The Index of Discontinuous Vector Fields

by

Daniel H. Gottlieb and Geetha Samaranayake

Introduction

At the frontier between the continuous and the discrete there is a naturally occurring additive, integral "quantum number" which is preserved under "collisions" of discontinuities. This quantum number depends only on the basic topological notions of compactness, connectedness, dimension, and the concept of pointing inside.

We assume we are in a smooth manifold N. A vector field is an assignment of tangent vectors to some, not necessarily all, of the points of N. We make no assumptions about continuity. We will call this N the *arena* for our vector fields. We consider the set of defects of a vector field V in N, that is the set D which is the closure of the set of all zeros, discontinuities and undefined points of V. That is we consider a *defect* to be a point of N at which V is either not defined, or is discontinuous, or is the zero vector, or which contains one of those points in every neighborhood.

We are interested in the connected components of the defects and how they change in time. Those connected components of D which are compact we will call topological *particles*. If we can find an open set about a particle which does not intersect any defect not in the particle itself, then we say the particle is *isolated*. If C is an isolated particle we can assign an integer which we call the index of C in V. We denote this by $\text{Ind}(C)$.

The key properties of $\text{Ind}(C)$ are that it is nontrivial, additive over particles, easy to calculate and is conserved under interactions with proper components as V varies under time. For example, let V be the electric vector field generated by one electron in R^3. Then the position of the electron e is the only defect and $\text{Ind}(e) = -1$. Now if V changes under time in such a way that there are only a finite number of particles at each time, all contained in some large fixed sphere, then the sum of the indices of the particles at each time t is equal to -1. Thus the electron vector field can change to the proton vector field only if the set of defects changing under time is unbounded, since the proton has index $+1$ which is different from the index of the electron. In this case we will say that the transformation of the electron to the proton involves " topological radiation".

Vector fields varying under time, and defect components interacting with each other, can be made precise by introducing the concept of *otopy*, which is a generalization of the concept of homotopy. An *otopy* is a vector field on $N \times I$ so that each vector is tangent to a slice $N \times t$. Thus an otopy is a vector field W on $N \times I$ so that $W(n, t)$ is tangent to $N \times t$. We say that V_0 is otopic to V_1 if $V_0(n) = W(n, 0)$ and $V_1(n) = W(n, 1)$. We say that a set of components C_i of defects on V_0 *transforms* into a set of components of defects D_j of V_1 if there is a connected component T of the defects of W so that $T \cap (N \times 0) = \cup C_i$

and $T \cap (N \times 1) = \cup D_j$. If T is a compact connected component of defects of W, which transforms a set of isolated particles C_i into isolated particles D_j, then we say there is no topological *radiation* and

$$(1) \qquad\qquad \sum \text{Ind}(C_i) = \sum \text{Ind}(D_j).$$

If T is not compact, we say there is topological radiation.

We define $\text{Ind}(C)$ as follows. Since C is an particle, there is an open set U containing C so that there are no defects in the closure of U except for C. We can define an index for any vector field defined on the closure of an open set so that the set of defects is compact and there is no defect on the frontier of the open set. We say such a vector field is *proper* with *domain* the open set. In the case at hand, V restricted to $\text{cl}(U)$ is proper with domain U. Hence we can define $\text{Ind}(V|U)$. We set $\text{Ind}(C) = \text{Ind}(V|U)$.

Next we define $\text{Ind}(V)$ with domain U to be equal to the index of $V|M$ where $M \subset U$ is a smooth compact manifold with boundary containing the defects of V in its interior. We can find such an M since the defects are a compact set in U.

We call a vector field V defined on a compact manifold M *proper* if there are no defects on the boundary. Consider the open set of the boundary where V points inside. We denote that set by $\partial_- M$. We define the vector field $\partial_- V$ with domain $\partial_- M$ in the arena ∂M by letting $\partial_- V$ be the end product of first restricting V to the boundary and then projecting each vector so that it is tangent to ∂M which results in a vector field ∂V tangent to ∂M, and then finally restricting ∂V to $\partial_- M$ to get $\partial_- V$. Then we define $\text{Ind}(V)$ by the equation

$$(*) \qquad\qquad \text{Ind}(V) = \chi(M) - \text{Ind}(\partial_- V)$$

where $\chi(M)$ denotes the Euler-Poincare number of M. We know that $\partial_- V$ is a proper vector field with domain $\partial_- M$ since the set of defects is compact unless there is a defect at the the frontier of $\partial_- M$. If there were such a defect, it would be a zero of V tangent to ∂M and hence a zero of V on the boundary, so V would not have been proper.

Now $\partial_- V$ is a proper vector field with domain the open set $\partial_- M$ which is one dimension lower than M. Then $\text{Ind}(\partial_- V)$ is defined in turn by finding a compact manifold containing the defects of $\partial_- V$ and using equation (*). We continue this process until either $\partial_- M$ is a zero dimensional manifold where every point is a defect and so $\text{Ind}(\partial_- V)$ is simply the number of points, or where $\partial_- M$ empty in which case $\text{Ind}(\partial_- V) = 0$.

To summarize, we define the index of a proper vector field V with domain U assuming that the index for vector fields is already defined for compact manifolds with boundary. Then the index of V is defined to be the index of V restricted to a compact smooth manifold with boundary of codimension zero containing all the defects of V in U. We will show this definition is well-defined, that is it does not depend on the chosen manifold with boundary, by showing that a vector field with no defects defined on a compact manifold with boundary has index zero.

2

The well-definedness of this definition will involve the first four sections of this paper. In section 5 we summarize the useful properties of the index which we have proved along the way, along with a few proved in other papers. The key property is that of a *proper otopy* described below.

Suppose that V is a proper vector field with open domain U. A *proper otopy* is a proper vector field W defined on $N \times I$ with domain an open set where we require W to be tangent to the slices. Then we say W is a proper otopy of V if V is the restriction of W to $N \times 0$ and the domain of W intersects $N \times 0$ in U. The key property of the index of proper vector fields with open domains is that the index is invariant under proper otopy. For connected manifolds the converse is true: Two proper vector fields are properly otopic if and only if they have the same index.

We may generalize the concept of otopy in two ways. Recall an otopy is an open set T on $N \times I$ with a vector field W which is tangent to the slices. Now this can be generalized by considering a fibre bundle $E \to B$ with fibre N and an open set T on E and a vector field W whose vectors are tangent to the fibre. It is clear that if W is a proper vector field, that is the defects form a compact set and there are no defects on the frontier of T, then W restricted to any fibre has an index. This index is the same for every fibre. In [B-G], for the case of continuous W, it is shown that there is an S-map which induces a transfer on homology with trace equal to this index.

The second way to generalize an otopy is to note that $N \times I$ can be thought of as a manifold S with a natural non-zero vector field. Then W is a vector field which is orthogonal to this vector field. In fact any vector field can be projected orthogonal to the natural vector field. If S is a space-time, there is a field of light cones. If we consider a space-like vector field W on S, it is like an otopy. W restricts to any space-like slice and projects tangent to it. The index of the defects at any event is thus an invariant of general relativity, it is invariant under any change of coordinate system. The defects propagate through space-time and the index satisfies a conservation law, just like the conservation law of electric charges under particle collisions. It is very easy to believe that the index of a vector field, as here exposed, must lead to an explanation of the conservation of physical properties under collision based on the idea of connectivity and continuity and pointing inside.

1. The definition for one-dimensional manifolds

The inductive definition begins with empty vector fields, that is domains which are empty. This could arise since $\partial_- M$ is empty if V never points inside from the boundary. We define the index of an empty vector field to be equal to zero. Zero dimensional manifolds consist of discrete sets of points. The only vectors are zero vectors, so for a vector field to be proper it must consist of a finite number of zeros. One-dimensional compact manifolds with boundary consist of a finite disjoint union of compact components which are compact

intervals. We use the definition (*), that is

$$\text{Ind}(V) = \text{ (number of components) } - \text{ (number of boundary}$$
$$\text{points where } V \text{ is pointing inwards)}.$$

In the case of components without boundaries, circles in this case, we define the index to be $\chi(\text{circle}) = 0$.

LEMMA 1.1. *Two vector fields V and V' are properly otopic if and only if*

$$\text{Ind}(\partial_- V) = \text{Ind}(\partial_- V') \text{ on each component of the boundary.}$$

Proof. Let W be a vector field so that $W(m) = V(m)/\|V(m)\|$ for m on the boundary of M. Assume that $W(m) = 0$ outside a collar of the boundary, and assume that W continuously decreases in size from the unit vectors on the boundary to the zero vectors at the other end of the collar. Then we define the homotopy $tV + (1-t)W$. This is a proper homotopy, since at any point m on the boundary $V(m)$ and $W(m)$ both point either inside or outside so no zero can arise on the boundary. If V should have a defect at some m in the interior, we may alter V by assigning $V(m) = 0$. Thus the homotopy is defined. Now both V and V' are properly otopic to W, hence they are otopic to each other.

LEMMA 1.2. *If M is a finite collection of manifolds with boundary and f is a diffeomorphism so that the related vector field is denoted by V^*, then*

$$\text{Ind}(V) = \text{Ind}(V^*).$$

Proof. Pointing inside is preserved under diffeomorphism.

LEMMA 1.3. *If V has no defects, then $\text{Ind}(V) = 0$.*

Proof. Each connected component of M is an interval. Since V has no defects on this interval, V must point outside on one end and inside on the other. Thus $\text{Ind}(V) = 1-1 = 0$ on this interval, and thus on all the intervals. So $\text{Ind}(V) = 0$ is true for M.

Now suppose that the arena is a connected manifold N with no boundary and not compact. Thus an open interval. Then we define the index of V with open domains to be the index of V restricted to a union of compact intervals which contain the defects of V. This is well-defined. If M and M' are two manifolds with boundary containing the defects, there is a compact manifold with boundary M'' containing both M and M'. The vector field V restricted to $M'' - \text{int}(M)$ is a nowhere zero vector field, and the previous lemma and the fact that the index is additive proves that the index is well-defined.

Next we deal with the case of the arena N being a closed manifold, in this case that is a finite set of circles. We will consider the case of a single circle, the general case will be given

by adding the indices for each connected component. The set of defects is closed. If the defects can be contained in a compact manifold with boundary, in this case diffeomorphic to a closed interval, we define the index of V to be the index of V restricted to the compact manifold. On the other hand, if the domain of V is the entire arena, then we define

$$\text{Ind}(V) = \chi(\text{arena}) - \text{Ind}(\partial_- V) = \chi(\text{circle}) - \text{Ind}(\text{empty vector field}) = 0.$$

These two definitions are consistent. If V has domain the entire circle, then it is properly homotopic to the zero vector field. Then we homotopic the zero vector field to V' which is zero inside a large closed interval and not zero around a point with the vectors thus forced to point in the same sense around the circle. Then V' restricted to the large closed interval has index zero which is just what the global definition gives.

We make a few more observations before we finish with the one-dimensional case.

LEMMA 1.4. *Given a connected arena N, two proper vector fields are properly otopic if and only if they have the same index. For every integer n there is a vector field whose index equals that integer.*

Proof. Suppose we have a proper otopy W with domain T on $N \times I$. Let V_t denote W restricted to $N \times t$. We show that there is some interval about t such that V_s has the same index for all s in the interval. Since the set of defects of the otopy is compact we can find a compact manifold M so that $M \times J$, for some closed interval J, lies in T and contains the defects inside $\partial M \times J$. Thus the proper homotopy V_t on $M \times J$ preserves the index on M, and hence the proper otopy on $N \times J$ preserves the index on N as t runs over J. Thus we have a finite sequence of vector fields each having the same index as the previous vector field. Hence the first and last vector fields have equal indices. Conversely, for any integer n, let W_n be the vector field consisting of $|n|$ vector fields defined on disjoint open intervals in N, each one of index 1 if $n > 0$ and of index -1 if $n < 0$. Thus $\text{Ind}(W_n) = n$. Now if V has index n, we must show that V is properly homotopic to W_n. Now the domain of V consists of open connected intervals, and only a finite number of them contain defects. Each of these intervals has index equal to 1, -1, or 0. Now V is properly otopic to the same vector field V whose domain is restricted to only those intervals which have nonzero indices. Now if two adjacent intervals have different indices, there is a proper otopy which leaves the rest of the vector field fixed, and removes the two intervals of opposite indices. After a finite number of steps we are left with either an empty vector field, if $n = 0$, or a W_n. The empty vector field is W_0. Thus V is properly otopic to W_n.

LEMMA 1.5. *The index of a vector field on an open manifold is invariant under diffeomorphism.*

Proof. Immediate from Lemma 1.2 and the definition of index for open manifolds.

LEMMA 1.6. *Let V be a vector field over a domain U and suppose that U is the disjoint union of U_1 and U_2. Then if V_1 and V_2 denote V restricted to U_1 and U_2 respectively, we have*

$$Ind(V) = Ind(V_1) + Ind(V_2).$$

2. The index defined for compact n-manifolds

THE OTOPY EXTENSION PROPERTY. *Let V be a continuous vector field on a closed manifold N. Let U be an open set in N. Any continuous proper otopy of V on the domain U can be extended to a continuous homotopy of V on all of N.*

Proof. The continuous proper otopy implies there is a continuous vector field W on an open set T in $N \times I$ which extends to the closure of T with no zeros on the frontier and which is V when restricted to $N \times 0$. This vector field W can be thought of as a cross-section to the tangent bundle over $N \times I$ defined over a closed subset. It is well known that cross-sections can be extended from closed sets to continuous cross-sections over the whole manifold.

We assume that the index is defined for $(n-1)$-manifolds in such a way that all the lemmas of section 1 hold.

First we consider the case of compact manifolds such that every component is a manifold with boundary. We suppose that V is a proper vector field on such a manifold M. We choose a vector field N on the boundary ∂M which points outside of M. Every vector v at a point m on ∂M can be uniquely written as $v = t + kN(m)$ where t is a vector tangent to ∂M and k is some real number. We say t is the *projection* of v tangent to ∂M. Then ∂V is the vector field obtained by projecting V tangent to ∂M. Now we define $\partial_- V$ by restricting ∂V to $\partial_- M$, the set of points such that V is pointing inward. Then we define

(*) $$\mathrm{Ind}(V) = \chi(M) - \mathrm{Ind}(\partial_- V).$$

LEMMA 2.1. *$Ind(V)$ is well-defined.*

Proof. We have already defined the index on $(n-1)$-dimensional manifolds with open domains for proper vector fields. Note that $\partial_- V$ is proper since V is, since the frontier of $\partial_- M$ is a subset of $\partial_0 M$ where V is tangent to ∂M. So a defect of $\partial_- V$ on the frontier must come from a defect of V on ∂M. Hence $\mathrm{Ind}(\partial_- V)$ is defined. Now the vector field $\partial_- V$ obviously depends upon the outward pointing N. If we had another outward pointing vector field N' we would project down to a different $\partial_- V$, call it W. Now the homotopy of

vector fields $N_t = tN + (t-1)N'$ always points outside of M for every t. Hence it induces a homotopy from $\partial_- V$ to W and this homotopy is proper. Thus $\text{Ind}(\partial_- V) = \text{Ind}(W)$.

We will also allow the case where N is not defined on a closed set of ∂M which is disjoint from the frontier of $\partial_- M$. Then ∂V has defects, but $\partial_- V$ is still proper. A homotopy between N and N', as in the lemma, still induces a proper otopy between $\partial_- V$ and W, so the $\text{Ind}(V)$ is still well-defined in this case also. This case arises when M is embedded as a co-dimension zero manifold in such a way that it has corners. Then the natural outward pointing normal in this situation is not defined on the corners. But we still have the index defined if none of the corners is on the frontier of $\partial_- M$.

Now our goal is to prove that non-zero vector fields have index equal to zero on compact manifolds with boundary.

THEOREM 2.2. *V is properly otopic to W if and only if*

$$\text{Ind}(\partial_- V) = \text{Ind}(\partial_- W)$$

for every connected component of ∂M. So as a corollary in the case that ∂M is connected, we have that V is properly otopic to W if and only if $\text{Ind}(V) = \text{Ind}(W)$. If V and W are both continuous, then "otopic" can be replaced by "homotopic" in the above statements.

Proof. The theorem is true for manifolds one dimension lower by lemma 1.1. A proper otopy of V to W induces a proper otopy from $\partial_- V$ to $\partial_- W$ in the arena ∂M. Hence $\text{Ind}(\partial_- V) = \text{Ind}(\partial_- W)$. Hence $\text{Ind}(V) = \text{Ind}(W)$ from $(*)$. Conversely, we can find a smooth collar $\partial M \times I$ of the boundary so that V restricted to this collar has no defects. Then we otopy V to V' where V' is defined by $V'(m, t) = tV(m)$ for a point in the collar and $V' = 0$ outside the collar. Now since $\text{Ind}(\partial_- V) = \text{Ind}(\partial_- W)$ for each connected component of the boundary, we can find a proper otopy from $\partial_- V$ to $\partial_- W$. Now this otopy can be extended to a homotopy of ∂V to ∂W by the otopy extension property. This homotopy in turn can be used to define a proper homotopy from V' to W'. Here we assume W' has the same definition relative to W as V' has to V. Thus W is properly otopic to V.

LEMMA 2.3. *Suppose V is a proper vector field on a compact manifold M each of whose components has a non-empty boundary. Let $\partial M \times I$ be a collar of the boundary so small so that V has no defects on the collar. Then V restricted to M minus the open collar $\partial M \times (0, 1]$ has the same index as V.*

Proof. Let ∂V_t denote the projection of V tangent to the submanifold $\partial M \times t$ for every t in I. Let W be the vector field on the collar defined by $W(m, t) = \partial_- V_t$ if (m, t) is a point in $\partial_- M \times t$. Then W is a proper otopy, proper since V has no defects on the collar. Thus $\text{Ind}(\partial_- V) = \text{Ind}(\partial_- V_0)$ and hence $\text{Ind}(V) = \chi(M) - \text{Ind}(\partial_- V)$ equals the index of V restricted to $M' = M - open\ collar$, because the indices of the ∂_- vector fields are the same on their respective boundaries and $\chi(M) = \chi(M')$.

LEMMA 2.4. *Let V be a proper continuous vector field on M. Suppose that $\partial_- V$ is properly otopic to some vector field W on ∂M. Then there is a proper homotopy of V to a proper continuous vector field X so that $\partial_- X = W$ and the zeros of each stage of the homotopy V_t are not changed.*

Proof. Use the otopy extension property to find a homotopy H_t from ∂V to a vector field on ∂M which we shall call ∂X. Let $n(m, t)$ be a continuous real valued function on $\partial M \times I$ which is positive on the open set T of the otopy between $\partial_- V$ and W, zero on the frontier of T, and negative in the complement of the closure of T, and so that $n(m, 0) = n(m)$ where $V(m) = n(m)N(m) + \partial V(m)$ defines $n(m)$. Such a function exists by the Tietze extension theorem. Using $n(m, t)$, we define a vector field X' on $\partial M \times I$ by $X'(m, t) = n(m, t)N(m) + H_t(m)$. We adjoin the collar to M as an external collar and extend the vector field V by X' to get the continuous vector field X. Now M with the external collar is diffeomorphic to M. Under this diffeomorphism X becomes a vector field which we still denote by X. We may assume this diffeomorphism was so chosen that $X = V$ outside of a small internal collar. Then the homotopy $tX + (1-t)V$ is the required homotopy which does not change the zeros of V.

LEMMA 2.5. *If V is a vector field with no defects on an n-ball, then $\mathrm{Ind}(V) = 0$.*

Proof. For the standard n-ball of radius 1 and center at the origin, we define the homotopy $W_t(r) = V(tr)$. This homotopy introduces no zeros and shows that V is homotopic to the constant vector field. The constant vector field has index equal to zero, as can be seen by using $(*)$. If we have a ball diffeomorphic to the standard ball, then the index of the vector field under the diffeomorphism is preserved, and hence it has the zero index. If the ball is embedded with corners so that the corners are not on the frontier of the set of inward pointing vectors of V, then the index is defined and by lemma 2.3 it is equal to the index of V restricted to a smooth ball slightly inside the original ball. This index is zero.

THEOREM 2.6. *If V is a vector field with no defects on a compact manifold such that all the components have non-empty boundary, then $\mathrm{Ind}(V) = 0$.*

Proof. Now M can be triangulated and suppose we have proved the theorem for manifolds triangulated by $k - 1$ n-simplicies. The previous lemma proves the case $k = 1$. We divide M by a manifold L of one lower dimension into manifolds M_1 and M_2 each covered by fewer than k n-simplicies so that the theorem holds for them.

We arrange it so that L is orthogonal to ∂M. We use lemma 2.4 to homotopy V to a vector field with no defects so that the new V is pointing outside orthogonally to ∂M at $L \cap \partial M$. Then a simple counting argument shows that $\mathrm{Ind}(V) = 0$ since the restrictions of V to M_1 and M_2 have index zero. This argument works if M has no corners. If M has corners we find a collar of M which is a smooth embedding of $\partial M \times t$ for all t but the last $t = 1$. Then by lemma 2.3 above, we find that V, restricted to the manifold bounded by $\partial M \times t$ for t close enough to 1, has the same index as V. That is zero.

The counting argument goes as follows. By induction, $\text{Ind}(V|M_1) = \text{Ind}(V|M_2) = 0$. Thus $\text{Ind}(\partial_- V_1) = \chi(M_1)$ and $\text{Ind}(\partial_- V_2) = \chi(M_2)$. Now $\text{Ind}(\partial_- V) = \text{Ind}(\partial_- V_1) + \text{Ind}(\partial_- V_2) - \text{Ind}(W)$ where W is the projection of V on the common part of the boundary of M_1 and M_2, that is L. This follows from repeated applications of lemma 1.6. Now $\text{Ind}(W) = \chi(L)$ since W points outwards at the boundary of L. Hence

$$\text{Ind}(\partial_- V) = \text{Ind}(\partial_- V_1) + \text{Ind}(\partial_- V_2) - \text{Ind}(W) = \chi(M_1) + \chi(M_2) - \chi(L) = \chi(M).$$

Hence $\text{Ind}(V) = 0$ from $(*)$.

3. The index for open n-manifolds

Let N be an n-manifold and let V be a proper vector field on N with domain U. Then the set of defects of V in U is compact. Thus we can find a compact manifold M which contains the defects of V. We define

$$(**) \qquad\qquad\qquad \text{Ind}(V) = \text{Ind}(V|M).$$

LEMMA 3.1. *$\text{Ind}(V)$ is well-defined.*

Proof. If M and M' are two manifolds with boundary containing the defects, there is a compact manifold with boundary M'' containing both M and M'. The vector field V restricted to $M'' - \text{int}(M)$ is a nowhere zero vector field. Then Theorem 2.6 implies that the index of V restricted to $M'' - \text{int}(M)$ is zero. Now the index of V restricted to M'' equals the index of V restricted to M by the following lemma.

LEMMA 3.2. *Suppose M is the union of two manifolds M_1 and M_2 where the three manifolds are compact manifolds with boundary so that the intersection of M_1 and M_2 consist of part of the boundary of M_1 and is disjoint from the boundary of M. Suppose that V is a proper vector field defined on M which has no defects on the boundaries of M_1 and M_2. Then $\text{Ind}(V) = \text{Ind}(V_1) + \text{Ind}(V_2)$ where $V_i = V|M_i$.*

Proof.

$$\begin{aligned} \text{Ind}(V) &= \chi(M) - \text{Ind}(\partial_- V) \\ &= \chi(M) - (\text{Ind}(\partial_- V_1) + \text{Ind}(\partial_- V_2) - \text{Ind}(\partial_- V_1|L) - \text{Ind}(\partial_- V_2|L)) \end{aligned}$$

where $L = M_1 \cap M_2$. Now

$$\text{Ind}(\partial_- V_1|L) + \text{Ind}(\partial_- V_2|L) = \text{Ind}(\partial_- V_1|L) + \text{Ind}(\partial_+ V_1) = \chi(L).$$

Thus

$$\text{Ind}(V) = \chi(M_1) + \chi(M_2) - \text{Ind}(\partial_- V_1) - \text{Ind}(\partial_- V_2) = \text{Ind}(V_1) + \text{Ind}(V_2),$$

as was to be proved.

LEMMA 3.3. *Let V be a proper vector field with domain U. Suppose U is the union of two open sets U_1 and U_2 such that the restriction of V to each of them and to $U_1 \cap U_2$ is a proper vector field denoted V_1 and V_2 and V_{12} respectively. Then*

$$(\text{***}) \qquad \qquad Ind(V) = Ind(V_1) + Ind(V_2) - Ind(V_{12}).$$

Proof. We choose disjoint compact manifolds M_1, M_2, and M_{12} containing the zeros of V which lie in $U_1 - U_{12}$ and $U_2 - U_{12}$ and U_{12} respectively. Then the index of V is equal to the index of V restricted to the union of M_1, M_2, and M_{12}. But the index of V_1 is the index of V restricted to M_1 and M_{12}, and the index of V_2 is the index of V restricted to M_2 and M_{12}, and the index of V_{12} is the index of V restricted to M_{12}. Hence counting the index gives the equation $(\ast\ast\ast)$.

THEOREM 3.4. *Given a connected arena N, two proper vector fields are properly otopic if and only if they have the same index. For every integer n there is a vector field whose index equals that integer.*

Proof. Suppose we have a proper otopy W with domain T on $N \times I$. Let V_t denote W restricted to $N \times t$. We show that there is some interval about t such that V_s has the same index for all s in the interval. Since the set of defects of the otopy is compact we can find a compact manifold M so that $M \times J$, for some closed interval J, lies in T and contains the defects so that the defects avoid $\partial M \times J$. Thus the proper homotopy V_t on $M \times J$ preserves the index on M, and hence the proper otopy on $N \times J$ preserves the index on N as t runs over J. Thus we have a finite sequence of vector fields each having the same index as the previous vector field. Hence the first and last vector fields have equal indices.

Conversely, for any integer k, let W_k be the vector field consisting of $|k|$ vector fields defined on disjoint open balls in N, each one of index 1 if $k > 0$ or of index -1 if $k < 0$. Thus $Ind(W_k) = k$. Now if V has index k, we must show that V is properly homotopic to W_k. Now the defects of V form a compact set which are contained in a compact manifold with boundary M so that V is defined and has no defects on the boundary. We may proper otopy V first to a continuous vector field, and then to a smooth vector field. Then we consider V as a cross-section to the tangent bundle of M. Using the transversality theorem, we can smoothly homotopy the cross-section so that it is transversal to the zero section of the tangent bundle keeping the cross-section fixed over the boundary. The dimensions are such that the intersection consists of a finite number of points. Thus we proper otopy V to a vector field with only a finite number of zeros. Now we put small open balls around each of these zeros. The index of the vector field on the ball around each of these zeros is either 1 or -1. This follows from transversality, but we do not need that fact. We may find a diffeomorphic n-ball which contains exactly $|k|$ zeros so that around these zeros the vector field restricts to W_k. The two vector fields have the same index on the n-ball and thus are properly homotopic, since from $(\ast)$ the index on the boundary of

10

the inward pointing ∂_- vector fields is the same, and so by induction they are properly otopic, hence by the otopy extension property the ∂ vector fields are homotopic. This homotopy can be extended to a homotopy of the two vector fields originally on the n-ball. Then using the sequence of homotopies and otopies, we can piece together a proper otopy of V to W_k.

COROLLARY 3.5. *The proper homotopy classes of continuous proper vector fields on a compact manifold with connected boundary is in one-to-one correspondence with the integers via the index.*

LEMMA 3.8. *The index of a vector field on an open manifold is invariant under diffeomorphism.*

LEMMA 3.9. *The index of a vector field V on a closed manifold M whose domain is the whole of M is equal to $\chi(M)$.*

Proof. First otopy V to the zero vector field. Then homotopy the zero vector field to a vector field V' so that it is a non-zero vector field on a small n-ball B about a point. Now let V_1 be V' on the n-ball and let V_2 be V' on the complement. Then $\text{Ind}(V_1) = 0$, so $\text{Ind}(\partial_- V_1) = 1$. Now $\text{Ind}(\partial_- V_2) = (-1)^{n-1}$. So

$$\text{Ind}(V_2) = \chi(M - B) - (-1)^{n-1} = \chi(M) - (-1)^n - (-1)^{n-1} = \chi(M).$$

Hence $\text{Ind}(V) = \text{Ind}(V_1) + \text{Ind}(V_2) = 0 + \chi(M)$.

4. The Index of particles

Let V be a vector field on an arena N. Let D be the set of defects of V. Then D breaks up into a set of connected components D_i. We define an index for each component D_i which is compact and is an open set in the subspace topology of D. That is, in the terminology of the Introduction, we define the index of an isolated particle. For isolated particles we can find a compact manifold M containing D_i and no other defects. Then we define

$$(****) \qquad\qquad \text{Ind}(D_i) = \text{Ind}(V|M).$$

Now if we have a finite number of particles D_i in the domain of V, then $\text{Ind}(V) = \sum_i \text{Ind}(D_i)$. However it is possible that V is a proper vector field and there are an infinite number of D_i. Then at least one of the D_i is not isolated in D. But the index of V is still defined. This event is very rare in practical situations. A one dimensional example occurs when M is the interval $[-1, 1]$ and the vector field V is defined by $V(x) = x\sin(1/x)$ for $x \neq 0$ and $V(0) = 0$. Then 0 is a connected component of the defects which is not open in the set of zeros of V.

If we have an otopy V_t, we imagine the components of the defects D_t as changing under time. We can say that D_{ti} at time t transforms without radiation into D_{sj} at time s if there is a compact connected component T of the defects of the otopy from time t to time s so that T intersects $N \times t$ in exactly D_{ti} and T intersects $N \times s$ exactly at D_{si}. The index of D_{ti} is the same as the index of D_{sj} if T is compact. In other words if a finite number of particles D_i at time t are transformed into a finite number of particles C_j at time s by a compact T, the sum of the indices are conserved. That is

$$(1) \qquad \sum \mathrm{Ind}(C_i) = \sum \mathrm{Ind}(D_j).$$

Thus the idea of otopy allows us to make precise the concept of defects moving with time and changing with time and undergoing collisions. The index is conserved under these collisions as long as the "world line" T of the component is compact. That is, as long as there are is no radiation.

5. Properties of the Index

$(2) \quad \mathrm{Ind}(V) + \mathrm{Ind}\,\partial_- V = \chi(M)$

This is in fact the equation (*) which defines the index.

$(3) \quad$ Let N be a connected arena. V is a properly otopic to W if and only if $\mathrm{Ind}\,V = \mathrm{Ind}\,W$. For any integer n there is a vector field W so that $n = \mathrm{Ind}\,W$.

$(4) \quad$ Suppose M is a compact manifold so that ∂M is connected, and suppose V and W are continuous proper vector fields on M. Then V is properly homotopic to W if and only if $\mathrm{Ind}\,V = \mathrm{Ind}\,W$. For any integer n there is a continuous proper vector field W so that $n = \mathrm{Ind}\,W$.

$(5) \quad$ If M is a closed compact manifold and V is a vector field whose domain is all of M, then $\mathrm{Ind}\,V = \chi(M)$.

Proof. Property (3) and (4) are Theorem 3.4 and Corollary 3.5 respectively for the homotopy part. For the fact that $n = \mathrm{Ind}\,W$ for some vector field W, we apply (2) and induction starting with Lemma 1.4. The proof of (5) is Lemma 3.9.

$(6) \quad$ Let A and B be open sets and let V be a proper vector field on $A \cup B$ so that $V|A$ and $V|B$ are also proper. Then $\mathrm{Ind}(V|A \cup B) = \mathrm{Ind}(V|A) + \mathrm{Ind}(V|B) - \mathrm{Ind}(V|A \cap B)$.

Proof of (6). Lemma 3.3

$(7) \quad$ Suppose V us a vector field with no defects. Then $\mathrm{Ind}\,V = 0$.

Proof. Theorem 2.6 for compact manifolds with boundary.

(8) Suppose V is a proper vector field and the set of defects consists of a finite number of connected components D_i. Then Ind $V = \sum_i \text{Ind}(D_i)$.

Proof. This follows from the definition of $\text{Ind}(D_i)$ and (3).

(9) Let V and W be proper vector fields on A and B respectively. Let $V \times W$ be a vector field on $A \times B$ defined by $V \times W(s,t) = (V(s), W(t))$. Then $\text{Ind}(V \times W) = (\text{Ind } V) \cdot (\text{Ind } W)$.

Proof. We can assume that A and B are open sets in their arenas. Then V is otopic to V_n where V_n is restricted to a finite set of open sets in A homeomorphic to the interior of I^k when $k = \dim A$ and so that $V_n(t_1, \ldots, t_k) = (\pm t_1, t_2, \ldots, t_k)$ where the $+t_1$ is taken if $\text{Ind } V$ is positive and $-t_1$ is taken if $\text{Ind } V$ is negative. The index of the $V_n|I_k$ is ± 1 respectively (by induction on (9)). So $\text{Ind } (V \times W) = (\text{Ind } V_n \times W_n) = \sum_{i,j} \text{Ind}(V_n|I_i^k) \times (W_n|I_j^\ell)$. Now it is easy to see that $\text{Ind}(V_n|I_i^k) \times (W_n|I_j^\ell)) = \text{Ind}(V_n|I_i^k) \cdot \text{Ind}(W_n|I_j^k))$.

(10) $(-1)^n \text{Ind}(V) = \text{Ind}(-V)$ where $n = \dim M$.

Proof. The theorem is true for $n = 1$. Assume it is true for $(n-1)$-manifolds. Now using (2) we have

$$\begin{aligned}
\text{Ind}(-V) &= \chi(M) - \text{Ind}(\partial_-(-V)) \quad \text{by (2)} \\
&= \chi(M) - \text{Ind}(-\partial_+ V) \quad \text{by definition of } \partial_- V \text{ and } \partial_+ V \\
&= \chi(M) - (-1)^{n-1}\text{Ind}(\partial_+(V)) \quad \text{by induction} \\
&= \chi(M) + (-1)^n(\chi(\partial M) - \text{Ind}(\partial_- V))
\end{aligned}$$

since

$$\chi(\partial M) = \text{Ind}(\partial_- V) + \text{Ind}(\partial_+ V).$$

If n is even then

$$\text{Ind}(-V) = \chi(M) + (0 - \text{Ind}(\partial_- V)) = \text{Ind } V \quad \text{by (2)}.$$

If n is odd then

$$\begin{aligned}
\text{Ind}(-V) &= \chi(M) - (2\chi(M) - \text{Ind}(\partial_- V)) \\
&= -(\chi(M) - \text{Ind}(\partial_- V)) = -\text{Ind } V \quad \text{by (2)}
\end{aligned}$$

(11) Suppose M is a compact sub-manifold of $\mathbb{R}^n$ of 0-codimension. Let $f : M \to \mathbb{R}^n$ be a map so that $f(\partial M)$ does not contain the origin. Define a proper vector field V^f on M by $V^f(m) = f(m)$. Then $\text{Ind } V^f = deg\ f'$ where $f' : \partial M \to S^{n-1}$ by $f'(m) = \frac{f(m)}{\|f(m)\|}$.

Proof. We homotopy f if necessary so that $\vec{0}$ is a regular value. Then $f^{-1}(\vec{0})$ is a finite set of points. There is a neighborhood of $f^{-1}(0)$ of small balls so that $f : \partial(\text{ball}) \to \mathbb{R}^n - 0 \cong$

S^{n-1}. Now, in each of these small balls, f has either degree 1 or -1. If degree equals 1, then $f|\partial(\text{ball})$ is homotopic to the identity. If degree $= -1$, then $f|\partial(\text{ball})$ is homotopic to reflection about the equator. In these cases $\text{Ind}(V^f|\text{ball}) = \pm 1 = deg\ f|\partial(\text{ball})$. Now

$$\text{Ind}(V^f) = \sum \text{Ind}\ V^f|(\text{ball}) \qquad \text{by proper otopy}$$
$$= \sum deg\ f|\partial(\text{balls}) = deg\ f'.$$

(12) Suppose $f : M \to \mathbb{R}^n$ where $M \subset \mathbb{R}^n$ is a codimension zero compact manifold. Define $V_f(m) = m - f(m)$. Then $\text{Ind}\ V_f$ = fixed point index of f (assuming no fixed points on ∂M)

   Proof. The fixed point index is defined to be the degree of the map $m \to \frac{m - f(m)}{\|m - f(m)\|}$ from $\partial M \to S^{n-1}$. Hence by (11) we have the result

(13) Let $f : M \to N$ where M and N are Riemannian manifolds and f is a smooth map. Let V be a vector field on M. Define the pullback vector field $f^*(V)$ by

$$\langle f^* V(m), \vec{v}_m \rangle = \langle V(f(m)), f_*(\vec{v}_m) \rangle.$$

Then if $f : M^m \to \mathbb{R}^n$ so that $f_*|\partial M$ has maximal rank and $f(\partial M)$ contains no zeros of V, then
$$\text{Ind}\ f^* V = \sum v_i w_i + (\chi(M) - deg\ \hat{N})$$

where $v_i = \text{Ind}(x_i)$ where x_i is the i$^{\text{th}}$ zero of V, w_i is the winding number of $f|\partial M$ about x_i, and $\hat{N} : \partial M \to S^{n-1}$ is the normal (or Gauss) map.

   Proof. In paper [G$_5$].

REFERENCES

[G$_1$] DANIEL H. GOTTLIEB, *A certain subgroup of the fundamental group*, Amer. J. Math., 87 (1966), pp. 1233–1237.

[G$_2$] ————————————, *A de Moivre formula for fixed point theory*, ATAS do 5° Encontro Brasiliero de Topologia, Universidade de São Paulo, São Carlos, S.P. Brasil, 31 (1988), pp. 59–67.

[G$_3$] ————————————, *A de Moivre like formula for fixed point theory*, Proceedings of the Fixed Point Theory Seminar at the 1986 International Congress of Mathematicians, R. F. Brown (editor), Contemporary Mathematics, AMS Providence, Rhode Island, 72, pp. 99–106.

[G$_4$] ————————————, *On the index of pullback vector fields*, Proc. of the 2nd Siegen Topology Symposium, August 1987, Ulrich Koschorke (editor), Lecture Notes of Mathematics, Springer Verlag, New York.

[G$_5$] ————————————, *Zeroes of pullback vector fields and fixed point theory for bodies*, Algebraic topology, Proc. of Intl. Conference March 21–24, 1988, Contemporary Mathematics, 96, pp. 168–180.

[G$_6$] ————————————, *Vector fields and classical theorems of topology*, Renconti del Seminario Matematico e Fisico, Milano.

[M] MARSTON MORSE, *Singular points of vector fields under general boundary conditions*, Amer. J. Math, 51 (1929), pp. 165–178.

[P] CHARLES C. PUGH, *A generalized Poincare index formula*, Topology, 7 (1968), pp. 217–226.

Purdue University