RIEMANNIAN MANIFOLDS WHOSE SKEW-SYMMETRIC CURVATURE OPERATOR HAS CONSTANT EIGENVALUES

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Abstract. A Riemannian metric on a manifold is said to be IP if the eigenvalues of the skew-symmetric curvature operator are pointwise constant, i.e. they depend upon the point of the manifold but not upon the particular 2 plane in the tangent bundle at that point. We classify the IP metrics in dimensions $m = 5$, $m = 6$, and $m \geq 9$.

§0 Introduction

Let R be the curvature of a connected Riemannian manifold M^m of dimension m. If X is a unit tangent vector in the tangent space T_PM^m to M^m at a point P, let $J(X) : T_PM^m \to T_PM^m$ be the Jacobi operator; $J(X) : Y \mapsto R(Y, X)X$. If M^m is a local 2-point homogeneous space, then the local isometries of M^m act transitively on the bundle of unit tangent vectors $S(M^m)$ so that the eigenvalues of $J(X)$ are constant as X varies in $S(M^m)$. Osserman conjectured [12] that the converse might hold. Chi [5] showed this to be the case if m is odd, if $m \equiv 2$ mod 4, or if $m = 4$; the case $m = 4k+4$ for $k \geq 1$ remains open in this conjecture. Chi's proof is a lovely blend of algebraic topology and differential geometry and uses in an essential fashion work of Adams [1] concerning vector fields on spheres.

If π is an oriented 2 plane in the tangent bundle, let $R(\pi) := R(X, Y)$ be the skew-symmetric curvature operator; here $\{X, Y\}$ is any oriented orthonormal basis for π and $R(\pi)$ is independent of the particular $\{X, Y\}$ chosen. In this paper, we study when the eigenvalues of $R(\pi)$ are independent of π; we are motivated at least partially by the results cited above for the Osserman conjecture. Let $R_{ijkl} := (R(e_i, e_j)e_k, e_l)$ be the components of the curvature tensor relative to a local orthonormal frame e_i where $(\cdot, \cdot)$ denotes the inner product. Then we have the following relations; these are the curvature symmetries and the first Bianchi identity:

$$(0.1) \qquad R_{ijkl} = -R_{jikl},\ R_{ijkl} = R_{klij},\ \text{and } R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

Definition. *A 4 tensor R defined at a point P of M^m is an algebraic curvature tensor if the identities of equation* (0.1) *hold at P.*

Note that if R is an algebraic curvature tensor, then there exists a metric $\tilde{g}$ extending the metric on T_PM^m so that R is the curvature tensor of $\tilde{g}$ at P.

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Definition. *Let $Gr_2^+(T_PM^m)$ be the Grassmanian of oriented 2 planes in T_PM^m. We say that an algebraic curvature tensor R is IP if the eigenvalues of $R(\pi)$ are constant on $Gr_2^+(T_PM^m)$; let the rank of R be $\dim\text{Range}(R(\pi))$.*

We have chosen this notation as the fundamental papers in this subject are due to Ivanov and Petrova [9]; see also related work in [3, 7, 8].

Definition. *We say a metric g is IP if the associated curvature tensor R is IP at every point; such a metric has rank at most k if $\text{rank}(R) \le k$ everywhere.*

Note that if g is IP, then the eigenvalues of $R(\pi)$ are constant on $Gr_2^+(T_PM^m)$ for each P but can vary with P. If $\text{rank}(R) = 0$, then $R = 0$; if g has rank 0 everywhere, then g is flat. Ivanov and Petrova [9] give the following examples of IP metrics:

Example 0.2.

(1) *Let g be a metric of constant sectional curvature C on a manifold M^m. The curvature tensor is $C\mathcal{R}$ for $\mathcal{R}(X,Y,Z,W) := (X,W)(Y,Z) - (X,Z)(Y,W)$. The eigenvalues of $R(\pi)$ are $\{\pm\sqrt{-1}C(t), 0, ..., 0\}$. Let $\{X,Y,Z\}$ be an orthonormal set. Then*

$$R(X,Y)X = -CY,\ R(X,Y)Y = CX,\ R(X,Y)Z = 0.$$

(2) *Let $M = I \times N$ be a product manifold where I is a subinterval of $\mathbb{R}$ and where ds_N^2 is a metric of constant sectional curvature K on N. Give M^m the metric $ds_M^2 := dt^2 + f(t)ds_N^2$ where $f(t) := \frac{\mathcal{K}t^2+At+B}{2} > 0$. The eigenvalues of $R(\pi)$ are $\{\pm\sqrt{-1}C(t), 0, ..., 0\}$ for $C(t) := \frac{4\mathcal{K}B-A^2}{4f(t)^2}$. If $\{\partial_t, X, Y, Z\}$ is an orthonormal set, then*

$$R(X,Y)X = -C(t)Y,\ R(X,Y)Y = C(t)X,\ R(X,Y)Z = 0,\ R(X,Y)\partial_t = 0$$
$$R(\partial_t,X)X = -C(t)\partial_t,\ R(\partial_t,X)Y = 0,\ R(\partial_t,X)Z = 0,\ R(\partial_t,X)\partial_t = C(t)X.$$

If M^m has constant sectional curvature, the local isometries of M^m act transitively on the set of all 2 planes in the tangent bundle of M^m so the metric is globally IP. In Lemma 3.3, we will show the metrics in (2) are IP; note that if $A^2 - 4B\mathcal{K} \ne 0$, then the metric in (2) does not have constant sectional curvature. We have the following examples of algebraic IP curvature tensors.

Example 0.3. *Let $\mathcal{R}$ be the algebraic curvature tensor associated to a metric of constant sectional curvature +1 given above. If ϕ is an isometry of $\mathbb{R}^m$ with $\phi^2 = Id$ and if $C \ne 0$, set $R_{\phi,C}(X,Y) = C\mathcal{R}(\phi X, \phi Y)$.*

We will show these algebraic curvature tensors are IP in Lemma 2.3. We can give a geometric realization of the algebraic curvature tensor $R_{\phi,C}$ as follows. Decompose $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ into the ±1 eigenvalues of ϕ. Let ds_x^2 and ds_y^2 be the flat metrics on $\mathbb{R}^p$ and $\mathbb{R}^q$. Then R is the curvature tensor of the following metric at the origin; as we shall not need this fact, we omit the details:

$$ds^2 := \{1 + \tfrac{1}{2}C(|y|^2 - |x|^2)\}ds_x^2 + \{1 + \tfrac{1}{2}C(|x|^2 - |y|^2)\}ds_y^2.$$

Note that the curvature tensor R given in Example 0.2 (2) corresponds to $R_{\phi,C}$ where ϕ has eigenvalue -1 with multiplicity 1 and where $C(t)$ is as given above.

The following Theorem is the main result of this paper; it shows that the IP metrics of Example 0.2 and the algebraic IP curvature tensors of Example 0.3 are the only examples if $m=5$, $m=6$, or $m\ge 9$.

Theorem A. *Let $m\ge 5$.*

1. *Let R be an algebraic IP curvature tensor. If $m\ne 7,8$, then* $\operatorname{rank}(R)\le 2$.
2. *An algebraic curvature tensor R is IP with* $\operatorname{rank}(R)=2$ *if and only if $R=R_{\phi,C}$ as in* Example 0.3. *Furthermore, $R_{\phi,C}=R_{\tilde\phi,\tilde C}$ if and only if $C=\tilde C$ and $\phi=\pm\tilde\phi$.*
3. *A metric g is IP of rank 2 everywhere if and only if g is locally isometric to one of the metrics given in* Example 0.2.
4. *If g is an IP metric of rank at most 2, then either g is flat or g has rank 2 everywhere.*

Ivanov and Petrova [9] constructed IP metrics in dimension $m=3$ which are not of the form given Example 0.2 and classified the IP metrics in dimensions 4. We therefore assume $m\ge 5$ henceforth. Gilkey and Petrova [6] have shown that Theorem A (1) holds if $m=8$; thus only the case $m=7$ is still open.

In contrast to the methods of [9] used in dimension 4 which are purely differential geometric, we will use topological methods to prove Theorem A (1). Let $\mathfrak{so}(\nu)$ be the Lie algebra of the orthogonal group; $\mathfrak{so}(\nu)$ is the vector space of skew-symmetric $\nu\times\nu$ real matrices. Let $Gr_2^+(m)$ be the Grassmanian of oriented 2 planes in $\mathbb{R}^m$.

Definition. *We say that $R: Gr_2^+(m)\to\mathfrak{so}(\nu)$ is admissible if R is continuous, if $R(-\pi)=-R(\pi)$, and if* $\dim\ker R(\pi)$ *is constant. If R is admissible, then let* $\operatorname{rank}(R):=\dim\operatorname{Range}(R(\pi))$.

Theorem A (1) will follow from the following result:

Theorem B. *Let $m\ge 5$ and let $R: Gr_2^+(m)\to\mathfrak{so}(\nu)$ be admissible.*

1. *If $\nu=m$ and if $m\ne 7$, 8, then* $\operatorname{rank}(R)\le 2$.
2. *If $\nu<m$ and if $m\ne 8$ then $R(\pi)=0$.*
3. *There exists an admissible $R: Gr_2^+(8)\to\mathfrak{so}(8)$ so that* $\operatorname{rank}(R)=8$.
4. *There exists an admissible $R: Gr_2^+(8)\to\mathfrak{so}(7)$ so that* $\operatorname{rank}(R)=6$.
5. *There exists an admissible $R: Gr_2^+(7)\to\mathfrak{so}(7)$ so that* $\operatorname{rank}(R)=6$.

There are similar investigations in the literature dealing with an analogous problem. Let $L(m,n,k)$ be the dimension of the largest linear subspace V in $\operatorname{Hom}(\mathbb{R}^m,\mathbb{R}^n)$ so that the rank of $0\ne v\in V$ is k. Various authors have tried to bound L for given k where k is relatively large. Adams [1] determines $L(m,m,m)$. Lam and Yiu [10] determine $L(m,m,m-1)$, $L(m,m-1,m-2)$, and $L(m,m,m-2)$. There is an extensive literature on the subject; see the bibliographies in Lam and Yiu [10] and in Meshulam [11] for further references.

Here is a brief guide to the paper. In §1, we will give the proof of Theorem B. If R is admissible, $\dim\ker(R(\pi))$ is constant so $W_0(\pi):=\ker R(\pi)$ and $W_1(\pi):=W_0(\pi)^\perp=\operatorname{Range}(R(\pi))$ define vector bundles over $Gr_2^+(m)$. Since $R(-\pi)=-R(\pi)$, the W_i descend to define vector bundles V_i over the unoriented

Grassmanian $Gr_2(m)$. The map which sends ξ to $\text{span}\{\xi, e_m\}$ defines the canonical embedding of $\mathbb{RP}^{m-2}$ in $Gr_2(m)$; let U_i be the restriction of V_i to $\mathbb{RP}^{m-2}$. The restriction of the non-trivial real line bundle

$$L := Gr_2^+(m) \times \mathbb{R}/(\pi, \lambda) \sim (-\pi, -\lambda)$$

over $Gr_2(m)$ to $\mathbb{RP}^{m-2}$ is the non-trivial real line bundle over $\mathbb{RP}^{m-2}$; thus we may use L to denote both line bundles without fear of confusion. Note that we have $W_0 \oplus W_1 = m \cdot 1$. We also have $R(-\pi) = -R(\pi)$. This equivariance property implies that R descends to induce an isomorphism between V_1 and $V_1 \otimes L$ over $Gr_2(m)$. We have:

$$\begin{aligned} &W_0(\pi) := \ker R \text{ and } W_1 := W_0^\perp = \text{Range}(R) \text{ over } Gr_2^+(m); \\ &V_0 \oplus V_1 = m \cdot 1 \text{ and } V_1 = V_1 \otimes L \text{ over } Gr_2(m); \\ &U_0 \oplus U_1 = m \cdot 1 \text{ and } U_1 = U_1 \otimes L \text{ over } \mathbb{RP}^{m-1}. \end{aligned} \tag{0.4}$$

We will use the Stieffel-Whitney classes and these observations to prove assertion (1) of Theorem B. If $m \geq 10$, we use the cohomology ring of $\mathbb{RP}^{m-2}$; if $m = 5$, if $m = 6$, or if $m = 9$, we use the cohomology ring of $Gr_2(m)$. We will derive assertion (2) similarly. We will use the spin representations to prove the remaining assertions of Theorem B by constructing suitable examples.

In §2, we will prove Theorem A (2). In §3, we will use the second Bianchi identity

$$R_{ijkl;n} + R_{ijln;k} + R_{ijnk;l} = 0 \tag{0.5}$$

to prove Theorem A (3). In §4, we prove Theorem A (4). Theorem A classifies the IP metrics in dimensions $m = 5$, $m = 6$, and $m \geq 9$; it also classifies the IP metrics of rank at most 2 in dimensions $m = 7$ and $m = 8$. The cases $m = 7$ and $m = 8$ are exceptional in Theorem B; we refer to [6] for a proof of Theorem A (1) if $m = 8$; we do not know if Theorem A (1) fails if $m = 7$.

We summarize below for the convenience of the reader some notational conventions we shall use. We do *not* adopt the Einstein convention; we do *not* sum over repeated indices in this paper.

(1) Let $\mathcal{A}_2(m)$ be the set of the algebraic IP curvature tensors of rank 2.
(2) Let $\mathcal{G}_2(m)$ be the set of the IP metrics of rank 2.
(3) Let $\mathcal{R}(X, Y, Z, W) := (X, W)(Y, Z) - (X, Z)(Y, W)$.
(4) Let $\mathcal{I}(m)$ be the set of isometries ϕ of $\mathbb{R}^m$ so ϕ^2 is the identity.
(5) For $\phi \in \mathcal{I}(m)$, let $p(\phi)$ and $q(\phi)$ be the dimensions of the $+1$ and -1 eigenspaces.
(6) Let $W_0(T) := \ker(T)$ and $W_1(T) := \ker(T)^\perp = \text{Range}(T)$ for $T \in \mathfrak{so}(m)$.
(7) Let $\mathfrak{so}_2(m) = \{T \in \mathfrak{so}(m) : \text{rank}(T) = 2\}$.

§1 The proof of Theorem B

If U is a real vector bundle over a topological space X, let $w(U)$ be the total Stiefel-Whitney class of U. We have:

a) $w(U) = 1 + w_1(U) + w_2(U) + \ldots$ for $w_i \in H^i(X;\mathbb{Z}_2)$.
b) $w_i(U) = 0$ for $i > \dim(U)$.
c) $w(U \oplus V) = w(U)w(V)$.
d) If U is a trivial bundle, then $w(U) = 1$.
e) w is natural with respect to restriction.

Let $[U]$ denote the corresponding element in the K theory group $KO(X)$. The work of Adams [1] shows the elements [1] and $[L]$ generate $KO(\mathbb{RP}^{m-2})$ and that $[L] - [1]$ is an element of order $2^{\phi(m-2)}$ where $\phi(m-2)$ is given below.

If $R : Gr_2^+(m) \to \mathfrak{so}(m)$ is admissible, let W_i, V_i, and U_i be as given in equation (0.4). We will show $\dim(U_1) \leq 2$ if $m = 5$, $m = 6$, or $m \geq 9$. Decompose

$$[U_i] = a_i([L] - [1]) + \dim(U_i)[1] \text{ in } KO(\mathbb{RP}^{m-2})$$

where a_i is well defined modulo $2^{\phi(m-2)}$.

1.1 Lemma. *Let $m = 5$, let $m = 6$, or let $m \geq 9$. Choose j so $2^j \leq m - 2 < 2^{j+1}$. Let R be admissible.*

(1) *If $x = w_1(L)$, then $H^*(\mathbb{RP}^{m-2};\mathbb{Z}_2) = \mathbb{Z}_2[x]/x^{m-1}$.*
(2) *We have $\phi(0) = 0$, $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 2$, $\phi(4) = 3$, $\phi(5) = 3$, $\phi(6) = 3$, $\phi(7) = 3$, and $\phi(8k + j) = 4k + \phi(j)$.*
(3) *We have $w(U_i) = (1 + x)^{a_i}$ and $a_0 + a_1 \equiv 0$ mod $2^{\phi(m-2)}$.*
(4) *We have $\dim(U_0) \neq 0$ and $2a_1 \equiv \dim(U_1)$ mod $2^{\phi(m-2)}$.*
(5) *If $2a_1 \equiv \dim(U_1)$ mod 2^{j+2}, then $\text{rank}(R) \leq 2$.*

Proof. Assertion (1) is well known. Assertion (2) follows from the work of Adams [1, Theorem 7.4]. Assertion (3) follows from the fact that $U_0 \oplus U_1 = m \cdot 1$ is trivial. If $\dim(U_0) = 0$, then $m([1] - [L]) = [U_1] - [U_1 \otimes L] = 0$ in $KO(\mathbb{RP}^{m-2})$ since $U_1 = U_1 \otimes L$ by equation (0.4). This implies $2^{\phi(m-2)}$ divides m. The following table is immediate from the definitions which we have given:

Table 1.2

m	5	6	6	8	9	10	11	12	13
$m-2$	3	4	5	6	7	8	9	10	11
$\phi(m-2)$	2	3	3	3	3	4	5	6	6
$j(m)$	1	2	2	2	2	3	3	3	3

It is clear that $2^{\phi(m-2)}$ does not divide m in this range for $m \neq 8$. Since $\phi(m-2)$ grows roughly linearly with slope $\frac{1}{2}$, $2^{\phi(m-2)} > m$ for $m \geq 9$ and hence $2^{\phi(m-2)}$ does not divide m. This shows that $\dim(U_0) > 0$. We complete the proof of assertion (4) by equating coefficients of $([L]-[1]) \bmod 2^{\phi(m-2)}$ in the equation:

$$\begin{aligned}[U_1] =& a_1([L]-1) + \dim(U_1)[1] \\ =& [U_1 \otimes L] = a_1([1]-[L]) + \dim(U_1)[L] \\ =& (\dim(U_1) - a_1)([L]-1) + \dim(U_1)[1].\end{aligned}$$

Assume $R \neq 0$. Decompose $m-2 = 2^{j_1} + ... + 2^{j_r} + \delta$ in a 2-adic expansion where $\delta = 0, 1$ and $j = j_1 > ... > j_r > 0$. Recall that a_i is defined mod $2^{\phi(m-2)}$ and since $2^j \leq m-2 < 2^{j+1}$, we have $2^{\phi(m-2)} \geq 2^{j+1}$. So we can take $\bar{a}_i$ with $0 \leq \bar{a}_i < 2^{j+1}$ so $a_i \equiv \bar{a}_i \bmod 2^{j+1}$; then $w(U_i) = (1+x)^{\bar{a}_i}$. If $2a_1 \equiv \dim(U_1) \bmod 2^{j+2}$, then $\bar{a}_1 \equiv \frac{1}{2}\dim(U_1) \bmod 2^{j+1}$. As $\dim(U_1)$ is even and as $1 \leq \dim(U_0)$, $2 \leq \dim(U_1) \leq 2^{j+1}$. Then

$$1 \leq \bar{a}_1 \leq 2^j,\ \bar{a}_0 = 2^{j+1} - \bar{a}_1, \text{ and } 2^j \leq \bar{a}_0 \leq 2^{j+1} - 1.$$

Choose s maximal with $1 \leq s \leq r$ so that the powers 2^{j_1}, ..., 2^{j_s} appear in the 2-adic expansion of $\bar{a}_0$. Let $\mu := 2^{j_1} + ... + 2^{j_s} \leq m-2$. Then the coefficient of x^μ is non-trivial in $w(U_0) = (1+x)^{\bar{a}_0}$. Thus for dimensional reasons $\mu \leq \dim(U_0)$. Suppose that $s < r$. As $\dim U_1$ is even, δ appears in the 2 adic expansion of $\dim(U_0)$. Then

$$\begin{aligned}\dim U_1 =& m - \dim U_0 \leq (m-2) - \mu - \delta + 2 \\ =& 2^{j_{s+1}} + \cdots + 2^{j_r} + 2 \leq 2 \cdot 2^{j_{s+1}} \text{ and} \\ \bar{a}_0 =& 2^{j+1} - \tfrac{1}{2}\dim(U_1) \geq 2^j + 2^{j-1} + \cdots + 2^{j_{s+1}}\end{aligned}$$

contradicting the maximality of s. Thus $s = r$, $\mu = m - 2 - \delta \leq \dim(U_0)$, and $\dim(U_1) \leq 2 + \delta$. Since $\dim(U_1)$ is even, $\dim(U_1) \leq 2$. □

Let $m \geq 11$. We use Table 1.2 to see that $\phi(m-2) \geq j(m) + 2$; the function ϕ grows linearly and the function j grows logarithmically. Consequently we have $2a_1 \equiv \dim(U_1) \bmod 2^{j+2}$ and we can apply Lemma 1.1 to see $\operatorname{rank}(R) \leq 2$. To complete the proof of Theorem B (1) if $m = 5$, $m = 6$, $m = 9$, or if $m = 10$, we must eliminate the possibility that $2a_1 \equiv \dim(U_1) + 2^{j+1} \bmod 2^{j+2}$.

If $m = 10$, $2^j = 8$, $H^*(\mathbb{RP}^8) = \mathbb{Z}_2[x]/x^9$ and $\phi(m-2) = 4$. There are 3 cases:

(1) $\dim(U_1) = 8$, $\dim(U_0) = 2$, $\bar{a}_1 = 12$. Then $w(U_1) = (1+x)^{12}$ so that $w(U_0) = (1+x)^4$. This contains x^4 and is impossible since $\dim(U_0) = 2$.

(2) $\dim(U_1) = 6$, $\dim(U_0) = 4$, $\bar{a}_1 = 11$. Then $w(U_1) = (1+x)^{11}$. This contains x^8 and is impossible since $\dim(U_1) = 6$.

(3) $\dim(U_1) = 4$, $\dim(U_0) = 6$, $\bar{a}_1 = 10$. Then $w(U_1) = (1+x)^{10}$. This contains x^8 and is impossible since $\dim(U_1) = 4$.

We will use the cohomology of the Grassmanian $Gr_2(m)$ to study the cases $m = 5$, 6, and 9. We decompose $m \cdot 1 = E_2 \oplus E_2^\perp$ where E_2 is the classifying 2 plane bundle over $Gr_2(m)$; $E_2 := \{(\pi, \xi) \in Gr_2(m) \times \mathbb{R}^m : \xi \in \pi\}$. We define $w := w(E_2) = 1 + w_1 + w_2$ and $\bar{w} := w(E_2^\perp) = 1 + \bar{w}_1 + \bar{w}_2 + ...$ We compute that:

Table 1.3

$\bar{w}_1 = w_1$	$\bar{w}_2 = w_1^2 + w_2$
$\bar{w}_3 = w_1^3$	$\bar{w}_4 = w_1^4 + w_1^2 w_2 + w_2^2$
$\bar{w}_5 = w_1^5 + w_1 w_2^2$	$\bar{w}_6 = w_1^6 + w_1^4 w_2 + w_2^3$
$\bar{w}_7 = w_1^7$	$\bar{w}_8 = w_1^8 + w_1^6 w_2 + w_1^4 w_2^2 + w_2^4$

We refer to Borel [4] for the proof of the following result:

1.4 Lemma. *$H^*(Gr_2(m);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/(\bar{w}_i = 0)$ for $m-1 \le i$.*

If $\dim(V_0) = 1$, then $V_0 = 1$ or $V_0 = L$; this is determined by $w(V_0)$ or equivalently by $w(U_0)$. There are 5 remaining cases to rule out to prove Theorem B (1); we use the calculations of Table 1.3.

(1) $m = 5$, $2^j = 2$, $\dim V_1 = 4$, $\dim V_0 = 1$, and $\bar{a}_1 = 0$. Then $w(U_1) = 1$ so $w(U_0) = 1$ and U_0 is the trivial line bundle; thus V_0 is the trivial line bundle. $5 = V_0 \oplus V_1$, so $5 \cdot L = V_0 \otimes L \oplus V_1$. Then we have $(1+w_1)w(V_1) = (1+w_1)^5$ in $H^*(Gr_2(5);\mathbb{Z}_2)$ so $w(V_1) = (1+w_1)^4$. This implies $1 = w(V_1) = (1+w_1)^4$ so w_1^4 belongs to the span of $w_1^4 + w_1^2 w_2 + w_2^2$ in $\mathbb{Z}_2[w_1, w_2]$ which is false.

(2) $m = 6$, $2^j = 4$, $\dim V_1 = 4$, $\dim V_0 = 2$, and $\bar{a}_1 = 6$. Then we have $w(U_1) = (1+x)^6$ so that $w(U_0) = (1+x)^2$. Thus U_0 and hence V_0 are orientable. Note that $\dim Gr_2(6) = 8$. We have $w(V_0) = 1 + w_1^2 + \alpha w_2$ and $w(V_1) = w(V_0)^{-1} = (1 + w_1^2 + \alpha w_2)^7$ so $(w_1^2 + \alpha w_2)^3 = 0$ in $H^6(Gr_2(6);\mathbb{Z}_2)$ since $\dim(V_1) = 4$. Thus $w_1^6 + \alpha(w_1^4 w_2 + w_1^2 w_2^2 + w_2^3)$ belongs to the span of $w_1 \bar{w}_5 = w_1^6 + w_1^2 w_2^2$ and $\bar{w}_6 = w_1^6 + w_1^4 w_2 + w_2^3$ in $\mathbb{Z}_2[w_1, w_2]$; this is false.

(3) $m = 9$, $2^j = 4$, $\dim(V_1) = 8$, $\dim(V_0) = 1$, $\bar{a}_1 = 0$. Then $w(U_1) = 1$ so $w(U_0) = 1$; Thus U_0 and V_0 are trivial. Since $9 \cdot L = V_0 \otimes L \oplus V_1$, we have $(1+w_1)w(V_1) = (1+w_1)^9$ in $H^*(Gr_2(9);\mathbb{Z}_2)$ so $w(V_1) = (1+w_1)^8$. This implies $1 = w(V_0)w(V_1) = (1+w_1)^8$ so $w_1^8 = 1$. This implies w_1^8 belongs to the span of $\bar{w}_8 = w_1^8 + w_1^6 w_2 + w_1^4 w_2^2 + w_2^4$ in $\mathbb{Z}_2[w_1, w_2]$ which is false.

(4) $m = 9$, $2^j = 4$, $\dim(V_1) = 6$, $\dim(V_0) = 3$, $\bar{a}_1 = 7$. Then $w(U_1) = (1+x)^7$ in $\mathbb{Z}_2[x]/x^8$. This contains x^7 and is impossible since $\dim(U_1) = 6$.

(5) $m = 9$, $2^j = 4$, $\dim(V_1) = 4$, $\dim(V_0) = 5$, $\bar{a}_1 = 6$. Then $w(V_1) = (1+x)^6$ in $\mathbb{Z}_2[x]/x^7$. This contains x^6 and is impossible since $\dim(V_1) = 4$.

This completes the proof of Theorem B (1). We now prove Theorem B (2). Let $m \ge 5$ and $m \ne 8$. Let $R : Gr_2^+(m) \to \mathfrak{so}(\nu)$ be admissible and non-trivial for $\nu < m$. By replacing R by $R \oplus 0$ if necessary we may suppose $\nu = m - 1$. Suppose first $m \ne 7$. Apply Theorem B (1) to $R \oplus 0$ to see $\dim U_1 = 2$, where we decompose $[m] = W_0 \oplus W_1$ using $R \oplus 0$ as in (0.4). Thus $\bar{a}_1 = 1$ or $\bar{a}_1 = 1 + 2^j$. If $\bar{a}_1 = 1$, then $\bar{a}_0 = 2^{j+1} - 1$ and $w(U_0) = (1+x)^{\bar{a}_0} = 1 + x + x^2 + ... + x^{m-2}$; write $U_0 = \tilde{U}_0 \oplus [1]$ since the decomposition given in equation (0.4) uses $R \oplus 0$. Then $w(\tilde{U}_0) = w(U_0)$, which implies $\dim \tilde{U}_0 \ge m - 2$. This is false since $\dim \tilde{U}_0 \le \nu - 2 < m - 2$. If $\bar{a}_1 = 1 + 2^j$, then $w(U_1) = (1+x)^{\bar{a}_1}$. This contains x^{2^j} so $2^j \le \dim(U_1) = 2$. Thus $j = 1$ and $m = 5$. Since $m - 2 = 3$, $x^{2^j+1} = x^3$ survives in $w(U_1)$. Thus $3 \le \dim(U_1) = 2$ which is false.

Suppose next $m = 7$. We have 5 cases to eliminate to complete the proof of Theorem B (2); note that $\dim(V_0) + \dim(V_1) = 6$.

(1) $\dim(V_1) = 6$, $\dim(V_0) = 0$. Then $6([L] - [1]) = 0$ in $\widetilde{KO}(\mathbb{RP}^5)$ which is false since $[L] - [1]$ has order 8.

(2) $\dim(V_1) = 4$, $\dim(V_0) = 2$, and $\bar{a}_1 = 2$. Then $w(U_1) = (1 + x)^2$ so that $w(U_0) = (1 + x)^6$ in $\mathbb{Z}_2[x]/x^6$. This contains x^4 and is impossible since $\dim(U_0) = 2$.

(3) $\dim(V_1) = 4$, $\dim(V_0) = 2$, and $\bar{a}(U_1) = 6$. Then $w(U_1) = (1 + x)^6$ so $w(U_0) = (1 + x)^2$ in $\mathbb{Z}_2[x]/x^6$. This shows that $w(V_0) = 1 + w_1^2 + \alpha w_2$ and that $w(V_1) = (1 + w_1^2 + \alpha w_2)^7$. Thus $0 = w_1^6 + \alpha(w_1^2 w_2^4 + w_1^4 w_2 + w_2^3)$ in $H^6(Gr_2(7))$ since $\dim(V_1) = 4$. Thus $w_1^6 + \alpha(w_1^2 w_2^4 + w_1^4 w_2 + w_2^3)$ belongs to the span of $\bar{w}_6 = w_1^6 + w_1^4 w_2 + w_2^3$ in $\mathbb{Z}_2[w_1, w_2]$ which is false.

(4) $\dim(V_1) = 2$, $\dim(V_0) = 4$, and $\bar{a}(U_1) = 1$. Then $w(U_1) = 1 + x$ so we have $w(U_0) = (1 + x)^7$. This contains x^5 and is impossible since $\dim(U_0) = 4$.

(5) $\dim(V_1) = 2$, $\dim(V_0) = 4$, and $\bar{a}(U_1) = 5$. Then $w(U_1) = (1 + x)^5$. This contains x^5 and is impossible since $\dim(U_1) = 2$.

We now prove assertions (3), (4), and (5) of Theorem B. We refer to Atiyah, Bott, and Shapiro [2] for further details concerning Clifford algebras and spinors. Recall that the Clifford algebra $\mathrm{Clif}(m)$ is the universal real unital algebra generated by $\mathbb{R}^m$ subject to the commutation relations $v * w + w * v = -2(v, w)1$. Let $\mathrm{Spin}(m)$ be the subset of $\mathrm{Clif}(m)$ generated by all products of the form $\omega = v_1 * ... * v_k$ where the v_i are unit vectors in $\mathbb{R}^m$ and k is even. This is a smooth submanifold of $\mathrm{Clif}(m)$ which has the structure of a Lie group under Clifford multiplication. If $\{v_1, v_2\}$ is an oriented orthonormal basis for a 2 plane π, let $\sigma(\pi) := v_1 * v_2 \in \mathrm{Spin}(m)$; this is independent of the particular oriented orthonormal basis for π and defines a smooth embedding $\sigma : Gr_2^+(m) \mapsto \mathrm{Spin}(m)$. There is a natural representation $\rho : \mathrm{Spin}(m) \mapsto \mathrm{SO}(m)$ that exhibits $\mathrm{Spin}(m)$ as the universal cover of $\mathrm{SO}(m)$ for $m \geq 3$; $\rho(\sigma(\pi))$ is -1 on π and $+1$ on $\pi^\perp$.

Suppose $m = 8$. The spin representation c gives a representation of the Clifford algebra $\mathrm{Clif}(8)$ on $\mathbb{R}^{16}$. There is a decomposition $\mathbb{R}^{16} = \mathbb{R}^8_+ \oplus \mathbb{R}^8_-$ so that Clifford multiplication by a vector $v \in \mathbb{R}^8$ interchanges the two summands, i.e. $c(v) : \mathbb{R}^8_\pm \to \mathbb{R}^8_\mp$. Clifford multiplication by an element $\omega \in \mathrm{Spin}(m)$ preserves the two summands; thus c restricts to define representations, called the half spin representations, $c_\pm$ of $\mathrm{Spin}(8)$ on $\mathbb{R}^8_\pm$. Let $R(\pi) = c_+(\sigma(\pi))$; R is an admissible map from $Gr_2^+(8)$ to $\mathfrak{so}(8)$ which has constant rank 8 since $R(\pi)^2 = -1$ and $R(-\pi) = -R(\pi)$.

Fix $e_8 \in \mathbb{R}^8_+$. Let $E(\pi) := \mathrm{span}\{e_8, R(\pi)e_8\}$. This is invariant under $R(\pi)$ so we may decompose $R = R_0 \oplus R_1$ where R_0 is the restriction of R to E and R_1 is the restriction of R to $E^\perp$. We have $E^\perp \subset e_8^\perp = \mathbb{R}^7$; the map $\pi \mapsto R_1(\pi)$ is an admissible map from $Gr_2^+(8)$ to $\mathfrak{so}(7)$ which has constant rank 6; the restriction of R_1 to $Gr_2^+(7) \subset Gr_2^+(8)$ defines an admissible map from $Gr_2^+(7)$ to $\mathfrak{so}(7)$ which has rank 6. This completes the proof of Theorem B.

§2 THE PROOF OF THEOREM A (2)

If $T \in \mathfrak{so}(m)$, define $\omega(T) \in \Lambda^2(\mathbb{R}^m)$ by $\omega(T)(\xi,\eta) = (T\xi,\eta)$; ω is an isomorphism from $\mathfrak{so}(m)$ to $\Lambda^2(\mathbb{R}^m)$. Let $(T_1,T_2) := -\frac{1}{2}\operatorname{Tr}(T_1T_2)$ be the *Killing metric* on $\mathfrak{so}(m)$. Let $T_{ij}^e := \mathcal{R}(e_i,e_j) = \omega^{-1}(de_j \wedge de_i)$ where e is an orthonormal basis for $\mathbb{R}^m$. Then $\{T_{ij}^e\}_{i<j}$ is an orthonormal basis for $\mathfrak{so}(m)$. We have

$$T_{ij}^e : e_j \mapsto e_i,\ T_{ij}^e : e_i \mapsto -e_j,\ T_{ij}^e : e_k \mapsto 0 \text{ for } k \neq i, k \neq j.$$

If we expand $T := \Sigma_{i<j} a_{ij}^e T_{ij}^e$, we then have

$$\omega(T) = \Sigma_{i<j} a_{ij}^e de^j \wedge de^i.$$

Definition. *Let $\mathfrak{so}_2(m)$ be the set of elements T which belong to $\mathfrak{so}(m)$ and which have rank 2. Note that $\mathfrak{so}_2(m)$ is not a linear space.*

We shall need the following technical Lemma.

2.1 Lemma.

(1) *We have $\mathfrak{so}_2(m) = \{T \in \mathfrak{so}(m) : T \neq 0 \text{ and } \omega(T)^2 = 0\}$.*

(2) *Let T be a linear isometry from $\mathbb{R}^k$ to $\mathfrak{so}(m)$ with $T(f) \in \mathfrak{so}_2(m)$ for $f \neq 0$. Let $\{f_1,f_2\}$ be an orthonormal set in $\mathbb{R}^k$; choose an orthonormal basis e for $\mathbb{R}^m$ so $T(f_1) = T_{12}^e$. Then $T(f_2) = \Sigma_{2<i}(a_{1i}^e T_{1i}^e + a_{2i}^e T_{2i}^e)$ where we have the relations $a_{1i}^e a_{2j}^e = a_{1j}^e a_{2i}^e$ for $2 < i < j$.*

Proof. Let $\{u_a, v_a\}$ be an orthonormal basis for $W_1(T) := \ker(T)^\perp$ so $Tu_a = \lambda_a v_a$ and $Tv_a = -\lambda_a u_a$ for $\lambda_a > 0$. Then $\omega(T) = \Sigma_{1\le a\le \operatorname{rank}(T)} \lambda_a du^a \wedge dv^a$. Consequently T has rank 2 if and only if $\omega(T)^2 = 0$ which proves assertion (1).

If the hypotheses of assertion (2) are satisfied, expand $T(f_2) = \Sigma_{k<l} a_{kl}^e T_{kl}^e$. Fix $2 < i < j$. Let $\xi = \xi_1 f_1 + \xi_2 f_2$. Then

$$\xi_1^2 + \xi_2^2 = |T(\xi)|^2 = (\xi_1 + a_{12}^e \xi_2)^2 + \xi_2^2 \Sigma_{k<l,(k,l)\neq(1,2)} (a_{kl}^e)^2, \text{ and}$$
$$0 = \tfrac{1}{2}\omega(T(\xi))^2(e_1,e_2,e_i,e_j) = \xi_1\xi_2 a_{ij}^e + \xi_2^2(a_{12}^e a_{ij}^e - a_{1i}^e a_{2j}^e + a_{1j}^e a_{2i}^e).$$

Therefore $a_{12}^e = 0$, $a_{ij}^e = 0$ and $a_{1i}^e a_{2j}^e = a_{1j}^e a_{2i}^e$. $\square$

Let $\mathcal{A}_2(m)$ be the set of algebraic curvature tensors of rank 2. If $R \in \mathcal{A}_2(m)$ and if X is a unit vector, then the map $Y \to R(X,Y)$ defines a 1-1 linear map T from $X^\perp \subset \mathbb{R}^m$ to $\mathfrak{so}(m)$ taking values in $\mathfrak{so}_2(m)$ for $Y \neq 0$. In the following Lemma, we give a normal form for such maps if $m \geq 5$.

2.2 Lemma. *Let $k \geq 4$. If T is a 1-1 linear map from $\mathbb{R}^k$ to $\mathfrak{so}(m)$ so that $T(f) \in \mathfrak{so}_2(m)$ for $f \neq 0$, then there exists a unit vector $\xi \in \mathbb{R}^m$ so that we have $T(f) = \mathcal{R}(\xi, -T(f)\xi)$.*

Proof. We pull-back the Killing form on $\mathfrak{so}(m)$ to define an inner product on $\mathbb{R}^k$ relative to which T is an isometry. Let $\{f_\nu\}$ be an orthonormal basis for $\mathbb{R}^k$. Then $\{T_\nu := T(f_\nu)\}$ is an orthonormal set in $\mathfrak{so}_2(m)$. Suppose there is a unit vector $\xi \in \cap_{\nu=1}^k W_1(T_\nu)$. Since $W_1(T_\nu)$ is the rotational 2 plane of T_ν, $T_\nu = \mathcal{R}(\xi, -T_\nu\xi)$. As T is linear, $T(f) = \mathcal{R}(\xi, -T(f)\xi) \forall f \in \mathbb{R}^k$. Thus we must show $\cap_{\nu=1}^k W_1(T_\nu) \neq 0$.

We first show $W_1(T_1) \cap W_1(T_2) \neq 0$. Choose the orthonormal basis e for $\mathbb{R}^m$ so $T_1 = T^e_{12}$. By Lemma 2.1, $T_2 = \Sigma_{2<i}(a^e_{1i}T^e_{1i} + a^e_{2i}T^e_{2i})$. Since $0 \neq T_2$, by changing the base e, we may suppose that $a^e_{13} \neq 0$ and that $a^e_{1i} = 0$ for $i > 3$. By Lemma 2.1, $a^e_{13}a^e_{2i} = a^e_{1i}a^e_{23}$. Thus $a^e_{1i} = 0$ implies $a^e_{2i} = 0$ for $i > 3$ and $T_2 = a^e_{13}T^e_{13} + a^e_{23}T^e_{23}$. Thus $\dim(W_1(T_1) + W_1(T_2)) \leq 3$ and $W_1(T_1) \cap W_1(T_2) \neq 0$.

Choose the orthonormal basis e for $\mathbb{R}^m$ so $e_1 \in W(T_1) \cap W(T_2)$, $T_1 = T^e_{12}$, and so that $T_2 = T^e_{13}$. Let f be a unit vector with $f \perp f_1$ and $f \perp f_2$. Expand $T(f) = \Sigma_{ij}a^e_{ij,f}T^e_{ij}$. By Lemma 2.1, $a^e_{12,f} = 0$ and $a^e_{ij,f} = 0$ for (i,j) disjoint from $(1,2)$; similarly $a^e_{13,f} = 0$ and $a^e_{ij,f} = 0$ for (i,j) disjoint from $(1,3)$. Consequently

$$T(f) = a^e_{23,f}T^e_{23} + \Sigma_{i>3}a^e_{1i,f}T^e_{1i}.$$

If $i > 3$, then $a^e_{23,f}a^e_{1i,f} = a^e_{13,f}a^e_{2i,f}$; thus $a^e_{23,f}a^e_{1i,f} = 0$ since $a^e_{2i,f} = 0$. Since T is an isometry, $\Sigma_{i<j}(a^e_{ij,f})^2 = 1$. Therefore $a^e_{23,f} \in \{0, \pm 1\}$. Since $f \mapsto a^e_{23,f}$ is continuous and S^{k-3} is connected, $a^e_{23,f}$ is constant; this uses the hypothesis that $k \geq 4$. Since $T(-f) = -T(f)$, $a^e_{23,f} = 0$, $T(f) = \Sigma_{i>3}a^e_{1i,f}T^e_{1i}$, and $e_1 \in W_1(T(f_\nu))$ for any ν. $\square$

We note that Lemma 2.2 can fail if $k = 3$. Let $T(\xi) = \xi_1 T_{23} - \xi_2 T_{13} + \xi_3 T_{12}$; T is an isometry from $\mathbb{R}^3$ to $\mathfrak{so}_2(3)$ with no common non-trivial eigenvector. Thus the restriction that $k \geq 4$ is an essential one. We establish the first equivalence in assertion (2) of Theorem A by proving:

2.3 Lemma. *Let $m \geq 5$. We have $R \in \mathcal{A}_2(m)$ if and only if $R = R_{\phi,C}$ for some $\phi \in \mathcal{I}(m)$ and for some $C \neq 0$.*

Proof. Suppose $R = R_{\phi,C}$. Since ϕ^2 is the identity, $R(X,Y) = \phi^{-1}C\mathcal{R}(X,Y)\phi$ so R has rank 2. Choose an orthonormal basis for $\mathbb{R}^m$ so that $\phi e_a = e_a$ for $a \leq p$ and so that $\phi e_\alpha = -e_\alpha$ for $\alpha > p$. The non-zero curvatures are

$$\begin{aligned}
&R(e_a, e_b)e_b = Ce_a \text{ and } R(e_a, e_b)e_a = -Ce_b \text{ for } a < b \leq p,\\
&R(e_\alpha, e_\beta)e_\beta = Ce_\alpha \text{ and } R(e_\alpha, e_\beta)e_\alpha = -Ce_\beta \text{ for } p < \alpha < \beta,\\
&R(e_a, e_\alpha)e_\alpha = -Ce_a \text{ and } R(e_a, e_\alpha)e_a = Ce_\alpha \text{ for } a \leq p < \alpha.
\end{aligned} \tag{2.4}$$

We note that $R_{ijkl} = 0$ unless $(i,j) = (k,l)$ or $(i,j) = (l,k)$. The curvature symmetries of equation (0.1) are now immediate so R is an algebraic curvature tensor.

Conversely, let $R \in \mathcal{A}_2(m)$. By rescaling R, we may assume the eigenvalues of R^2 are 0 and -1; this means that $|R(\pi)| = 1$. Let $f = \{f_i\}$ be an orthonormal basis for $\mathbb{R}^m$. Let $T(f) := R(f_1, f)$ for $f \perp f_1$. Then $|T(f)| = |R(f_1, f)| = |f|$ so T is a linear isometry. We apply Lemma 2.2 to choose a unit vector $\xi \in \mathbb{R}^m$ so that $T(f) = \mathcal{R}(-T(f)\xi)$ for all $f \in f_1^\perp$. Let $e_1 = \xi$. For $i > 1$, let $T(f_i)e_1 = e_i$; $e_1 \perp e_i$. Since T is an isometry, $e := \{e_1, ..., e_m\}$ is an orthonormal basis for $\mathbb{R}^m$. With this normalization, we have

$$R(f_1, f_i) = T(f_i) = -T^e_{1i} \text{ for } i > 1.$$

Expand $R(f_\nu, f_\mu) = \Sigma_{i<j}a^e_{ij,\nu\mu}T^e_{ij}$ for $2 \leq \nu < \mu$. Let $T(\xi) := R(f_\nu, \xi_1 f_1 + \xi_2 f_\mu)$ for $\xi \in \operatorname{span}\{f_1, f_\mu\}$. Since $T(f_1) = -T^e_{1\nu}$, we apply Lemma 2.1 to see $a^e_{1\nu,\nu\mu} = 0$,

$a^e_{ij,\nu\mu} = 0$ for i and j distinct from 1 and ν, and $a^e_{1i,\nu\mu}a^e_{\nu\mu,\nu\mu} = a^e_{1\mu,\nu\mu}a^e_{\nu i,\nu\mu}$. Similarly, $a^e_{1\mu,\nu\mu} = 0$ and $a^e_{ij,\nu\mu} = 0$ for i and j distinct from 1 and μ. We may now expand

$$R(f_\nu, f_\mu) = a^e_{\nu\mu,\nu\mu}T^e_{\nu\mu} + \Sigma_{1<j\neq\nu\mu}a^e_{1j,\nu\mu}T^e_{1j}.$$

Since $a^e_{1i,\nu\mu}a^e_{\nu\mu,\nu\mu} = a^e_{1\mu,\nu\mu}a^e_{\nu i,\nu\mu}$ and since $a^e_{1\mu,\nu\mu} = 0$, we have $a^e_{\nu\mu,\nu\mu}a^e_{1i,\nu\mu} = 0$ for i distinct from $1, \nu, \mu$. Since $|R(f_\nu, f_\mu)|^2 = 1$, the sum of the squares of the coefficients is 1. This implies $a^e_{\nu\mu,\nu\mu} \in \{0, \pm 1\}$.

Let $\mathcal{S}(f_1)$ be the Stieffel variety of orthonormal 2 frames $\{\xi, \eta\}$ for $f_1^\perp$. Let $a(\xi, \eta) := -(R(f_1, \xi)R(f_1, \eta), R(\xi, \eta))$. Then

$$\begin{aligned} a(f_\nu, f_\mu) =& -(R(f_1, f_\nu)R(f_1, f_\mu), R(f_\nu, f_\mu)) = -(T^e_{1\nu}T^e_{1\mu}, R(f_\nu, f_\mu)) \\ =&(T^e_{\nu\mu}, R(f_\nu, f_\mu)) = a^e_{\nu\mu,\nu\mu}. \end{aligned}$$

Thus the argument given above shows a takes values in $\{0, \pm 1\}$. Since a is continuous and since $\mathcal{S}$ is connected, a is constant. This yields 3 cases:

(1) We have $a^e_{\nu\mu,\nu\mu} = 1$ for all $1 < \nu < \mu$. Then $R(f_i, f_j) = T^e_{ij}$ for all i, j.
(2) We have $a^e_{\nu\mu,\nu\mu} = -1$ for all $1 < \nu < \mu$. Set $\tilde{e}_1 = -e_1$ and set $\tilde{e}_i = e_i$ for $i > 1$. Then $R(f_i, f_j) = -T^{\tilde{e}}_{ij}$ for all i and j.
(3) We have $a^e_{\nu\mu,\nu\mu} = 0$ for all $1 < \nu < \mu$. Then $R(f_\nu, f_\mu) = \Sigma_{i>1,i\neq\nu\mu}a^e_{i,\nu\mu}T^e_{1i}$. Consequently $e_1 \in W_1(R(\pi))$ for all π. Let $\{e_1, f_2, f_3\}$ be an orthonormal set and let $e_4 := R(f_2, f_3)e_1$. Since $(R(f_2, f_3)e_1, e_4) = 1$, the curvature symmetries given in equation (0.1) imply $(R(e_1, e_4)f_2, f_3) = 1$. Thus $f_2 \in W_1(R(e_1, e_4))$ and $f_3 \in W_1(R(e_1, e_4))$ so $e_1 \notin W_1(R(e_1, e_4))$; this contradiction eliminates this case from consideration.

Let $\phi(f_i) := e_i$ define an isometry of $\mathbb{R}^m$. Then $R(X, Y) = \pm\mathcal{R}(\phi X, \phi Y)$. We complete the proof by showing ϕ^2 is the identity. If $|\xi| = 1$, let

$$\mathcal{P}(\xi) := \cap_{|\eta|=1,\eta\perp\xi}W_1(R(\xi, \eta)).$$

By Lemma 2.2, $\dim(\mathcal{P}(\xi)) = 1$. Furthermore $\phi(\xi) \in \mathcal{P}(\xi)$. Note that $\{\phi(\xi), \phi(\eta)\}$ is an orthonormal basis for $W_1(R(\xi, \eta))$. The curvature symmetry

$$(R(\xi, \eta)\phi(\xi), \phi(\eta)) = (R(\phi(\xi), \phi(\eta))\xi, \eta)$$

shows that $\{\xi, \eta\}$ is an orthonormal basis for $W_1(R(\phi(\xi), \phi(\eta)))$ so $\xi \in \mathcal{P}(\phi(\xi))$. Consequently $\xi = \delta\phi(\phi(\xi))$ for $\delta = \pm 1$. To complete the proof, we must show $\delta = 1$. We assume $\delta = -1$ and argue for a contradiction. If $\delta = -1$, then ϕ defines a complex structure on $\mathbb{R}^m$. Choose an orthonormal basis for $\mathbb{R}^m$ so $\phi(\alpha_a) = \beta_a$ and so $\phi(\beta_a) = -\alpha_a$. As R satisfies the Bianchi identities given in equation (0.1),

$$\begin{aligned} 0 =&R(\alpha_a, \beta_b)\alpha_b + R(\beta_b, \alpha_b)\alpha_a + R(\alpha_b, \alpha_a)\beta_b \\ =&\pm\{-\mathcal{R}(\beta_a, \alpha_b)\alpha_b - \mathcal{R}(\alpha_b, \beta_b)\alpha_a + \mathcal{R}(\beta_b, \beta_a)\beta_b\} \\ =&\pm\{-\beta_a + 0 - \beta_a\} \neq 0. \quad \square \end{aligned}$$

The second equivalence in assertion (2) of Theorem A will follow from the following Lemma:

2.5 Lemma. *Let $m \geq 5$.*

(1) *We have $R_{\phi,C} = R_{\tilde{\phi},\tilde{C}}$ if and only if $C = \tilde{C}$ and $\phi = \pm\tilde{\phi}$.*

(2) *Let $R_i \in \mathcal{A}_2$. Then R_1 and R_2 are orthogonally equivalent if and only if $C_1 = C_2$ and $p(\phi_1) = p(\phi_2)$ or $p(\phi_1) = q(\phi_2)$.*

(3) *We have $\mathcal{A}_2(m)$ is a smooth submanifold of $\otimes^4\mathbb{R}^m$.*

(4) *Let R be a smooth map from a simply connected manifold to $\mathcal{A}_2(m)$. Then there exists (ϕ, C) smooth so $R = R_{\phi,C}$.*

Proof. If $R \in \mathcal{A}_2(m)$, let $V_1^R = \text{Range}(R(\pi))$ be the 2 plane bundle over $Gr_2(m)$ discussed in the introduction. We define a continuous map α_R from $Gr_2(m)$ to $Gr_2(m)$ by setting $\alpha_R(\pi) = V_1^R(\pi)$. Let $-C^2 \neq 0$ be the non-trivial eigenvalue of $R(\pi)^2$.

If $\{X, Y\}$ is an orthonormal basis for π, then $\{Z, W\}$ is an orthonormal basis for $\alpha_R(\pi)$ if and only if $(R(X,Y)Z,W) = \pm C$. Thus the curvature symmetry $(R(X,Y)Z,W) = (R(Z,W)X,Y)$ shows that $V_1^R(V_1^R(\pi)) = \pi$ so α_R^2 is the identity. We use equation (2.4) to see that the fixed point set of α_R is the disjoint union of $Gr_2(p)$, $Gr_2(q)$, and $\mathbb{RP}^{p-1} \times \mathbb{RP}^{q-1}$. We use these two Grassmanians to decompose $\mathbb{R}^m$ into complementary orthogonal subspaces of dimensions p and q. If $R = R_{\phi,C}$, these subspaces are the ± 1 eigenspaces of ϕ. Thus R determines ϕ up to sign.

Let ρ_R be the Ricci tensor of R; ρ_R is diagonal with respect to any basis which diagonalizes ϕ. By equation (2.4), $\rho_R(e_a, e_a) = (p - q - 1)C$ for $a \leq p$ and $\rho_R(e_\alpha, e_\alpha) = (q - p - 1)C$ for $p < \alpha$. If $p = q$ or $p = 0$ or $q = 0$, then $-C$ is the only eigenvalue of ρ_R. Otherwise, there are two distinct eigenvalues; the eigenvalue with the greater multiplicity is $(|q - p| - 1)C$ and the eigenvalue with the lesser multiplicity is $(-|q - p| - 1)C$. Thus R also determines C; this proves assertion (1). Two isometries $\phi_i \in \mathcal{I}(m)$ are orthogonally equivalent if and only if $p(\phi_1) = p(\phi_2)$; assertion (2) now follows.

We normalize the choice of ϕ so $p(\phi) \leq q(\phi)$; distinct values of p define different components $\mathcal{A}_2^p(m)$ of $\mathcal{A}_2(m)$ so we may fix p in proving assertion (3). Let

$$\mathcal{I}_p(m) := \{\phi \in \mathcal{I}(m) : p(\phi) = p\}.$$

The orthogonal group acts transitively on $\mathcal{I}_p(m)$ by conjugation so $\mathcal{I}_p(m)$ is a smooth homogeneous manifold. The parameter C ranges over $\mathbb{R}^* := \mathbb{R} - 0$. If $p < q$, then $\mathcal{A}_2^p(m) = \mathcal{I}_p(m) \times \mathbb{R}^*$. If $p = q$, the map $\phi \mapsto -\phi$ defines a fixed point free action of $\mathbb{Z}_2$ on $\mathcal{I}_p(m)$ and $\mathcal{A}_2^p(m) = (\mathcal{I}_p(m)/\mathbb{Z}_2) \times \mathbb{R}^*$. Assertion (3) now follows.

Let $R(P)$ satisfy the hypothesis of assertion (4). The function $C(P)$ is uniquely defined and is smooth. If $p < q$, we can define ϕ uniquely by requiring that $p(\phi(P)) < q(\phi(P))$. If $p = q$, we must define a lifting from $\mathcal{A}_2^p(m)$ to the double cover $\mathcal{I}_p(m) \times \mathbb{R}^*$; this is possible as the domain of R is simply connected. $\square$

§3 The proof of Theorem A (3)

In this section, we adapt arguments of Ivanov and Petrova [9]. Let $g \in \mathcal{A}_2(m)$ for $m \geq 5$ be an IP metric of rank 2. Let R be the curvature tensor of g. We are interested in local questions so we may assume M^m is a ball in $\mathbb{R}^m$ and hence simply connected. Thus by Theorem A (2) and Lemma 2.5, $R = R_{\phi,C}$ for smooth ϕ and C. Let indices i, j etc. range from 1 through m, let indices a, b, etc. range from 1 through p, and let indices α, β range from $p+1$ through m. Choose a local frame e diagonalizing ϕ; this means $\phi e_a = e_a$ and $\phi e_\alpha = -e_\alpha$. Let ϕ_{ij}, $\phi_{ij;k}$, R_{ijkl}, and $R_{ijkl;n}$ be the components of ϕ, $\nabla\phi$, R, and ∇R. Let $\mathcal{F}_\pm$ be the distributions defined by the ± 1 eigenspaces of ϕ; the e_a span $\mathcal{F}_+$ and the e_α span $\mathcal{F}_-$. We begin with a technical Lemma:

3.1 Lemma. *Let $m \geq 5$, let $g \in \mathcal{G}_2(m)$, and let $R = R_{\phi,C}$.*

(1) $R_{ijkl;n} = C_{;n}(\phi_{il}\phi_{jk} - \phi_{ik}\phi_{jl}) + C(\phi_{il;n}\phi_{jk} + \phi_{il}\phi_{jk;n} - \phi_{ik;n}\phi_{jl} - \phi_{ik}\phi_{jl;n})$.

(2) *We have $\phi_{ij;k} = \phi_{ji;k}$ for any i, j, and k.*

(3) *We have $\phi_{ab;i} = 0$ and $\phi_{\alpha\beta;i} = 0$ for any a, b, α, β, and k.*

(4) *If i, j, and k are distinct, then $\phi_{ij;k} = \phi_{ik;j}$.*

(5) *If $a \neq b$, then $\phi_{a\alpha;b} = 0$; if $\alpha \neq \beta$, then $\phi_{a\alpha;\beta} = 0$.*

(6) *The Christoffel symbols $\Gamma_{ia\alpha} = -\frac{1}{2}\phi_{a\alpha;i}$.*

(7) *The distributions $\mathcal{F}_\pm$ are integrable.*

(8) *If there exists $\alpha \neq \beta$, then $C_{;a} = -C\phi_{\beta a;\beta} - C\phi_{\alpha a;\alpha}$ and $C_{;\beta} = C\phi_{a\beta;a}$.*

(9) *If there exists $a \neq b$, then $C_{;\alpha} = C\phi_{\alpha a;a} + C\phi_{\alpha b;b}$, and $C_{;b} = -C\phi_{\alpha b;\alpha}$.*

(10) *If $q \geq 3$, then $C_{;\alpha} = 0$. If $p \geq 3$, then $C_{;a} = 0$.*

(11) *Either $p \leq 1$ or $q \leq 1$.*

Proof. We covariantly differentiate the identity $R_{ijkl} = C(\phi_{il}\phi_{jk} - \phi_{ik}\phi_{jl})$ to establish assertion (1). Since $\phi \in \mathcal{I}(m)$, $\phi_{ij} = \phi_{ji}$ and assertion (2) follows.

We covariantly differentiate the relation $\delta_{ij} = \Sigma_l \phi_{il}\phi_{lj}$ to establish the identity $0 = \Sigma_l\{\phi_{il;k}\phi_{lj} + \phi_{il}\phi_{lj;k}\}$. By our hypotheses, $\phi_{ab} = \delta_{ab}$, $\phi_{\alpha\beta} = -\delta_{\alpha\beta}$, and $\phi_{a\alpha} = 0$. Thus we may set $i = a$ and $j = b$ in the previous identity to see $0 = 2\phi_{ab;k}$. Similarly we may set $i = \alpha$ and $j = \beta$ to see that $0 = -2\phi_{\alpha\beta;k}$. This proves assertion (3). (If $i = a$ and if $j = \beta$, then the two terms cancel and we gain no new information).

Let i, j, and k be distinct. Since $m \geq 5$, we may choose l distinct from i, j, and k. By assertion (1) $R_{illj;k} = C\phi_{ij;k}\phi_{ll}$, $R_{iljk;l} = 0$, and $R_{ilkl;j} = -C\phi_{ik;j}\phi_{ll}$. By the second Bianchi identity of equation (0.5), $C(\phi_{ij;k} - \phi_{ik;j})\phi_{ll} = 0$; assertion (4) follows as $C \neq 0$ and $\phi_{ll} = \pm 1$. If $a \neq b$, then $\phi_{a\alpha;b} = \phi_{ab;\alpha} = 0$ by assertions (3) and (4). Similarly, if $\alpha \neq \beta$, then $\phi_{a\alpha;\beta} = \phi_{\alpha\beta;a} = 0$. This proves assertion (5).

Let $\Pi_\pm := \frac{1}{2}(1 \pm \phi)$ be orthogonal projection on $\mathcal{F}_\pm$; $e_\alpha = \Pi_- e_\alpha$ and $e_a = \Pi_+ e_a$. We prove assertion (6) by computing:

$$\begin{aligned}\Gamma_{ia\alpha} =& (\nabla_{e_i} e_a, e_\alpha) = (\nabla_{e_i} e_a, \Pi_- e_\alpha) = (\Pi_- \nabla_{e_i} e_a, e_\alpha)\\ =& (\Pi_- \nabla_{e_i} e_a - \nabla_{e_i}\Pi_- e_a, e_\alpha) = \Pi_{-;a\alpha i} = -\tfrac{1}{2}\phi_{a\alpha;i}.\end{aligned}$$

Since $\Gamma_{ab\alpha} = -\frac{1}{2}\phi_{a\alpha;b} = 0$ for $a \neq b$, $\Pi_-([e_a, e_b]) = \Pi_-(\nabla_{e_a} e_b - \nabla_{e_b} e_a) = 0$. This shows $\mathcal{F}_+$ is integrable; similarly $\mathcal{F}_-$ is integrable and assertion (7) follows.

Let $\alpha \neq \beta$. We use assertion (1) to compute

$$R_{\alpha\beta\beta\alpha;a} = C_{;a},\ R_{\alpha\beta\alpha a;\beta} = C\phi_{\beta a;\beta},\ R_{\alpha\beta a\beta;\alpha} = C\phi_{\alpha a;\alpha},$$
$$R_{\alpha aa\alpha;\beta} = -C_{;\beta},\ R_{\alpha a\alpha\beta;a} = C\phi_{a\beta;a},\ \text{and } R_{\alpha a\beta a;\alpha} = 0.$$

Assertion (8) now follows from the second Bianchi identity and from assertion (1). We replace ϕ by $-\phi$ and interchange the roles of the greek and roman indices to derive assertion (9) from assertion (8). If $q \geq 3$, we may choose α, β, and γ distinct and compute:

$$R_{\gamma\beta\beta\gamma;\alpha} = C_{;\alpha},\ \text{and } R_{\gamma\beta\gamma\alpha;\beta} = R_{\gamma\beta\alpha\beta;\gamma} = 0.$$

The second Bianchi identity now implies $C_{;\alpha} = 0$; similarly $C_{;a} = 0$ if $p \geq 3$. If $p \geq 2$ and $q \geq 2$, we use assertions (2) through (9) to show $\nabla C = 0$, $\nabla\phi = 0$, and $\Gamma_{ia\alpha} = 0$. Thus the distribution $\mathcal{F}_+$ is parallel and $0 = (R(e_\alpha, e_a)e_a, e_\alpha) = -C$; this is false. □

By replacing ϕ by $-\phi$ if necessary, we may suppose that $p(\phi) \leq q(\phi)$. Thus $p(\phi) = 0$ or $p(\phi) = 1$. If $p = 0$, then M^m has constant sectional curvature C. We therefore suppose $p(\phi) = 1$. Let $y = (y^1, ..., y^{m-1})$ be local coordinates on a leaf of the foliation $\mathcal{F}_-$. Let $T(t, y) := \exp_y(te_1(y))$ define local coordinates on M^m.

3.2 Lemma. *Let $m \geq 5$, let $g \in \mathcal{G}_2(m)$, let $R = R_{\phi,C}$, and let $p(\phi) = 1$.*

1. *We have $C_{;\alpha} = 0$, $C_{;1} = -2C\phi_{1\alpha;\alpha}$ for any α, and $\Gamma_{\alpha1\beta} = \frac{1}{4}\delta_{\alpha\beta}C^{-1}C_{;1}$.*
2. *For fixed y_0, the curves $t \to T(t, y_0)$ are unit speed geodesics in M^m which are leaves of the foliation $\mathcal{F}_+$.*
3. *For fixed t_0, the surfaces $T(t_0, y)$ are leaves of the foliation $\mathcal{F}_-$ and inherit metrics of constant sectional curvature.*
4. *Locally $ds^2 = dt^2 + f ds^2_{\mathcal{K}}$ where $f(t)$ is a positive smooth function and $ds^2_{\mathcal{K}}$ is a metric of constant sectional curvature $\mathcal{K}$.*

Proof. We apply Lemma 3.1. Since $p = 1$, $a = 1$. Since $q \geq 3$, $C_{;\alpha} = 0$ and $C_{;1} = -C(\phi_{1\alpha;\alpha} + \phi_{1\beta;\beta}) = -C(\phi_{1\alpha;\alpha} + \phi_{1\gamma;\gamma})$ so $C_{;1} = -2C\phi_{1\alpha;\alpha}$ for any α and $\Gamma_{\alpha1\beta} = \frac{1}{4}\delta_{\alpha\beta}C^{-1}C_{;1}$. Since $\Gamma_{111} = 0$ and $\Gamma_{11\alpha} = -\frac{1}{2}\phi_{1\alpha;1} = -\frac{1}{4}C^{-1}C_{;\alpha} = 0$, the integral curves for e_1 are unit speed geodesics; assertion (2) now follows. We compute

$$\partial_t(\partial_t, \partial^y_\alpha) = (\partial_t, \nabla_{\partial_t}\partial^y_\alpha) = (\partial_t, \nabla_{\partial^y_\alpha}\partial_t) = \tfrac{1}{2}\partial^y_\alpha(\partial_t, \partial_t) = 0.$$

This shows $(\partial_t, \partial^y_\alpha) = 0$ so the ∂^y_α span the perpendicular distribution $\mathcal{F}_-$ and the manifolds $T(t_0, y)$ are leaves of the foliation $\mathcal{F}_-$. Since $C_{;\alpha} = 0$, we have that C and $\Gamma_{\alpha1;\beta} = \frac{1}{4}\delta_{\alpha\beta}C^{-1}C_{;1}$ are constant on the leaves of $\mathcal{F}_-$. Let R_- be the curvature of the induced metric on the leaves of $\mathcal{F}_-$. Assertion (3) now follows from the identity $R_-(e_\alpha, e_\beta, e_\gamma, e_\sigma) = R(e_\alpha, e_\beta, e_\gamma, e_\sigma) - \Gamma_{\beta\gamma1}\Gamma_{\alpha1\sigma} - \Gamma_{\alpha\gamma1}\Gamma_{\beta1\sigma}$. Let $\partial^y_\alpha = \Sigma_\gamma a^e_{\alpha\gamma}e_\gamma$. Since $C^{-1}C_{;1}$ only depends on the parameter t, we show that the metric is a twisted product by computing:

$$\begin{aligned}(\nabla_{\partial_t}\partial^y_\alpha, \partial^y_\beta) &= (\nabla_{\partial^y_\alpha}\partial_t, \partial^y_\beta) = \Sigma_{\gamma\sigma}a^e_{\alpha\gamma}a^e_{\beta\sigma}(\nabla_{e_\gamma}\partial_t, e_\sigma)\\ &= \tfrac{1}{4}\Sigma_{\gamma\sigma}a^e_{\alpha\gamma}a^e_{\beta\sigma}\delta_{\alpha\beta}C^{-1}C_{;1} = \tfrac{1}{4}C^{-1}C_{;1}g_{\alpha\beta},\\ \partial_t g_{\alpha\beta} &= \tfrac{1}{2}C^{-1}C_{;1}g_{\alpha\beta}. \qquad \square\end{aligned}$$

Theorem A (3) will follow from Lemma 3.1, from Lemma 3.2, and from the following Lemma that determines which warping functions give IP metrics. Let $\dot f = \partial_t f$ and $\ddot f = \partial_t^2 f$.

3.3 Lemma. *Let $ds^2 := dt^2 + f(t)ds^2_{\mathcal{K}}$ where $ds^2_{\mathcal{K}}$ is a metric of constant sectional curvature $\mathcal{K}$. Then ds^2 is an IP metric of rank 2 if and only if $\ddot f = 2\mathcal{K}$ i.e. $f = \mathcal{K}t^2 + At + B$ where A and B are constants. Let $C = \frac{1}{4f^2}(4\mathcal{K}B - A^2)$. Let $\phi(\partial_t) = \partial_t$ and $\phi(X) = -X$ for $X \perp \partial_t$. Then $R = R_{\phi,C}$. The metric has constant sectional curvature if and only if $4\mathcal{K}B - A^2 = 0$.*

Proof. Let $y = (y^2, ..., y^m)$ be geodesic polar coordinates on a leaf of the foliation $\mathcal{F}_-$ near some point y_0; $g_{\alpha\beta} = \delta_{\alpha\beta} + O(|y|^2)$. Let $\tilde\Gamma$ be the Christoffel symbols relative to a coordinate frame ∂_t, ∂^y_α; $2 \le \alpha \le m$. We have $e_\alpha(y_0) = (\partial^y_\alpha/\sqrt{f})(y_0)$ and

(1) $\tilde\Gamma_{\alpha 1\beta} = \tilde\Gamma_{1\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta}\dot f + O(y^2)$, $\tilde\Gamma_{\alpha 1}{}^{\beta} = \tilde\Gamma_{1\alpha}{}^{\beta} = \frac{1}{2f}\delta^\beta_\alpha \dot f + O(y^2)$,

(2) $\tilde\Gamma_{\alpha\beta 1} = -\frac{1}{2}\delta_{\alpha\beta}\dot f + O(y^2)$, $\tilde\Gamma_{\alpha\beta}{}^{1} = -\frac{1}{2}\delta_{\alpha\beta}\dot f + O(y^2)$,

(3) $R(\partial^y_\alpha, \partial^y_\beta, \partial^y_\gamma, \partial^y_\sigma)(y_0) = f\{R_-(\partial^y_\alpha, \partial^y_\beta, \partial^y_\gamma, \partial^y_\sigma) + \tilde\Gamma_{\alpha 1}{}^{\sigma}\tilde\Gamma_{\beta\gamma}{}^{1} - \tilde\Gamma_{\beta 1}{}^{\sigma}\tilde\Gamma_{\alpha\gamma}{}^{1}\}$
$= f(\mathcal{K} - \frac{1}{4f}\dot f^2)(\delta_{\alpha\sigma}\delta_{\beta\gamma} - \delta_{\beta\sigma}\delta_{\alpha\gamma})$,

(4) $R(e_\alpha, e_\beta, e_\gamma, e_\delta)(y_0) = \frac{1}{f}(\mathcal{K} - \frac{1}{4f}\dot f^2)(\delta_{\alpha\sigma}\delta_{\beta\gamma} - \delta_{\beta\sigma}\delta_{\alpha\gamma})$.

(5) $R(\partial^y_\alpha, \partial_t, \partial_t, \partial^y_\beta)(y_0) = f\{-\partial_t\tilde\Gamma_{\alpha 1}{}^{\alpha} - \tilde\Gamma_{\alpha 1}{}^{\alpha}\tilde\Gamma_{1\alpha\alpha}\}\delta_{\alpha\beta}$
$= f\{-\frac{1}{2f}\ddot f + \frac{1}{2f^2}\dot f^2 - \frac{1}{4f^2}\dot f^2\}\delta_{\alpha\beta}$.

(6) $R(e_\alpha, e_1, e_1, e_\beta)(y_0) = \{-\frac{1}{2f}\ddot f + \frac{1}{4f^2}\dot f^2\}\delta_{\alpha\beta}$.

The remaining curvatures vanish. Suppose g is an IP metric with $p = 1$. Then

$$\frac{\mathcal{K}}{f} - \frac{\dot f^2}{4f^2} = -(-\frac{\ddot f}{2f} + \frac{\dot f^2}{4f^2}).$$

This shows $\mathcal{K} = \frac{1}{2}\ddot f$ so we may expand $f = \mathcal{K}t^2 + At + B$ where A and B are constant. Conversely, if ds^2 has this form, we use equation (2.4) to see $R = R_{\phi,C}$ where $\phi(\partial_t) = \partial_t$, $\phi(\partial^y_\alpha) = -\partial^y_\alpha$, and

$$C = \frac{\mathcal{K}}{f} - \frac{\dot f^2}{4f^2} = \frac{4\mathcal{K}(t^2\mathcal{K} + At + B) - (2\mathcal{K}t + A)^2}{4f^2} = \frac{4\mathcal{K}B - A^2}{4f^2}.$$

It is now immediate that ds^2 has constant sectional curvature if and only if C is constant or equivalently if $4\mathcal{K}B - A^2 = 0$; furthermore, the metric is flat in this setting. $\square$

§4 The proof of Theorem A (4)

We suppose that Theorem A (4) fails. Let g be an IP metric of rank at most 2. Assume that g has rank 0 at some point and that g has rank 2 at some other point. Let $\pm\sqrt{-1}\mathcal{C}$ for $\mathcal{C} \geq 0$ be the non-zero eigenvalues of R. As the manifold is connected and $\mathcal{C}$ is continuous, there exists a unit speed geodesic γ defined for $s \in [a, b]$ so that $\mathcal{C}(s) := \mathcal{C}(R(\gamma(s)))$ satisfies $\mathcal{C}(a) = 0$ and $\mathcal{C}(b) \neq 0$. The set of s where $\mathcal{C}(s) = 0$ is closed; let s_0 be the least upper bound of this set. Then $\mathcal{C}(s) \neq 0$ for $s \in (s_0, b]$ while $\mathcal{C}(s_0) = 0$.

Let $p = p(R(\gamma(s)))$; this is independent of $s \in (s_0, b]$. If $p = 0$, the manifold has constant sectional curvature near $\gamma(s)$ so $\mathcal{C}(s) \neq 0$ is constant and $\mathcal{C}(s)$ does not tend to zero as $s \downarrow s_0$. Thus $p = 1$.

Let $t(s)$ measure the distance along the foliation $\mathcal{F}_+$ for s in the range $(s_0, b]$;

$$t(s) := -\int_s^b (\dot{\gamma}(u), e_1(\gamma(u))du$$

Note that $|t(s)| \leq |b - a|$ so t is uniformly bounded. Let $R(\gamma(s)) = R_{\phi(s), C(s)}$. The parameter t determines $C(s) = C(t(s))$; $\mathcal{C}(s) = |C(s)|$. The numerator of $C(t)$ in Lemma 3.3 is constant. The denominator $4f^2$ can have zeros; near these zeros, C tends to infinity. Although $4f^2 \to \infty$ as $t \to \pm\infty$, the denominator is bounded since $|t(s)| \leq |b - a|$ is uniformly bounded. Thus C is uniformly bounded away from 0 for $s \in (s_0, b]$ and thus C does not tend to 0 as $s \downarrow s_0$. This provides the desired contradiction and completes the proof of Theorem A. $\square$

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