A remark on N. Kuhn's unbounded strong realization conjecture

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Abstract

N. Kuhn has given several conjectures on the special features satisfied by the singular cohomology of topological spaces with coefficients in a finite prime field, as modules over the Steenrod algebra [Ku]. The so-called *realization conjecture* was solved in special cases in [Ku] and in complete generality by L. Schwartz [Sc2]. The more general *strong realization conjecture* has been settled at the prime 2, as a consequence of the work of L. Schwartz [Sc3], and the subsequent work of F.-X. Dehon and the author [DG]. In this note, we are interested in the even more general *unbounded strong realization conjecture.* We shall prove that it holds at the prime 2 for the class of spaces whose cohomology has a trivial Bockstein action in high degrees.

1 Introduction

The singular cohomology of a topological space with coefficients in a finite prime field is naturally endowed with the structure of an unstable algebra over the Steenrod algebra. That is, a graded ring structure together with a compatible action of the Steenrod algebra [Sc1, p. 21].

If an unstable module is isomorphic to the cohomology of some space, we say that this module is *topologically realizable.*

N. Kuhn's conjectures [Ku] tell that the realizable unstable modules have rather special algebraic features. Namely, these conjectures tell us that the action of the Steenrod algebra on the cohomology of a topological space has to be *'either very big or very small'.*

The first of these conjectures [Ku, Realization conjecture, p. 321] was settled by L. Schwartz. We quote it as the following theorem.

Theorem 1.1 *[Sc2, Théorème 0.1] Let X be a topological space. If the singular cohomology of X with coefficients in a finite prime field is finitely generated as a module over the Steenrod algebra, then it is finite (as a graded vector space).*

The more general *strong realization conjecture* [Ku, p.324] was only settled at the prime 2, as consequence of the work of L. Schwartz [Sc3], and of F.-X. Dehon and the author [DG]. Let $\mathcal{U}_d$ be the full subcategory of unstable modules annihilated by $\bar{\mathrm{T}}^{d+1}$, the reduced Lannes' functor iterated $(d+1)$ times. The subcategory $\mathcal{U}_0$ is the subcategory of *locally finite* modules [Sc1, Sc3].

Theorem 1.2 *[Sc2, DG] Let X be a topological space. If the singular cohomology $\bar{\mathrm{H}}^*X$ of X with coefficients in $\mathbb{F}_2$ is in $\mathcal{U}_d$ for some d, then $\bar{\mathrm{H}}^*X$ is in $\mathcal{U}_0$.*

In this note, we turn our attention to the more general *unbounded strong realization conjecture.*

Note. In the sequel, $\bar{\mathrm{H}}^*X$ always means the *modulo* 2 singular cohomology of the space X.

1.1 The unbounded strong realization conjecture

We denote by $\mathcal{U}$ the category of unstable modules over the *modulo* 2 Steenrod algebra. Every object M of $\mathcal{U}$ is equipped with a natural decreasing filtration, the so-called *nilpotent filtration* [Ku, Sc3]:

$$M = M_0 \supset M_1 \supset M_2 \dots M_s \supset M_{s+1} \supset \dots 0 \quad .$$

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We recall some properties of the nilpotent filtration. We say that an unstable module is *reduced* if the operator

$$\mathrm{Sq}_0 \ : \ M \longrightarrow M, \ m \longmapsto \mathrm{Sq}^{|m|} \ m$$

is injective. If M has a compatible algebra structure, then M is reduced if and only if it has no nilpotent elements. For each s, the module M_s/M_{s+1} is of the form $\Sigma^s \mathrm{R}_s M$ where $\mathrm{R}_s M$ is a reduced module.

Another important property of the nilpotent filtration is that any unstable module is complete with respect to its nilpotent filtration. This means that the natural map $M \longrightarrow \lim_s M/M_s$ is an isomorphism. This can be seen from the fact that for each s, the module M_s is s-connected.

An unstable module such that $M = M_s$ is called s-nilpotent. A 1-nilpotent module is simply called *nilpotent*. An element of an unstable module is s-nilpotent provided it spans a s-nilpotent submodule.

The functors $\bar{\mathrm{T}}$ commute with the nilpotent filtration in the following sense [Ku, prop. 2.5, p. 331]. Let M be any unstable module. Let

$$M = M_0 \supset M_1 \supset M_2 \ldots M_s \supset M_{s+1} \supset \ldots 0 \quad .$$

be the nilpotent filtration of M. Then the induced filtration of $\bar{\mathrm{T}}M$

$$\bar{\mathrm{T}}M = \bar{\mathrm{T}}M_0 \supset \bar{\mathrm{T}}M_1 \supset \bar{\mathrm{T}}M_2 \ldots \bar{\mathrm{T}}M_s \supset \bar{\mathrm{T}}M_{s+1} \supset \ldots 0$$

is the nilpotent filtration of $\bar{\mathrm{T}}M$, *i.e.* for all s,

$$\bar{\mathrm{T}}M_s = (\bar{\mathrm{T}}M)_s \quad .$$

Furthermore, by exactness and commutation of $\bar{\mathrm{T}}$ with suspensions, we have a sequence of equalities and natural isomorphisms

$$\Sigma^s \mathrm{R}_s \bar{\mathrm{T}}M = (\bar{\mathrm{T}}M)_s/(\bar{\mathrm{T}}M)_{s+1} = \bar{\mathrm{T}}(M_s)/\bar{\mathrm{T}}(M_{s+1}) \cong \bar{\mathrm{T}}(M_s/M_{s+1}) = \bar{\mathrm{T}}\Sigma^s \mathrm{R}_s M \cong \Sigma^s \bar{\mathrm{T}} \mathrm{R}_s M \quad ,$$

that is, the functor $\bar{\mathrm{T}}$ and R_s commute for all s.

Let n be an integer. Let $n = \sum_{i=1}^{\ell} 2^{n_i}$ be the diadic expansion of n. We attach to n the integer $\alpha(n) = \ell$.

Definition 1.3 *Let M be a reduced unstable module. We say that M is of weight less than t if M is trivial in all degrees ℓ such that $\alpha(\ell) > t$.*

The weight $w(M)$ of M is the integer (maybe infinite) such that M is of weight less than $w(M)$ but not $w(M) - 1$.

A reduced module is of weight 0 if and only if it is concentrated in degree zero. In this case, we say that M is *constant.*

To understand the definition, we give the following examples.

Example 1.4 *Let $\mathrm{F}(1)$ be the unstable submodule generated by the non zero degree one class in $\bar{\mathrm{H}}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[u]$. It is exactly the submodule of primitive elements of the Hopf algebra $\mathrm{H}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[u]$. A graded $\mathbb{F}_2$-basis for $\mathrm{F}(1)$ is given by the elements $\{u^{2^i}\}_{i \in \mathbb{N}}$. So $\mathrm{F}(1)$ is zero in degrees ℓ such that $\alpha(\ell)$ is strictly more that one. So the weight $w(\mathrm{F}(1))$ equals* 1.

Example 1.5 *It is easy to see that $w(\mathrm{F}(1)^{\otimes n}) = n$.*

Example 1.6 *The reduced cohomology ring $\bar{\mathrm{H}}^* B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[u]$ is of infinite weight.*

The next proposition shows the importance of the notion of weight.

Proposition 1.7 *[FS] A reduced unstable module M is in $\mathcal{U}_n$ if and only if its weight $w(M)$ is less or equal to n.*

We can state the *unbounded strong realization conjecture* [Ku, p. 326] in a slightly modified form.

unbounded strong realization conjecture. *Let M be an unstable module such that $\mathrm{R}_s M$ is of finite weight for each s. If M is topologically realizable, then the module $\mathrm{R}_s M$ is constant for all s.*

The original conjecture of N. Kuhn is not stated in terms of weight, but in terms of functors [Ku, p. 325-326]. Let $\mathcal{N}il$ be the full subcategory of $\mathcal{U}$ of *nilpotent* unstable modules. One can form the quotient category $\mathcal{U}/\mathcal{N}il$. By [HLS], it is known that $\mathcal{U}/\mathcal{N}il$ is equivalent to the full subcategory $\mathcal{F}_\omega$ of *analytic functors* of the category $\mathcal{F}$ of functors from finite dimensional $\mathbb{F}_2$-vector spaces to all $\mathbb{F}_2$-vector spaces (with natural tranformations as morphisms). In the category $\mathcal{F}$, one has a notion of polynomial functor of degree n.

Let $q : \mathcal{U} \longrightarrow \mathcal{F}_\omega$ denote the quotient functor $\mathcal{U} \longrightarrow \mathcal{U}/\mathcal{N}il$ composed with the equivalence of categories $\mathcal{U}/\mathcal{N}il \cong \mathcal{F}_\omega$.

The point is that a reduced unstable module is of weight n if and only if $q(M)$ is polynomial of degree n.

For the reader's convenience, we shall underline the proof of the fact that the strong realization conjecture 1.2 is a consequence of the unbounded strong realization conjecture. It relies on the following lemma.

Lemma 1.8 *An unstable module M is in $\mathcal{U}_n$ if and only if $R_s M$ is in $\mathcal{U}_n$ for all s.*

Proof of lemma 1.8. Suppose M is in $\mathcal{U}_n$. As $\mathcal{U}_n$ is a Serre subcategory [Sc3], the modules M_s and M_s/M_{s+1} are also in $\mathcal{U}_n$ for each s. So $M_s/M_{s+1} = \Sigma^s R_s M$ is in $\mathcal{U}_n$. But the functor $\bar{\mathrm{T}}$ commutes with suspensions, so $R_s M$ is also in $\mathcal{U}_n$.

Conversely, if $R_s M$ is in $\mathcal{U}_n$ for all s, by exactness of $\bar{\mathrm{T}}$ it follows that M/M_s (recall that the nilpotent filtration is decreasing) is in $\mathcal{U}_n$ for each s. In other words,

$$\bar{\mathrm{T}}^{n+1}(M)/(\bar{\mathrm{T}}^{n+1}(M))_s = \bar{\mathrm{T}}^{n+1}(M)/\bar{\mathrm{T}}^{n+1}(M_s) \cong \bar{\mathrm{T}}^{n+1}(M/M_s) = 0$$

for each s. But $\bar{\mathrm{T}}^{n+1}(M)$ is complete with respect to its nilpotent filtration, hence

$$\bar{\mathrm{T}}^{n+1}(M) = 0 \quad .$$

It follows that M is in $\mathcal{U}_n$. □

Now suppose we have an unstable module M which is realizable and is in $\mathcal{U}_n$, *i.e.* such that $\bar{\mathrm{T}}^{n+1}M = 0$. By the preceding lemma, the module $\mathrm{R}_s M$ is also in $\mathcal{U}_n$. But an unstable module is of finite weight n if and only if it is in $\mathcal{U}_n$.

So, the unbounded strong realization conjecture implies that $\mathrm{R}_s M$ is constant for $s \geq 0$. But a reduced module is constant if and only if it is in $\mathcal{U}_0$. Hence, by the lemma, the module M is in $\mathcal{U}_0$ and so the strong realization conjecture holds for M.

Another consequence of lemma 1.8 is to give another form of the unbounded strong realization conjecture:

unbounded strong realization conjecture. *Let M be an unstable module such that $\mathrm{R}_s M$ is of finite weight for each s. If M is topologically realizable, then M is locally finite.*

1.2 The main result

Our main result is the following.

Theorem 1.9 *Let X be a topological space such that $\bar{\mathrm{H}}^*X$, the modulo 2 cohomology of X has a trivial action of the Bockstein operator in high degrees. If $\bar{\mathrm{H}}^*X$ is non constant, the first $\mathrm{R}_s\bar{\mathrm{H}}^*X$ which is non zero is of infinite weight.*

So we get in particular that the *strong realization conjecture* holds for the class of spaces such that the Bockstein acts trivially in high degrees.

Theorem 1.10 *Let M be an unstable module such that $\mathrm{R}_s M$ is of finite weight for every s. Suppose moreover that the Bockstein acts trivially on M in high degrees. If M is topologically realizable then M is locally finite.*

One should compare theorem 1.10 with [Ku, theorem 0.1 and theorem 0.3] which was our first motivation to study the importance of Bocksteins in this situation.

To give background for our result, we recall the important example 0.11 of [Ku, p. 326]. Let X be the t^{th} bar filtration of $\mathrm{B}\mathbb{C}\mathrm{P}^{\infty}$. Then X is a nilpotent space such that for $1 \leq s < t$, the module $\mathrm{R}_s\bar{\mathrm{H}}^*X$ is of finite weight s. Our result shows that all cohomology classes which reduce non trivially in $\mathrm{R}_1\bar{\mathrm{H}}^*X$ have a non zero Bockstein.

Assume the unbounded conjecture is true. If the cohomology ring $\bar{\mathrm{H}}^*X$ of a space X is not locally constant, then for some integer s, the reduced module $\mathrm{R}_s\bar{\mathrm{H}}^*X$ has to be of infinite weight. L. Schwartz has provided precise conjectures [Sc3, conjecture 0.2, conjecture 0.3] about the value of s in special cases. Our main theorem says that in the case of the vanishing of Bocksteins in high degrees, *the first* non constant R_i has to be of infinite weight. However, the example of N. Kuhn shows that in general, the value of s can be arbitrary high.

To prove theorem 1.9, we shall use, as in [DG] the theory of profinite spaces to be free of any finiteness hypotheses. The Theorem 1.9 is a consequence of the more general

Theorem 1.11 *Let X be a profinite space such that the modulo 2 cohomology of X has no action of the Bockstein operator in high degrees. If $\bar{\mathrm{H}}^*X$ is non constant, the first $\mathrm{R}_s\bar{\mathrm{H}}^*X$ which is non zero is of infinite weight.*

Theorem 1.11 implies Theorem 1.9 because the cohomology of a space is naturally isomorphic to that of its profinite completion (which is a profinite space) as an unstable algebra [DG, p. 404, section 2.3].

2 Proof of the theorem 1.2

The proof of theorem 1.11 is by contradiction. Suppose there exists a profinite space X such that

(i) the cohomology of X is not locally constant and for the lowest d such that $\mathrm{R}_d\bar{\mathrm{H}}^*X$ is non constant, the module $\mathrm{R}_s\bar{\mathrm{H}}^*X$ is of finite weight,

(ii) the action of the Bockstein is trivial in high degrees in $\bar{\mathrm{H}}^*X$.

We shall then find a contradiction. To this end, we use the same line of proof that was used in [Sc1, Sc2, DG]. Let us recall how it goes. First, we let d be the minimal integer such that $\mathrm{R}_s\bar{\mathrm{H}}^*X$ is non constant.

We necessarily have that $d \geq 1$ [Ku, prop. 0.8 and cor. 0.9]

We can suppose that $\bar{\mathrm{H}}^*X$ is d-nilpotent (see the discussion of [DG, beginning of section 7.2]) and as connected as we want.

It follows from the hypotheses that $\mathrm{R}_d\bar{\mathrm{H}}^*X$ is of finite weight $f > 0$. We shall then perform *Kuhn's reduction* [DG, section 7.1]. That is to say, by using Lannes' theory in the framework of profinite spaces, we can suppose that $f = 1$.

Using this fact, we shall then construct a family $(\alpha_{i,d})_{i\geq\kappa}$ of special classes in $\bar{\mathrm{H}}^*X$ satisfying a certain set of conditions $(\mathcal{H}_d)$.

Then we follow this classes for $0 \leq s \leq d$ in the iterated loop spaces $\Omega^s X$ using the map induced in cohomology by the evaluation map $\Sigma\Omega Z \longrightarrow Z$. We shall show that the induced classes $(\alpha_{i,s})_{i\geq\kappa}$ in $\Omega^s X$ satisfy a similar set of conditions $(\mathcal{H}_s)$.

We show that for $s = 1$, the set of conditions $(\mathcal{H}_1)$ implies that the cup square of $\alpha_{i,1}$ is trivial for $i \geq \kappa$, for some integer κ.

The final step is to show that we also have that the cup square of $\alpha_{i,0}$ is trivial for $i \geq \kappa$. A carefull reading of [DG] should convince the reader that the last step of our proof is in some sense already contained in [DG]. However, [DG] was not written in a way to make evident this assertion. Furthermore, we feel that we have been a little sketchy at some important points [DG, p. 425]. So we offer here to give some of the details lacking in [DG].

The contradiction follows from the fact that the set of condition $(\mathcal{H}_0)$ says in particular that the cup square of $\alpha_{i,0}$ is non trivial for $i \geq \kappa$, thus giving a contradiction.

2.1 Kuhn's reduction with trivial Bocksteins

Let Y be a profinite space. Let RY be the Bousfield-Kan functorial fibrant replacement of Y ([Mo], see also [DG, section 2.4]). We denote by ΔY the homotopy cofiber (in the homotopical algebra of profinite spaces) of the natural map

$$Y \longrightarrow \mathrm{Map}(B(\mathbb{Z}/2\mathbb{Z}), RY) \quad .$$

Let $f \geq 1$ be the weight of $\mathrm{R}_d\bar{\mathrm{H}}^*X$. We consider the space $\Delta^{f-1}X$.

Lemma 2.1 *The space $\Delta^{f-1}X$ satisfies*

(i) *the unstable module $\mathrm{R}_d\bar{\mathrm{H}}^*\Delta^{f-1}X$ is of weight 1,*

(ii) *the action of the Bockstein is trivial in high degrees in $\bar{\mathrm{H}}^*\Delta^{f-1}X$.*

Proof. It follows from [DG, section 5] that

$$\bar{\mathrm{T}}\bar{\mathrm{H}}^*X \cong \bar{\mathrm{H}}^*\Delta X$$

as unstable modules.

As the nilpotent filtration commutes with $\bar{\mathrm{T}}$, it follows that for all s and t

$$\bar{\mathrm{T}}^s\mathrm{R}_t\bar{\mathrm{H}}^*X \cong \mathrm{R}_t\bar{\mathrm{T}}^s\bar{\mathrm{H}}^*X \quad .$$

On the other hand, we know that M is of weight k if and only if

$$\bar{\mathrm{T}}^{k+1}M = 0 \text{ and } \bar{\mathrm{T}}^kM \neq 0 \quad .$$

We only need to prove that the action of the Bockstein is also trivial in $\bar{\mathrm{T}}^{f-1}\bar{\mathrm{H}}^*X \cong \bar{\mathrm{H}}^*\Delta^{f-1}X$. But this is a consequence of proposition A.2. □

2.2 Technicalities

We need the following lemma.

Lemma 2.2 *For $1 \leq \ell \leq d$, the module $\mathrm{R}_\ell\bar{\mathrm{H}}^*X_\ell$ is of weight one.*

Proof of lemma 2.2. If $d = 1$ the lemma is clearly true from the hypotheses, otherwise we prove lemma 2.2 by induction on :

Lemma 2.3 *Let Y be a profinite space such that $\bar{\mathrm{H}}^*Y$ is h-nilpotent, $h \geq 2$. Then $\mathrm{R}_{h-1}\bar{\mathrm{H}}^*\Omega Y$ and R_hY have the same weight.*

Proof of lemma 2.3. We use the Eilenberg-Moore spectral sequence which calculates $\bar{\mathrm{H}}^*\Omega Y$ from $\bar{\mathrm{H}}^*Y$. Its $E_2^{s,*}$-term is a subquotient of $\bar{\mathrm{H}}^*Y^{\otimes s}$, which is sh-nilpotent. Because the subcategory of t-nilpotent modules is a Serre subcategory, it happens that $E_\infty^{-s,*}$ is also sh-nilpotent.

Let $\mathrm{F}_s\bar{\mathrm{H}}^*\Omega Y$ be the Eilenberg-Moore filtration, whose associated graded is the abutment of the Eilenberg-Moore spectral sequence. We have

$$E_\infty^{s,*} = \Sigma^s(\mathrm{F}_s/\mathrm{F}_{s-1})\bar{\mathrm{H}}^*\Omega Y \quad ,$$

as unstable modules, hence $(\mathrm{F}_s/\mathrm{F}_{s-1})\bar{\mathrm{H}}^*\Omega Y$ is $(hs - s)$-nilpotent.

Because the Eilenberg-Moore filtration is convergent, we have that $\bar{\mathrm{H}}^*\Omega Y/\mathrm{F}_s\bar{\mathrm{H}}^*\Omega Y$ is at least $(hs - s)$-nilpotent.

From the short exact sequence

$$\mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega Y \longrightarrow \bar{\mathrm{H}}^*\Omega Y \longrightarrow \bar{\mathrm{H}}^*\Omega Y/\mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega Y$$

and from [DG, Corollaire A.3], we have that $\mathrm{R}_{h-1}\bar{\mathrm{H}}^*\Omega Y$ and $\mathrm{R}_{h-1}\mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega Y$ are of the same weight.

We know that

$$\mathrm{R}_{h-1}\mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega Y \cong \mathrm{R}_h\Sigma\mathrm{F}_{-1}\bar{\mathrm{H}}^*\Omega Y/\mathrm{F}_0\bar{\mathrm{H}}^*\Omega Y = \mathrm{R}_h E_\infty^{-1,*}$$

and so we need to compare $\mathrm{R}_h E_\infty^{-1,*}$ and $\mathrm{R}_h\bar{\mathrm{H}}^*Y$.

But $E_\infty^{-1,*}$ is isomorphic to the quotient of $\bar{\mathrm{H}}^*Y$ by B, the union of the images of the differentials. The image of the differential d^r is easily seen to be at least $((r+1)(h-1)+2)$-nilpotent (see [DG]). Hence, the union of the image of the differentials is at least $(2h-1)$-nilpotent. We have a short exact sequence:

$$B \longrightarrow \bar{\mathrm{H}}^*Y \longrightarrow E_\infty^{-1,*} \quad .$$

A new application of [DG, Corollaire A.3] gives that $\mathrm{R}_h E_\infty^{-1,*}$ and $\mathrm{R}_h\bar{\mathrm{H}}^*Y$ are of the same weight and the lemma follows. □

Lemma 2.4 *The module* $\mathrm{R}_0\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0$ *is of weight* 1. *The module* $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$ *is of weight* 2.

Proof of lemma 2.4. We have an isomorphism

$$\mathrm{R}_0\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0 \cong \mathrm{R}_1\Sigma(\mathrm{F}_{-1}/\mathrm{F}_0)\bar{\mathrm{H}}^*X_0 \cong \mathrm{R}_1 E_\infty^{-1,*} \quad .$$

The module $E_\infty^{-1,*}$ is a quotient of $\bar{\mathrm{H}}^*X_1$ by an at least 2-nilpotent submodule B.

So we have an exact sequence

$$B \longrightarrow \bar{\mathrm{H}}^*X_1 \longrightarrow E_\infty^{-1,*} \quad .$$

By the lemma 2.2, the module $\mathrm{R}_1\bar{\mathrm{H}}^*X_1$ is of wght 1 which proves the first assertion.

The module $\mathrm{R}_0(\mathrm{F}_{-2}/\mathrm{F}_{-1})\bar{\mathrm{H}}^*X_0$ is isomorphic to $\mathrm{R}_2\Sigma^2(\mathrm{F}_{-2}/\mathrm{F}_{-1})\bar{\mathrm{H}}^*X_0 = \mathrm{R}_2E_\infty^{-2,*}$. The module $E_\infty^{-2,*}$ is a subquotient of $(\bar{\mathrm{H}}^*X_1)^{\otimes 2}$. So we have modules $B \subset C \subset (\bar{\mathrm{H}}^*X_1)^{\otimes 2}$ such that $C/B = E_\infty^{-2,*}$. The module B is the union of all the images of the differentials and C is the submodule of infinite cycles. One estimates that B is at least 3-nilpotent. Hence by [DG, corollaire A.2] implies that $\mathrm{R}_2E_\infty^{-2,*}$ is isomorphic to R_2C. On the other hand the functor R_2 preserves monomorphisms [DG, proposition A.1] and so $\mathrm{R}_2E_\infty^{-2,*}$ is isomorphic to some submodule of $\mathrm{R}_2((\bar{\mathrm{H}}^*X_1)^{\otimes 2}$. We note at last that

$$\mathrm{R}_2(\bar{\mathrm{H}}^*X_1)^{\otimes 2} = \oplus_{i+j=2}\mathrm{R}_i(\bar{\mathrm{H}}^*X_1)\otimes\mathrm{R}_j(\bar{\mathrm{H}}^*X_1) = \mathrm{R}_1(\bar{\mathrm{H}}^*X_1)\otimes\mathrm{R}_1(\bar{\mathrm{H}}^*X_1)$$

As $\mathrm{R}_1(\bar{\mathrm{H}}^*X_1)$ is of weight one, the module $\mathrm{R}_2(\bar{\mathrm{H}}^*X_1)^{\otimes 2}$ is of weight 2, and so are $\mathrm{R}_2E_\infty^{-2,*}$ and $\mathrm{R}_0\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0$.

From the short exact sequence

$$\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0 \longrightarrow \mathrm{F}_{-2} \longrightarrow \mathrm{F}_{-2}/\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0\Sigma^2E_\infty^{-2,*}$$

[DG, Corollaire A.3], and the preceding remarks, we find that $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$ is of weight 2. □

2.3 Construction of classes

This lemma is a special case of [DG, proposition 7.2].

Lemma 2.5 *Let* M *be a reduced module of weight* 1. *Let* η *be the unity of the adjunction* $M \to \bar{\mathrm{T}}M\otimes\bar{\mathrm{H}}^*\mathrm{B}(\mathbb{Z}/2\mathbb{Z})$. *Then* η *factorizes by the submodule* $\bar{\mathrm{T}}M\otimes\mathrm{F}(1)$. *Moreover, the kernel and cokernel of*

$$\eta : M \to \bar{\mathrm{T}}M\otimes\mathrm{F}(1)$$

are locally finite.

We apply this lemma to $M = \mathrm{R}_d\bar{\mathrm{H}}^*X$, which we can suppose to be of weight 1 by proposition 2.1. Then it follows that there is a cyclic submodule of the form $\mathrm{F}(1)^{\geq 2^\xi}$ in M, generated by some $\bar{\alpha}_\xi$ of degree 2^ξ. We can suppose ξ as big as we want. So we pick up some $\kappa \geq \xi$.

We lift up $\Sigma^s\bar{\alpha}_\kappa$ to a class $\alpha_{\kappa,d}$ of degree $2^\kappa + d$ through the epimorphism $(\bar{\mathrm{H}}^*X)_s \longrightarrow \Sigma^s\mathrm{R}_s(\bar{\mathrm{H}}^*X)$, and we define recursively, for $i \geq \kappa$

$$\alpha_{i+1,d} = \mathrm{Sq}^{2^i}\alpha_{i,d} \quad .$$

We get some classes $(\alpha_{i,d})_{i\geq\kappa}$ satisfying the following set of conditions:

$$(\mathcal{H}_d)\begin{cases} \text{the class } \alpha_{i,d} \text{ is defined for } i \geq \kappa \text{ and is of degree } 2^i + d \text{ in } \bar{\mathrm{H}}^*X \ , \\ \text{the class } \alpha_{i,d} \text{ reduces non trivially in } \mathrm{R}_d(\bar{\mathrm{H}}^*X) \ \ \text{(hence is non zero)}, \\ \text{the Bockstein acts trivially on } \alpha_{i,d} \ , \\ \text{for } i \geq \kappa, \ \text{we have } \mathrm{Sq}^{2^i}\alpha_{i,d} = \alpha_{i+1,d} \ . \end{cases}$$

Now, define for $0 \leq \ell \leq d$,

$$X_\ell = \Omega^{d-\ell}X$$

so that $X_d = X$ and $X_0 = \Omega^d X$.

The evaluation morphism $\Sigma\Omega Y \longrightarrow Y$ induces for any profinite space Y a map $ev_Y : \bar{\mathrm{H}}^*Y \longrightarrow \bar{\mathrm{H}}^*\Sigma\Omega Y \cong \Sigma\bar{\mathrm{H}}^*\Omega Y$. For $0 \leq i \leq d-1$, we define recursively classes $(\alpha_{i,\ell})_{i\geq\kappa}$ in $\bar{\mathrm{H}}^*X_\ell$ by

$$\Sigma\alpha_{i,\ell} = ev_{X_{\ell+1}}\ (\alpha_{i,\ell})$$

We prove by downward induction that

Proposition 2.6 *The classes $(\alpha_{i,\ell})_{i\geq\kappa}$ satisfy, for $0 \leq \ell \leq d$ and $i \geq \kappa$:*

$$(\mathcal{H}_\ell)\begin{cases} \text{the class } \alpha_{i,\ell} \text{ is of degree } 2^i + \ell \text{ in } \bar{\mathrm{H}}^*X \ , \\ \text{the class } \alpha_{i,\ell} \text{ reduces non trivially in } \mathrm{R}_\ell(\bar{\mathrm{H}}^*X) \ , \\ \text{the Bockstein acts trivially on } \alpha_{i,\ell} \ , \\ \text{for } i \geq \kappa, \ \text{we have } \mathrm{Sq}^{2^i}\alpha_{i,\ell} = \alpha_{i+1,\ell} \ . \end{cases}$$

Proof of proposition 2.6. The assertion on the degree of $(\alpha_{i,\ell})_{i\geq\kappa}$ follows from the definitions. The second point is a consequence of the following lemma (see [DG, proposition A.4]).

Lemma 2.7 *Let Y be a profinite space such that $\bar{\mathrm{H}}^*Y$ is ℓ-nilpotent for $\ell \geq 1$. Then $\bar{\mathrm{H}}^*Y$ is $(\ell-1)$-nilpotent and the evaluation morphism induces a* monomorphism

$$\mathrm{R}_d\bar{\mathrm{H}}^*Y \hookrightarrow \mathrm{R}_d\Sigma\bar{\mathrm{H}}^*\Omega Y \cong \mathrm{R}_{d-1}\Omega Y$$

The third and fourth points are consequences of the Steenrod algebra linearity of the evaluation morphism. Namely, it follows from

$$\Sigma(\mathrm{Sq}^1\ \alpha_{i,\ell-1}) = \mathrm{Sq}^1\ \Sigma\alpha_{i,\ell-1} = \mathrm{Sq}^1\ ev_{X_\ell}\ (\alpha_{i,\ell}) = ev_{X_\ell}\ (\mathrm{Sq}^1\ \alpha_{i,\ell}) = 0$$

that the Bockstein acts trivially on $\alpha_{i,\ell}$, and

$$\Sigma(\mathrm{Sq}^{2^i}\ \alpha_{i,\ell-1}) = \mathrm{Sq}^{2^i}\ \Sigma\alpha_{i,\ell} = \mathrm{Sq}^{2^i}\ ev_{X_\ell}\ (\alpha_{i,\ell}) = ev_{X_\ell}\ (\mathrm{Sq}^{2^i}\ \alpha_{i,\ell}) = ev_{X_\ell}\ (\alpha_{i+1,\ell}) = \Sigma\alpha_{i+1,\ell-1}$$

shows how Sq^{2^i} acts on $\alpha_{i,\ell}$.

2.4 The cup-square of $\alpha_{i,1}$ is trivial

For $\ell = 1$, the classes $\alpha_{i,1}$ have degree $2^i + 1$, and the unstable algebra structure gives for $i \geq \kappa$,

$$\alpha_{i,1} \cup \alpha_{i,1} = \mathrm{Sq}^{2^i+1}\ \alpha_{i,1} = \mathrm{Sq}^1\mathrm{Sq}^{2^i}\ \alpha_{i,1} = \mathrm{Sq}^1\ \alpha_{i+1,1} = 0 \quad .$$

So to sum up the situation, we have a profinite space $X_1 = \Omega^{d-1}X$ and classes $(\alpha_{i,1})_{i\geq\kappa}$ such that for $i \geq \kappa$,

(i) the class $\alpha_{i,1}$ is of degree 2^i+1 in $\bar{\mathrm{H}}^*X_1$,

(ii) the class $\alpha_{i,1}$ reduces non trivially in $\mathrm{R}_1(\bar{\mathrm{H}}^*X_1)$,

(iii) the Bockstein acts trivially on $\alpha_{i,1}$,

(iv) we have $\mathrm{Sq}^{2^i}\alpha_{i,1}=\alpha_{i+1,1}$,

(v) the cup square $\alpha_{i,1}\cup\alpha_{i,1}$ is trivial.

Suppose that we are able to prove that the same set of conditions holds for $(\alpha_{i,0})_{i\geq\kappa'}$, then, we have the following contradiction

$$0=\alpha_{i,0}\cup\alpha_{i,0}=\mathrm{Sq}^{2^i}\ \alpha_{i,0}=\alpha_{i+1,0}\neq 0 \quad .$$

So we need to prove that $\alpha_{i,0}\cup\alpha_{i,0}=0$. To this end, we proceed exactly as in [DG, Sc3]. But before, we need some technical results.

Note. We could have define directly the classes $\alpha_{i,1}$. However, it would not have shorten much the proof. So we proceed as in [Sc2, Sc3, DG]

2.5 The cup square $\alpha_{i,0}$ is trivial

We use the Eilenberg-Moore spectral sequence which relates $\bar{\mathrm{H}}^*X_1$ to $\bar{\mathrm{H}}^*X_0=\bar{\mathrm{H}}^*\Omega X_1$.

We know that for $i\geq\kappa$, the cup square $\alpha_{i,1}\cup\alpha_{i,1}$ is trivial. So $\alpha_{i,1}\cup\alpha_{i,1}$ defines an element of $E_2^{-1,*}$. For degree reasons, the higher differentials coming from $E_2^{-1,*}$ are trivial and so, the cup square $\alpha_{i,1}\cup\alpha_{i,1}$ induces a permanent cycle, which never bounds for nilpotence reasons (see [DG, section 7.4]). Let $w_{i,\ell}$ an element of $\bar{\mathrm{H}}^*X_0$ detected by this permanent cycle.

We want to compare $\mathrm{Sq}^{2^i}\ w_{i,0}$ to $\alpha_{i+1,0}\cup\alpha_{i,0}$. In the very same way than the 1-cycle $\alpha_{i,1}\otimes\alpha_{i,1}$ induces an infinite cycle, the 1-cycle $\alpha_{i,1}\otimes\alpha_{i+1,1}+\alpha_{i+1,1}\otimes\alpha_{i,1}$ induces also a permanent non trivial cycle. On the other hand, the shuffle product on the E_2-term of the Eilenberg-Moore spectral sequence converges to the cup product on the E_∞-term, so this cycle detects the cup product $\alpha_{i+1,0}\cup\alpha_{i,0}$. By Cartan's formula,

$$\mathrm{Sq}^{2^i}(\alpha_{i,1}\otimes\alpha_{i,1})=\alpha_{i,1}\otimes\alpha_{i+1,1}+\alpha_{i+1,1}\otimes\alpha_{i,1}$$

is a permanent cycle, and the compatibility of the Eilenberg-Moore spectral sequence with Steenrod operations shows that $\mathrm{Sq}^{2^i}(\alpha_{i,1}\otimes\alpha_{i,1})$ converges to $\mathrm{Sq}^{2^i}\ w_{i,0}$.

The Cartan formula gives

$$\mathrm{Sq}^{2^i}(\alpha_{i,0}\cup\alpha_{i+1,0}) \quad = \quad \textstyle\sum_{p+q=2^i}\mathrm{Sq}^p\ \alpha_{i,0}\cup\mathrm{Sq}^q\ \alpha_{i+1,0}$$

so

$$\begin{aligned}\mathrm{Sq}^{2^i}(\alpha_{i,0}\cup\alpha_{i+1,0}) &= (\mathrm{Sq}^{2^i}\alpha_{i,0})\cup\alpha_{i+1,0}+\textstyle\sum_{p<2^i}\mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}\\ &= \alpha_{i+1,0}\cup\alpha_{i+1,0}+\textstyle\sum_{p<2^i}\mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0} \quad .\end{aligned}$$

Let z be defined by

$$z=\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}w_{i,0}-\alpha_{i+1,0}\cup\alpha_{i+1,0}=\sum_{p<2^i}\mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0} \quad .$$

If $0<p<2^i$, then $\mathrm{Sq}^p\ \alpha_{i,0}$ is in degree ℓ such that $\alpha(\ell)>1$. the element $\alpha_{i,0}$ is in $\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0$ by definition, thus so is $\mathrm{Sq}^p\ \alpha_{i,0}$. But $\mathrm{R}_0\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0$ is of weight one and this implies that $\mathrm{Sq}^p\alpha_{i,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-1}\bar{\mathrm{H}}^*X_0$. In other words, $\mathrm{Sq}^p\alpha_{i,0}$ is nilpotent.

If $p=0$, then $\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ is of weight greater than 2. The same argument shows that if $p=0$, the element $\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ is nilpotent.

So for $p<2^i$, either $\mathrm{Sq}^p\alpha_{i,0}$ or $\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ is nilpotent and so is the cup product $\mathrm{Sq}^p\alpha_{i,0}\cup\mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$.

We conclude that for $p < 2^i$, the element $\mathrm{Sq}^p\alpha_{i,0} \cup \mathrm{Sq}^{2^i-p}\alpha_{i+1,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$. So $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}w_{i,0}$ and $\alpha_{i,0} \cup \alpha_{i,0}$ project to *equal* elements of $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$.

On the other hand, the decomposition of $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}$ [Sc3, lemme 5.7, p. 554] implies that $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}w_{i,0}$ projects to an element of weight greater than three in $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$. But $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$ is of weight 2 by lemma 2.4, so $\mathrm{Sq}^{2^i}\mathrm{Sq}^{2^i}w_{i,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$. Thus $\alpha_{i+1,0} \cup \alpha_{i+1,0}$ reduces to zero in $\mathrm{R}_0\mathrm{F}_{-2}\bar{\mathrm{H}}^*X_0$ for $i \geq \kappa$.

In other words, the element $\alpha_{i,0} \cup \alpha_{i,0}$ is nilpotent for $i \geq \kappa$. Thus for some t

$$\mathrm{Sq}_0^t(\alpha_{i,0} \cup \alpha_{i,0}) = \mathrm{Sq}_0^t\alpha_{i,0} \cup \mathrm{Sq}_0^t\alpha_{i,0} = 0$$

but

$$\mathrm{Sq}_0^t\alpha_{i,0} \cup \mathrm{Sq}_0^t\alpha_{i,0} = \alpha_{i+t,0} \cup \alpha_{i+t,0} = \mathrm{Sq}^{2^{i+t}}\alpha_{i+t,0} = \alpha_{i+t+1,0} \neq 0 \quad .$$

This is a contradiction. □

A Trivial Bockstein actions and Lannes' functor

The material of this section is well-known. It is already used in [Ku, proposition 1.3, p. 328] and first proved by M. Winstead [W]. We thank gratefully J. Lannes who explained us the following proof.

Let M be an unstable module. The notation $M^{\geq n}$ stands for the submodule of M of elements of degree greater than n. We say that the action of the Bockstein is *trivial in degree greater than* n if $\mathrm{Sq}^1\ M^{\geq n} = 0$.

Proposition A.1 *Let M be an unstable module. The action of the Bockstein in M is trivial in degree greater than n if and only if the action of the Bockstein in $\mathrm{T}M$ is trivial in degrees greater than n.*

Because $\bar{\mathrm{T}}M$ is a submodule of $\mathrm{T}M$ we have :

Corollary A.2 *Let M be an unstable module. If the action of the Bockstein in M is trivial in degree greater than n, then the action of the Bockstein in $\bar{\mathrm{T}}M$ is also trivial in degrees greater than n.*

Before proving the proposition A.1, we recall (see [LZ] or [Sc1, p. 27]) the definition of the *double* ΦM of an unstable module M. The module ΦM is the unique unstable module ΦM such that:

(i) the module ΦM is zero in odd degrees,

(ii) for any ℓ, $\Phi M^{2\ell}$ is M^ℓ,

(iii) the natural map $\Phi : \Phi M \longrightarrow M$ which maps m to $\Phi m = \mathrm{Sq}_0\ m$ is linear with respect to the Steenrod algebra.

In other words:

$$\mathrm{Sq}^{2\ell}\ \Phi m = \Phi \mathrm{Sq}^\ell\ m \quad .$$

It is evident from the definition that the action of the Bockstein is trivial on ΦM. Conversely, we have

Lemma A.3 *Let M be an unstable module such that the action of the Bockstein is trivial in each degree. We denote by M^{odd} and M^{even} the odd and even degree parts of M as graded vector spaces. Then M splits as a module over the Steenrod algebra as*

$$M = M^{odd} \oplus M^{even} \quad .$$

proof of lemma A.3. This lemma is the consequence of the following facts:

(i) the Steenrod algebra is generated as an algebra by the squares Sq^i,

(ii) we have for any odd square the Adem relation

$$\mathrm{Sq}^{2n+1} = \mathrm{Sq}^1\mathrm{Sq}^{2n} \quad .$$

When the action of the Bockstein is trivial, it follows that M^{odd} and M^{even} are unstable submodules and that the vector space decomposition $M = M^{odd} \oplus M^{even}$ is in fact a Steenrod algebra module deccomposition. □

Lemma A.4 *Let M be a module such that M is zero in odd degrees. Then M is of the form ΦM_1 for a unique unstable module M_1. Let M be an unstable module such that M is zero in even degrees. Then M is of the form $M = \Sigma\Phi M_2$ for a unique module M_2.*

proof of lemma A.3. Let us prove the first assertion. It follows from the definitions that M_1 has to be defined by $M_1^\ell = M^{2\ell}$. Furthermore, we also have no choice for the Steenrod algebra structure on M_1. It remains only to show that this actually defines an action of the Steenrod algebra, which amounts to the definition of Φ.

To prove the second assertion, we remark that for any module M concentrated in odd degrees, the operator Sq_0 is trivial. But The triviality of this operator is the exactly the obstruction for the algebraically desuspending an unstable module. So M is of the form $M = \Sigma M'$ for a unique M'. Now M' is concentrated in even degree and by the first part, we have that $M' = \Phi M_2$ for a unique M_2. So, we have:

$$M = \Sigma M' = \Sigma\Phi M_2 \quad .$$

□

We return to the proof of proposition A.1.

Proof of proposition A.1. Let M be an unstable module having trivial action of the Bockstein in degrees greater than n.

We have a short exact sequence of unstable modules

$$M^{\geq n} \longrightarrow M \longrightarrow M/M^{\geq n} \quad .$$

By exactness of the T functor, we get an exact sequence:

$$\mathrm{T}M^{\geq n} \longrightarrow \mathrm{T}M \longrightarrow \mathrm{T}M/M^{\geq n} \quad . \tag{1}$$

Lannes' T functor admits a natural splitting as

$$T \cong \bar{\mathrm{T}} \oplus \mathrm{Id}$$

hence the exact sequence 1 splits into two short exact sequences

$$M^{\geq n} \longrightarrow M \longrightarrow M/M^{\geq n} \text{ and } \bar{\mathrm{T}}M^{\geq n} \longrightarrow \bar{\mathrm{T}}M \longrightarrow \bar{\mathrm{T}}M/\bar{\mathrm{T}}M^{\geq n} = \bar{\mathrm{T}}(M/M^{\geq n})$$

Now $M/M^{\geq n}$ is a bounded module, so $\bar{\mathrm{T}}(M/M^{\geq n}) = 0$. On the other hand, $M^{\geq n}$ has trivial action of Bocksteins and so, by lemma A.3,

$$M^{\geq n} = (M^{\geq n})^{even} \oplus (M^{\geq n})^{odd} \quad .$$

Now, lemma A.4 ensures that

$$M^{\geq n} = \Phi M_1 \oplus \Sigma\Phi M_2$$

and so

$$\bar{\mathrm{T}}M^{\geq n} = \bar{\mathrm{T}}(\Phi M_1 \oplus \Sigma\Phi M_2) = (\Phi\bar{\mathrm{T}}M_1 \oplus \Sigma\Phi\bar{\mathrm{T}}M_2)$$

because the functor $\bar{\mathrm{T}}$ commutes to suspensions and to Φ.

It follows that $\bar{\mathrm{T}}M^{\geq n}$ has trivial action of Bocksteins in each degrees. Finally, $\mathrm{T}M = M \oplus \bar{\mathrm{T}}M$ has trivial action of Bocksteins in degrees greater than n.

The converse is a consequence of the aforementioned splitting of the T functor. □

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