CLASSIFYING SPACES OF COMPACT LIE GROUPS AND FINITE LOOP SPACES

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GÖTTINGEN

PROBLEM-SESSION

Classifying Spaces

The following problems were proposed by Guido Mislin.

1. Let G and H be compact connected Lie groups with BG homotopy equivalent to BH. Are G and H isomorphic as Lie groups?

The following problem was proposed by Clarence Wilkerson.

1. Given a p-adic reflection group $W \leq GL(n, \mathbb{Z}_{\hat{p}})$ which extensions

$$1 \longrightarrow (\mathbb{Z}_{p^\infty})^n \longrightarrow NT \longrightarrow W \longrightarrow 1 \ ,$$

with W acting via the given inclusion can occur as the normalizer of a maximal torus of a finite loop space at p? That is, for which such extensions is there a p-complete finite loop space X and a map

$$BNT \longrightarrow BX$$

with homotopy fibre F the homotopy type of the p-completion of a finite complex and $\chi(F) = 1$?

The following problem was proposed by Jaumé Aguade.

In number 2 of the list of Shephard and Todd of reflection groups there is an infinite family of groups $G(m, n, r)$ defined as follows: $G(m, n, r)$ is the group of all linear maps of the form

$$x_i \mapsto \theta^{\nu_i} x_{\sigma(i)},$$

where σ is a permutation of the basis $x_1, \ldots, x_n$, θ is an m-root of unity and $\nu_1, \ldots, \nu_n$ are such that $\nu_1 + \cdots + \nu_n \equiv 0\,(r)$. If $r|m|p-1$, $G(m, n, r)$ can be realized over the p-adic integers and the invariants mod p form a polynomial algebra

$$R = F_p[y_1, \ldots, y_{n-1}, z],$$

where $y_1, \ldots, y_n$ are the elementary symmetric functions on $x_1^m, \ldots, x_n^m$ and $z = (x_1 \cdots x_n)^{m/r}$.

If we choose a prime p which does not divide $|G(m, n, r)|$ then one can easily construct a space X such that $H^*(X; F_p) \cong R$. In the modular case, Quillen and Zabrodsky (by different methods) constructed spaces realizing the invariants of the groups $G(m, n, 1)$. It is an open problem to realize the invariants of the groups with $r > 1$.

There seems to be two approaches to the problem: One could try to construct X inductively as a homotopy limit of a large diagram. Alternatively, there may be some discrete group G such that BG has the desired mod p cohomology. (This is Quillen's method of realizing the invariants of $G(m, n, 1)$ as the cohomology of some general linear group over some infinite field of positive characteristic.)

The following problems were proposed by Stefan Jackowski and Bob Oliver.

1. Construct unstable Adams operations for a compact connected Lie group G, using the methods in [JMO]. If T is a maximal torus in G, and $N_p(T)$ is a maximal p -toral subgroup, then the main problem is to extend the k -th power map on the torus (for any k prime to the order of the Weyl group) to an "R_p -invariant representation"

$$f_k : N_p(T) \longrightarrow G.$$

In other words, for any p-toral subgroup $P \subset N_p(T)$ and any g such that $gPg^{-1} \subset N_p(T)$, the two restrictions of f_k to homomorphisms $P \longrightarrow G$ must be conjugate in G.

2. Find more explicit definitions of the exotic maps which so far are constructed using completions, homotopy colimits, and/or computations of higher limits. Closely related to this are the following more explicit problems:

2a. Construct a pair of homomorphisms $r, s : G \longrightarrow G'$ (e.g., with G finite and G' connected), such that r and s are equal on a maximal p-toral subgroup (or Sylow p-subgroup) of G, but such that Br and Bs are not homotopic as maps $BG \longrightarrow BG'$. Note that this situation requires a nonvanishing higher limit (and probably would require the explicit computation of an obstruction in that group).

2b. Find a nonvanishing existence obstruction. More precisely, find an example where one of the obstruction groups to the existence of a map defined on a homotopy colimit is nonzero, and where the only way to determine the existence or nonexistence of the map is by explicitly determining the obstruction.

3. Let G, G' be compact simple Lie groups with maximal tori T and T' respectively. Assume dim $(T') \leq$ dim $(T) + 1$. Let $[BG, BG']_i$ denote the set of homotopy classes of maps which are "rationally injective" in some appropriate sense: e.g. maps $f : BG \longrightarrow BG'$ such that $H^*(BG\ ;\ \mathbb{Q})$ is finitely generated as a module over $H^*(BG'\ ;\ \mathbb{Q})$.

— Describe $[BG, BG']_i$.

— Is homotopy detected by rational cohomology in this case?

Comments. The idea of "rationally injective" should be that f lifts to a map between the maximal tori which has finite kernel. Apply methods of [JMO].

4. Determine (or study) the space of maps map(BG, X_G), when G is a compact connected simple Lie group and X is a G-complex (and X_G is the Borel construction). This is closely related to the problem of describing the homotopy fixed point set X^{hG} in this situation.

5. For any pair of compact [connected?] Lie groups P, G, and any $f : P \longrightarrow G$, does the

set

$$Rep(P,G)_f = \{f' : P \longrightarrow G : f'(g)\ G\text{–conjugate } f(g)\ \forall g \in P\}/\text{Inn}(G)$$

have an abelian group structure in some natural way?

Comment: If this could be done in general, then the problem of constructing $\mathcal{R}_p$ -invariant representations would be in part "reduced" to studying an obstruction in $H^1(\mathcal{R}_p(G); --)$, where the coefficients are these $Rep(P,G)_f$ for appropriate f. (Reduced in quotes because there would still also be the problem of showing that appropriate pairs of homomorphims are elementwise conjugate.)

6. Show the rigidity of the [JM2] diagram: i.e., the functor $\mathcal{A}_p(G) \longrightarrow Top$ which sends a nontrivial elementary abelian subgroup $A \subset G$ to $BC_G(A)$. Here, rigidity means that any diagram (functor) $\mathcal{A}_p(G) \longrightarrow Top$, which is equivalent to the first one in the homotopy category, is weakly equivalent to it as functors to Top. This means showing the vanishing of appropriate higher limits of the functor

$$A \longrightarrow \pi_*(Z(C_G(A)))$$

defined on elementary abelian p -subgroups of G.

7. Find useful decompositions of spaces as homotopy inverse limits. If a space has a hocolim decomposition, is there a "dual" holim decomposition?

8. [Rothenberg] Let G', G be a pair of compact Lie groups, where G is simple. Describe $[B(G',G), B(G',G)]$. Here, $B(G',G)$ is the classifying space for G -fiber bundles with structure group G'. An answer should generalize corresponding results in [JMO]. Of main interest is the case where $G' = (S)O(n)$, $(S)U(n)$, etc. (for studying equivariant vector bundles).

References:

[JMO] S.Jackowski, J.McClure, and R.Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math (to appear)

[JM2] S.Jackowski and J.McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, Topology (to appear)

[McG] C.McGibbon, Self maps of projective spaces, TAMS vol.271 (1982), 325-346

[FG] Feder and Gitler, Mappings of quaternionic projective space, Bol. Soc. Mat. Mexicana, vol.18 (1973), 33-37

Spaces Related to Classifying Spaces

[McGibbon] Describe self-maps of $\mathbb{HP}(n)$, using some homotopy decomposition. (Or, at least, determine which degrees can be realized for self-maps.) See [McG] and [FG] for background.

Problems proposed by Larry Smith.

Let G be a compact connected Lie group with $T \hookrightarrow G$ a maximal torus. Denote by $HE(G/T)$ the monoid of homotopy equivalences of the homogenous space G/T . Let $SHE(G/T)$ denote the identity component of $HE(G/T)$, and $HE_{id^*}(G/T)$ the submonoid of maps inducing the identity map in cohomology ($\mathbb{Z}$ or $\mathbb{Q}$).

1. How many components does $HE(G/T)$ have?

2. How many components does $HE_{id^*}(G/T)$ have? (Expermental evidence shows that there can be several.)

Problem proposed by Clarence Wilkerson.

1. For which simply connected finite complextes is $\dim H^k(SHE(X),\ \mathbb{Q}) < \infty$. The spirit

of the problem is to decide for which spaces X $SHE(X)$ has properties analagous to Lie groups or arithmetic groups.

Finite Loop Spaces

Problems proposed by Larry Smith.

1. If a finite loop space X has a maximal torus, is BX homotopy equivalent to BG for some Lie group G ?

2. If a finite loop space X has a maximal torus, must the Weyl group be crystalographic?

3. If a finite loop space has a maximal torus, what should be the normalizer of that torus be?

References

1. Smith, L. , Finite Loop Spaces with maximal Tori have Finite Weyl Groups, Math. Göttingensis 1991.

The following problems were proposed by Richard Kane.

Let p be a prime and X a finite loop space at the prime p.

1. Does $H^*(BX,\ \mathbb{Z})$ have only elementary torsion: i.e. is all the p-torsion of at most order p?

Comment: Kono has raised this question for Lie groups. He has shown that $H^*(BSpin(n);\mathbb{Z})$ has only elementary 2-torsion. He also observes that transfer arguments severely restrict torsion in $H^*(BG,\mathbb{Z})$.

2. Does $k(n)^*(BX)$ have only v_n torsion: i.e. does v_n annhilate the torsion submodule. Here $K(N)^*(--)$ denotes Morave k-theory, and $k(n)^*(*) = \mathbb{F}_p[v_n]$, $\deg(v_n) = 2p^n - 2$.

3. Let $BT \longrightarrow BX$ be the Dwyer-Wilkerson maximal torus and let X/T be the fibre of this map (everything completed at p). Determine $H^*(X/T;\ \mathbb{F}_p)$ and more generally the

Serre spectral sequence of the fibration:

$$X \longrightarrow X/T \longrightarrow BT$$

This was done by Kac and Peterson for the Lie group case. In particular well behaved Chevalley Δ operators play a major role in their argument. In the following two problems p denotes an it odd prime.

4. In particular Kac and Peterson show define the generalized invariants in $H^*(BT;‘\mathbb{F}_p)$ and show that these generalized invariants determine $H^*(G;\ \mathbb{F}_p)$ as an algebra. Show that the coalgebra structure is also determined.

5. The generalized invariants are generated by a regular sequence $\{x_1, \ldots, x_n\}$, so there is a *non-canonical* inclusion $\mathbb{F}_p[x_1, \ldots x_n] \subset H*(BT;\ \mathbb{Z})$. Is there a choice that makes this a ring of invariants? The example of E_6 shows that if the choice exists then the group in question need not be a subgroup of the Weyl group.

Invariant Theory

The following problems were proposed by David Benson.

Let $k = \mathbf{F}_q$, and G be a subgroup of the finite general linear group $GL_n(F_q) = GL(V)$. Let c_0 be the highest degree Dickson invariant.

1. When is $k[V]^G[c_0^{-1}]$ a localized polynomial ring? Is it always Cohen–Macaulay*?

2. If G is generated by pseudoreflections (elements fixing a hyperplane pointwise), what properties does $k[V]^G$ have? Is it always a complete intersection? (Example: In the case of the finite symplectic groups in their natural representation, Carlisle and Kropholler [1] have shown that $k[V]^G$ is a complete intersection)

3. Is $k[V]^G$ generated by elements of degree $\leq |G|$? (A theorem of Noether asserts that this is true in the case where $|G|$ is invertible in k.) Find better bounds (in the case where $|G|$ is invertible in k, see the 1989 Comptes Rendues paper of Barbara Schmid).

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* I.e. free and finitely generated as a module over some polynomial subalgebra

4. Carlisle and Kropholler [1] have made the following conjecture. If we form the Poincaré series $p(t) = \sum_{i \geq 0} t^i \dim_k(k[V]^G)_i$, then the Laurent expansion begins

$$p(t) = \frac{1}{|G|}(1-t)^{-n} + \frac{r}{2|G|}(1-t)^{-n+1} + \cdots$$

Here, the first term is correct by general degree considerations, and the content of the conjecture is the value of r. This is given as follows. For each maximal subspace W of V, let the stabiliser have order $p^a h$ where h is coprime to p. Then the contribution from W is $a(p-1) + (h-1)$, and the value of r is the sum of the contributions from all the maximal subspaces.

Is there a similar interpretation of the further terms in the Laurent expansion in terms of stabilisers of smaller subspaces? Is there a closed expression for the Poincaré series?

References

1. D. Carlisle and P. H. Kropholler, Modular invariants of finite symplectic groups, Preprint

The following problems were proposed by Jaume Aguadé and Carlos Broto.

These problems are connected with our *lack* of understanding the modular invariants of the Weyl groups.

Let R be a root system and W its Weyl group. Let M be the root lattice generated by R and let S denote the symmetric algebra functor. Then W acts on $S(M \otimes F_p)$ for any prime p and one is interested in the algebra of invariants $S(M \otimes F_p)^W$. There are three types of primes p:

a) If p does not divide $|W|$ then everything is as in the characteristic zero case and $S(M \otimes F_p)^W$ is a polynomial algebra (Chevalley-Shephard-Todd theorem).

b) If p does divide $|W|$ but p is not a *torsion prime* for R then $S(M \otimes F_p)^W$ is also polynomial (Demazure). Here p is a torsion prime for R if the compact Lie group associated to R has torsion at the prime p. Equivalently, there is a purely algebraic description of the torsion primes in the following way: Let $J : S(M) \to S(M)$, $J = \sum(\det \omega)\, \omega$ and let $d \in S$ be the product of all positive roots of R. Then if t is the smallest integer such that $td \in \mathrm{Im}(J)$ then the torsion primes are those dividing t (Demazure).

c) If p is a torsion prime for R, there are examples with $S(M \otimes F_p)^W$ polynomial and other examples with $S(M \otimes F_p)^W$ not polynomial. For instance, if R is the root lattice associated to $PSU(n)$ then the torsion primes are those dividing n. For $n = p = 3$ the invariants are polynomial. For $n = p = 5$ they are not.

1. The question of which torsion primes produce polynomial algebras of invariants is only settled in a few cases.

2. Extend Demazure's theory of torsion primes to complex reflection groups.

Since this statement is too imprecise, let us state a very concrete question. Let R be a root system and let R' be a sub-root-system of R. Let W and W' be the respective Weyl groups and assume that p is prime to the index of W' in W. Then Demazure proves that if p is a torsion prime for R then it is also a torsion prime for R'. Is this also true for more general groups generated by pseudo-reflections? Demazure's proof uses at a crucial point the existence of a unique element of maximal length in any Coxeter group and this fails in more general groups.

The following problems were proposed by Larry Smith.

Let $G \leq GL(n, \mathbb{F})$ be a (finite) subgroup acting on $V = \oplus \mathbb{F}$. There is then the ring of ***coinvariants***

$$P(V)_G := \mathbb{F} \otimes_{P(V)^G} P(V) = P(V)/I^G(V)$$

where $P(V)^G$ is the ring of invariants and $I^G(V) \subset P(V)$ the ideal generated by the invariants of positive degree. The group G acts on $P(V)_G$, and so has a ring of invariants

References

1. Demazure, M. Invariants symétriques des groupes de Weyl et torsion, Inv. Math 21 (1973), pp. 287 - 301.

2. Speerlich, T., Automorphisms of Rings of Coinvariants, Math Göttingensis 1991.

${P(V)_G}^G$, which may or may not be trivial, and hence a ring of coinvariants

$$P(V)_{G,2} := \mathbb{F} \otimes_{{P(V)_G}^G} P(V)_G$$

Continuing in this way we define inductively

$$P(V)_{G,i} := \mathbb{F} \otimes_{(P(V)_{G,i-1})^G} P(V)_{G,i-1}.$$

If we introduce the notations

$$P(V)_{G,0} := P(V)$$

$$P(V)_{G,1} := P(V)_G$$

we then have a sequence of epimorphisms:

$$P(V) \longrightarrow P(V)_{G,1} \longrightarrow P(V)_{G,2} \longrightarrow \cdots \longrightarrow P(V)_{G,i} \longrightarrow \cdots$$

Let

$$I_i^G(V) := ker\{P(V) \longrightarrow P(V)_{G,i}\} \ .$$

By Noether's theorem this stops. Define s to be the first integer at which it stops:

1. When is $P(V)_{G,s}$ a Poincaré duality algebra?

2. When is $I_s^G(V)$ generated by a regular sequence?

3. For Weyl Groups in characteristic zero Papadima and Speerlich have shown that the group of algebra automorphisms of $P(V)_G$ is the normalizer of the Weyl group in $GL(V)$. What happens for the other pseudo reflection groups? What happens in finite characteristic?

3. Kac, Victor G., Dale N. Peterson, Generalized Invariants of Groups generated by Reflections , Journees de Geometria, Birkhauser Verlag 1985.

Cohomology of Groups

The following problems were proposed by David Benson.

1. Suppose that n is a positive integer and p is a prime. Can one construct a finite group G and an element $x \in H^*(G, \mathbf{F}_p)$ with $x^n = 0$ but $x^{n-1} \neq 0$? If $p = 2$, Avrunin and Carlson have found a family of 2-groups of nilpotence class 2 which answer this problem. If p is odd, no example is known of a nilpotent element of $H^*(G, \mathbf{F}_p)$ with $x^p \neq 0$.

2. Suppose that G is a finite group and k is a field of characteristic p. Suppose further that $k[\zeta_1, \ldots, \zeta_r]$ is a polynomial subring over which $H^*(G, k)$ is finitely generated as a module, with $\deg(\zeta_i) = n_i$, so that in particular by a theorem of Quillen, r is equal to the p-rank of G. Then a theorem of [BC] states that there is a first quadrant spectral sequence whose E_2 term is

$$H^*(G, k) \otimes \Lambda^*(\tilde{\zeta}_1, \ldots, \tilde{\zeta}_r).$$

Here, $H^*(G, k)$ appears on the base, and the $\tilde{\zeta}_i$ are exterior generators on the fibre of degree $n_i - 1$. The spectral sequence converges to the cohomology of a finite Poincaré duality complex, where the dualising class is in degree $\sum_{i=1}^{r}(n_i - 1)$. In particular, if $H^*(G, k)$ is Cohen–Macaulay, then the Poincaré series $p(t) = \sum_{i \geq 0} t^i \dim_k H^i(G, k)$ (as a rational function of t) satisfies the functional equation

$$p(1/t) = (-t)^r p(t).$$

(i) Is the converse true? In other words, if the above functional equation holds, does it follow that $H^*(G, k)$ is Cohen–Macaulay?

(ii) Is there a generalisation of this to compact Lie groups? In this case, one would expect the dualising class to have degree $-\dim G + \sum_{i=1}^{r}(n_i - 1)$, where $\dim G$ denotes the dimension of G as a manifold. In particular, in the Cohen–Macaulay case, one would obtain the functional equation

$$p(1/t) = (-t)^r t^{\dim G} p(t).$$

This has been checked for $Spin(10)$ at the conference.

3. Let Z be a central subgroup of (say, finite) groups G_1 and G_2, and let G be the central product of G_1 and G_2 along Z. Then there is a pullback square of fibrations

$$\begin{array}{ccc} BG & \to & BG_1/Z \\ \downarrow & & \downarrow \\ BG_2/Z & \to & K(Z,2). \end{array}$$

and hence an Eilenberg–Moore spectral sequence

$$\mathrm{Tor}^{**}_{H^*(K(Z,2))}(H^*(G_1/Z), H^*(G_2/Z)) \Rightarrow H^*(G).$$

Is there an algebraic construction of this spectral sequence? It is not hard to construct an algebraic model for $K(Z,2)$. Namely, one regards a (strictly coassociative) projective resolution for Z as a differential graded augmented algebra, and constructs a projective resolution over this. But it seems harder to make a sensible construction of a resolution for G_i/Z as a differential graded module over this.

4. Is there a sensible definition of Steenrod operations in Tate cohomology of finite groups, $\hat{H}^*(G, \mathbf{F}_p)$? Preferably, the Steenrod operations should commute with Tate duality.

Is there a sensible definition of an unstable algebra over the Steenrod algebra which takes into account the fact that Tate cohomology is graded by both the positive and negative integers?

References

[BC]. D. J. Benson and J. F. Carlson. Projective resolutions and Poincaré duality complexes. Submitted to Trans. A.M.S.

Problem proposed by Larry Smith.

1. If $H^*(BG, \mathbb{F}_p))$ is Cohen-MacCauley, is it Cohen-MacCauley over the Steenrod algebra? I.E. Does there exist a polynomial subalgebra closed under the action of the Steenrod algebra over which it is finitely generate.

Steenrod Operations

Problems proposed by Clarence Wilkerson.

1. In $H^*(\mathbb{R}\mathbb{P}(\infty)^n,\ \mathbb{Z}_2)$ find ideals that are $\mathcal{A}(2)$ invariant and generated by a regular sequence, with generators of the same degree.

2. In $H^*(\mathbb{R}\mathbb{P}(\infty)^N,\ \mathbb{Z}_2)$ let $\mathcal{I}$ be an ideal generated by elements $\{f_i \mid i = 1, \dots n\}$ and $\{g_k \mid k = 1, \dots m\}$ of dimension 2. Suppose $Sq^1(f_i) \subset \mathcal{I} \ \ \forall\ i$. Let $\mathcal{J}$ be the smallest $\mathcal{A}(2)$ ideal containing $\mathcal{I}$. Show that for $2m + n < N$ there are non-nilpotent elements in $H^*(\mathbb{R}\mathbb{P}(\infty)^N, \mathbb{Z}_2)/\mathcal{J}$.