [SS70] or Proposition 3.36 of [Ca85] we have $N_G(T_1)/T_1 \cong W_\sigma \cong C_W(w)$ (see also [GLSv3, Theorem 2.1.2(d)] and the techniques in the proof of Theorem 4.10.2 in that volume).

In the special case where G is a classical group (linear, unitary, symplectic, orthogonal) the normalizer of S can be computed explicitly by its action on the underlying natural module, V, as described in the proof of Lemma 3.4. In the notation of this lemma, the semisimple component of order prime to p comes from the normal subgroup $\mathrm{Isom}(V_0)$ in $\mathrm{Isom}(V)$, where $V_0 = C_V(S)$, and S is the direct product of the cyclic groups $S \cap \mathrm{Isom}(V_i)$ for $i = 1, 2, \ldots, s$. The Weyl group normalizing S acts as the symmetric group S_s permuting the subgroups $\mathrm{Isom}(V_i)$. The orders of the centralizer and normalizer of a (cyclic) Sylow p-subgroup in each subgroup $\mathrm{Isom}(V_i)$ depend on p and the nature of G — Chapter 3 of [Ca85] gives techniques for computing these.

For an easy explicit example of this let $G = SL_{n+1}(q)$ where $q = r^m$ and $p > n+1$, and assume $p \mid q-1$. In this case we may choose S contained in the group of diagonal matrices T of determinant 1, which is an abelian group of type $(q-1, \ldots, q-1)$ of rank n (here T is the split torus). In this case $T = C_G(S)$ and $N_G(S) = N_G(T) = TW$, where $W \cong S_{n+1}$ is the group of permutation matrices permuting the entries of matrices in T in the natural fashion (as the "trace zero" submodule of the natural action on the direct product of $n+1$ copies of the cyclic group of order $q-1$). To obtain the Sylow p-normalizer in the simple group $PSL_{n+1}(q)$ factor out the subgroup of scalar matrices of order $(n+1, q-1)$.

5.2. **Split extensions.** In this subsection we consider some non-simple groups. Here we give some explicit examples of $B\mathbb{Z}/p$-cellularization of split extensions which are beyond the scope of [FS07], and show the usefulness of Theorems 4.2 and 4.3.

In [FS07] the $B\mathbb{Z}/p$-cellularization of BG is described when G is generated by elements of order p, $\mathfrak{A}_1(S)$ is a proper subgroup of S, and the normalizer of S controls strong fusion in S. No example was given there of a group for which the first two conditions hold but not the third. George Glauberman suggested an example of a group of the latter type: a wreath product $(\mathbb{Z}/2) \wr Sz(2^n)$. In this section we generalize this example, showing that many split extensions for which these conditions hold can be constructed. The computation of the cellularization of their classifying space is then easy from Corollaries 2.4 and 2.5. This construction demonstrates that even when $N_{\overline{G}}(\overline{S})$ (or $N_{\overline{G}}(\overline{A})$) controls $\overline{G}$-fusion in $\overline{S}$, where overbars denote passage to $G/O_A(G)$, it need *not* be the case that $N_G(A)$ controls fusion in S (or in A), even when $N_{\overline{G}}(\overline{A}) = \overline{N_G(A)}$. This highlights the importance of "recognizing" the subgroup $\mathcal{O}_A(G)$ as well as the isomorphism types of the components of $G/O_A(G)$ in our classifications.

Proposition 5.3. *Let R be any group that is not a p-group but is generated by elements of order p. Assume also that $\mathfrak{A}_1(T) \neq T$ for some Sylow p-subgroup T of R. Let E be any elementary abelian p-group on which R acts in such a way that $R/C_R(E)$ is not a p-group. Let G be the semidirect product $E \rtimes R$, and let $S = ET$ be a Sylow p-subgroup of G. Then G is generated by elements of order p, $\mathfrak{A}_1(S) \neq S$, and $N_G(S)$ does not control fusion in S.*

Proof. Note that the split extension $G = ER$ is clearly generated by elements of order p since both E and R are. Also, $\mathfrak{A}_1(S)$ contains E, and by Lemma 2.8, since the extension is split we obtain $\mathfrak{A}_1(S)/E \cong \mathfrak{A}_1(T) < T$, so $\mathfrak{A}_1(S) \neq S$. It remains to show that $N_G(S)$ does not control fusion in S.

Let $0 = E_0 < E_1 < \cdots < E_{n-1} < E_n = E$ be a chief series through E, so that each factor E_i/E_{i-1} is an irreducible $\mathbb{F}_pR$-module. If each such factor is one-dimensional, then R is represented by upper triangular matrices in its action on E. Since R is generated by elements of order p, it must be represented by unipotent matrices, hence $R/C_R(E)$ is a p-group, a contradiction.

Thus there is some chief factor E_i/E_{i-1} that is not one-dimensional. If a Sylow normalizer controlled fusion in S, then by Lemma 2.8 the same would be true in the quotient group G/E_{i-1}; we show this is not the case. To do so, we may pass to the quotient and therefore assume E_1 is a minimal normal, noncentral subgroup of G. Now $Z_1 = Z(S) \cap E_1 \neq 1$ and Z_1 is invariant under $N_G(S)$. However, R acts irreducibly and nontrivially on E_1 and R is generated by conjugates of S, so $Z_1 \neq E_1$ and hence Z_1 is not R-invariant. Thus for some $z \in Z_1$ and $g \in G$ we must have $z^g \in E_1 - Z_1$, which shows $N_G(S)$ does not control fusion in S. $\square$

This proposition can be invoked to create a host of examples: Let R be any of the simple groups L_i (or their quasisimple universal covers) in the conclusion to Theorem 2.2 and let E be an $\mathbb{F}_pR$-module on which R acts nontrivially (for example, any nontrivial permutation module). More specifically, for p odd let q be any prime power such that $p^2 \mid q-1$, so that Sylow p-subgroups of $R = SL_2(q)$ are cyclic of order $\geq p^2$ (for example, $p = 3$ and $q = 19$). Then R permutes the $q+1$ lines in a 2-dimensional space over $\mathbb{F}_q$, and so permutes $q+1$ basis vectors in a $q+1$-dimensional vector space E over $\mathbb{F}_p$. Then $G = E \rtimes R$ gives a specific realization for Proposition 5.3.

If $G = E \rtimes R$ is any group satisfying the conditions of the previous proposition with R simple and $A = E\mathfrak{A}_1(T)$, in the notation of Section 2 it is clear that $E = \mathcal{O}_A(G)$. As the extension is split, the canonical projection $G \to R$ sends $\mathfrak{A}_1(S)$ to $\mathfrak{A}_1(T)$. Then, according to Theorem 4.2, the $B\mathbb{Z}/p$-cellularization of $BG^\wedge_p$ has the homotopy type of the fiber of the composition

$$BG^\wedge_p \longrightarrow BR^\wedge_p \longrightarrow B(N_R(\mathfrak{A}_1(T))/\mathfrak{A}_1(T))^\wedge_p.$$

As R is simple, the base of the fibration was studied in the previous subsection.

Building on the preceding example where $R = SL(2,q)$ for any prime power q such that $p^2 \mid q-1$: then T may be represented by diagonal matrices over $\mathbb{F}_q$, so is cyclic of order $p^n = |q-1|_p$; moreover, $C_R(T)$ is the group of all diagonal matrices of determinant 1, hence is cyclic of order $q-1$. In particular, $\mathfrak{A}_1(T) =$

$\Omega_1(T) \cong \mathbb{Z}/p$. Furthermore, $N_R(T) = N_R(\mathfrak{A}_1(T))$ is of index 2 in $C_R(T)$ and an involution in $N_R(T)$ inverts $C_R(T)$. Thus $N_R(\mathfrak{A}_1(T))/\mathfrak{A}_1(T)$ is isomorphic to the dihedral group of order $2(q-1)/p$. Again comparing Chachólski fibrations, we obtain that $B(N_R(\mathfrak{A}_1(T)))/\mathfrak{A}_1(T))^\wedge_p$ is homotopy equivalent to $B(N_R(T)/\mathfrak{A}_1(T))^\wedge_p$.

5.3. **Exotic extensions of $G_2(q)$.** When G is the simple group $G_2(q)$ for some q with $(q, 3) = 1$, although a Sylow 3-subgroup S contains a strongly closed subgroup $A = Z(S)$ of order $p = 3$, when we impose the additional hypothesis that our strongly closed subgroup must contain all elements of order 3 the strongly closed subgroup A does not arise in our considerations because $S = \Omega_1(S)$. For the same reason, if $G = ER$ is any split extension of $R = G_2(q)$ by an elementary abelian 3-group and $S = ET$ for $T \in Syl_3(R)$, then again $S = \Omega_1(S) = \mathfrak{A}_1(S)$. In this subsection we describe a family of extensions that we call "half-split" in the sense that they split over a certain conjugacy class of elements of R but do not split over another. In this way we construct extensions G of $R = G_2(q)$ by certain elementary abelian 3-groups E such that for $S \in Syl_3(G)$ we have $\Omega_1(S)/E$ mapping onto the strongly closed subgroup of order 3 in a Sylow 3-subgroup S/E of $G_2(q)$. In particular, these "exotic" extensions show that the exceptional case of Corollary 2.4 cannot be removed: when $9 \mid q^2 - 1$ these groups G are generated by elements of order 3, have $\mathfrak{A}_1(S) \neq S$, but $N_{G/E}(S/E)$ does not control fusion in S/E (here $E = \mathcal{O}_A(G)$ where $A = \mathfrak{A}_1(S)$).

The following general proposition will construct such extensions.

Proposition 5.4. *Let p be a prime dividing the order of the finite group R and let X be a subgroup of order p in R. Then there is an $\mathbb{F}_pR$-module E and an extension*

$$1 \longrightarrow E \longrightarrow G \longrightarrow R \longrightarrow 1$$

of R by E such that the extension of X by E does not split, but the extension of Z by E splits for every subgroup Z of order p in R that is not conjugate to X. In particular, for nonidentity elements $x \in X$ and $z \in Z$ every element in the coset xE has order p^2 whereas zE contains elements of order p in G.

Proof. Let E_0 be the one-dimensional trivial $\mathbb{F}_pX$-module. By the familiar cohomology of cyclic groups ([Bro82], Section III.1):

$$H^2(X, E_0) \cong \mathbb{Z}/p\mathbb{Z} \tag{5.1}$$

and a non-split extension of X by E_0 is just a cyclic group of order p^2. Now let

$$E = \mathrm{Coind}_X^R E_0 = \mathrm{Hom}_{\mathbb{Z}X}(\mathbb{Z}R, E_0)$$

be the coinduced module from X to R (which is isomorphic to the induced module $E_0 \otimes_{\mathbb{F}_p X} \mathbb{F}_p R$ in the case of finite groups), so that E has $\mathbb{F}_p$-dimension $\frac{1}{p}|R|$. By Shapiro's Lemma ([Bro82], Proposition III.6.2)

$$H^2(R, E) \cong H^2(X, E_0). \tag{5.2}$$

Thus by (5.1) there is a non-split extension of R by E — call this extension group G and identify E as a normal subgroup of G with quotient group $G/E = R$.

The isomorphism in Shapiro's Lemma, (5.2), is given by the compatible homomorphisms $\iota : X \hookrightarrow R$ and $\pi : \mathrm{Coind}_X^R E_0 \to E_0$, where π is the natural map $\pi(f) = f(1)$. In particular, this isomorphism is a composition

$$H^2(R, E) \xrightarrow{\text{res}} H^2(X, E) \xrightarrow{\pi^*} H^2(X, E_0).$$

Thus the 2-cocycle defining the non-split extension group G, which maps to a non-trivial element in $H^2(X, E_0)$, by restriction gives a non-split extension of X by E as well.

For any subgroup Z of R of order p with Z not conjugate to X, by the Mackey decomposition for induced representations

$$\mathrm{Res}_Z^R \mathrm{Ind}_X^R E_0 = \bigoplus_{g \in \mathcal{R}} \mathrm{Ind}_{Z \cap gXg^{-1}}^Z \mathrm{Res}_{Z \cap gXg^{-1}}^{gXg^{-1}} gE_0 \tag{5.3}$$

where $\mathcal{R}$ is a set of representatives for the (Z, X)-double cosets in R. By hypothesis, $Z \cap gXg^{-1} = 1$ for every $g \in R$, hence each term in the direct sum on the right hand side is an $\mathbb{F}_p Z$-module obtained by inducing a one-dimensional trivial $\mathbb{F}_p$-module for the identity subgroup to a p-dimensional $\mathbb{F}_p Z$-module, i.e., is a free $\mathbb{F}_p Z$-module of rank 1. (Alternatively, E is the $\mathbb{F}_p$-permutation module for the action of R by left multiplication on the left cosets of X; by the fusion hypothesis, Z acts on a basis of E as a product of disjoint p-cycles with no 1-cycles.) This shows E is a free $\mathbb{F}_p Z$-module, and hence the extension of Z by E splits. This completes the proof. □

The p^{th}-power map on elements in the lift of X to G can be described more precisely. By the Mackey decomposition in (5.3) inducing from X but rather restricting to X

instead of Z, or by direct inspection of the action of X on the $\mathbb{F}_p$-permutation module E, we see that E decomposes as an $\mathbb{F}_pX$-module direct sum as

$$E = E_1 \oplus E_2,$$

where E_1 is a trivial $\mathbb{F}_pX$-module and E_2 is a free $\mathbb{F}_pX$-module. Since X splits over the free summand E_2, we see that X does not split over E_1, and hence

$$XE_1 \cong (\mathbb{Z}/p^2) \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p \qquad \text{with} \qquad E_1 = \Omega_1(XE_1).$$

Thus for every element x in $G - E$ mapping to an element of X in G/E, x^p has a nontrivial component in E_1.

One may also observe that by taking direct sums we can arrange more generally that if $X_1, X_2, \ldots, X_n$ are representatives of the distinct conjugacy classes of subgroups of order p in R, then for any $i \in \{1, 2, \ldots, n\}$ there is an $\mathbb{F}_pR$-module E and an extension of R by E such that in the extension group each of $X_1, \ldots, X_i$ splits over E but none of $X_{i+1}, \ldots, X_n$ do.

We are particularly interested in the case $R = G_2(q)$ with $p = 3$ and $(q, 3) = 1$. The normalizer of a Sylow 3-subgroup of R is described in Proposition 2.12: Let $T \in Syl_3(R)$ and let $Z = Z(T) = \langle z \rangle$. In the notation preceding Proposition 2.10, $N_R(Z) \cong SL_3^\epsilon(q) \cdot 2$ according as $3 \mid q - \epsilon$. Moreover, if $9 \mid q - \epsilon$ then $N_R(T)$ does not control fusion in T: all elements of order 3 in $T - Z$ are conjugate in $C_R(Z)$ whereas by Proposition 2.12, $N_R(T)/T$ has order 4 for this congruence of q. Thus $BN_R(T)^\wedge_3$ is not homotopy equivalent to $BG_2(q)^\wedge_3$.

Now consider the extension group G constructed in Proposition 5.4 with $p = 3$, $R = G_2(q)$, $Z = \langle z \rangle$ and $X = \langle x \rangle$ for any $x \in T - Z$ of order 3. Let $S \in Syl_3(G)$ with S mapping onto T in $G/E \cong R$. Since Proposition 2.12 shows all elements of order 3 in $T - Z$ are conjugate to x but not to z, the structure of the extension implies that $A = \Omega_1(S) = \mathfrak{A}_1(S)$ contains E and maps to Z in S/E. Thus $\mathcal{O}_A(G) = E$ and $\overline{A} = \overline{Z}$. By Corollary 2.4, the normalizer of Z in $R = G_2(q)$ controls 3-fusion in $G_2(q)$, so in particular $SL_3^*(q)$ has the same mod 3 cohomology as $G_2(q)$, where $SL_3^*(q)$ denotes the group $SL_3^\epsilon(q)$ together with the outer (graph) automorphism of order 2 inverting its center ($N_R(Z) \cong SL_3^*(q)$). On the other hand, Z is normal in $SL_3^*(q)$, and $SL_3^*(q)/Z$ is isomorphic to $PSL_3^*(q)$. Hence, by Theorem 4.2, the $B\mathbb{Z}/3$-cellularization of $(BG)^\wedge_3$ is the homotopy fibre of the map $(BG)^\wedge_3 \to BPSL_3^*(q)^\wedge_3$, but not of the map $(BG)^\wedge_3 \to B(N_R(T)/T)^\wedge_3$ when $9 \mid q^2 - 1$.

From this computation one can deduce that the object that determines the $\Sigma B\mathbb{Z}/p$-nullification of the cofibre of the Chachólski map in the case of finite groups is the normalizer of $\overline{A}$, and not the Sylow normalizer, as might be inferred from the particular cases studied in [FS07]. This example also highlights the importance of having a classification of *all* groups possessing a nontrivial strongly closed p-subgroup that is not Sylow — not just the simple groups having such a subgroup that contains $\Omega_1(S)$ — since the subgroup $\mathfrak{A}_1(S)$ does not pass in a transparent fashion to quotients.

In conclusion, some interesting open questions remain. We have characterized with precision $\mathbf{CW}_{B\mathbb{Z}/p}BG$ for every finite G, and in the course of the proof we have also described $\mathbf{CW}_{B\mathbb{Z}/p}BG^\wedge_p$ when G is generated by order p elements. However we do not address the issue of what happens in general with the cellularization of $BG^\wedge_p$ if we remove the generation hypothesis. There are some cases that can be deduced from the previous developments — for example if G is not equal to $\Omega_1(G)$ but is mod p equivalent to a group that is so — but it would be nice to have a general statement.

The extensions of our techniques and results to more general p-local spaces with a notion of p-fusion seem to be the natural next step of our study; in particular, classifying spaces of p-local finite groups and some families of non-finite groups offer enticing possibilities.

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