Symmetric topological complexity of projective and lens spaces

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Dedicated to the memory of Bob Stong

Abstract

For real projective spaces, (a) the Euclidean immersion dimension, (b) the existence of axial maps, and (c) the topological complexity are known to be three facets of the same problem. This paper describes the corresponding relationship between the symmetrized versions of (b) and (c) to the Euclidean embedding dimension of projective spaces. Extensions to the case of 2^e torsion lens spaces and complex projective spaces are discussed.

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1 Introduction and main result: real projective spaces

The Euclidean immersion and embedding questions for projective spaces were topics of intense research during the beginning of the second half of the last century. In the case of real projective spaces, the immersion problem has recently received a fresh push, partly in view of a surprising reformulation in terms of a basic concept arising in robotics, namely, the motion planning problem of mechanical systems. In this first section we start with a brief review of such an interpretation for the immersion problem of real projective spaces, and then we continue to describe our main result (Theorem 1.7 below) on the corresponding interpretation for the embedding problem.

1.1 *Axial maps.* Activities were launched with Hopf's early work [17] constructing, for $n > r$, a Euclidean n-dimensional embedding for the r-dimensional real projective space P^r out of a given symmetric nonsingular bilinear map $\alpha\colon \mathbb{R}^{r+1}\times\mathbb{R}^{r+1}\to\mathbb{R}^{n+1}$. By restricting to unit vectors (and normalizing), this yields a (symmetric) $\mathbb{Z}/2$-biequivariant map $\widetilde{\alpha}\colon S^r\times S^r\to S^n$ covering a (symmetric) axial map $\widehat{\alpha}\colon \mathrm{P}^r\times\mathrm{P}^r\to \mathrm{P}^n$, that is, a map which is homotopically non-trivial over each axis (and satisfies the extra condition $\widehat{\alpha}(x,y)=\widehat{\alpha}(y,x)$ for $x,y\in\mathrm{P}^r$).

Using Hirsch's characterization of (Euclidean) immersions in terms of the normal bundle's geometric dimension, the relevance of (not necessarily symmetric) axial maps was settled in [1] by showing:

Theorem 1.1. *For $n>r$, the existence of an axial map $\mathrm{P}^r\times\mathrm{P}^r\to\mathrm{P}^n$ is equivalent to the existence of a smooth immersion $\mathrm{P}^r\looparrowright\mathbb{R}^n$.* ■

The hypothesis $n>r$ is needed only for $r=1,3,7$. In those cases P^r is parallelizable and has an optimal Euclidean immersion in codimension 1; however the complex, quaternion, and octonion multiplications yield axial maps with $n=r$.

The connection with robotics was established almost 30 years latter with M. Farber's work (initiated in [7, 8]) on the motion planning problem.

1.2 *Topological complexity.* The Schwarz genus ([19]) of a fibration $p\colon E\to B$, denoted by $\mathfrak{genus}(p)$, is the smallest number of open sets U covering B in such a way that p admits a (continuous) section over each U. The topological complexity of a space X, $\mathrm{TC}(X)$, is defined as the genus of the end-points evaluation map $\mathrm{ev}\colon P(X)\to X\times X$, where $P(X)$ is the free path space $X^{[0,1]}$ with the compact-open topology. $\mathrm{TC}(X)$ is a homotopy invariant of X. Thinking of X as the space of configurations of a given mechanical system, $\mathrm{TC}(X)$ gives a measure of the topological instabilities in a motion planning algorithm for X —a perhaps discontinuous section of the map ev. We refer the reader to [9] for a very useful survey of results in this area.

Remark 1.2. As shown in [12], for $r\neq 1,3,7$, $\mathrm{TC}(\mathrm{P}^r)$ can be characterized as the smallest integer n such that there is an axial map $\mathrm{P}^r\times\mathrm{P}^r\to\mathrm{P}^{n-1}$. Consequently, $\mathrm{TC}(\mathrm{P}^r)-1$ is the smallest dimension of Euclidean spaces where P^r can be immersed. This assertion holds for the three exceptional values of r provided $\mathrm{TC}(\mathrm{P}^r)-1$ is replaced by $\mathrm{TC}(\mathrm{P}^r)$.

1.3 *Symmetric axial maps.* As shown in [4], the embedding problem for projective spaces can be closely modeled by keeping Hopf's original symmetry condition for axial maps. We give below a quick review of some of the main ideas in [4], but for the sake of fluidity in reading, we first state the following basic result of Haefliger's ([15, Théorème 1′]).

Theorem 1.3. *Let $2m \geq 3(n+1)$. For a smooth compact n-dimensional manifold M, there is a surjective map from the set of isotopy classes of smooth embeddings $M \subset \mathbb{R}^m$ onto the set of $\mathbb{Z}/2$ equivariant homotopy classes of maps $M^* \to S^{m-1}$. Here $\mathbb{Z}/2$ acts antipodally on S^{m-1}, and by interchanging coordinates on $M^* = M \times M - \Delta_M$, where Δ_M stands for the diagonal in $M \times M$.* ■

All we need from Theorem 1.3 is the fact that, under the stated hypothesis, the existence of a smooth embedding $M \subset \mathbb{R}^m$ is equivalent to the existence of a $\mathbb{Z}/2$ equivariant map $M^* \to S^{m-1}$. Although it is not relevant for our purposes, it is worth remarking that the surjective map in Theorem 1.3 is explicit, and that it is in fact bijective when $2m > 3(n+1)$.

As for the application in [4], we start by observing that a symmetric axial map $\widehat{\alpha}\colon \mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^s$ is covered by a map $\widetilde{\alpha}\colon S^r \times S^r \to S^s$ satisfying

$$-\widetilde{\alpha}(x,y) = \widetilde{\alpha}(-x,y) = \widetilde{\alpha}(x,-y) \quad \text{and} \quad \widetilde{\alpha}(x,y) = \widetilde{\alpha}(y,x) \tag{1}$$

for $x, y \in S^r$ —what we call a symmetric $\mathbb{Z}/2$ biequivariant map. Under these conditions it is elementary to check that the composite

$$V_{r+1,2} \xrightarrow{\Psi} S^r \times S^r \xrightarrow{\widetilde{\alpha}} S^s, \qquad \Psi(x,y) = \left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) \tag{2}$$

is a D_4 equivariant map. Here D_4 is the dihedral group written as the wreath product $(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts on $\mathbb{Z}/2 \times \mathbb{Z}/2$ by interchanging factors. This group acts on S^s via the canonical projection $(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$, and on $V_{r+1,2}$ (the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{r+1}$) via the restricted D_4 action in $S^r \times S^r$, where $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/2$ act on $S^r \times S^r$ by the product antipodal action and by switching coordinates, respectively.

On the other hand, setting $\widetilde{\Delta} = \{(x,y) \in S^r \times S^r \mid x \neq \pm y\}$, the map $H\colon (S^r \times S^r - \widetilde{\Delta}) \times [0,1] \to S^r \times S^r - \widetilde{\Delta}$ defined by $H(u_1,u_2,t) = (\widetilde{u}_1, \widetilde{u}_2)$ where

$$\begin{aligned}
\widetilde{u}_1 &= \frac{u_1 + t(v_1 - u_1)}{\|\, u_1 + t(v_1 - u_1)\, \|} & \qquad \widetilde{u}_2 &= \frac{u_2 + t(v_2 - u_2)}{\|\, u_2 + t(v_2 - u_2)\, \|} \\
v_1 &= w_1 + w_2 & v_2 &= w_1 - w_2 \\
w_1 &= \frac{u_1 + u_2}{\sqrt{1 + \langle u_1, u_2 \rangle}} & w_2 &= \frac{u_1 - u_2}{\sqrt{1 - \langle u_1, u_2 \rangle}}
\end{aligned}$$

gives a D_4 equivariant deformation retraction of $S^r \times S^r - \widetilde{\Delta}$ onto $V_{r+1,2}$. (Figure 1 depicts the case in which the angle between u_1 and u_2 is less than 90 degrees; the situation for an angle between 90 and 180 degrees is similar, but lowering the angle

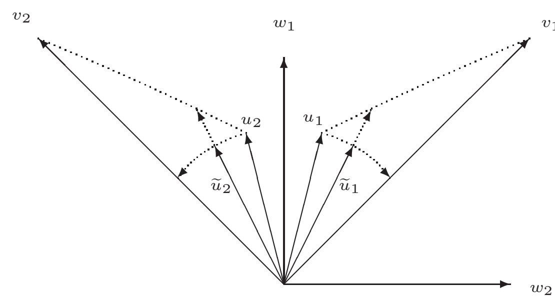

Figure 1: The D_4 equivariant deformation retraction H

to be 90 degrees.) Then, composing the retraction $H(-,1)$ with $\widetilde{\alpha} \circ \Psi$ and passing to $\mathbb{Z}/2 \times \mathbb{Z}/2$ orbit spaces, we get a $\mathbb{Z}/2$ equivariant map $(\mathrm{P}^r)^* \to S^s$.

In view of Theorem 1.3, the above construction settles the first part in Theorem 1.4 below. The bulk of the work in [4] uses Haefliger and Hirsch's fundamental work [15, 16] on embeddings and immersions in the stable range to establish the (unrestricted) converse.

Theorem 1.4. *For $2s > 3r$, the existence of a symmetric axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^s$ implies the existence of a smooth embedding $\mathrm{P}^r \subset \mathbb{R}^{s+1}$. Conversely, the existence of a smooth embedding $\mathrm{P}^r \subset \mathbb{R}^s$ implies the existence of a symmetric axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^s$.* ■

The arguments in [4] go a bit further; using the full power of Theorem 1.3, they explicitly relate, for instance, isotopy classes of embeddings to symmetric homotopy classes of symmetric axial maps. We will not make use of these more complete results, though.

From [1, Lemma 2.1], an axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^s$ can only hold within Haefliger's stable range $2s > 3r$, when $r > 15$. In fact, we will need to consider the following slight improvement (to be proved at the end of Section 2).

Proposition 1.5. *For $r \in \{8, 9, 13\}$ or $r > 15$, an axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^s$ can hold only when $2s > 3r + 2$.*

Our main interest in Theorem 1.4 arises as follows. Let $E(r)$ denote the Euclidean embedding dimension of P^r, and let $a_S(r)$ denote the smallest integer k for which there exists a symmetric axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^k$. It is immediate from Theorem 1.4 that, at least in the range of Proposition 1.5,

$$E(r) = a_S(r) + \delta \quad \text{with} \quad \delta = \delta(r) \in \{0, 1\}. \tag{3}$$

To the best of our knowledge, the explicit value of δ (as a function of r) remains to be an open question. As an alternative, our main result, Theorem 1.7 below, will avoid this δ indeterminacy by replacing $a_S(r)$ with the symmetric topological complexity of P^r.

1.4 *Symmetric topological complexity.* A slightly weaker form (Theorem 1.6 below) of our main result extends the E vs. a_S relation in (3) within the topological complexity viewpoint, giving our symmetric interpretation of the first statement in Remark 1.2. We start by recalling the basic definitions in [10].

For a topological space X, let $\mathrm{ev}_1\colon P_1(X) \to X \times X - \Delta_X$ be the restriction of the fibration ev in Subsection 1.2 where, as in Theorem 1.3, Δ_X is the diagonal in $X \times X$, and $P_1(X)$ is the subspace of $P(X)$ consisting of paths $\gamma : [0,1] \to X$ with $\gamma(0) \neq \gamma(1)$. Note that ev_1 is a $\mathbb{Z}/2$ equivariant map, where $\mathbb{Z}/2$ acts freely on both $P_1(X)$ and $X \times X - \Delta_X$, by running a path backwards in the former, and by switching coordinates in the latter. Let $P_2(X)$ and $B(X,2)$ denote the corresponding orbit spaces, and let $\mathrm{ev}_2\colon P_2(X) \to B(X,2)$ denote the resulting fibration. The symmetric topological complexity of X is defined by $\mathrm{TC}^S(X) = \mathfrak{genus}(\mathrm{ev}_2) + 1$.

Corollary 9 in [10] establishes the inequality

$$\mathrm{TC} \leq \mathrm{TC}^S. \tag{4}$$

The proof of this result suggests that the "+1" part in the definition of TC^S is intended to cover (a neighborhood of) the diagonal Δ_X when describing symmetric local sections s for ev with the property that, for any $x \in X$, $s(x,x)$ is the constant path at $x \in X$.

Theorem 1.6. *$E(r)+1 \leq \mathrm{TC}^S(\mathrm{P}^r) \leq a_S(r)+2$, the first inequality holding provided $2\,\mathrm{TC}^S(\mathrm{P}^r) > 3r+4$.*

Remark 1.2, Proposition 1.5, and (4) assure that the condition $2\,\mathrm{TC}^S(\mathrm{P}^r) > 3r+4$ holds for all $r > 15$ as well as for $r \in \{8, 9, 13\}$. In those cases, and in view of (3), the end terms in the inequalities of Theorem 1.6 are off at most by 1.

We can actually do a bit better, but we have chosen to give Theorem 1.6 first in order to compare with the work in [12] for the non-symmetric case. Let ξ be the Hopf line bundle over P^r and consider the exterior tensor product $\xi \otimes \xi$ over $\mathrm{P}^r \times \mathrm{P}^r$. Let $I(r)$ denote the smallest integer k such that the iterated $(k+1)$-fold Whitney multiple of $\xi \otimes \xi$ admits a nowhere vanishing section. Finally, let $a(r)$ denote the smallest integer k for which there is an axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^k$. The main results in [12], Corollary 5 and Proposition 17, give

$$I(r) + 1 \leq TC(\mathrm{P}^r) \leq a(r) + 1, \tag{5}$$

an assertion a bit sharper than its symmetric analogue in Theorem 1.6. The punch line then comes from the classical fact that, for $n \neq 1, 3, 7$, both $I(r)$ and $a(r)$ agree with the dimension of the smallest Euclidean space where P^r admits an immersion. However, there is no sectioning-Whitney-multiples interpretation available for the symmetric version of (5). Instead, an adaptation of the ideas in [4] allows us to refine Theorem 1.6.

Theorem 1.7. *$E(r) + 1 = \mathrm{TC}^S(\mathrm{P}^r)$ when $2\,\mathrm{TC}^S(\mathrm{P}^r) > 3r + 4$.*

This is our symmetric interpretation of the second statement in Remark 1.2. As discussed after Theorem 1.6, the hypothesis in Theorem 1.7 holds for all $r > 15$ as well as for $r \in \{8, 9, 13\}$. Of course, Theorem 1.6 is a direct consequence of (3) and Theorem 1.7.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.7. The situation for 2^e torsion lens spaces is discussed in Section 3. Here our results are weaker than the case $e = 1$, due in part to the fact that, as the torsion increases, the end terms in (5) start measuring different phenomena; this prevents us from closing the cycle of inequalities. The numerical value of the symmetric topological complexity of complex projective spaces is computed in Section 4.

2 Proof of Theorem 1.7

As a warm up, we start by showing that the second inequality in Theorem 1.6 can be settled with a straightforward adaptation of the idea in the first part of the proof of [12, Proposition 17].

Let $\widetilde{\alpha}\colon S^r \times S^r \to S^s$ satisfy (1), with $s = a_S(r)$. For $1 \leq i \leq s+1$, let $\alpha_i\colon S^r \times S^r \to \mathbb{R}$ be the ith real component of $\widetilde{\alpha}$, and set U_i to be the open subset of $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ consisting of pairs (L_1, L_2) of lines with $\alpha_i(\ell_1, \ell_2) \neq 0$, for representatives $\ell_j \in L_j \cap S^r$, $j = 1, 2$. Consider the function $s_i\colon U_i \to P_1(\mathrm{P}^r)$ defined as follows: given $(L_1, L_2) \in U_i$, choose elements ℓ_j as above with $\alpha_i(\ell_1, \ell_2) > 0$. The two such possibilities (ℓ_1, ℓ_2) and $(-\ell_1, -\ell_2)$ give the same orientation for the 2-plane $P(L_1, L_2)$ generated by L_1 and L_2. Under these conditions $s_i(L_1, L_2)$ is the path rotating L_1 to L_2 in the oriented plane $P(L_1, L_2)$.

Evidently s_i is a continuous section of the fibration ev_1 over U_i. It is also $\mathbb{Z}/2$ equivariant, in view of the last condition in (1). Therefore, it induces a corresponding (continuous) section $\bar{s}_i$ of the fibration ev_2 over the image of U_i under the canonical (open) projection $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r} \to B(\mathrm{P}^r, 2)$. But $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ is covered by the U_i's, so we deduce $\mathfrak{genus}(\mathrm{ev}_2) \leq s + 1$. Adding 1, we get the second inequality in Theorem 1.6. ■

In proving Theorem 1.7, it will be convenient to recall a few of the constructions and methods used in [12] for handling the non-symmetric topological complexity (see Remark 1.2).

Let $S^r \times_{\mathbb{Z}/2} S^r$ denote the quotient space of $S^r \times S^r$ by the diagonal action of $\mathbb{Z}/2$. The 2-fold Cartesian product of the canonical projection $S^r \to \mathrm{P}^r$ factors through $S^r \times_{\mathbb{Z}/2} S^r$ yielding a $\mathbb{Z}/2$ covering space $\pi\colon S^r \times_{\mathbb{Z}/2} S^r \to \mathrm{P}^r \times \mathrm{P}^r$. It is well known that π is the sphere bundle associated to $\xi \otimes \xi \to \mathrm{P}^r \times \mathrm{P}^r$, the line bundle in Subsection 1.4 (see for instance [21, Lemma 3.1]). Now, using Theorem 3 together with the final remarks in Chapter II of [19], we know that the Whitney multiple $k(\xi \otimes \xi)$ admits a global nowhere trivial section for $k = \mathfrak{genus}(\pi)$. The relevance of these observations comes from the standard fact that the smallest such k agrees with the smallest n for which there is an axial map $\mathrm{P}^r \times \mathrm{P}^r \to \mathrm{P}^{n-1}$ (see for instance the proof of [14, Proposition 2.7]). In these terms, half of the work in [12] comes from observing that the topological complexity of P^r is bounded from below by $\mathfrak{genus}(\pi)$ —the other half being a sharpening of the argument at the beginning of this section. This lower bound is easily settled in [12, Theorem 3] from a fiber-preserving map of the form

$$\begin{array}{ccc} P(\mathrm{P}^r) & \xrightarrow{\;f\;} & S^r \times_{\mathbb{Z}/2} S^r \\ & {\scriptstyle \mathrm{ev}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathrm{P}^r \times \mathrm{P}^r & \end{array} \tag{6}$$

In fact, as a byproduct of the methods in [12], we know

$$\mathrm{TC}(\mathrm{P}^r) = \mathfrak{genus}(\pi). \tag{7}$$

But for the the symmetric situation it will be necessary to settle the analogous equality in a more direct way. To this end, we start by recalling the definition of f. For a path $\gamma \in P(\mathrm{P}^r)$, let $\widehat{\gamma}\colon [0,1] \to S^r$ be any lifting of γ through the canonical projection $S^r \to \mathrm{P}^r$, and then set $f(\gamma)$ to be the class of $(\widehat{\gamma}(0), \widehat{\gamma}(1))$. For our purposes, all we need to know is that (6) is a commutative $\mathbb{Z}/2$ equivariant diagram, where the $\mathbb{Z}/2$ action on $S^r \times_{\mathbb{Z}/2} S^r$ switches coordinates (and the $\mathbb{Z}/2$ actions on $P(\mathrm{P}^r)$ and $\mathrm{P}^r \times \mathrm{P}^r$ are the obvious extensions of the respective $\mathbb{Z}/2$ actions on $P_1(\mathrm{P}^r)$ and $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ described in Subsection 1.4). In particular, by restricting to $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ and then passing to $\mathbb{Z}/2$ orbit spaces, (6) yields corresponding triangles

$$\begin{array}{ccc} P_1(\mathrm{P}^r) & \xrightarrow{\;f_1\;} & E_1 \\ & {\scriptstyle \mathrm{ev}_1} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & \mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r} & \end{array} \qquad \begin{array}{ccc} P_2(\mathrm{P}^r) & \xrightarrow{\;f_2\;} & E_2 \\ & {\scriptstyle \mathrm{ev}_2} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & B(\mathrm{P}^r, 2) & \end{array} \tag{8}$$

The next result is the symmetric analogue of (7).

Proposition 2.1. *For $i \in \{1, 2\}$, $\mathfrak{genus}(\mathrm{ev}_i) = \mathfrak{genus}(\pi_i)$.*

Proof. It suffices to construct a fiber-preserving $\mathbb{Z}/2$ equivariant map $g_1\colon E_1 \to P_1(\mathrm{P}^r)$ running backwards in the left triangle of (8). To this end, we use the idea at the beginning of this section. An explicit model for E_1 is the quotient of $S^r \times S^r - \widetilde{\Delta}$ by the relation $(x, y) \sim (-x, -y)$, where $\widetilde{\Delta} \subset S^r \times S^r$ is defined in Subsection 1.3. In these terms, the $\mathbb{Z}/2$ action on E_1 interchanges coordinates. Then, the required map g_1 takes the class of a pair (x_1, x_2) into the curve $[0, 1] \to S^r \to \mathrm{P}^r$, where the second map is the canonical projection, and the first map is given by $t \mapsto \nu(tx_2 + (1-t)x_1)$. Here $\nu\colon \mathbb{R}^{r+1} - \{0\} \to S^r$ is the normalization map. ■

Remark 2.2. The continuity problems avoided in the proof of [12, Proposition 17] with their Lemma 11 are simply not an issue here, as the diagonal Δ_{P^r} has been removed.

The next ingredient replaces the axial map characterization for the sectioning problem of multiples of $\xi \otimes \xi$ (discussed in the paragraph previous to (6)) by Schwarz's identification of the space classifying $\mathbb{Z}/2$ covers of a given genus.

Lemma 2.3 (Corollary 1 pg. 97 in [19])**.** *The canonical* $\mathbb{Z}/2$ *cover* $S^{n-1} \to \mathrm{P}^{n-1}$ *classifies* $\mathbb{Z}/2$ *covers of genus at most* n*.* ■

The word *classifies* in Lemma 2.3 is used in the sense that the $\mathbb{Z}/2$ principal actions on a space X which admit a $\mathbb{Z}/2$ equivariant map $X \to S^{n-1}$ are precisely those for which the canonical projection $X \to X/(\mathbb{Z}/2)$ has $\mathfrak{genus} \leq n$. A similar (ab)use of this word is made in Lemma 3.7.

Proof of Theorem 1.7. Think of the space $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ as the quotient of $S^r \times S^r - \widetilde{\Delta}$ by the relations

$$(-x, y) \sim (x, y) \sim (x, -y). \tag{9}$$

Likewise, and just as in the proof of Proposition 2.1, E_2 is the quotient of $S^r \times S^r - \widetilde{\Delta}$ by the relations

$$(-x, -y) \sim (x, y) \sim (y, x). \tag{10}$$

In these terms, Berrick-Feder-Gitler's trick described in Subsection 1.3 translates into the observation that the extension of (2)

$$S^r \times S^r - \widetilde{\Delta} \xrightarrow{\Psi} S^r \times S^r - \widetilde{\Delta}, \qquad \Psi(x, y) = \big(\nu(x + y)\,, \nu(x - y)\big), \tag{11}$$

where $\nu\colon \mathbb{R}^{r+1} - \{0\} \to S^r$ is the normalization map at the end of the proof of Proposition 2.1, sends relations (9) into relations (10) and vice versa. Moreover, the resulting maps $\Psi'\colon E_2 \to \mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ and $\Psi''\colon \mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r} \to E_2$ are easily seen to be equivariant with respect to the $\mathbb{Z}/2$ action on E_2 coming from π_2 in the right triangle of (8), and on $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r}$ coming from interchanging coordinates.

The result is now a direct consequence of Proposition 2.1 and Lemma 2.3, which characterize $\mathrm{TC}^S(\mathrm{P}^r)-1$ as the smallest integer t for which there is a $\mathbb{Z}/2$ equivariant map $E_2 \to S^{t-1}$, and from Theorem 1.3, which (under the current hypothesis) characterizes $E(r)$ as the smallest integer e for which there is a $\mathbb{Z}/2$ equivariant map $\mathrm{P}^r \times \mathrm{P}^r - \Delta_{\mathrm{P}^r} \to S^{e-1}$. ■

Proof of Proposition 1.5. In view of Theorem 1.1, the axial map hypothesis can be replaced by an immersion $\mathrm{P}^r \looparrowright \mathbb{R}^s$, and we need to prove that, for r as stated, the smallest such s satisfies $2s > 3r+2$. Cases with $r \in \{8, 9, 13\}$ follow from inspection of [6]. For $r > 15$ we revisit the argument in the proof of [1, Lemma 2.1]. Pick $\rho \geq 4$ with $2^\rho \leq r < 2^{\rho+1}$. Each of the cases

- $r \leq 2^\rho + 3$
- $r = 2^{\rho+1} - 1$
- $2^\rho + 2^{\rho-1} + 2 \leq r \leq 2^{\rho+1} - 3$

can be dealt with by the corresponding non-immersion result stated in [1].

Assume $2^\rho + 4 \leq r \leq 2^\rho + 2^{\rho-1} + 1$ and choose $\sigma \in \{1, 2, \cdots, \rho - 2\}$ with $2^\rho + 2^\sigma + 2 \leq r \leq 2^\rho + 2^{\sigma+1} + 1$. From [5], $\mathrm{P}^{2^\rho+2^\sigma+2}$ does not immerse in Euclidean space of dimension $2^{\rho+1} + 2^{\sigma+1} - 4$. Therefore, in the optimal immersion $\mathrm{P}^r \looparrowright \mathbb{R}^s$, we must have $s \geq 2^{\rho+1} + 2^{\sigma+1} - 3$, and this easily yields the required inequality $2s > 3r + 2$ when $\sigma \geq 3$ or $\rho \geq 5$. For the smaller cases with $\rho = 4$ and $1 \leq \sigma \leq 2$, the required $2s > 3r + 2$ follows, as above, from direct inspection of [6].

It remains to consider the case $r = 2^{\rho+1}-2$. As the case $\rho = 4$ follows again from inspection of [6], we assume further $\rho \geq 5$. Let $m = 2^{\rho-1} + 2^{\rho-2} + 2^{\rho-3}$. From [5] we know that $P^{2(m+\alpha(m)-1)}$ does not immerse $\mathbb{R}^{4m-2\alpha(m)}$, where $\alpha(m)$ is the number of ones appearing in the dyadic expansion of m. Therefore, in the optimal immersion $\mathrm{P}^r \looparrowright \mathbb{R}^s$, we must have $s \geq 4m - 2\alpha(m) + 1 = 2^{\rho+1} + 2^\rho + 2^{\rho-1} - 5$, from which one easily deduces the required inequality $2s > 3r + 2$. ■

3 Lens spaces

Unlike the case of real projective spaces, the symmetric topological complexity of a lens space does not seem to be related to its embedding dimension. In retrospect, the problem arises from the fact that the (non-symmetric) topological complexity of $L^{2n+1}(2^e)$ actually differs from the immersion dimension of this manifold, and the difference gets larger as e increases (until it attains a certain stable value, see Remark 3.9). Following the non-symmetric lead, in this section we (a) indicate how one can characterize $\mathrm{TC}^S(L^{2n+1}(2^e))$, and (b) point out concrete differences with respect to a similar characterization for the embedding dimension of $L^{2n+1}(2^e)$. To better appreciate the picture, it will be convenient to make a summary of, and compare to, the known situation in the non-symmetric case.

3.1 *e-axial maps, their symmetric analogues, and embeddings of lens spaces.* The well known relationship (in Theorem 1.1) between Euclidean immersions of real projective spaces and (not necessarily symmetric) axial maps has been generalized in [2] for 2^e torsion lens spaces to prove that, with the possible exceptions of $n = 2, 3, 5$, the existence of an immersion $L^{2n+1}(2^e) \looparrowright \mathbb{R}^m$ is equivalent to the existence of an e-axial map $\mathrm{P}^{2n+1} \times_{\mathbb{Z}/2^{e-1}} \mathrm{P}^{2n+1} \to \mathrm{P}^m$, that is, a map yielding a standard axial map when precomposed with the canonical projection $\mathrm{P}^n \times \mathrm{P}^n \to \mathrm{P}^n \times_{\mathbb{Z}/2^{e-1}} \mathrm{P}^n$ —the notation $\mathrm{P}^n \times_{\mathbb{Z}/2^{e-1}} \mathrm{P}^n$ refers to the usual Borel construction with respect to the standard free $\mathbb{Z}/2^{e-1}$ action on P^{2n+1} with orbit space $L^{2n+1}(2^e)$.

At the level of covering spaces, the e-axial map condition translates into having a map $\widetilde{\alpha}\colon S^{2n+1} \times S^{2n+1} \to S^m$ satisfying the relations

$$\widetilde{\alpha}(\omega x, y) = \widetilde{\alpha}(x, \omega y) \ \text{ and } \ \widetilde{\alpha}(-x, y) = -\widetilde{\alpha}(x, y) \tag{12}$$

for $x, y \in S^{2n+1}$ and $\omega \in \mathbb{Z}/2^e \subset S^1$ —these correspond to the first group of conditions in (1). Our first objective is to indicate how the slight variation in (13) below of the obvious symmetrization of these conditions describes the Euclidean embedding dimension for 2^e torsion lens spaces. To this end, start by observing that the product action of $\mathbb{Z}/2^e \times \mathbb{Z}/2^e$ on the Cartesian product $S^{2n+1} \times S^{2n+1}$ extends to an action of the wreath product $G_e = (\mathbb{Z}/2^e \times \mathbb{Z}/2^e) \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts on $S^{2n+1} \times S^{2n+1}$ by interchanging axes. This action is stable on the orbit configuration space $F_{\mathbb{Z}/2^e}(S^{2n+1}, 2)$ consisting of pairs in $S^{2n+1} \times S^{2n+1}$ generating different $\mathbb{Z}/2^e$ orbits (this is the obvious generalization of the space $S^r \times S^r - \widetilde{\Delta}$ found in Subsection 1.3 and in the proof of Proposition 2.1). The quotient $F_{n,e} = F_{\mathbb{Z}/2^e}(S^{2n+1}, 2)/(\mathbb{Z}/2^e \times \mathbb{Z}/2^e)$ has an involution induced by the action of G_e on the orbit configuration space, and this gives a $\mathbb{Z}/2$ equivariant model for $L^{2n+1}(2^e) \times L^{2n+1}(2^e) - \Delta_{L^{2n+1}(2^e)}$, where $\mathbb{Z}/2$ acts by switching coordinates. In these terms, Theorem 1.3 translates into:

Lemma 3.1. *Assume $m \geq 3(n+1)$. $L^{2n+1}(2^e)$ can be smoothly embedded in $\mathbb{R}^m$ if and only if there is a $\mathbb{Z}/2$ equivariant map $F_{n,e} \to S^{m-1}$.* ∎

Of course, having a $\mathbb{Z}/2$ equivariant map as above is equivalent to having a G_e equivariant map $\widetilde{\alpha}\colon F_{\mathbb{Z}/2^e}(S^{2n+1}, 2) \to S^{m-1}$, where G_e acts on S^{m-1} via the canonical projection $(\mathbb{Z}/2^e \times \mathbb{Z}/2^e) \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2$. Explicitly, $\widetilde{\alpha}$ must satisfy

$$\widetilde{\alpha}(\omega x, y) = \widetilde{\alpha}(x, y) = \widetilde{\alpha}(x, \omega y) \ \text{ and } \ \widetilde{\alpha}(x, y) = -\widetilde{\alpha}(y, x) \tag{13}$$

for $x, y \in S^{2n+1}$ and $\omega \in \mathbb{Z}/2^e \subset S^1$. In the case $e = 1$, the key connection between (13) and the symmetrized version of (12) is given by the ideas in [4], which teach us how to take care of the deleted "equivariant diagonal" in $F_{\mathbb{Z}/2^e}(S^{2n+1}, 2)$. Unfortunately, we have not succeeded in obtaining such a connection for larger values of e. The major problem seems to be given by the apparent lack of a suitable equivariant deformation retraction of $L^{2n+1}(2^e) \times L^{2n+1}(2^e) - \Delta_{L^{2n+1}(2^e)}$ that plays the role of $V_{2n+2,2}$ in the $e = 1$ arguments of [4] described in Subsection 1.3. This problem will reappear, in a slightly different form, in regards to a potential characterization for the symmetric topological complexity of lens spaces in terms of the $\mathbb{Z}/2^e$ biequivariant maps of the next subsection (Remark 3.5 below). It is worth remarking that, in the symmetric $e = 1$ situation of Section 2, we do make an indirect use of this equivariant deformation retraction through the extended map Ψ in (11).

3.2 *$\mathbb{Z}/2^e$ biequivariant maps, their symmetric analogues, and symmetric topological complexity of lens spaces.* As shown in [14], the (non-symmetric) topological complexity of $L^{2n+1}(2^e)$ turns out to be (perhaps one more than) the smallest odd integer $2k-1$ for which there is a $\mathbb{Z}/2^e$ biequivariant map $\widetilde{\alpha}\colon S^{2n+1} \times S^{2n+1} \to S^{2k-1}$, that is, a map satisfying the (stronger than (12)) requirements

$$\widetilde{\alpha}(\omega x, y) = \widetilde{\alpha}(x, \omega y) = \omega\, \widetilde{\alpha}(x, y),$$

for $x, y \in S^{2n+1}$ and $\omega \in \mathbb{Z}/2^e \subset S^1$. Alternatively, if $c\colon S^{2n+1} \to S^{2n+1}$ stands for complex conjugation in every complex coordinate, then by precomposing with $1 \times c$, a $\mathbb{Z}/2^e$ biequivariant map as above can equivalently be defined through the requirements

$$\widetilde{\alpha}(\omega x, y) = \omega\, \widetilde{\alpha}(x, y) = \widetilde{\alpha}(x, \omega^{-1} y). \tag{14}$$

In analogy to the a_S notation introduced at the end of Subsection 1.3 to measure the existence of symmetric axial maps, the following definition (which, up to composition with $1 \times c$, corresponds to the symmetrized version of the number $s(n, e)$ defined in [14]) is intended to measure the existence of symmetric $\mathbb{Z}/2^e$ biequivariant maps.

Definition 3.2. For integers n and e, let $b_S(n, e)$ denote the smallest integer k such that there is a map $\widetilde{\alpha}\colon S^{2n+1} \times S^{2n+1} \to S^{2k-1}$ satisfying (14) and

$$\widetilde{\alpha}(x, y) = \widetilde{\alpha}(y, x) \tag{15}$$

for $x, y \in S^{2n+1}$ and $\omega \in \mathbb{Z}/2^e \subset S^1$.

The next result gives our characterization for the symmetric topological complexity of lens spaces. The proof will be postponed to the end of the subsection.

Theorem 3.3. *The integral part of* $\frac{1}{2}\mathrm{TC}^S(L^{2n+1}(2^e))$ *agrees with the smallest integer* m *such that there is a map* $\widetilde{\alpha}\colon F_{\mathbb{Z}/2^e}(S^{2n+1},2) \to S^{2m-1}$ *satisfying* (14) *and* (15) *for* $x, y \in S^{2n+1}$ *and* $\omega \in \mathbb{Z}/2^e \subset S^1$.

Most of the work in [10] goes into the direction of giving strong lower bounds for TC^S. However, there seems to be a relative lack of suitable upper bounds; the only ones we are aware of are derived, some way or other, from Schwarz's general estimate for the genus of a fibration $F \to E \to B$ in terms of the dimension of B and the connectivity of F ([19, Theorems 5 and 5′]). For instance, in [10, Proposition 10] the upper bound

$$\mathrm{TC}^S(M) \leq 2m+1 \tag{16}$$

is derived for any m-dimensional closed smooth manifold M. In the case $M = L^{2n+1}(2^e)$, Corollary 3.4 below (which is an immediate consequence of Theorem 3.3) offers an alternative to (16) that takes not only dimension into account, but also torsion. Theorem 3.10 below gives a typical example (in the non-symmetric setting, though) of the potential use of this kind of result.

Corollary 3.4. *The integral part of* $\frac{1}{2}\mathrm{TC}^S(L^{2n+1}(2^e))$ *is no greater than* $b_S(n,e)$. ■

Remark 3.5. In the direction of exploring a possible symmetric analogue of the main result in [14], it would be useful to make precise how much the above upper bound differs from being an equality. The main obstruction for such a goal seems to be the apparent lack of an e-analogue of the map Ψ in (2) and (11).

We close this subsection with the proof of Theorem 3.3. As will quickly become clear, the details are formally the same as in the $e = 1$ case. The e analogue of (6), first considered in [12, Theorem 3], reads

$$\begin{array}{ccc} P(L^{2n+1}(2^e)) & \xrightarrow{\quad f \quad} & \left(S^{2n+1} \times S^{2n+1}\right) \big/ \mathbb{Z}/2^e \\ & {\scriptstyle \mathrm{ev}} \searrow \qquad \swarrow {\scriptstyle \pi} & \\ & L^{2n+1}(2^e) \times L^{2n+1}(2^e) & \end{array}$$

The orbit space in the upper right corner is taken with respect to the $\mathbb{Z}/2^e$ diagonal action. The map f, whose definition is the obvious generalization of that in the case $e = 1$, is $\mathbb{Z}/2$ equivariant. In these conditions, the analogue of (8) and the proof of Proposition 2.1 generalize in a straightforward way to produce the following alternative definition of $\mathrm{TC}^S(L^{2n+1}(2^e))$.

Proposition 3.6. $\mathrm{TC}^S\left(L^{2n+1}(2^e)\right) - 1 = \mathfrak{genus}\left(\pi_{2,e}\colon E_{2,e} \longrightarrow B(L^{2n+1}(2^e),2)\right)$. *Here* $E_{2,e}$ *is the quotient of* $F_{\mathbb{Z}/2^e}(S^{2n+1},2)$ *by the relations* $(x,y) \sim (\omega x, \omega y)$ *and* $(x,y) \sim (y,x)$. *Moreover,* $\pi_{2,e}$ *is a* $\mathbb{Z}/2^e$ *cover with* $\mathbb{Z}/2^e$ *acting on* $E_{2,e}$ *as* $\omega\cdot(x,y) \mapsto (\omega x, y)$, *for* $x, y \in S^{2n+1}$ *and* $\omega \in \mathbb{Z}/2^e \subset S^1$. ■

Theorem 3.3 is now a direct consequence of Proposition 3.6 and the following e analogue of Lemma 2.3 (proved in full generality in [19, Corollary 1, pg. 97]).

Lemma 3.7. *The canonical* $\mathbb{Z}/2^e$ *cover* $S^{2n-1} \to L^{2n-1}(2^e)$ *classifies* $\mathbb{Z}/2^e$ *covers of genus at most* $2n$. ■

3.3 *Topological complexity of high torsion lens spaces.* We say that a lens space $L^{2n+1}(2^e)$ is of high torsion when e is larger than $\alpha(n)$, the number of ones in the dyadic expansion of n. The (non-symmetric) topological complexity of a high torsion lens space has recently been settled in [11].

Theorem 3.8. *For* $e > \alpha(n)$, $\mathrm{TC}(L^{2n+1}(2^e)) = 4n + 2$. ■

Remark 3.9. This result is the analogue of the following situation. For a fixed n, the immersion dimension of $L^{2n+1}(2^e)$ is a bounded non-decreasing function of e which, therefore, becomes stable for large e. As explained in [13] and [14, Section 6], the stable value of the immersion dimension is expected to be attained roughly for $e > \alpha(n)$ (with an expected value close to the immersion dimension of the complex projective n-dimensional space). A very concrete situation, which compares TC to the immersion dimension of lens spaces, is illustrated in Example 3.11 below.

We extend Theorem 3.8 to the first case outside the high-torsion range by combining the techniques in [11] with the $\mathbb{Z}/2^e$ biequivariant map characterization of $\mathrm{TC}(L^{2n+1}(2^e))$ discussed at the beginning of Subsection 3.2. The result arose from an e-mail exchange between the first author and Professor Farber in regards to the results in [11].

Theorem 3.10. *For* $e = \alpha(n)$, $\mathrm{TC}(L^{2n+1}(2^e)) = 4n$.

Proof. Proposition 2.2 and Theorem 2.9 in [14] yield $\mathrm{TC}(L^{2n+1}(2^e)) \leq 4n$. The opposite inequality follows by taking $m = 2^{\alpha(n)}$, $k = n$, and $\ell = n - 1$ in [11, Theorem 11], and using the easily verified fact that $\alpha(n) - 1$ is the largest power of 2 dividing the binomial coefficient $\binom{2n-1}{n}$. ■

Example 3.11. The table below summarizes the topological complexity and immersion dimension for $L^{2n+1}(2^e)$ and $\mathbb{C}\mathrm{P}^n$ in the case $n = 2^r + 1$ with $r \geq 1$. The information is taken from [6, 12] in the case of P^{2n+1}, from [13, 20] in the case of the immersion dimension of $L^{2n+1}(2^e)$ for $e \geq 2$, from [12, Corollary 2] in the case of $\mathrm{TC}(\mathbb{C}\mathrm{P}^n)$, and from [3, 18] in the case of the immersion dimension of $\mathbb{C}\mathrm{P}^n$. Note that in the case under consideration $\mathrm{TC}(\mathbb{C}\mathrm{P}^n)$ is just half the stable value of $\mathrm{TC}(L^{2n+1}(2^e))$ (i.e., for $e \geq 3$). Such a behavior comes from the fact that $\mathbb{C}\mathrm{P}^n$ is simply connected and from Schwarz's estimates [19, Theorem 5] for the genus of a fibration.

	P^{2n+1}	$L^{2n+1}(4)$	$L^{2n+1}(2^e)\ \ e \geq 3$	$\mathbb{C}\mathrm{P}^n$
TC	$4n-3\ \ (r \geq 2)$ $4n-4\ \ (r = 1)$	$4n$	$4n+2$	$2n+1$
Imm	$4n-4$	$4n-3$	$4n-2$	$4n-3$

We close this section with what we believe should be an accessible challenge: Determine the symmetric topological complexity of high 2 torsion lens spaces. We remark that the inequalities

$$4n+2 \leq \mathrm{TC}^S(L^{2n+1}(2^e)) \leq 4n+3,$$

for $e > \alpha(n)$, follow from (4), (16), and Theorem 3.8.

4 Complex projective spaces

The (non-symmetric) topological complexity of the n-dimensional complex projective space was computed in [12, Section 3] to be $\mathrm{TC}(\mathbb{C}\mathrm{P}^n) = 2n+1$. In this brief final section we show that the same value holds in the symmetric case.

Theorem 4.1. $\mathrm{TC}^S(\mathbb{C}\mathrm{P}^n) = 2n+1$.

Proof. In view of (4), we only need to show that $\mathrm{TC}^S(\mathbb{C}\mathrm{P}^n) \leq 2n+1$. The diagram of pull-back squares

$$\begin{array}{ccccc}
P(\mathbb{C}\mathrm{P}^n) & \longleftarrow & P_1(\mathbb{C}\mathrm{P}^n) & \longrightarrow & P_2(\mathbb{C}\mathrm{P}^n) \\
\downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_2} \\
\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n & \longleftarrow & \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n - \Delta_{\mathbb{C}\mathrm{P}^n} & \longrightarrow & B(\mathbb{C}\mathrm{P}^n, 2)
\end{array}$$

where horizontal maps on the left are inclusions, and horizontal maps on the right are canonical projections onto $\mathbb{Z}/2$ orbit spaces, shows that the common fiber for the three vertical maps is the path connected space $\Omega\mathbb{C}\mathrm{P}^n$. In particular, Theorem $5'$ in [19] applied to ev_2 gives

$$\mathrm{TC}^S(\mathbb{C}\mathrm{P}^n) = \mathfrak{genus}(\mathrm{ev}_2) + 1 \leq \frac{\dim(Y)}{2} + 2$$

where Y is any CW complex having the homotopy type of $B(\mathbb{C}\mathrm{P}^n, 2)$. The required inequality follows since, as indicated below, there is such a model Y having $\dim(Y) = 4n-2$. ■

In the proof of [10, Proposition 10] it is observed that, for a smooth closed m-dimensional manifold M, $B(M, 2)$ has the homotopy type of a $(2m-1)$-dimensional

CW complex. Although this is certainly enough for completing the proof of Theorem 4.1, we point out that an explicit (and smaller) model for $B(\mathbb{C}\mathrm{P}^n, 2)$ was described by Yasui in [22, Proposition 1.6]. We recall the details. The unitary group U(2) has the two subgroups

T^2: diagonal matrices, and

G : matrices in T^2 together with those of the form $\begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix}$ for $z_1, z_2 \in S^1$.

Consider the standard action of U(2) on the complex Stiefel manifold $W_{n+1,2}$ of orthonormal 2-frames in $\mathbb{C}^{n+1}$ with quotient the Grassmann manifold of complex 2-planes in $\mathbb{C}^{n+1}$. Yasui's model for $B(\mathbb{C}\mathrm{P}^n, 2)$ is the corresponding quotient $W_{n+1,2}/G$. Note that $\dim(G) = \dim(T^2) = 2$, so that the dimension of Yasui's model is

$$\dim(W_{n+1,2}) - 2 = 4n - 2.$$

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