Augmental Homology and the Künneth Formula for Topological Joins

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Abstract. The "simplicial complexes" and "join" ($*$) today used within combinatorics isn't the classical concepts, cf. [11] p. 108-9, but, except for $\emptyset$, complexes having $\{\emptyset\}$ as a subcomplex resp. $\Sigma_1 * \Sigma_2 := \{\sigma_1 \cup \sigma_2 | \sigma_i \in \Sigma_i\}$, implying a tacit change of unit element w.r.t. the join operation, from $\emptyset$ to $\{\emptyset\}$. Now, the classical (co)homology theory automatically becomes obsolete, since it is inseparable from its uniquely determined source category. Obeying the Eilenberg-Steenrod formalism, this paper is dedicated to the reconstruction, and product/join exploration of the relative simplicial and singular homology theories in this new setting. Explicitly; $\mathbf{H}_\star(X,\{\emptyset\}) = \mathbf{H}_\star(X)$, $\mathbf{H}_{-1}(\{\emptyset\},\emptyset;\mathbf{G}) = \mathbf{G}$, $\mathbf{H}_\star(X,\emptyset) = \widetilde{\mathbf{H}}_\star(X)$ and $\mathbf{H}_\star(X,Y) = \mathbf{H}_\star(X,Y)$ else. Defining pair joins through $(X_1,X_2)*(Y_1,Y_2) := (X_1*Y_1,\ X_1*Y_2 \cup X_2*Y_1)$, the *relative Künneth formula for joins* reads; (Th. 4 p. 6)

$$\mathbf{H}_{q+1}((X_1,X_2)*(Y_1,Y_2);\mathbf{G}\otimes_{\mathbf{R}}\mathbf{G}') \overset{\mathbf{R}}{\cong} \bigoplus_{i+j=q}(\mathbf{H}_i(X_1,X_2;\mathbf{G})\otimes_{\mathbf{R}}\mathbf{H}_j(Y_1,Y_2;\mathbf{G}')) \bigoplus_{i+j=q-1}\mathrm{Tor}_1^{\mathbf{R}}(\mathbf{H}_i(X_1,X_2;\mathbf{G}),\mathbf{H}_j(Y_1,Y_2;\mathbf{G}')).$$

Complying with Whitehead's vocabulary below we'll call the $\mathbf{H}$-functor the *Augmental Homology Functor*.

1 Introduction

Classical Simplicial and Singular Homology Theories have always been accompanied by the *Reduced Homology Functor* in a mix governed only by the the skill of the individual mathematician. When, within Combinatorics, the classical category of simplicial complexes $\mathcal{K}$ were abandoned some 25 years ago in favor of the contemporary $\{\emptyset\}$-extended category of simplicial complexes $\mathcal{K}_\circ$ it would have been quite natural, as in p. 3, to define a Simplicial Homology Theory formalizing this mix and through the realization functor p. 3 also to define an associated Singular Homology Theory, as in p. 4.

The join operation is extremely important within algebraic topology. For instance it is fundamental to Milnor's construction of the universal principal fibre bundle in [8] where he also formulate the non-relative Künneth formula for joins as; $\widetilde{\mathrm{H}}_{q+1}(X_1 * Y_1) \overset{\mathbf{Z}}{\cong} \bigoplus_{i+j=q}(\widetilde{\mathrm{H}}_i(X_1)\otimes_{\mathbf{R}}\widetilde{\mathrm{H}}_j(Y_1)) \bigoplus_{i+j=q-1}\mathrm{Tor}_1^{\mathbf{Z}}(\widetilde{\mathrm{H}}_i(X_1),\widetilde{\mathrm{H}}_j(Y_1))$ i.e. the "$X_2 = Y_2 = \emptyset$"-case in our Th. 4 p. 6. These results, apparently, inspired G.W. Whitehead to introduce the *Augmental Total Chain Complex* $\widetilde{\mathbf{S}}(\circ)$ and *Augmental Homology*, $\widetilde{H}_\star(\circ)$, in [12].

Whitehead's introduction do differ, for reasons now given, from that of ours. G.W. Whitehead states that $\widetilde{\mathbf{S}}(X*Y)$ and $\widetilde{\mathbf{S}}(X)\otimes\widetilde{\mathbf{S}}(Y)$ are chain equivalent, $\approx$, cf. our Theorem 3 p. 6 and then in a footnote he points out, referring to [8] p. 431 Lemma 2.1, that; -"*This fact does not seem to be stated explicitly in the literature but is not difficult to deduce from Milnor's proof of the "Künneth theorem" for ... join...*". But, $X*\emptyset = X$, [12] p. 56, and -"...$\widetilde{H}_0(X,\emptyset)$ is the reduced (0)-dimensional homology group of X", [12] p. 57, implying $\widetilde{\mathbf{S}}_i(\emptyset) \equiv \mathbf{0}\ \forall i \in \mathbf{Z}$ and so; $\widetilde{\mathbf{S}}(X) = \widetilde{\mathbf{S}}(X*\emptyset) \approx \widetilde{\mathbf{S}}(X)\otimes\widetilde{\mathbf{S}}(\emptyset) \equiv \mathbf{0}$.

The above contradiction steams from the fact that G.W. Whitehead gave the empty space, $\emptyset$, the status of a (-1)-dimensional standard simplex but never took into account that $\emptyset$ then would get the identity map, $\mathrm{Id}_\emptyset$, as a generator for its (-1)-dimensional singular augmental chain group.

On the chain level it's indeed "*not difficult*" to see what's needed to achieve $\widetilde{\mathbf{S}}(X*Y) \approx \widetilde{\mathbf{S}}(X)\otimes\widetilde{\mathbf{S}}(Y)$, but then to actually do it for the classical category of topological spaces and within the frames of the Eilenberg-Steenrod formalism is, unfortunately, impossible, since the need for a (-1)-dimensional standard simplex is indisputable and the initial object $\emptyset$ just won't do. Instead we'll use a familiar routine from Homotopy Theory providing free spaces with a common base point, i.e. we disjointly add to each topological space an element $\wp$ and choose $\{\wp\}$ to be our desired (-1)-dimensional standard simplex and join unit. To this collection we add $\emptyset$ without which really nothing would have been achieved.

G.W. Whitehead has been one of the most distinguished typologists of all times and the rest of [12] is, of course, reliable, for instance (2.3) p. 57, the "$X_2 = \emptyset$ or $Y_2 = \emptyset$"-case of our Theorem 4 p. 6. Whitehead statement, almost 50 years ago, that no proof was *stated explicitly in the literature* for $\widetilde{\mathbf{S}}(X*Y) \approx \widetilde{\mathbf{S}}(X)\otimes\widetilde{\mathbf{S}}(Y)$ has been true ever since except that our Theorem 3 p. 6 now provides such a proof. Whitehead uses a "built in" dimension shift in $\widetilde{\mathbf{S}}(X)$ while we're using a suspension operator $\mathbf{s}$.

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2. Augmental homology theory

2.1. Definitions of underlying categories and notations

Let $\mathcal{K}$ be the classical category of simplicial complexes and simplicial maps, with vertices belonging to a common universe $\mathbf{W}$. The typical morphisms of $\mathcal{K}$ are the simplicial maps, as defined in [11] p. 109, which, in particular, implies; $\mathbf{Mor}_{\mathcal{K}}(\emptyset,\Sigma)=\{\emptyset\}=\{0_{\emptyset,\Sigma}\}$, $\mathbf{Mor}_{\mathcal{K}}(\Sigma,\emptyset)=\emptyset$ if $\Sigma\neq\emptyset$ and $\mathbf{Mor}_{\mathcal{K}}(\emptyset,\emptyset)=\{\emptyset\}=\{0_{\emptyset,\emptyset}\}=\{\mathrm{id}_\emptyset\}$ where $0_{\Sigma,\Sigma'}=\emptyset=$ the empty function from Σ to Σ'. So; $0_{\Sigma,\Sigma'}\in\mathbf{Mor}_{\mathcal{K}}(\Sigma,\Sigma')\iff\Sigma=\emptyset$. If in a category we have $\varphi_i\in\mathbf{Mor}(R_i,S_i)$ $(i=1,2)$ we define

$$\varphi_1\sqcup\varphi_2:R_1\sqcup R_2\longrightarrow S_1\sqcup S_2:r\mapsto\begin{cases}\varphi_1(r) & \text{if } r\in R_1\\ \varphi_2(r) & \text{if } r\in R_2\end{cases},\ \text{where “}\sqcup\text{”} := \text{“disjoint union”}.$$

Definition of the objects in $\mathcal{K}_\circ$**:** *An (abstract) simplicial complex Σ on a vertex set V_Σ is a collection (empty or non-empty) of finite (**empty** or non-empty) subsets σ of V_Σ satisfying;*
(a) *If $\mathrm{v}\in V_\Sigma$, then $\{\mathrm{v}\}\in\Sigma$.* **(b)** *If $\sigma\in\Sigma$ and $\tau\subset\sigma$ then $\tau\in\Sigma$.*

So, $\{\emptyset\}$ is allowed as an object in $\mathcal{K}_\circ$. We will write “concept$_\circ$” or “concept$_\wp$” when to stress that a concept relates to our extended categories. If $|\sigma|=\#\sigma:=\mathrm{card}(\sigma)=q+1$ then $\dim\sigma:=q$ and σ is said to be a q-face$_\circ$ or a q-simplex$_\circ$ of $\Sigma_\circ$ and $\dim\Sigma_\circ:=\sup\{\dim(\sigma)|\sigma\in\Sigma_\circ\}$. Writing $\emptyset_\circ$ when using $\emptyset$ as a simplex, we get $\dim(\emptyset)=-\infty$ and $\dim(\{\emptyset_\circ\})=\dim(\emptyset_\circ)=-1$.

Note that each object in the category $\mathcal{K}_\circ$ of simplicial complexes$_\circ$ except for $\emptyset$, includes $\{\emptyset_\circ\}$ as a subcomplex. A typical object$_\circ$ in $\mathcal{K}_\circ$ is $\Sigma\sqcup\{\emptyset_\circ\}$ or $\emptyset$ where $\Sigma\in\mathcal{K}$ and ψ is a morphism in $\mathcal{K}_\circ$ if;

(**a**) $\psi=\varphi\sqcup\mathrm{id}_{\{\emptyset\}}$ for some $\varphi\in\mathbf{Mor}_{\mathcal{K}}(\Sigma,\Sigma')$ or

(**b**) $\psi=0_{\emptyset,\Sigma_o}$. (In particular, $\mathbf{Mor}_{\mathcal{K}_\mathbf{o}}(\Sigma_o,\{\emptyset\})=\emptyset$ if and only if $\Sigma_o\neq\{\emptyset\},\emptyset$.)

A functor $\mathbf{E}$: $\mathcal{K}_o\longrightarrow\mathcal{K}$:

Set $\mathbf{E}(\Sigma_o)=\Sigma_o\setminus\{\emptyset\}\in Obj(\mathcal{K})$ and given a morphism ψ: $\Sigma_o\to\Sigma_o'$ we put;

$\mathbf{E}(\psi)=\varphi$ if ψ fulfills **(a)** above and
$\mathbf{E}(\psi)=0_{\emptyset,\mathbf{E}(\Sigma_o)}$ if ψ fulfills **(b)** above.

A functor $\mathbf{E}_o$: $\mathcal{K}\longrightarrow\mathcal{K}_o$:

Set $\mathbf{E}_o(\Sigma)$=$\Sigma\sqcup\{\emptyset\}\in Obj(\mathcal{K}_o)$ and given simplicial φ:$\Sigma\to\Sigma'$, putting $\psi:=\varphi\sqcup\mathrm{id}_{\{\emptyset\}}$, gives;

— $\mathbf{E}\mathbf{E}_o=\mathbf{id}_{\mathcal{K}}$
— $\mathbf{im}\mathbf{E}_o=Obj(\mathcal{K}_o)\setminus\{\emptyset\}$
— $\mathbf{E}_o\mathbf{E}=\mathrm{id}_{\mathcal{K}}$ except for $\mathbf{E}_o\mathbf{E}(\emptyset)=\{\emptyset\}$

Similarly, let $\mathcal{C}$ be the category of topological spaces and continuous maps. Consider the category $\mathcal{D}_\wp$ with objects: $\emptyset$ together with $X_\wp:=X+\{\wp\}$, for all $X\in\mathrm{Obj}(\mathcal{C})$, i.e. the set $X_\wp:=X\sqcup\{\wp\}$ equipped with the weak topology, $\tau_{X_\wp}$, with respect to X and $\{\wp\}$, cf. [3] Def. 8.4 p. 132.

$f_\wp\in\mathbf{Mor}_{\mathcal{D}_\wp}(X_\wp,Y_\wp)$ if; $f_\wp=\begin{cases}\textbf{a)}\ f+\mathrm{id}_{\{\wp\}}\ (:=f\sqcup\mathrm{id}_{\{\wp\}}) \text{ with } f\in\mathbf{Mor}_{\mathcal{C}}(X,Y) \text{ and } f \text{ is } \underline{\text{on } X} \text{ to } Y, \text{ i.e}\\ \text{the domain of } f \text{ is the whole of } X \text{ and } X_\wp=X+\{\wp\},\ Y_\wp=Y+\{\wp\}\ \textbf{ or}\\ \textbf{b)}\ 0_{\emptyset,Y_\wp}\ (=\emptyset=\text{the empty function from } \emptyset \text{ to } Y_\wp).\end{cases}$

There are functors $\mathcal{F}_\wp:\begin{matrix}\mathcal{C}\longrightarrow\mathcal{D}_\wp\\ X\mapsto X+\{\wp\}\end{matrix}$, $\mathcal{F}:\begin{matrix}\mathcal{D}_\wp\longrightarrow\mathcal{C}\\ X+\{\wp\}\mapsto X\\ \emptyset\mapsto\emptyset\end{matrix}$ resembling $\mathbf{E}_o$ resp. $\mathbf{E}$.

Note. The “$\mathcal{F}_\wp$-lift topologies”, $\tau_{X_o}:=\tau_X\cup\{X_\circ\}=\{\mathcal{O}_\circ|\mathcal{O}_\circ=X_\circ\setminus(\mathcal{N}\sqcup\{\wp\})\,;\ \mathcal{N} \text{ closed in } X\}\cup\{X_\circ\}$ and $\tau_{X_o}:=\mathcal{F}_\wp(\tau_X)\cup\{\emptyset\}=\{\mathcal{O}_\circ=\mathcal{O}\sqcup\{\wp\}\,|\,\mathcal{O}\in\tau_X\}\cup\{\emptyset\}$ would also give $\mathcal{D}_\wp$ due to the domain restriction in **a**, making $\mathcal{D}_\wp$ a link between the two constructions of partial maps referred to in [1] pp. 184-6. No extra morphisms$_\wp$ has been allowed into $\mathcal{D}_\wp$ ($\mathcal{K}_o$) in the sense that the morphisms$_\wp$ are all pictures under $\mathcal{F}_\wp$ ($\mathbf{E}_o$) except $0_{\emptyset,Y_\wp}$ defined through item **b**, re-establishing $\emptyset$ as the unique initial object.

NOTATIONS: We have used w.r.t.:=with respect to, and τ_X:=the topology of X. We'll also use; **PID**:= Principal Ideal Domain, l.h.s.(r.h.s.):=left (right) hand side, iff:=if and only if, cp.:=compare (cf. = cp.!), **LHS**:=Long **H**omology Sequence and **M-Vs**:=Mayer-Vietoris sequence. Let $\mathbf{\Delta}$=$\{\Delta^{\wp},\partial\}$ be the classical singular chain complex and let "$\simeq$" denote "homeomorphism" or "chain isomorphism". Let $\bullet$ ($\bullet\bullet$) denote the one (two) point space $\{\bullet,\wp\}$ ($\{\bullet\bullet,\wp\}$).

If $X_\wp \neq \emptyset,\ \{\wp\}$ then $(X_\wp,\tau_{X_\wp})$ is a non-connected space, and it therefore seems adequate to define $X_\wp \neq \emptyset,\ \{\wp\}$ to have a certain point set topological "property$_\wp$" if $\mathcal{F}(X_\wp)$ has the "property" in question. For instance; $X_\wp$ is connected$_\wp$ if and only if X is connected.

2.2. Simplicial augmental homology theory and realizations

We will let $\mathbf{H}$ denote the simplicial as well as the singular augmental (co)homology functor$_o$.

Choose oriented q-simplices to generate $C^o_q(\Sigma_o;\mathbf{G})$, where the coefficient module $\mathbf{G}$ is a unital ($\leftrightarrow 1_{\mathbf{A}}\circ g = g$) module over any commutative ring $\mathbf{A}$ with unit.
$\mathbf{C}^o(\emptyset;\mathbf{G})$ is identically 0 in all dimensions, where 0 denotes the additive unit-element.
$\mathbf{C}^o(\{\emptyset_o\};\mathbf{G})$ is identically 0 in all dimensions except in dimension -1 where $C^o_{-1}(\{\emptyset_o\};\mathbf{G}) \tilde{=} \mathbf{G}$.
$\mathbf{C}^o(\Sigma_o;\mathbf{G}) \equiv \widetilde{\mathbf{C}}(\mathbf{E}(\Sigma_o);\mathbf{G}) \equiv$ "$\{\emptyset_o\}$-augmented chain" as defined in ordinary algebraic topology.

By just hanging on to the "$\{\emptyset_o\}$-augmented chains", also when defining relative chains$_o$, we get the *Relative Simplicial Augmental Homology Functor for $\mathcal{K}_o$-pairs*, denoted $\mathbf{H}_*$ and fulfilling;

$$\mathbf{H}_i(\Sigma_{o1},\Sigma_{o2};\mathbf{G}) = \begin{cases} \mathrm{H}_i(\mathbf{E}(\Sigma_{o1}),\mathbf{E}(\Sigma_{o2});\mathbf{G}) & \text{if } \Sigma_{o2}\neq\emptyset \\ \widetilde{\mathrm{H}}_i(\mathbf{E}(\Sigma_{o1}),\mathbf{G}) & \text{if } \Sigma_{o1}\neq\{\emptyset_o\},\emptyset, \text{ and } \Sigma_{o2}=\emptyset \\ \begin{cases} \tilde{=}\,\mathbf{G} & \text{if } i=-1 \\ =0 & \text{if } i\neq -1 \end{cases} & \text{when } \Sigma_{o1}=\{\emptyset_o\} \text{ and } \Sigma_{o2}=\emptyset \\ 0 & \text{for all } i \text{ when } \Sigma_{o1}=\Sigma_{o2}=\emptyset. \end{cases}$$

$\emptyset \neq \{\emptyset_o\}$ and both lacks final sub-objects, which under any useful definition of the *realization of a simplicial complex* implies that $|\emptyset| \neq |\{\emptyset_o\}|$ demanding the addition of a non-final object $\{\wp\} = |\{\emptyset_o\}|$ into the classical category of topological spaces as *join-unit* and (-1)-dimensional standard simplex. This approach conforms Homology Theory and considerably simplifies the study of manifolds, cf. p. 8.

We will use Spanier's definition of the "function space realization" $|\Sigma_o|$ as given in [11] p. 110, unaltered, except for the "$_o$"'s and the underlined <u>addition</u> where $\wp := \alpha_0$ and $\alpha_0(\mathrm{v}) \equiv 0\ \forall\ \mathrm{v} \in V_\Sigma$:

–" We now define a covariant functor from the category of simplicial complexes$_o$ and simplicial maps$_o$ to the category of topological spaces$_o$ and continuous maps$_o$. Given a nonempty simplicial complex$_o$ Σ_o, let $|\Sigma_o|$ be the set of all functions α from the set of vertices of Σ_o to $\mathbf{I} := [0,1]$ such that;
(a) For any α, $\{\mathrm{v} \in V_{\Sigma_o} | \alpha(\mathrm{v}) \neq 0\}$ is a simplex$_o$ of Σ_o (in particular, $\alpha(\mathrm{v}) \neq 0$ for only a finite set of vertices).
(b) For any $\alpha \underline{\neq \alpha_0}$, $\sum_{\mathrm{v}\in V_{\Sigma_o}} \alpha(\mathrm{v}) = 1$.
If $\Sigma_o = \emptyset$, we define $|\Sigma_o| = \emptyset$. **"**

The *barycentric coordinates$_o$* α, defines a metric $d(\alpha,\beta) = \sqrt{\sum_{v\in \mathbf{V}_{\Sigma_o}} [\alpha(\mathrm{v})-\beta(\mathrm{v})]^2}$ on $|\Sigma_o|$ inducing the topological space $|\Sigma_o|_d$ with *the metric topology.* We'll equip $|\Sigma_o|$ with another topology and for this purpose we define the *closed simplex$_o$* $|\sigma_o|$ of $\sigma_o \in \Sigma_o$ i.e. $|\sigma_o| := \{\alpha \in |\Sigma_o| \mid [\alpha(\mathrm{v}) \neq 0] \Longrightarrow [\mathrm{v} \in \sigma_o]\}$.

Definition. *For $\Sigma_o \neq \emptyset$, $|\Sigma_o|$ is topologized through $|\Sigma_o| := |\mathbf{E}(\Sigma_o)| + \{\alpha_0\}$, which is equivalent to give $|\Sigma_o|$ the weak topology w.r.t. the $|\sigma_o|$'s, naturally imbedded in $\mathbf{R}^n + \{\wp\}$ and we define Σ_o to be connected if $|\Sigma_o|$ is, i.e. if $\mathcal{F}(|\Sigma_o|) \simeq |\mathbf{E}(\Sigma_o)|$* is. ($\tau_{|\Sigma_o|} = \tau_{|\Sigma_o|_d}$ <u>iff</u> Σ_o is locally finite by [11] p. 119 Th. 8.)

Proposition. ([5] pp. 115, 226) *The function space realization$_o$ $|\Sigma_o|$ is homotopy* equivalent to $|\Sigma_o|_d$. □

$|\sigma_\wp|$ imbedded in $\mathbf{R}^n+\{\wp\}$ generates a satisfying set of "standard simplices$_\wp$" and "singular simplices$_\wp$". This implies in particular that the "p-standard simplices$_\wp$", denoted $\Delta^{\wp p}$, are defined by $\Delta^{\wp p} := \Delta^p + \{\wp\}$ where Δ^p denotes the usual p-dimensional standard simplex and $+$ is the topological sum, i.e. $\Delta^{\wp p} := \Delta^p \sqcup \{\wp\}$ with the weak topology w.r.t. Δ^p and $\{\wp\}$. In particular: $\Delta^{\wp(-1)} := \{\wp\}$.

Let T^p denote an classical, arbitrary singular p-simplex ($p \geq 0$). The "singular p-simplex$_\wp$", denoted $\sigma^{\wp p}$, now stands for a function of the following kind:

$$\sigma^{\wp p} : \Delta^{\wp p} = \Delta^p + \{\wp\} \longrightarrow X + \{\wp\} \quad \text{where} \quad \sigma^{\wp p}(\wp) = \wp \quad \text{and}$$

$$\sigma^{\wp p}_{|\Delta^p} = T^p \text{ for some ordinary } p\text{-dimensional singular simplex } T^p \text{ for all } p \geq 0.$$

In particular; $\qquad \sigma^{\wp(-1)} : \{\wp\} \longrightarrow X_\wp = X + \{\wp\} : \wp \mapsto \wp.$

The boundary function $\partial_\wp$ is defined by $\partial_{\wp p}(\sigma^{\wp p}) := \mathcal{F}_\wp(\partial_p(T^p))$ if $p > 0$ where ∂_p is the ordinary singular boundary function, and $\partial_{\wp 0}(\sigma^{\wp 0}) \equiv \sigma^{\wp(-1)}$ for every singular 0-simplex$_\wp$ $\sigma^{\wp 0}$. Let $\boldsymbol{\Delta}^\wp = \{\Delta^{\wp p}, \partial_\wp\}$ denote the singular augmental chain complex$_\wp$. Observation; $|\Sigma_o| \neq \emptyset \Longrightarrow |\Sigma_o| = \mathcal{F}_{\alpha_0}(|\mathbf{E}(\Sigma_o)|) \in \mathcal{D}_{\alpha_0}$.

By the strong analogy to classical singular homology, H, we omit the proof of the next lemma.

Lemma. (Analogously for co**H**omology.)

$$\mathbf{H}_i(X_{\wp 1}, X_{\wp 2}; \mathbf{G}) = \begin{cases} \mathrm{H}_i(\mathcal{F}(X_{\wp 1}), \mathcal{F}(X_{\wp 2}); \mathbf{G}) & \textit{if } X_{\wp 2} \neq \emptyset \\ \widetilde{\mathrm{H}}_i(\mathcal{F}(X_{\wp 1}); \mathbf{G}) & \textit{if } X_{\wp 1} \neq \{\wp\}, \emptyset \textit{ and } X_{\wp 2} = \emptyset \\ \begin{cases} \cong \mathbf{G} & \textit{if } i = -1 \\ = 0 & \textit{if } i \neq -1 \end{cases} & \textit{when } X_{\wp 1} = \{\wp\} \textit{ and } X_{\wp 2} = \emptyset \\ 0 & \textit{for all } i \textit{ when } X_{\wp 1} = X_{\wp 2} = \emptyset. \end{cases} \qquad \square$$

Note. i. $\Delta^i(X_1, X_2; \mathbf{G}) \cong \Delta^{\wp i}(\mathcal{F}_\wp(X_1), \mathcal{F}_\wp(X_2); \mathbf{G})$ always. In particular, $\underline{\mathbf{H}}_{-1}(\mathcal{F}_\wp(X_1), \mathcal{F}_\wp(X_2); \mathbf{G}) \equiv 0$.

ii. $\boldsymbol{\Delta}^\wp(X_{\wp 1}, X_{\wp 2}) \simeq \boldsymbol{\Delta}(\mathcal{F}(X_{\wp 1}), \mathcal{F}(X_{\wp 2}))$ except <u>iff</u> $X_{\wp 1} \neq X_{\wp 2} = \emptyset$ when the only non-isomorphisms occur for $\Delta^{\wp(-1)}(X_{\wp 1}, \emptyset) \cong \mathbf{Z} \ncong \mathbf{0} \cong \Delta^{(-1)}(\mathcal{F}(X_{\wp 1}), \emptyset)$ and $\mathbf{H}_0(X_{\wp 1}, \emptyset) \oplus \mathbf{Z} \cong \mathrm{H}_0(\mathcal{F}(X_{\wp 1}), \emptyset)$ if $X_{\wp 1} \neq \{\emptyset\}$.

iii. $\mathbf{C}^o(\Sigma_{o1}, \Sigma_{o2}; \mathbf{G}) \approx \boldsymbol{\Delta}^\wp(|\Sigma_{o1}|, |\Sigma_{o2}|; \mathbf{G})$ (see p. 1 for $\approx$)connects the simplicial and singular functor$_o$.

iv. $\mathbf{H}_0(X_\wp +_o Y_\wp, \{\wp\}; \mathbf{G}) = \mathbf{H}_0(X_\wp, \{\wp\}; \mathbf{G}) \oplus \mathbf{H}_0(Y_\wp, \{\wp\}; \mathbf{G})$ but $\mathbf{H}_0(X_\wp +_o Y_\wp, \emptyset; \mathbf{G}) = \mathbf{H}_0(X_\wp, \emptyset; \mathbf{G}) \oplus$ $\oplus\, \mathbf{H}_0(Y_\wp, \{\wp\}; \mathbf{G}) = \mathbf{H}_0(X_\wp, \{\wp\}; \mathbf{G}) \oplus \mathbf{H}_0(Y_\wp, \emptyset; \mathbf{G})$ with $X_\wp +_o Y_\wp := \mathcal{F}_\wp(\mathcal{F}(X_\wp) + \mathcal{F}(Y_\wp))$ if $X_\wp \neq \emptyset \neq Y_\wp$.

Definition: *The "pth Singular Augmental Homology Group of* $X_\wp$ *w.r.t.* $\mathbf{G}$"$:= \mathbf{H}_p(X_\wp, \emptyset; \mathbf{G})$. The *Coefficient Group*$_o := \underline{\mathbf{H}}_{-1}(\{\wp\}, \emptyset; \mathbf{G})$. *Using* $\mathcal{F}_\wp$ ($\mathbf{E}_o$), *we "lift" the concepts of homotopy, excision and point in* $\mathcal{C}$ ($\mathcal{K}$) *into* $\mathcal{D}_\wp$*-concepts* ($\mathcal{K}_o$*-concepts*) *homotopy*$_\wp$, *excision*$_\wp$ *and point*$_\wp$ (=: $\bullet$), *respectively.*

So; $f_o, g_o \in \mathcal{D}_\wp$ are *homotopic*$_\wp$ if and only if $f_o = g_o = 0_{\emptyset, Y_o}$ or there are homotopic maps $f_1, g_1 \in \mathcal{C}$ such that $f_o = f_1 + \mathrm{Id}_{\{\wp\}}, g_o = g_1 + \mathrm{Id}_{\{\wp\}}$.

An inclusion$_o$ $(i_o, i_{o_{A_o}}) : (X_o \setminus U_o, A_o \setminus U_o) \longrightarrow (X_o, A_o)$ is an *excision*$_\wp$ if and only if there is an excision $(i, i_A) : (X \setminus U, A \setminus U) \longrightarrow (X, A)$ such that $i_o = i + \mathrm{Id}_{\{\wp\}}$ and $i_{o_{A_o}} = i_A + \mathrm{Id}_{\{\wp\}}$.

$\{\mathbf{P}, \wp\} \in \mathcal{D}_\wp$ is a *point*$_\wp$ <u>iff</u> $\{\mathbf{P}\} + \{\wp\} = \mathcal{F}_\wp(\{\mathbf{P}\})$ and $\{\mathbf{P}\} \in \mathcal{C}$ is a *point*. So, $\{\wp\}$ is **not** a point$_\wp$.

Conclusion: $(\mathbf{H}, \partial_\wp)$, abbreviated $\mathbf{H}$, is a *homology theory on the h-category of pairs from* $\mathcal{D}_\wp$ ($\mathcal{K}_o$), c.f. [4] p. 117, i.e. $\mathbf{H}$ fulfills the h-category analogues, given in [4] §§8-9 pp. 114-118, of the seven Eilenberg-Steenrod axioms from [4] §3 pp. 10-13. The necessary verifications are either equivalent to the classical or completely trivial. E.g. the *dimension axiom* is fulfilled since $\{\wp\}$ is not a point$_\wp$.

Since the exactness of the relative Mayer-Vietoris sequence of a proper triad, follows from the axioms, cf. [4] p. 43 and, paying proper attention to Note **iv**, we'll use it without further motivation.

$\widetilde{\mathrm{H}}(X) = \mathbf{H}(\mathcal{F}_\wp(X), \emptyset)$ explains most of the ad-hoc reasoning surrounding the classical $\widetilde{\mathrm{H}}$-functor.

3 3:1 Augmental homology modules for joins and products

Let ∇ be one of the classical topological product/join operations "$\times$"// "$*$"// "$\hat{*}$", defined in [11] p. 4 // in [3] p. 128 Ex. 3 including [3] p. 135 Problem 6:1, [9] p. 373 and [12] p. 128 // in [1] pp. 159-160 and [9] resp. Recall; $X * \emptyset = X = \emptyset * X$ classically. We will let $\ominus$ stand for "$\cup$" or "$\cap$".

Definition. $X_{\wp 1} \nabla_{\!\circ} X_{\wp 2} := \begin{cases} \emptyset & \text{if } X_{\wp 1} = \emptyset \text{ or } X_{\wp 2} = \emptyset \\ \mathcal{F}_{\wp}(\mathcal{F}(X_{\wp 1}) \nabla \mathcal{F}(X_{\wp 2})) & \text{if } X_{\wp 1} \neq \emptyset \neq X_{\wp 2} \end{cases}$.

From now on we'll delete the $\wp/\circ$-indices. So, e.g. "X connected" now means "$\mathcal{F}(X)$ connected".

Equivalent Join Definition. *Put* $\emptyset \sqcup_1 X = X \sqcup_1 \emptyset := \emptyset$. If $X \neq \emptyset$; $\{\wp\} \sqcup_1 X = \{\wp\}$, $X \sqcup_1 \{\wp\} = X$. *For* $X, Y \neq \emptyset, \wp$ *let* $X \sqcup_1 Y$ *denote the set* $X \times Y \times (0,1]$ *pasted to the set* X *by* $\varphi_1 : X \times Y \times \{1\} \longrightarrow X; (x, y, 1) \mapsto x$, *i.e. the quotient set of* $\big(X \times Y \times (0,1]\big) \sqcup X$, *under the equivalence relation* $(x, y, 1) \sim x$ *and let* $p_1 : \big(X \times Y \times (0,1]\big) \sqcup X \longrightarrow X \sqcup_1 Y$ *be the quotient function. For* $X, Y = \emptyset$ or $\{\wp\}$ *let* $X \sqcup_0 Y := Y \sqcup_1 X$ *and else the set* $X \times Y \times [0,1)$ *pasted to the set* Y *by the function* $\varphi_2 : X \times Y \times \{0\} \longrightarrow Y; (x, y, 0) \mapsto y$, *and let* $p_2 : \big(X \times Y \times [0,1)\big) \sqcup Y \longrightarrow X \sqcup_0 Y$ *be the quotient function. Put* $X \circ Y := \big(X \sqcup_1 Y\big) \cup \big(X \sqcup_0 Y\big)$.

$(x, y, t) \in X \times Y \times [0,1]$ *specifies the point* $(x, y, t) \in X \sqcup_1 Y \cap X \sqcup_0 Y$, *one-to-one, if* $0 < t < 1$ *and the equivalence class containing* x *if* $t = 1$ (y *if* $t = 0$), *which we denote* $(x, 1)$ $((y, 0))$. *This allows "coordinate functions"* $\xi : X \circ Y \to [0,1]$, $\eta_1 : X \sqcup_1 Y \to X$, $\eta_2 : X \sqcup_0 Y \to Y$ *extendable to* $X \circ Y$ *through* $\eta_1(y, 0) :\equiv x_0 \in X$ *resp.* $\eta_2(x, 1) :\equiv y_0 \in Y$ *and a projection* $p : X \sqcup \big(X \times Y \times [0,1]\big) \sqcup Y \to X \circ Y$.

Let $X \hat{*} Y$ *denote* $X \circ Y$ *equipped with the smallest topology making* ξ, η_1, η_2 continuous and $X * Y$, $X \circ Y$ *with the quotient topology w.r.t. p, i.e. the largest topology making p continuous* $(\Rightarrow \tau_{X \hat{*} Y} \subset \tau_{X * Y})$.

Pair-definitions: $(X_1, X_2) \nabla_{\ominus} (Y_1, Y_2) := (X_1 \nabla_{\!\circ} Y_1, (X_1 \nabla_{\!\circ} Y_2) \ominus (X_2 \nabla_{\!\circ} Y_1))$, *where, if either* X_2 *or* Y_2 *is not closed* cf. [3] p. 122 Th. 2.1(1), $(X_1 * Y_1, (X_1 * Y_2) \ominus (X_2 * Y_1))$ *has to be interpreted as* $(X_1 * Y_1, (X_1 \circledast Y_2) \ominus (X_2 \circledast Y_1))$ *i.e.* $(X_1 \circ Y_2) \ominus (X_2 \circ Y_1)$ *with the subspace topology in the 2:nd component. Analogously for simplicial complexes with "$\times$" ("$*$") from [4] p. 67 Def. 8.8 ([11] p. 109 Ex. 7,* cf. p. 1).

Note: $(X_1, \{\wp\}) \times (Y_1, Y_2) = (X_1, \emptyset) \times (Y_1, Y_1)$ if $Y_2 \neq \emptyset$ and $(X_1 \circ Y_2) \cap (X_2 \circ Y_1) = X_2 \circ Y_2$. $(X \sqcup_1 Y)^{t \geq 0.5}$ in $X * Y$ is homeomorphic to the mapping cylinder w.r.t. the coordinate map $q_1 : X \times Y \to X$. $X_2 \hat{*} Y_2$ is always a subspace of $X_1 \hat{*} Y_1$ by [1] 5.7.3 p. 163. $X_2 * Y_2$ is a subspace of $X_1 * Y_1$ if X_2, Y_2 are closed. $\hat{*}$ is associative by [1] p. 161 and $*$ is claimed (non-)associative in ([1] p. 162) [9] p. 373 Ex. 4. $*$ and $\hat{*}$ are both commutative. "$\times_{\cap}$" is (still, cf. [2] p. 15) the categorical product on pairs from $\mathcal{D}_{\wp}$.

Through Lemma + Note **ii** p. 4 we convert the classical Künneth formula ($\equiv$ line 4), cf. [11] p. 235, mimicking what Milnor did, partially ($\equiv$ line 1), at the end of his proof of [8] p. 431 Lemma 2.1.

Theorem 1. *For* $\{X_1 \times Y_2, X_2 \times Y_1\}$ *excisive,* $\mathbf{q} \geq \mathbf{0}$, $\mathbf{R}$ *a* **PID**, *and assuming* $\mathrm{Tor}_1^{\mathbf{R}}(\mathbf{G}, \mathbf{G}') = 0$ *then;*
$\mathbf{H}_q((X_1, X_2) \times (Y_1, Y_2); \mathbf{G} \otimes_{\mathbf{R}} \mathbf{G}') \overset{\mathbf{R}}{\cong}$

$$\overset{\mathbf{R}}{\cong} \begin{cases} [\mathbf{H}_i(X_1; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{H}_j(Y_1; \mathbf{G}')]_q \oplus (\mathbf{H}_q(X_1; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{G}') \oplus (\mathbf{G} \otimes_{\mathbf{R}} \mathbf{H}_q(Y_1; \mathbf{G}')) \oplus \mathbf{T}_1 & \text{if } \mathbf{C} \text{ and } X_2 = \emptyset = Y_2 \\ [\mathbf{H}_i(X_1; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{H}_j(Y_1, Y_2; \mathbf{G}')]_q \oplus (\mathbf{G} \otimes_{\mathbf{R}} \mathbf{H}_q(Y_1, Y_2; \mathbf{G}')) \oplus \mathbf{T}_2 & \text{if } \mathbf{C} \text{ and } X_2 = \emptyset \neq Y_2 \\ [\mathbf{H}_i(X_1, X_2; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{H}_j(Y_1; \mathbf{G}')]_q \oplus (\mathbf{H}_q(X_1, X_2; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{G}') \oplus \mathbf{T}_3 & \text{if } \mathbf{C} \text{ and } X_2 \neq \emptyset = Y_2 \\ [\mathbf{H}_i(X_1, X_2; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{H}_j(Y_1, Y_2; \mathbf{G}')]_q \oplus \mathbf{T}_4 & \text{if } X_1 \times Y_1 = \emptyset, \{\wp\} \text{ or } X_2 \neq \emptyset \neq Y_2 \end{cases} \quad (\mathbf{1})$$

The $\mathbf{T}$-terms splits as those ahead of them, resp., e.g. $\mathbf{T}_4 = [\mathrm{Tor}_1^{\mathbf{R}}(\mathbf{H}_i(X_1, X_2; \mathbf{G}), \mathbf{H}_j(Y_1, Y_2; \mathbf{G}'))]_{q-1}$ and $\mathbf{T}_1 = [\mathrm{Tor}_1^{\mathbf{R}}\big(\mathbf{H}_i(X_1; \mathbf{G}), \mathbf{H}_j(Y_1; \mathbf{G}')\big)]_{q-1} \oplus \mathrm{Tor}_1^{\mathbf{R}}\big(\mathbf{H}_{q-1}(X_1; \mathbf{G}), \mathbf{G}'\big) \oplus \mathrm{Tor}_1^{\mathbf{R}}\big(\mathbf{G}, \mathbf{H}_{q-1}(Y_1; \mathbf{G}')\big)$. $\mathbf{C} :=$ "$X_1 \times Y_1 \neq \emptyset, \{\wp\}$" and $[\ldots]_q$ is still, i.e. as in [11] p. 235 Th. 10, to be interpreted as $\bigoplus_{i+j=q \,\&\, i,j \geq 0} \cdots$. □

Lemma. *Let* $f : (X, A) \to (Y, B)$ *be a relative homeomorphism, i.e.,* $f : X \to Y$ *is continuous and* $f : X \setminus A \to Y \setminus B$ *is a homeomorphism. If* $F : N \times \mathbf{I} \to N$ *is a strong (neighborhood) deformation retraction of* N *down onto* A *and* B *and* $f(N)$ *are closed in* $N' := f(N \setminus A) \cup B$, *then* B *is a strong (neighborhood) deformation retract of* N' *through;* $F' : N' \times \mathbf{I} \to N'$; $\begin{cases} (y,t) \mapsto y & \text{if } y \in B,\ t \in \mathbf{I} \\ (y,t) \mapsto f \circ F\,(f^{-1}(y),\ t) & \text{if } y \in f(N) \setminus B = f(N \setminus A),\ t \in \mathbf{I}. \end{cases}$

Proof. F' is continuous as being so when restricted to $f(N) \times \mathbf{I}$ resp. $B \times \mathbf{I}$, cf. [1] p. 34; 2.5.12. □

Theorem 2. (Analogously for $\hat{*}$ instead of $*$.) *If* $(X_1, X_2) \neq (\{\wp\}, \emptyset) \neq (Y_1, Y_2)$ *and* $\mathbf{G}$ *an* $\mathbf{A}$*-module*;

$$\mathbf{H}_q((X_1, X_2) \times (Y_1, Y_2); \mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_{q+1}((X_1,X_2)*(Y_1,Y_2); \mathbf{G}) \oplus \mathbf{H}_q((X_1,X_2)*(Y_1,Y_2)^{t\geq 0.5} + (X_1,X_2)*(Y_1,Y_2)^{t\leq 0.5}; \mathbf{G}) =$$

$$\overset{\mathbf{A}}{\cong} \begin{cases} \mathbf{H}_{q+1}(X_1 * Y_1; \mathbf{G}) \oplus \mathbf{H}_q(X_1; \mathbf{G}) \oplus \mathbf{H}_q(Y_1; \mathbf{G}) & \textit{if } X_1 \times Y_1 \neq \emptyset, \{\wp\} \textit{ and } X_2 = \emptyset = Y_2 \\ \mathbf{H}_{q+1}((X_1, \emptyset) * (Y_1, Y_2); \mathbf{G}) \oplus \mathbf{H}_q(Y_1, Y_2; \mathbf{G}) & \textit{if } X_1 \times Y_1 \neq \emptyset, \{\wp\} \textit{ and } X_2 = \emptyset \neq Y_2 \\ \mathbf{H}_{q+1}((X_1, X_2) * (Y_1, \emptyset); \mathbf{G}) \oplus \mathbf{H}_q(X_1, X_2; \mathbf{G}) & \textit{if } X_1 \times Y_1 \neq \emptyset, \{\wp\} \textit{ and } X_2 \neq \emptyset = Y_2 \\ \mathbf{H}_{q+1}((X_1, X_2) * (Y_1, Y_2); \mathbf{G}) & \textit{if } X_1 \times Y_1 = \emptyset, \{\wp\} \text{ or } X_2 \neq \emptyset \neq Y_2 \end{cases} \qquad (\mathbf{2})$$

Proof. Split $X*Y$ at $t=0.5$ then; X (Y) is a strong deformation retract of $(X*Y)^{t\geq 0.5}$ $((X*Y)^{t\leq 0.5})$. The relative **M-Vs** w.r.t. the excisive couple of pairs $\{(X_1,X_2)*(Y_1,Y_2)^{t\geq 0.5}; (X_1,X_2)*(Y_1,Y_2)^{t\leq 0.5}\}$ splits since the inclusion of their topological sum into $(X_1,X_2)*(Y_1,Y_2)$ is pair null-homotopic, cf. [9] p. 141 Ex. 6c, and since the 1:st (2:nd) pair is acyclic if Y_2 $(X_2) \neq \emptyset$ we get Theorem 2. Equivalently for $\hat{*}$ by the Lemma. □

5.7.4 [1] p. 164: *There is a homeomorphism:* $\nu : X \hat{*} Y \hat{*} \mathbf{E^0} \to (X \hat{*} \mathbf{E^0}) \times (Y \hat{*} \mathbf{E^0})$ *which restricts to a homeomorphism:* $X \hat{*} Y \to (X \hat{*} \mathbf{E^0}) \times Y \cup X \times (Y \hat{*} \mathbf{E^0})$. (Here $\mathbf{E^0}$ is a symbol for a point.) □

Corollary 5.7.9 [7] p. 210: *If* ϕ: $\mathbf{C} \approx \mathbf{E}$ *with inverse* ψ *and* ϕ': $\mathbf{C'} \approx \mathbf{E'}$ *with inverse* ψ', *then*

$$\phi \otimes \phi' : \mathbf{C} \otimes \mathbf{C'} \approx \mathbf{E} \otimes \mathbf{E'} \textit{ with inverse } \psi \otimes \psi'. \qquad \square$$

Theorem 46.2 [9] p. 279: *For free chain complexes* $\mathcal{C}, \mathcal{D}$ *vanishing below a certain dimension and if a chain map* $\lambda : \mathcal{C} \to \mathcal{D}$ *induces homology isomorphisms in all dimensions, then* λ *is a chain equivalence.* □

Theorem 3. (The relative Eilenberg-Zilber theorem for joins.) *For an excisive couple* $\{X \hat{*} Y_2, X_2 \hat{*} Y\}$ *from the category of ordered pairs* $((X, X_2), (Y, Y_2))$ *of topological pairs*$_\wp$, $\mathbf{s}(\boldsymbol{\Delta}^\wp(X, X_2) \otimes \boldsymbol{\Delta}^\wp(Y, Y_2))$ is *naturally chain equivalent to* $\boldsymbol{\Delta}^\wp((X, X_2) \hat{*} (Y, Y_2))$. ("**s**" stands for *suspension* i.e. the suspended chain equals the original except that dimension i in the original is dimension $i+1$ in the suspended chain.)

Proof. The second isomorphism is the key and is induced by the pair homeomorphism in [1] 5.7.4 p. 164. For the 2:nd last isomorphism we use [7] p. 210 Corollary 5.7.9 and that **LHS**-homomorphisms are "chain map"- induced. Note that the second component in the third module is an exicive union.

$$\mathbf{H}_q(X \hat{*} Y) \overset{\mathbf{Z}}{\cong} \mathbf{H}_{q+1}(X \hat{*} Y \hat{*} \{\mathbf{v}, \wp\}, X \hat{*} Y) \overset{\mathbf{Z}}{\cong} \mathbf{H}_{q+1}((X \hat{*} \{\mathbf{u}, \wp\}) \times (Y \hat{*} \{\mathbf{v}, \wp\}), ((X \hat{*} \{\mathbf{u}, \wp\}) \times Y) \cup (X \times (Y \hat{*} \{\mathbf{v}, \wp\}))) =$$

$$= \mathbf{H}_{q+1}((X \hat{*} \{\mathbf{u}, \wp\}, X) \times (Y \hat{*} \{\mathbf{v}, \wp\}, Y)) \overset{\mathbf{Z}}{\cong} \left[\begin{array}{l} \text{Motivation: The underlying chains on the l.h.s. and r.h.s. are,} \\ \text{by Note } \mathbf{ii} \text{ p. 4, isomorphic to their classical counterparts on} \\ \text{which we use the classical Eilenberg-Zilber Theorem.} \end{array} \right] \overset{\mathbf{Z}}{\cong}$$

$$\overset{\mathbf{Z}}{\cong} \mathbf{H}_{q+1}(\boldsymbol{\Delta}^\wp(X \hat{*} \{\mathbf{u}, \wp\}, X) \otimes_{\mathbf{Z}} \boldsymbol{\Delta}^\wp(Y \hat{*} \{\mathbf{v}, \wp\}, Y)) \overset{\mathbf{Z}}{\cong} \mathrm{H}_{q+1}(\mathbf{s}\boldsymbol{\Delta}^\wp(X) \otimes_{\mathbf{Z}} \mathbf{s}\boldsymbol{\Delta}^\wp(Y)) \overset{\mathbf{Z}}{\cong} \mathrm{H}_q(\mathbf{s}[\boldsymbol{\Delta}^\wp(X) \otimes_{\mathbf{Z}} \boldsymbol{\Delta}^\wp(Y)]).$$

Now the non-relative Eilenberg-Zilber Theorem for joins follows from [9] p. 279 Th. 46.2. ▷

Substituting, in the $\times$-original proof [11] p. 234, "$\hat{*}$", "$\boldsymbol{\Delta}^\wp / \mathbf{s}\boldsymbol{\Delta}^\wp$", "Th. 3, 1:st part" for "$\times$", "Δ", "Theorem 6" resp. will do since; $\mathbf{s}(\boldsymbol{\Delta}^\wp(X_1) \otimes \boldsymbol{\Delta}^\wp(Y_1)) / [\mathbf{s}(\boldsymbol{\Delta}^\wp(X_1) \otimes \boldsymbol{\Delta}^\wp(Y_2)) + \mathbf{s}(\boldsymbol{\Delta}^\wp(X_2) \otimes \boldsymbol{\Delta}^\wp(Y_1))] =$
$= \mathbf{s}\{\boldsymbol{\Delta}^\wp(X_1) \otimes \boldsymbol{\Delta}^\wp(Y_1) / [(\boldsymbol{\Delta}^\wp(X_1) \otimes \boldsymbol{\Delta}^\wp(Y_2)) + (\boldsymbol{\Delta}^\wp(X_2) \otimes \boldsymbol{\Delta}^\wp(Y_1))]\} = \mathbf{s}\{[\boldsymbol{\Delta}^\wp(X_1)/\boldsymbol{\Delta}^\wp(X_2)] \otimes [\boldsymbol{\Delta}^\wp(Y_1)/\boldsymbol{\Delta}^\wp(Y_2)]\}$. □

[11] Cor. 4 p. 231 now gives Th. 4 since; $\mathbf{H}_\star(\circ) \cong \mathbf{s}\mathbf{H}_{\star+1}(\circ) \cong \mathbf{H}_{\star+1}(\mathbf{s}(\circ))$ and $\boldsymbol{\Delta}^\wp((X, X_2) * (Y, Y_2)) \approx$
$\approx \boldsymbol{\Delta}^\wp((X, X_2) \hat{*} (Y, Y_2))$ by Th. 2. and [9] p. 279 Th. 46.2. ($\{X_1 * Y_2, X_2 * Y_1\}$ is excisive <u>iff</u> $\{X_1 \times Y_2, X_2 \times Y_1\}$ is.)

Theorem 4. (cp. [11] p. 235.) *If* $\{X_1 \hat{*} Y_2, X_2 \hat{*} Y_1\}$ *is an excisive couple in* $X_1 \hat{*} Y_1$, $\mathbf{R}$ *a* **PID**, $\mathbf{G}$, $\mathbf{G'}$ $\mathbf{R}$*-modules and* $\mathrm{Tor}_1^{\mathbf{R}}(\mathbf{G}, \mathbf{G'}) = 0$, *then the functorial sequences below are* (non-naturally) split exact;

$$0 \longrightarrow \bigoplus_{i+j=q} [\mathbf{H}_i(X_1, X_2; \mathbf{G}) \otimes_{\mathbf{R}} \mathbf{H}_j(Y_1, Y_2; \mathbf{G'})] \longrightarrow$$
$$\longrightarrow \mathbf{H}_{q+1}((X_1, X_2) \hat{*} (Y_1, Y_2); \mathbf{G} \otimes_{\mathbf{R}} \mathbf{G'}) \longrightarrow \bigoplus_{i+j=q-1} \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{H}_i(X_1, X_2; \mathbf{G}), \mathbf{H}_j(Y_1, Y_2; \mathbf{G'})) \longrightarrow 0 \qquad (\mathbf{3})$$

Analogously with "$*$" substituted for "$\hat{*}$" and [11] p. 247 Th. 11 gives the co**H**omology-analog. □

Theorem 5. [The Universal Coefficient Theorem for (co)**H**omology.] (Put $(X_1, X_2) = (\{\wp\}, \emptyset)$ in Th. 4.)

$$\mathbf{H}_i(Y_1, Y_2; \mathbf{G}) \overset{\mathbf{R}}{\cong} [[11] \text{ p. } 214] \overset{\mathbf{R}}{\cong} \mathbf{H}_i(Y_1, Y_2; \mathbf{R} \otimes_{\mathbf{R}} \mathbf{G}) \overset{\mathbf{R}}{\cong} [\mathbf{H}_i(Y_1, Y_2; \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{G}] \oplus \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{H}_{i-1}(Y_1, Y_2; \mathbf{R}), \mathbf{G}),$$

for *any* **R-PID** *module* $\mathbf{G}$. *If all* $\mathbf{H}_*(Y_1, Y_2; \mathbf{R})$ *are of finite type or* $\mathbf{G}$ *is finitely generated, then;*

$$\mathbf{H}^i(Y_1, Y_2; \mathbf{G}) \overset{\mathbf{R}}{\cong} \mathbf{H}^i(Y_1, Y_2; \mathbf{R} \otimes_{\mathbf{R}} \mathbf{G}) \overset{\mathbf{R}}{\cong} [\mathbf{H}^i(Y_1, Y_2; \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{G}] \oplus \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{H}^{i+1}(Y_1, Y_2; \mathbf{R}), \mathbf{G}). \qquad \square$$

Lemma. x (y) closed in X $(Y) \Longrightarrow \{X\times(Y\setminus y),(X\setminus x)\times Y\}, \{X*(Y\setminus y),(X\setminus x)*Y\}$ *both excisive.*

Proof. [11] p. 188 Th. 3 since $X\times(Y\setminus y_0)$ $\big(X*(Y\setminus y_0)\big)$ is open in $(X\times(Y\setminus y_0))\cup((X\setminus x_0)\times Y)$ $\big((X*(Y\setminus y_0))\cup((X\setminus x_0)*Y)\big)$ which, here, is equivalent to being open in $X\times Y$ $\big(X*Y\big)$, cf. [3] p. 122. □

Our definition of a 'setminus", "$\setminus_o$", in $\mathcal{D}_\wp$ is motivated by Proposition 1 below and reflects the non-local i.e. the non-geometrical nature of the (-1)-element $\wp$.

Definition. $X_\wp \setminus_o X'_\wp := \begin{cases} \emptyset & \text{if } X_\wp=\emptyset,\ X_\wp \subsetneqq X'_\wp \text{ or } X'_\wp=\{\wp\} \\ \mathcal{F}_\wp\big(\mathcal{F}(X_\wp)\setminus\mathcal{F}(X'_\wp)\big) & \text{else}\end{cases}$

Proposition 1 is our key motivation for introducing a topological (-1)-object. "$X\setminus x$" usually stands for "$X\setminus\{x\}$" and we will write x for $\{x,\wp\}$ as a notational convention. See p. 3 for Def. of α_0.

Proposition 1. *Let* $\mathbf{G}$ *be any module over a commutative ring* $\mathbf{A}$ *with unit. With* $\alpha\in\mathrm{Int}\sigma$ *and* $\alpha=\alpha_0$ iff $\sigma=\emptyset$, *the following module isomorphisms are all induced by chain equivalences*, cf. Th. 46.2 p. 6.

$$\mathbf{H}_{i\text{-}\#\sigma}(\mathrm{Lk}_\Sigma\sigma;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_i(\Sigma,\mathrm{cost}_\Sigma\sigma;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_i(|\Sigma|,|\mathrm{cost}_\Sigma\sigma|;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_i(|\Sigma|,|\Sigma|\setminus_o\alpha;\mathbf{G})$$
$$\mathbf{H}^{i\text{-}\#\sigma}(\mathrm{Lk}_\Sigma\sigma;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}^i(\Sigma,\mathrm{cost}_\Sigma\sigma;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}^i(|\Sigma|,|\mathrm{cost}_\Sigma\sigma|;\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}^i(|\Sigma|,|\Sigma|\setminus_o\alpha;\mathbf{G}).$$

Proof. (Cf. "More definitions" p. 8.) The "$\setminus_o$"-definition above, [9] Th. 46.2 p. 279 + pp. 194-199 Lemma 35.1-35.2 + Lemma 63.1 p. 374 gives the two ending isomorphisms since $|\mathrm{cost}_\Sigma\sigma|$ is a deformation retract of $|\Sigma|\setminus_o\alpha$, while already on the chain level; $C^o_\star(\Sigma,\mathrm{cost}_\Sigma\sigma) = C^o_\star(\overline{\mathrm{st}}_\Sigma(\sigma),\dot\sigma*\mathrm{Lk}_\Sigma\sigma) = C^o_\star(\overline{\sigma}*\mathrm{Lk}_\Sigma\sigma,\dot\sigma*\mathrm{Lk}_\Sigma\sigma) \simeq C^o_{\star-\#\sigma}(\mathrm{Lk}_\Sigma\sigma)$. (This result is, somewhat specialized, found in [6] p. 162 and partially also in [10] p. 116 Lemma 3.3.) □

Proposition 2. *If* $x\in X$ $(y\in Y)$ *closed and* $(t_1,\widetilde{x*y},t_2):=\{(x,y,t)\,|\,0<t_1\le t\le t_2<1\}$;

i. $\mathbf{H}_{q+1}(X\bar{*}Y,X\bar{*}Y\setminus_o(x,y,t);\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_{q+1}(X\bar{*}Y,X\bar{*}Y\setminus_o(t_1,\widetilde{x*y},t_2);\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_q(X\times Y,X\times Y\setminus_o(x,y);\mathbf{G}) \overset{\mathbf{A}}{\cong}$
$\overset{\mathbf{A}}{\cong} \left[\begin{smallmatrix}\text{A simple}\\ \text{calculation}\end{smallmatrix}\right] \overset{\mathbf{A}}{\cong} \mathbf{H}_q((X,X\setminus_o x)\times(Y,Y\setminus_o y);\mathbf{G}) \overset{\mathbf{A}}{\cong} \left[\begin{smallmatrix}\text{Th. 2 p. 6}\\ \text{line four}\end{smallmatrix}\right] \overset{\mathbf{A}}{\cong} \mathbf{H}_{q+1}((X,X\setminus_o x)\bar{*}(Y,Y\setminus_o y);\mathbf{G})$.

ii. $\mathbf{H}_{q+1}(X\bar{*}Y,X\bar{*}Y\setminus_o(y,0);\mathbf{G}) \overset{\mathbf{A}}{\cong} \mathbf{H}_{q+1}((X,\emptyset)\bar{*}(Y,Y\setminus_o y);\mathbf{G})$ and equivalently for the $(x,1)$-points.

All isomorphisms are induced by chain equivalences, cf. p. 6. Analogously with "$*$" substituted for "$\bar{*}$".

Proof. i. $\begin{cases} A := X\,\sqcup\!\!\!\!\sqcup\, Y\setminus_o\{(x_0,y_0,t)\,|\,t_1\le t<1\} \\ B := X\,\boxdot\, Y\setminus_o\{(x_0,y_0,t)\,|\,0<t\le t_2\}\end{cases} \Longrightarrow \begin{cases} A\cup B = X\bar{*}Y\setminus_o(t_1,\widetilde{x*y},t_2) \\ A\cap B = X\times Y\times(0,1)\setminus_o\{x_0\}\times\{y_0\}\times(0,1),\end{cases}$

with $x_0\times y_0\times(0,1) := \{x_0\}\times\{y_0\}\times\{t\,|\,t\in(0,1)\}\cup\{\wp\}$ and $(x_0,y_0,t_0) := \{(x_0,y_0,t_0),\wp\}$.

Now, using the null-homotopy in the relative **M-Vs**, w.r.t. $\{(X\sqcup\!\!\!\!\sqcup Y,A),(X\boxdot Y,B)\}$ the resulting splitting of it and also the involved pair deformation retractions as in the proof of Th. 2, we get;

$$\mathbf{H}_{q+1}(X\bar{*}Y,X\bar{*}Y\setminus_o(t_1,\widetilde{x_0*y_0},t_2)) \overset{\mathbf{Z}}{\cong} \mathbf{H}_q(X\times Y\times(0,1),X\times Y\times(0,1)\setminus_o\{x_0\}\times\{y_0\}\times(0,1)) \overset{\mathbf{Z}}{\cong}$$
$$\overset{\mathbf{Z}}{\cong}\left[\begin{smallmatrix}\text{The pare on the r.h.s. is a pair}\\ \text{deformation retract of the l.h.s.}\end{smallmatrix}\right] \overset{\mathbf{Z}}{\cong} \mathbf{H}_q(X\times Y\times\{t_0,\wp\},X\times Y\times\{t_0,\wp\}\setminus_o(x_0,y_0,t_0)) \overset{\mathbf{Z}}{\cong} \mathbf{H}_q(X\times Y,X\times Y\setminus_o(x_0,y_0)) =$$
$$= \mathbf{H}_q(X\times Y,(X\times(Y\setminus_o y_0))\cup((X\setminus_o x_0)\times Y)) = \mathbf{H}_q((X,X\setminus_o x_0)\times(Y,Y\setminus_o y_0)).$$ ▷

ii. $\begin{cases} A := X\sqcup\!\!\!\!\sqcup Y, \\ B := X\boxdot Y\setminus_o X\times\{y_0\}\times[0,1)\end{cases} \Longrightarrow \begin{cases} A\cup B = X\bar{*}Y\setminus_o(y_0,0) \\ A\cap B = X\times(Y\setminus_o y_0)\times(0,1)\end{cases}$ where $(x_0,y_0,t)\in X\times\{y_0\}\times[0,1)$

is independent of x_0 and $(x_0,y_0,t_0) := \{(x_0,y_0,t_0),\wp\}$. Continue as in **i** using Th. 1 p. 5 line 2 and that;

$$(X\times Y\times(0,1),X\times(Y\setminus_o y_0)\times(0,1)) \sim \left[\begin{smallmatrix}\text{Motivation: The r.h.s. is a pair}\\ \text{deformation retract of the l.h.s.}\end{smallmatrix}\right] \sim (X\times Y,X\times(Y\setminus_o y_0)) =$$
$$= (X,\emptyset)\times(Y,Y\setminus_o y_0).$$ □

Definitions

In the definitions below, we will use the same notations as in Chapter 2. The underlying principle for our definitions is that a concept in $\mathcal{C}$ ($\mathcal{K}$) is carried over to $\mathcal{D}_\wp$ ($\mathcal{K}_o$) by $\mathcal{F}_\wp$ ($\mathbf{E}_o$) with addition of definitions of the concept$_o$ for cases that isn't a proper image under $\mathcal{F}_\wp$ ($\mathbf{E}_o$). The definitions of the product/join operations "$\times_o$", "$*_o$", "$\hat{*}_o$" in page 5 and "$\setminus_o$" in page 7 certainly follows this principle.

Definition. $\emptyset/_{X_\wp} := \emptyset$ else; $X_{\wp 1}/_{X_{\wp 2}} := \mathcal{F}_\wp\big(\mathcal{F}(X_{\wp 1})/_{\mathcal{F}(X_{\wp 2})}\big)$ if $X_{\wp 2} \neq \emptyset$ in $\mathcal{D}_{o\wp}$ for $X_{\wp 2} \subset X_{\wp 1}$, **(i)**
where "/" is the classical "quotient" except that $\mathcal{F}(X_{\wp 1})/_\emptyset := \mathcal{F}(X_{\wp 1})$, cp. [1] p. 102.

Definition. $X_{\wp 1} +_o X_{\wp 2} := \begin{cases} X_{\wp 1} & \text{if } X_{\wp 2} = \emptyset \text{ (or vice versa)} \\ \mathcal{F}_\wp(\mathcal{F}(X_{\wp 1}) + \mathcal{F}(X_{\wp 2})) & \text{if } X_{\wp 1} \neq \emptyset \neq X_{\wp 2} \end{cases}$ **(ii)**
where "+" is the classical "topological sum" defined in [3] p. 127 under the name "free union".

More definitions. "The *link*$_o$ of σ_o w.r.t. Σ_o" $= \mathrm{Lk}_{o\Sigma_o}\sigma_o := \{\tau_o \in \Sigma_o | [\sigma_o \cap \tau_o = \emptyset] \wedge [\sigma_o \cup \tau_o \in \Sigma_o]\}$.
"The *closed star*$_o$ of σ_o w.r.t. Σ_o" $= \overline{\mathrm{st}}_{o\Sigma_o}\sigma_o := \{\tau_o \in \Sigma_o | \sigma_o \cup \tau_o \in \Sigma_o\}$.
"The contrastar$_o$ of σ_o w.r.t. Σ_o" $= \mathrm{cost}_{o\Sigma_o}\sigma_o := \{\tau_o \in \Sigma_o | \tau_o \not\supseteq \sigma_o\}$.
The set of all subsets of a simplex$_o$ σ_o is a simplicial complex$_o$ denoted $\bar{\sigma}_o$, while the *boundary of* σ_o, $\dot{\sigma}_o$, is the set of all proper subsets. ($\dot{\sigma}_o := \{\tau \mid \tau \subsetneq \sigma_o\} = \bar{\sigma}_o \setminus \{\sigma_o\}$, $\bar{\emptyset}_o = \{\emptyset_o\}$ and $\dot{\emptyset}_o = \emptyset$.)

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Let $\mathbf{G}$ be a unital module over a commutative ring $\mathbf{A}$.

Definition. *$X = \bullet\bullet$ is a homology$_{\mathbf{G}}$ 0-manifold. Else, $X \in \mathcal{D}_\wp$ is a (singular) homology$_{\mathbf{G}}$ n-manifold if it is a connected, locally compact Hausdorff space with* $\mathbf{H}_i(X \setminus_o x; \mathbf{G}) = 0$ $\forall i \geq n$ and $\forall x \in X$ *and;*

$$\mathbf{H}_i(X, X \setminus_o x; \mathbf{G}) \tilde{=} \begin{cases} 0 & \text{if } i \neq n \text{ for all } \wp \neq x \in X, \\ 0 \text{ or } \mathbf{G} & \text{if } i = n \text{ for all } \wp \neq x \in X. \end{cases} \qquad \textbf{(iii)}$$

The boundary $\mathrm{Bd}_{\mathbf{G}}X := \{x \in X | \mathbf{H}_n(X, X \setminus_o x; \mathbf{G}) = 0\}$. (So, $\mathrm{Bd}_{\mathbf{G}}X \neq \emptyset$ <u>iff</u> $\wp \in \mathrm{Bd}_{\mathbf{G}}X$, e.g. $\mathrm{Bd}_{\mathbf{G}}(\bullet) = \{\wp\}$.)

Note: For a classical manifold $M \neq \bullet$ with $\mathbf{Bd}M = \emptyset$; $\mathbf{Bd}\mathcal{F}_\wp(M) = \emptyset$ if M is compact and orientable and $\mathbf{Bd}\mathcal{F}_\wp(M) = \{\wp\}$ else. $\emptyset, \{\wp\}, \bullet$ and $\bullet\bullet$ are the only manifolds$_\wp$ in $\dim \leq 0$. $\mathbf{Bd}_{\mathbf{G}}\emptyset = \mathbf{Bd}_{\mathbf{G}}(\{\wp\}) = \mathbf{Bd}_{\mathbf{G}}\bullet\bullet = \emptyset$.

Observation: $\mathcal{M}_1 * \mathcal{M}_2$ is a homology$_{\mathbf{G}}$ (n_1+n_2+1)-manifold, $n_i \geq 0$, **<u>iff</u>** **iii** holds also for $x = \wp$ for the compact homology$_{\mathbf{G}}$ n_i-manifold $\mathcal{M}_i$, $i = 1, 2$. $\mathrm{Bd}(\mathcal{M}_1 * \mathcal{M}_2) = ((\mathrm{Bd}\mathcal{M}_1) * \mathcal{M}_2) \cup (\mathcal{M}_1 * (\mathrm{Bd}\mathcal{M}_2))$, cp. [6] p. 171. $\mathcal{M}_1 * \mathcal{M}_2$ is orientable <u>iff</u> both $\mathcal{M}_1$ and $\mathcal{M}_2$ are. E.g.: Put $\mathcal{P}^{\mathbf{2}}_{\mathbf{i}}$:= "the Projective Plane", $i = 1, 2$ and $\mathbf{Z_p}$:= "the Prime Field w.r.t. $\mathbf{p}$". Then $\dim(\mathcal{P}^{\mathbf{2}}_{\mathbf{1}} * \mathcal{P}^{\mathbf{2}}_{\mathbf{2}}) = 5$ while $\dim(\mathbf{Bd}_{\mathbf{Z_p}}(\mathcal{P}^{\mathbf{2}}_{\mathbf{1}} * \mathcal{P}^{\mathbf{2}}_{\mathbf{2}})) = \dim(\mathcal{P}^{\mathbf{2}}_{\mathbf{2}} \cup \mathcal{P}^{\mathbf{2}}_{\mathbf{1}}) = 2$ if $\mathbf{p} \neq 2$. Moreover: No n-manifold has an $(n-2)$-dimensional boundary, including classical manifolds.

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