Composition Methods in the homotopy groups of ring spectra

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1.

Progress in calculating the homotopy groups of spheres has seen two major breakthroughs. The first was Toda's work, culminating in his book [11] in which the EHP sequences of James and Whitehead were used inductively; "composition methods" were used to construct elements and evaluate homomorphisms. The second was the Adams spectral sequence. Each method has advantages and disadvantages. Toda's method has the advantage that unstable homotopy groups are calculated along with stable groups. It has the disadvantage that it applies only to spheres and in particular, naturality under maps between spaces cannot be applied. The Adams method has the advantage that much of the bookkeeping work is accomplished in advance during the calculation of the Ext groups. The disadvantage that it does not calculate unstable groups has been eliminated, for certain nice spaces, by the work of [3]. This work implies that for certain spaces, the unstable homotopy is as accessible as the stable homotopy. It is the purpose of this work to examine how, in certain cases, the methods of Toda can be used for spaces other than spheres. We will begin by summarizing the methods used by Toda. We will discuss the possibility of using these methods for other spectra and work out the example of the Moore space spectrum $S^0 \cup_{p^r} e^1$ for $p > 3$.

2.

The main tool Toda used was the EHP sequence. Localized at 2, this is a long exact sequence for each $n \geq 1$

$$\cdots \longrightarrow \pi_{r+2}(S^{2n+1}) \xrightarrow{P}$$
$$\pi_r(S^n) \xrightarrow{E} \pi_{r+1}(S^{n+1}) \xrightarrow{H} \pi_{r+1}(S^{2n+1}) \xrightarrow{P} \pi_{r-1}(S^n) \longrightarrow \cdots$$

By induction first on the stem $\sigma = r - n$ and then on n, the determination of $\pi_{r+1}(S^{n+1})$ above is done when the other 4 groups are known. Compositions are used, and formulas for P and H on compositions allow one to calculate the groups involved. As an example, we cite the following formulas:

Proposition 2.1.

a) $H(\alpha \circ E\beta) = H(\alpha) \circ E\beta$.
b) $H(E\alpha \circ \beta) = E(\alpha \wedge \alpha) \circ H(\beta)$.
c) $P(\alpha \circ E^2\beta) = P(\alpha) \circ \beta$.

Proposition 2.2 (Barratt-Hilton formula)**.** *If $\alpha \in \pi_r(S^n)$ and $\beta \in \pi_s(S^m)$ then $E^m\alpha \circ E^r\beta = (-1)^{(r-n)(s-m)}E^n\beta \circ E^s\alpha$. (In other words, on S^{n+m}, $\alpha\beta = (-1)^{(r-n)(s-m)}\beta\alpha$.)*

We will demonstrate how to calculate the 4 stem with these methods. We first need knowledge of previous stems.

Proposition 2.3.

	group	*generators*	*relations*
$\pi_{n+1}(S^n) =$	$\mathbf{Z}$, $n = 2$	η	
	$\mathbf{Z}/2$, $n > 2$	η	2η
$\pi_{n+2}(S^n) =$	$\mathbf{Z}/2$	η^2	
$\pi_{n+3}(S^n) =$	$\mathbf{Z}/2$, $n = 2$	η^3	
	$\mathbf{Z}/4$, $n = 3$	ω	$2\omega = \eta^3$
	$\mathbf{Z} \oplus \mathbf{Z}/4$, $n = 4$	ν, ω	
	$\mathbf{Z}/8$, $n \geq 5$	ν	$2\nu = \omega$

Now the Hopf map $\eta\colon S^3 \longrightarrow S^2$ induces an isomorphism in homotopy:

$$\pi_r(S^3) \xrightarrow{\cong} \pi_r(S^2)$$

when $r > 2$, so $\pi_6(S^2) \simeq \mathbf{Z}/4$ generated by the composition $\eta\omega$. The map $\pi_5(S^5) \xrightarrow{P} \pi_3(S^2)$ generates the kernal of E so $P(\iota) = 2\eta$. We wish to evaluate

$$\pi_8(S^5) \xrightarrow{P} \pi_6(S^2)$$

$\pi_8(S^5)$ is generated by ν which is not a double suspension. However $2\nu = \omega$ is a double suspension so we apply 2.1 c)

$$P(2\nu) = P(\omega) = P(\iota) \circ \omega = 2\eta \circ \omega = \eta \circ 2\iota \circ \omega = \eta \circ \omega \circ 2\iota = 2(\eta\omega)$$

since $2\iota \circ \omega = \omega \circ 2\iota$ stably by 2.2 and hence on S^3 since $\pi_6(S^3) \longrightarrow \pi_{6+n}(S^{3+n})$ is a monomorphism. But $P(2\nu) = 2(\eta\omega)$ implies that $P(\nu) = \pm\eta\omega$ since P is a homomorphism. It follows that $E(\eta\omega) = 0$ and $0 \longrightarrow \pi_7(S^3) \longrightarrow \pi_7(S^5)$ is exact. We can define two compositions in $\pi_7(S^3)$: $\omega\eta$ and $\eta\nu$. We apply 2.1 (a) in the first case to get $H(\omega\eta) = H(\omega) \circ \eta = \eta^2$ and 2.1 (b) in the second case to get $H(\eta \circ \nu) = \eta^2 \circ H(\nu) = \eta^2$. Thus $\omega\eta = \eta\nu$ generates $\pi_7(S^3)$ and has order 2. Since ν has Hopf invariant 1, the EHP sequence splits when $n = 3$ and $\pi_{n+1}(S^4) \simeq \pi_n(S^3) \oplus \pi_{n+1}(S^7)$; in particular $\pi_8(S^4) \simeq \mathbf{Z}/2 \oplus \mathbf{Z}/2$ generated by $\omega\eta = \eta\nu$ and $\nu\eta$. By 2.2, on $S^5, \omega\eta = \eta\omega = 0$, so in the sequence

$$\pi_{10}(S^9) \xrightarrow{P} \pi_8(S^4) \xrightarrow{E} \pi_9(S^5) \xrightarrow{H} \pi_9(S^9)$$

$P(\eta) = \omega\eta$. Since $\pi_9(S^5)$ is finite, $H = 0$ and $\pi_9(S^5) = \mathbf{Z}/2$ generated by $\nu\eta$. By 2.2, on S^6 $\nu\eta = \eta\nu = \omega\eta = 0$, so $E\big(\pi_9(S^5)\big) = 0$. Furthermore, $\pi_{10}(S^{11}) = 0$ so $\pi_{10}(S^6) = 0$ and $\pi_{n+4}(S^n) = 0$ for $n \geq 6$. The interested reader will find that the calculation of the 5 stem is just as easy. There is one other useful result which we did not need here.

Proposition 2.4 (Barratt-Toda formula). *If $\alpha \in \pi_r(S^n)$ and $\beta \in \pi_s(S^m)$, the difference*

$$[\alpha, \beta] = E^{m-1}\alpha \circ E^{r-1}\beta - (-1)^{(r-n)(s-m)} E^{n-1}\beta \circ E^{s-1}\alpha \colon S^{r+s-1} \longrightarrow S^{m+n-1}$$

is equal to $P\big(EH(\alpha) \wedge H(\beta)\big)$.

Using this one can see, for example, that

$$\nu\eta = \eta\nu + \nu\eta = P(\iota) \text{ on } S^5$$
$$\omega\eta = \omega\eta + \eta\omega = P(\eta) \text{ on } S^4 (\text{since } H(\omega) = \eta).$$

This method would continue very successfully if the homotopy groups of spheres were finitely generated. They clearly are not and one soon runs out of compositions. As a partial remedy to this Toda defined "secondary compositions", or what are commonly called Toda brackets and proved formulas similar to 2.1 for these operations.

When localized at $p > 2$, there are two types of EHP sequences. Define S_n by the formula

$$S_n = \begin{cases} S^{2k+1} & \text{if } n = 2k+1 \\ \widehat{S}^{2k} = J_{p-1}(S^{2k}) & \text{if } n = 2k \end{cases}$$

where J_m is the m^{th} stage of the James construction.

Then the EHP sequences are defined by fibrations:

$$S_{2n} \xrightarrow{E} \Omega S_{2n+1} \xrightarrow{H_p} \Omega S_{2np+1}$$
$$S_{2n-1} \xrightarrow{E} \Omega S_{2n} \xrightarrow{H} \Omega S_{2np-1}$$

where H_p is the p^{th} James Hopf invariant and H is the Toda-Hopf invariant ([7], [11]). Note that in case $p = 2$, these both degenerate into a single fibration

$$S_n \longrightarrow \Omega S_{n+1} \xrightarrow{H_2} \Omega S_{2n+1}$$

and $S_n = S^n$. Composition methods have been applied for $p > 2$ [6] with similar success. Proposition 2.1 generalizes directly while 2.2 requires some consideration. We define a modified suspension σ as follows: if $f \colon S^r \longrightarrow S_n$ let us write $\sigma(f)$ for the adjoint of $E \circ f \colon S^r \longrightarrow \Omega S_{n+1}$. We will write $SX = X \wedge S^1$ for the classical suspension of X and if $f\colon X \longrightarrow Y$, we write $Sf\colon SX \longrightarrow SY$ for the suspension of f.

Corresponding to Proposition 2.2 (at $p = 2$) we have

Theorem 2.5 (p local Barrat-Hilton Theorem)**.** *Suppose that $\alpha \in \pi_r(S_n)$ and $\beta \in \pi_s(S_m)$. Then*

$$\sigma^m\alpha \circ \sigma^r\beta = (-1)^{(r-n)(s-m)}\sigma^n\beta \circ \sigma^s\alpha$$

in $\pi_*(S_{n+m})$.

Proof. We consider 3 cases:

a) if $n = 2k+1$ and $m = 2\ell+1$, the formula holds on $S^{2k+2\ell+2}$; since the suspension $E\colon \pi_{r+s}(S^{2k+2\ell+1}) \longrightarrow \pi_{r+s+1}(S^{2k+2\ell+2})$ is a monomorphism, this formula holds on $S^{2k+2\ell+1} = S_{m+n-1}$ and hence on S_{m+n}

b) if $n = 2k$ and $m = 2\ell + 1$, apply case a) to $\sigma\alpha$ and β to demonstrate the formula on $S^{2k+2\ell+1} = S_{m+n}$

c) if $n = 2k$ and $m = 2\ell$, new arguments are needed.

Let $\iota\colon S^{2n} \longrightarrow \widehat{S}^{2n}$ be the inclusion. To discuss case c) we rely on

Lemma 2.6. *Suppose $g\colon S^r \longrightarrow \widehat{S}^{2n}$. Then the diagram:*

$$\begin{array}{ccc} S^{r+1} & \xrightarrow{Sg} & S\widehat{S}^{2n} \\ & {\scriptstyle\sigma(g)}\searrow & \uparrow{\scriptstyle S\iota} \\ & & S^{2n+1} \end{array}$$

commutes up to homotopy.

Proof. Let $K_n = S^{2n-1} \cup_{w_n} e^{2np-2}$ be the $2np-2$ skeleton of $\Omega^2 S^{2n+1}$. According to [7], [10], there is a map $\gamma\colon SK_n \longrightarrow \widehat{S}^{2n}$ such that $\Omega\gamma$ has a right homotopy inverse. In particular, there is a lifting of g to a map $g'\colon S^r \longrightarrow SK_n$

$$\begin{array}{ccc} S^{r} & \xrightarrow{g} & \widehat{S}^{2n} \\ & {\scriptstyle g'}\searrow & \uparrow{\scriptstyle \gamma} \\ & & SK_n \end{array}$$

Write $\sigma(\gamma)\colon S^2K_n \longrightarrow S^{2n+1}$ for the adjoint of $E \circ \gamma\colon SK_n \longrightarrow \Omega S^{2n+1}$. We now show that

$$\begin{array}{ccc} S^2K_n & \xrightarrow{S\gamma} & S\widehat{S}^{2n} \\ & {\scriptstyle\sigma(\gamma)}\searrow & \uparrow{\scriptstyle S\iota} \\ & & S^{2n+1} \end{array}$$

commutes up to homotopy. The Lemma then follows by combining these two diagrams. □

Now $S\widehat{S}^{2n} \simeq S^{2n+1} \vee S^{4n+1} \vee \cdots \vee S^{2(p-1)n+1}$; let $H_i\colon \widehat{S}^{2n} \longrightarrow S^{2n}_{\infty} \longrightarrow S^{2ni}_{\infty}$ be the restriction for the i^{th} James-Hopf invariant for $1 \le i \le p-1$. Combining, we get a map:

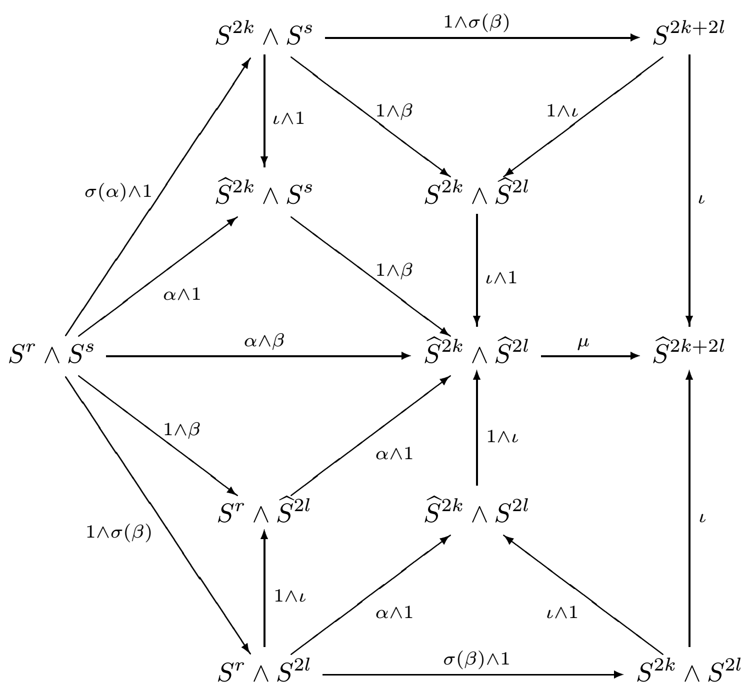
$$\theta\colon \widehat{S}^{2n} \longrightarrow \prod_{i=1}^{p-1} S^{2ni}_{\infty} \simeq \Omega(S^{2n+1} \times \cdots \times S^{2n(p-1)+1}) \longrightarrow \Omega(S\widehat{S}^{2n}),$$

where the last map is the splitting map from the loops on a product to the loops on a wedge. Since ΩH_i has order prime to p for $1 < i \le p-1$,

$$H_i \circ \gamma\colon SK_n \longrightarrow \widehat{S}^{2n} \longrightarrow S^{2ni}_{\infty}$$

is null homotopic if $1 < i \le p-1$. Therefore $\theta \circ \gamma$ factors through ΩS^{2n+1}. Since the adjoint of θ is an equivalence, $S\gamma$ factors through S^{2n+1}. This factorization can be identified as $\sigma(\gamma)$ by applying the inclusion $\widehat{S}^{2n} \longrightarrow \Omega S^{2n+1}$ and taking adjoints.

To finish the proof of 2.5, choose a map $\mu\colon \widehat{S}^{2k} \wedge \widehat{S}^{2\ell} \longrightarrow \widehat{S}^{2k+2\ell}$ which is degree 1 in dimension $2k+2\ell$. Then consider the diagram:



where the lemma is applied in 4 of the 6 triangular regions. If we identify $\sigma^t(f)$ with $1 \wedge f$, we introduce signs $(-1)^{(s-1)(r-2k)}$ and $(-1)^{(r-2k)(2l-1)}$ which combine to give $(-1)^{rs}$. □

3.

The utility of the EHP method applies when what you are studying is not a single space, but a sequence of spaces $\{X_n\}$ together with a suspension homomorphism $\pi_r(X_n) \xrightarrow{\sigma} \pi_{r+1}(X_{n+1})$. If σ is induced by a map $X_n \xrightarrow{e_n} \Omega X_{n+1}$, what we are in fact dealing with is a spectrum $X = \{X_n, e_n\}$. The utility of an EHP approach depends on the extent to which one has control over the fiber of e_n:

$$F_n \longrightarrow X_n \xrightarrow{e_n} \Omega X_{n+1}$$

From this point of view, it is clear that one can form two stably equivalent spectra $\{X_n, x_n\} \simeq \{Y_n, y_n\}$ with the spaces X_n and Y_n vastly different. We think of the sequence of spaces $\{X_n\}$ as an unstable development of the spectrum X. We seek a favorable unstable development of X. Ideally, one might seek an unstable development in which $F_n = \Omega^{k(n)} X_{\ell(n)}$ for some functions $k(n), \ell(n)$. This is explored in [9] and it appears formally that the spectra $V(m)$ of Smith and Toda are good candidates for this. In particular, $V(-1) = S^0$ and $V(0) = S^0 \cup_p e^1$ for $p > 2$. Considerable attention to this spectrum and the related spectra $S^0 \cup_{p^r} e^1$ will be deferred to section 4.

To apply "composition methods," unstable classes need to be composed. This suggests a ring structure in $\pi_*^S(X)$ which is associative (composition is associative) and commutative (if we wish to have a Barratt-Hilton formula). Thus one might begin by considering a homotopy associative homotopy commutative ring spectrum X. For Toda brackets we may need to consider higher homotopies of associativity ahnd commutativity. However, in order to have a successful unstable composition theory it is first necessary to develop a stable composition theory. To do this we will find certain self maps of a ring spectrum X which correspond to elements in the stable homotopy of X in such a way that the product in $\pi_* X$ corresponds to composition.

Suppose now that X is a ring spectrum. Using the multiplication $X \wedge X \xrightarrow{\mu} X$ we can think of X as an "X-module spectrum".

Definition 3.1. *Call a map $\phi\colon S^r X \longrightarrow X$ right modular if there is a commutative diagram:*

$$\begin{array}{ccc} S^r X \wedge X & \xrightarrow{\phi \wedge 1} & X \wedge X \\ \downarrow{\scriptstyle S^r\mu} & & \downarrow{\scriptstyle \mu} \\ S^r X & \xrightarrow{\phi} & X \end{array}$$

Note that the composition of modular maps is modular. Write $\mathrm{Mod}_r(X)$ for the group of homotopy classes of right modular maps $\rho\colon S^r X \longrightarrow X$. Then $\{\mathrm{Mod}_r(X)\}$ is a graded ring with unit under composition.

Theorem 3.2. *If X is homotopy associative, $\pi_*^S(X) \cong Mod_*(X)$ as graded rings.*

Proof. Define $F\colon \pi_r^S(X) \longrightarrow \mathrm{Mod}_r(X)$ by $F(\alpha) = \widehat{\alpha}$ where $\hat{\alpha}$ is the composition:

$$S^r X \xrightarrow{\alpha\wedge 1} X \wedge X \xrightarrow{\mu} X$$

to check modularity, consider the diagram:

$$\begin{array}{ccccc} S^r X \wedge X & \xrightarrow{\alpha\wedge 1\wedge 1} & X \wedge X \wedge X & \xrightarrow{\mu\wedge 1} & X \wedge X \\ \downarrow{\scriptstyle S^r\mu} & & \downarrow{\scriptstyle 1\wedge\mu} & & \downarrow{\scriptstyle \mu} \\ S^r X & \xrightarrow{\alpha\wedge 1} & X \wedge X & \xrightarrow{\mu} & X \end{array}$$

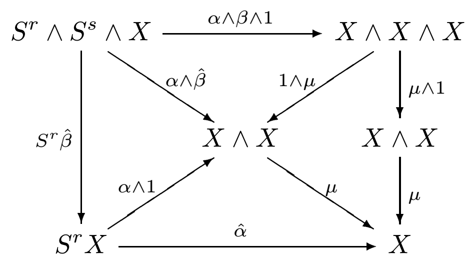
Now define $G\colon\ \mathrm{Mod}_r(X) \longrightarrow \pi_r^S(X)$ by $G(\phi) = \phi \circ S^r\iota$

$$S^r = S^r \wedge S^0 \xrightarrow{S^r\iota} S^r \wedge X \xrightarrow{\phi} X.$$

Clearly $G \circ F = Id$. To see that $F \circ G = Id$, consider the diagram:

$$\begin{array}{ccccc} S^r = S^r \wedge S^0 \wedge X & \xrightarrow{1\wedge\iota\wedge 1} & S^r \wedge X \wedge X & \xrightarrow{\phi\wedge 1} & X \wedge X \\ & \searrow^{1} & \downarrow{\scriptstyle 1\wedge\mu} & & \downarrow{\scriptstyle \mu} \\ & & S^r X & \xrightarrow{\phi} & X \end{array}$$

where $(F \circ G)(\phi)$ is the upper right hand composite. Furthermore, F is a ring homomorphism. To see this consider the following diagram where $\alpha \in \pi_r^S(X)$ and $\beta \in \pi_s^S(X)$. The top right hand composition represents $F(\alpha \circ \beta)$ while the lower left hand composition represents $F(\alpha) \circ F(\beta)$



This completes the proof. □

Now suppose in addition that X is homotopy commutative. Then by 3.2, composition is graded commutative, i.e.;

$$\widehat{\alpha} \circ S^r \widehat{\beta} \sim (-1)^{rs} \widehat{\beta} \circ S^s \widehat{\alpha}$$

4.

In this section we will develop the special case of the Moore space spectrum $S^0 \cup_{p^r} e^1$, $p > 2$. By the results of [9], this spectrum can be represented as $\{T_m, \sigma_n\}$, where $T_{2n} = S^{2n+1}\{p^r\}$, the fiber of the degree p^r map on S^{2n+1} and T_{2n-1} is a space constructed by Anick [1] and developed further in [2] and [12]. T_{2n-1} is the total space in a fibration

$$S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}$$

where the connecting map $\Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ has degree p^r. We will write $T_n(p^r)$ when we wish to keep the exponent in mind. There are EHP fibrations:*

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$T_{2n} \xrightarrow{E} \Omega T_{2n+1} \xrightarrow{H} BW_{n+1}$$

The spaces T_m are a homotopically simple unstable representation of the Moore space spectrum and we seek an unstable version of modularity. To do this we need to find a functorial extension of a map $S^m \xrightarrow{\alpha} T_n$ to a "modular map" $T_m \xrightarrow{\widehat{\alpha}} T_n$.

This is accomplished in 2 steps:

a) extend α to a map $P_m = S^m \cup_{p^r} e^{m+1} \xrightarrow{\alpha'} T_n$ so that it is "modular" in the appropriate sense.
b) extend α' to $\widehat{\alpha}$ so that it is an H map.

The idea behind point a) is to use the fact that $p^r\pi_*(T_n) = 0$ to define an extension. This has some technical complexity. Part b) is easily obtained from

Theorem 4.1 ([9, 2, 12])**.** a) *T_n is a Homotopy commutative and homotopy associative H space and $p^r \circ \pi_*(T_n) = 0$.*
b) *Let $\alpha' \colon P_m \longrightarrow X$ where X is a homotopy commutative and homotopy associative H space and $p^r\pi_r(X) = 0$. Then there is a unique H map $\widehat{\alpha} \colon T_m \longrightarrow X$ extending α'.*

Proof. The case n even was worked out in [9] and the existence but not the uniqueness when n is odd appear in [2]. The uniqueness and H property in case n is odd appears in [12]. □

Proposition 4.2. *Given an H map $\phi \colon T_m \longrightarrow T_n$, there is a unique H map $\sigma\phi \colon T_{m+1} \longrightarrow T_{n+1}$ such that the diagram:*

$$\begin{array}{ccc} T_m & \xrightarrow{\phi} & T_n \\ \downarrow E & & \downarrow E \\ \Omega T_{m+1} & \xrightarrow{\Omega\sigma\phi} & \Omega T_{n+1} \end{array}$$

is homotopy commutative.

*Note that it is an open conjecture [8] that $BW_n = \Omega T_{2np-1}(p)$ making these sequences formally similar to those in section 2.

Proof. There is a unique H extension of the composite:

$$SP_m \longrightarrow ST_m \xrightarrow{\Sigma\phi} ST_n \longrightarrow T_{n+1}$$

to an H map $\sigma(\phi)\colon T_{m+1} \longrightarrow T_{n+1}$. It then follows that the diagram in question commutes when restricted to P_r. Since all maps are H maps, it commutes to homotopy by 4.1 b). □

Definition 4.3. *Suppose X, Y, Z are H spaces, and $f\colon X \wedge Y \longrightarrow Z$. We will call f an H map in two variables (or an H map for short) if the adjoints of f:*

$$X \longrightarrow Z^Y$$

$$Y \longrightarrow Z^X$$

are H maps with the induced H space structure on the function space.

Proposition 4.4. *There is a unique H map in two variables*

$$\mu\colon T_r \wedge T_s \longrightarrow T_{r+s}$$

extending the composition $P_r \wedge P_s \xrightarrow{\pi} P_{r+s} \xrightarrow{\iota} T_{r+s}$ where $\pi_*(1 \otimes 1) = 1$ in homology.

Proof. The adjoint map $P_r \longrightarrow T_{r+s}^{P_s}$ has a unique extension to an H map by 4.1. This gives a map $T_r \longrightarrow T_{r+s}^{P_s}$ and hence $T_r \wedge P_s \longrightarrow T_{r+s}$. The extension to a map $T_r \wedge T_s \longrightarrow T_{r+s}$ is similar.

Clearly we have commutative diagrams:

$$\begin{array}{ccc} T_r \wedge T_s & & \\ \big\downarrow{\scriptstyle\tau} & \overset{\mu}{\searrow} & \\ & (-1)^{rs} & T_{r+s} \\ & \underset{\mu}{\nearrow} & \\ T_s \wedge T_r & & \end{array}$$

and a similar associativity diagram if $p > 3$ or $r > 1$. □

Given spaces A, B, we will often encounter the map $A \wedge \Omega B \xrightarrow{e} \Omega(A \wedge B)$ which is adjoint to the evaluation map. e is an H map in the second variable. In particular we can then extend

$$\begin{array}{ccccc} P_r \wedge \Omega T_s & \xrightarrow{e} & \Omega(P_r \wedge T_s) & \xrightarrow{\Omega\mu} & \Omega T_{r+s} \\ \big\downarrow & & & \overset{e'}{\nearrow} & \\ T_r \wedge \Omega T_s & & & & \end{array}$$

so that e' is an H map in two variables, and

$$\begin{array}{ccc} T_r \wedge \Omega T_s & \xrightarrow{e'} & \Omega T_{r+s} \\ {\scriptstyle 1\wedge E}\uparrow & & \uparrow{\scriptstyle E} \\ T_r \wedge T_{s-1} & \xrightarrow{\mu} & T_{r+s-1} \end{array}$$

commutes up to homotopy, and similarly in the other variable.

In order to define modularity, we require the following.

Lemma 4.5. *There is a map $\epsilon\colon P_1 \wedge \Omega T_n \longrightarrow T_n$ so that the diagram:*

$$\begin{array}{ccc} P_1 \wedge \Omega T_n & \xleftarrow{1\wedge\sigma} & P_1 \wedge T_{n-1} \\ \uparrow & \overset{\epsilon}{\searrow} & \downarrow{\scriptstyle \mu} \\ S^1 \wedge \Omega T_n & \xrightarrow[ev]{} & T_n \end{array}$$

commutes up to homotopy.

Proof. Since the identity map of T_n has order p, there is a lifting $T_n \longrightarrow T_n\{p\}$ where $T_n\{p\}$ is the fiber of the p^{th} power map:

$$\dots \longrightarrow T_n\{p\} \longrightarrow T_n \xrightarrow{p} T_n$$

However $\Omega T_n\{p\}$ is homotopy equivalent to the fiber of the map $\Omega T_n \longrightarrow \Omega T_n$ induced by the degree p map on S^1; i.e., $\Omega T_n\{p\} \simeq T_n^{P_1}$ as H spaces. It follows that there is an H map $\epsilon'\colon \Omega T_n \longrightarrow T_n^{P_1}$ with the composite

$$\Omega T_n \xrightarrow{\epsilon'} T_n^{P_1} \longrightarrow T_n^{S^1} \simeq \Omega T_n$$

homotopic to the identity. Furthermore, the restriction

$$T_{n-1} \xrightarrow{E} \Omega T_n \xrightarrow{\epsilon'} T_n^{P_1}$$

is an H map and hence is the adjoint to $\mu\colon P_1 \wedge T_{n-1} \longrightarrow T_n$. Taking adjoints yields the lemma. □

Our next step is to define, for each homotopy class $\alpha \in \pi_r(T_n)$ an H map $T_r \xrightarrow{\widehat{\alpha}} T_n$ extending α. It suffices to construct a map $P_r \longrightarrow T_n$ and this is accomplished as the composition:

$$P_r = P_1 \wedge S^{r-1} \xrightarrow{1\wedge d^*} P_1 \wedge \Omega T_n \xrightarrow{\epsilon} T_n$$

$\widehat{\alpha}$ is then the unique H extension. Consequently we have defined a homomorphism

$$F_\epsilon\colon \pi_r(T_n) \longrightarrow [T_r, T_n]_H$$

by $F_\epsilon(\alpha) = \widehat{\alpha}$.

Clearly we have a left inverse

$$G\colon [T_r, T_n]_H \longrightarrow \pi_r(T_n)$$

by restriction. We wish to determine the image of F_ϵ as the maps which are in some sense "modular." We will call a map ϕ k-modular if the diagram:

$$\begin{array}{ccc} T_k \wedge T_r & \xrightarrow{1\wedge\phi} & T_k \wedge T_n \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ T_{k+r} & \xrightarrow{\sigma^k(\phi)} & T_{k+n} \end{array}$$

commutes. Clearly ϕ is k-modular iff the composition $d_k(\phi)$:

$$P_{k+r+1} \xrightarrow{\nabla} P_k \wedge P_r \longrightarrow P_k \wedge T_r \xrightarrow{1\wedge\phi} P_k \wedge T_n \xrightarrow{\mu} T_{k+n}$$

is null homotopic. Now $\sigma\big(d_k(\phi)\big) = d_{k+1}(\phi)$, so if ϕ is k-modular, it is $(k+1)$-modular.

Strong Conjecture 4.6. *There is a choice of ϵ so that the diagram:*

$$\begin{array}{ccccc} P_1 \wedge P_1 \wedge \Omega T_n & & \xrightarrow{\quad 1\wedge\epsilon \quad} & & P_1 \wedge T_n \\ \downarrow{\scriptstyle \mu\wedge 1} & & & & \downarrow{\scriptstyle \mu} \\ P_2 \wedge \Omega T_n & \xrightarrow{ev} & P_1 \wedge T_n & \xrightarrow{\mu} & T_{n+1} \end{array}$$

commutes up to homotopy.

Proposition 4.7. *The strong conjecture implies that $\widehat{\alpha}$ is k-modular for each $k \geq 1$.*

Proof. We prove that $\widehat{\alpha}$ is 1-modular by factoring $d_1(\widehat{\alpha})$

$$\begin{array}{ccccccc} P_{r+2} & \xrightarrow{\nabla} & P_1 \wedge P_r & \xrightarrow{1\wedge\hat{\alpha}} & P_1 \wedge T_n & \xrightarrow{\mu} & T_{n+1} \\ \updownarrow & & \updownarrow & & \uparrow & & \uparrow \\ P_3 \wedge S^{r-1} & \longrightarrow & P_1 \wedge P_1 \wedge S^{r-1} & \xrightarrow{1\wedge\alpha^*} & P_1 \wedge P_1 \wedge \Omega T_n & & \\ & \searrow{\scriptstyle *} & \downarrow{\scriptstyle \mu\wedge 1} & & \downarrow{\scriptstyle \mu\wedge 1} & & \\ & & P_2 \wedge S^{r-1} & \xrightarrow{1\wedge\alpha^*} & P_2 \wedge \Omega T_n & \longrightarrow & P_1 \wedge T_n \end{array}$$ □

By including $S^1 \wedge P_1 \wedge \Omega T_n$ into $P_1 \wedge P_1 \wedge \Omega T_n$, we see that the strong conjecture implies the

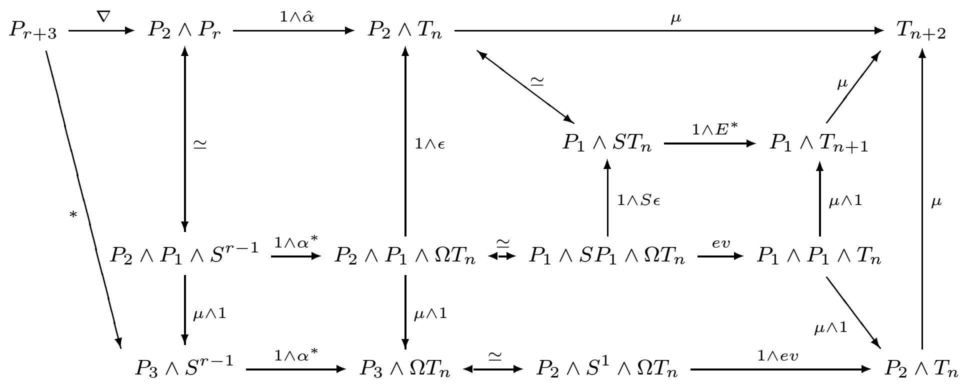
Weak Conjecture 4.8. *There is a choice of ϵ so that the diagram:*

$$\begin{array}{ccc} S^1 \wedge P_1 \wedge \Omega T_n & \xrightarrow{1\wedge\epsilon} & S^1 \wedge T_n \\ \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle \sigma} \\ P_1 \wedge T_n & \xrightarrow{\mu} & T_{n+1} \end{array}$$

commutes up to homotopy.

Proposition 4.9. *The weak conjecture implies that $\widehat{\alpha}$ is modular for each $k \geq 2$.*

Proof. We factor $d_2(\widehat{\alpha})$:



□

We call ϕ 0-modular if the composition $d_0(\phi)$:

$$P_{r+1} \xrightarrow{\nabla} P_1 \wedge P_{r-1} \xrightarrow{1\wedge\phi^*} P_1 \wedge \Omega T_n \xrightarrow{\epsilon} T_n$$

is null homotopic. It is easy to see that the weak conjecture implies that if ϕ is 0-modular, it is 1-modular and that if ϕ is 0-modular and $G(\phi) = *$ then $\phi \sim *$. If ϵ can be chosen so that $\widehat{\alpha}$ is 0-modular, there is then an isomorphism between the 0-modular H maps $T_r \longrightarrow T_n$ and $\pi_r(T_n)$. An even more complicated conjecture about ϵ will imply that $\widehat{\alpha}$ is always 0-modular. We will not pursue this line.

Proposition 4.10. *The weak conjecture holds when n is even.*

Proof. In this case we have an H-fibration sequence:

$$T_{2n} \longrightarrow \Omega T_{2n+1} \longrightarrow BW_n;$$

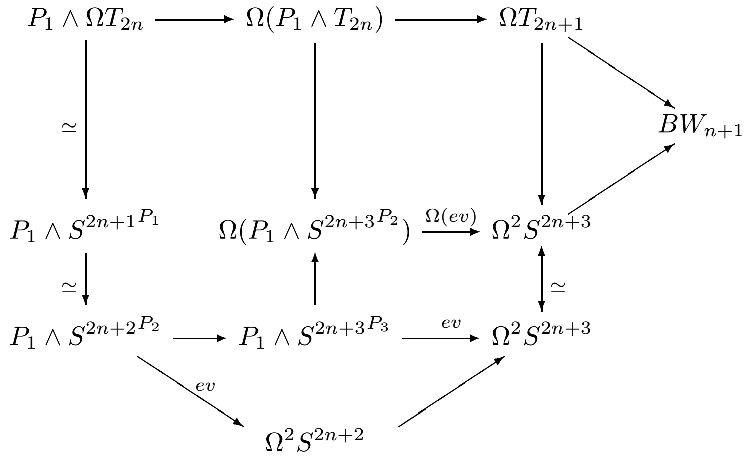
we will prove that such an ϵ exists by showing that the composition

$$P_1 \wedge \Omega T_{2n} \xrightarrow{e} \Omega(P_1 \wedge T_{2n}) \xrightarrow{\Omega\mu} \Omega T_{2n+1} \longrightarrow BW_{n+1}$$

is null homotopic. We begin by considering the diagram:

$$\begin{array}{ccc} P_1 \wedge T_{2n} & \xrightarrow{\quad\mu\quad} & T_{2n+1} \\ \downarrow & & \downarrow \\ P_1 \wedge \Omega^2 T_{2n+2} \simeq P_1 \wedge (S^{2n+3})^{P_2} & \xrightarrow{ev} & \Omega S^{2n+3} \end{array}$$

since both compositions are H maps of the second variable, they are homotopic iff they are homotopic when restricted to $P_1 \wedge P_{2n}$. This is clear. We now form a diagram which contains the loops on this diagram:



However the composition $\Omega S^{2n+2} \longrightarrow \Omega^2 S^{2n+3} \longrightarrow BW_{n+1}$ is null homotopic since $\Omega S^{2n+2} \longrightarrow \Omega^2 S^{2n+3} \longrightarrow BW_{n+1} \times S^{4n+3}$ is a fibration sequence [8]. This completes the proof. □

Theorem 4.11 (Barratt-Hilton Theorem)**.** *Suppose that* $\alpha\colon T_r \longrightarrow T_n$ *and* $\beta\colon T_s \longrightarrow T_m$ *are* k *modular where* $n, m \geq k$*. Then*

$$\sigma^m(\alpha) \circ \sigma^r(\beta) = (-1)^{(n-r)(m-s)} \sigma^n(\beta)\sigma^s(\alpha)\colon T_{r+s} \longrightarrow T_{m+n}.$$

Proof. Exactly as in the case of spheres, we have $(\alpha \wedge 1) \circ (1 \wedge \beta) = \alpha \wedge \beta = (1 \wedge \beta) \circ (\alpha \wedge 1)$.

Now we also have:

$$\begin{array}{ccc} T_k \wedge T_s & \xrightarrow{1\wedge\beta} & T_k \wedge T_m \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ T_{k+s} & \xrightarrow{\sigma^n(\beta)} & T_{k+m} \end{array}$$

for each k, while:

$$\begin{array}{ccc} T_r \wedge T_k & \xrightarrow{\alpha\wedge 1} & T_n \wedge T_k \\ \downarrow & & \downarrow \\ T_{r+k} & \xrightarrow{(-1)^{k(n-r)}\sigma(\alpha)} & T_{n+k}\,. \end{array}$$

It follows that the indicated equation holds when preceded by $\mu\colon T_r \wedge T_s \longrightarrow T_{r+s}$. However, the inclusion of P_{r+s} into T_{r+s} factors through μ, so the equation holds when restricted to P_{r+s}. Since both composites are H maps, they are homotopic by the universal property. □

Appendix A.

So far we have discussed composition behavior which mimics the behavior of the sphere spectrum localized at 2. Here we will discuss compositions in the sphere spectrum localized at a prime $p > 2$. Recall that we have:

$$S_\ell = \begin{cases} S^{2k+1}\,, & \text{if } \ell = 2k+1 \\ \widehat{S}^{2k} = J_{p-1}(S^{2k})\,, & \text{if } \ell = 2k. \end{cases}$$

The fact that half of the spaces in this spectrum are not spheres presents a difficulty in forming compositions. The question we deal with first is this: is there a sensible way of forming a "composite" of maps $f\colon S^r \longrightarrow \widehat{S}^{2n}$ and $g\colon S^{2n} \longrightarrow S_\ell$ to obtain a map $g * f\colon S^r \longrightarrow S_\ell$? If we allow ourselves to suspend once this can be done:

$$S^{r+1} \xrightarrow{\sigma f} S^{2n+1} \xrightarrow{\sigma g} S_{\ell+1}$$

It is remarkable that the case $p = 3$ is easier than that of larger primes.

Proposition A.1. *Suppose $p = 3$ and X is a space such that ΩX is homotopy commutative in the loop space structure. Then:*

a) *ΩS_ℓ is homotopy commutative in the loop space structure.*
b) *Each map $g\colon S^{2n} \longrightarrow X$ extends to a map*

$$\widehat{g}\colon \widehat{S}^{2n} \longrightarrow X$$

c) *If $\widehat{g}, \tilde{g}\colon \widehat{S}^{2n} \longrightarrow X$ are any two extensions of g, $\Omega\widehat{g} \sim \Omega\tilde{g}$.*
d) *The diagram:*

$$\begin{array}{ccc} \Omega\widehat{S}^{2k} & \xrightarrow{\Omega\widehat{g}} & \Omega X \\ \downarrow & & \downarrow \\ \Omega^2 S^{2k+1} & \xrightarrow{\Omega^2\Sigma g} & \Omega^2\Sigma X \end{array}$$

commutes up to homotopy.

Corollary A.2. *Suppose $p = 3$. If $f\colon S^r \longrightarrow \widehat{S}^{2k}$ and $g\colon S^{2k} \longrightarrow S_\ell$ we can define a composition*

$$g * f\colon S^r \longrightarrow S_\ell$$

by $g * f = \widehat{g} \circ f$. This is well defined and

$$\sigma(g * f) = \sigma(g) \circ \sigma(f)\colon S^{r+1} \longrightarrow S_{\ell+1}.$$

Proof. a) is well known if ℓ is odd and is proven in [7] in case ℓ is even. b) is due to Hua Feng [5]. It follows since when $p = 3$, $\widehat{S}^{2k} = S^{2k} \cup e^{4k}$ where the attaching map is the Whitehead product. The obstruction to defining $\widehat{g}$ is thus the composition $S^{4k-1} \longrightarrow S^{2k} \longrightarrow X$. The adjoint of this is the composition $S^{4k-2} \longrightarrow \Omega S^{2k} \longrightarrow \Omega X$ where the first map is a commutator. To prove c), note that two extensions $\widehat{g}$ and $\tilde{g}$ agree on S^{2n}. Let $\delta\colon S^{4k} \longrightarrow X$ be the difference element so that $\widehat{g}$ is homotopic to the composition:

$$\widehat{S}^{2n} \xrightarrow{\Delta} \widehat{S}^{2n} \vee S^{4n} \xrightarrow{\tilde{g}\vee\delta} X$$

Consider the diagram of fibrations:

$$\begin{array}{ccccc}
\Omega SK * \Omega S^{4n} & \xrightarrow{\Omega\gamma * 1} & \Omega\widehat{S}^{2n} * \Omega S^{4n} & & \\
\downarrow & & \downarrow & & \\
SK_n \vee S^{4n} & \xrightarrow{\gamma\vee 1} & \widehat{S}^{2n} \vee S^{4n} & \xrightarrow{\tilde{g}\vee\delta} & X \\
\downarrow & & \downarrow & & \\
SK_n \times S^{4n} & \xrightarrow{\gamma\times 1} & \widehat{S}^{2n} \times S^{4n} & &
\end{array}$$

where $K_n = S^{2n-1} \cup e^{2np-2}$ is the $2np-2$ skeleton of $\Omega\widehat{S}^{2n}$ and $\gamma\colon SK_n \longrightarrow \widehat{S}^{2n}$ is the adjoint of the inclusion. Since ΩX is homotopy commutative, $(\tilde{g}\vee\delta)(\gamma\vee 1)$ extends over $SK_n \times S^{4n}$, as this space is the mapping cone of a Whitehead product. It follows that the composite along the top is null-homotopic. But since $\Omega\gamma$ has a right homotopy inverse, the composite:

$$\Omega\widehat{S}^{2n} * \Omega S^{4n} \longrightarrow \widehat{S}^{2n} \vee S^{4n} \longrightarrow X$$

is null-homotopic. Since the right hand sequence is a fibration, we get an extension:

$$\begin{array}{ccc}
\Omega(\widehat{S}^{2n} \vee S^{4n}) & \xrightarrow{\Omega(\tilde{g}\vee\delta)} & \Omega X \\
\downarrow & \nearrow_{\Gamma} & \\
\Omega(\widehat{S}^{2n} \times S^{4n}) & &
\end{array}$$

and Γ must be $\Omega\tilde{g} \times \Omega\delta$. Fitting these diagrams together, we get

$$\begin{array}{ccccc}
 & & \Omega(\widehat{S}^{2n} \vee S^{4n}) & & \\
 & \nearrow^{\Omega\Delta} & \downarrow & \searrow^{\Omega(\tilde{g}\vee\delta)} & \\
\Omega\widehat{S}^{2n} & & & & \Omega X \\
 & \searrow_{1\times\Omega\pi} & & \nearrow_{\Omega(\tilde{g})+\Omega\delta} & \\
 & & \Omega\widehat{S}^{2n} \times \Omega S^{4n} & &
\end{array}$$

Hence $\Omega\widehat{g} = \Omega\tilde{g} + \Omega(\delta)\circ\Omega\pi$. But by [8, Proposition 7] $\Omega\pi \sim *$ so $\Omega\widehat{g} \sim \Omega\tilde{g}$. Finally, to prove d) observe both composites:

$$\widehat{S}^{2n} \xrightarrow{\widehat{g}} X \xrightarrow{\iota} \Omega\Sigma X$$

$$\widehat{S}^{2n} \longrightarrow \Omega S^{2n+1} \xrightarrow{\Omega\Sigma g} \Omega\Sigma X$$

extend $S^{2n} \longrightarrow X \longrightarrow \Omega\Sigma X$, so we may apply c) replacing X by $\Omega\Sigma X$. $\square$

Theorem A.3. *For any p and any map $g\colon S^{2m} \longrightarrow S_\ell$. There is an extension $\widehat{g}\colon \widehat{S}^{2m} \longrightarrow S_\ell$ such that the diagram:*

$$\begin{array}{ccc} \widehat{S}^{2m} & \xrightarrow{\widehat{g}} & S_\ell \\ \downarrow E & & \downarrow E \\ \Omega S^{2m+1} & \xrightarrow{\Omega(\sigma g)^*} & \Omega S_{\ell+1} \end{array}$$

commutes up to homotopy.

Note A.4. *We make no statement about the uniqueness of $\Omega\widehat{g}$ in the general case.*

Proof. In case ℓ is odd, S_ℓ is an H space and g can then be extended to a map $g_\infty\colon S^{2m}_\infty \longrightarrow S_\ell$. The diagram clearly commutes in this case. Suppose now that $\ell = 2n$. We first lift g to a map $g'\colon S^{2m} \longrightarrow SK_n$ so that $g' \sim g$. Then consider the diagram:

$$\begin{array}{ccccc} S^{2m} & \xrightarrow{g'} & SK_n & \xrightarrow{\gamma} & \widehat{S}^{2n} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{S}^{2m} = J_{p-1}(S^{2m}) & \xrightarrow{J_{p-1}(g')} & J_{p-1}(SK_n) & \xrightarrow{\phi} & \widehat{S}^{2n} \\ \downarrow & & \downarrow & & \downarrow \\ J_\infty(S^{2m}) & \longrightarrow & J_\infty(SK_n) & \longrightarrow & S^{2n}_\infty \end{array}$$

The proof of A3 is complete when we construct ϕ so that the two diagrams on the right homotopy commute. First we must prove

Proposition A.5. *Suppose $f\colon SX \longrightarrow SY$ and the composite:*

$$X \xrightarrow{f^*} \Omega SY \xrightarrow{H_k} \Omega SY^k$$

is null homotopic for each $k > 1$. Then there is a commutative diagram:

$$\begin{array}{ccc} \Omega SX & \xrightarrow{\Omega f} & \Omega SY \\ \downarrow H_k & & \downarrow H_k \\ \Omega SX^{(k)} & & \Omega SY^{(k)} \\ \downarrow & & \downarrow \\ \Omega^k S^k X^{(k)} & \xrightarrow{\Omega(f\wedge\cdots\wedge f)} & \Omega^k S^k Y^{(k)} \end{array}$$

Proof. Of course, if $f = Sf'$, this is clear by naturality of the Hopf invariant. We will prove this using the results of Boardman-Steer [4]. They describe Hopf invariants $\lambda_n\colon [SA, SB] \longrightarrow$ for each $n \geq 1$. Let $ev\colon S\Omega SX \longrightarrow SX$ be the evaluation map. Then [4, 3.15] $\lambda_n(ev)\colon S^n(\Omega SX) \longrightarrow S^n X^{(n)}$ is the adjoint of the composite $\Omega SX \xrightarrow{H_n} \Omega SX^{(n)} \longrightarrow \Omega^n S^n X^{(n)}$. Consequently, the diagram in question is equivalent to the diagram:

$$\begin{array}{ccc} S^k\Omega SX & \xrightarrow{S^k\Omega f} & S^k\Omega SY \\ {\scriptstyle \lambda_k(ev)}\downarrow & & \downarrow{\scriptstyle \lambda_k(ev)} \\ S^k X^{(k)} & \xrightarrow{f\wedge\cdots\wedge f} & S^k Y^{(k)} \end{array}$$

To establish this, we apply the composition formula [4, 3.16] to the composites in the square:

$$\begin{array}{ccc} S\Omega SX & \xrightarrow{ev} & SX \\ {\scriptstyle S\Omega f}\downarrow & & \downarrow{\scriptstyle f} \\ S\Omega SY & \xrightarrow{ev} & SY \end{array}$$

Since $\lambda_q(f) \sim *$ for each $q > 1$, $\lambda_k(f \circ ev) = (f \wedge \cdots \wedge f) \circ \lambda_k(ev)$. However $\lambda_q(S\Omega f) \sim *$ for each $q > 1$, so $\lambda_k(ev \circ S\Omega f) = \lambda_k(ev) \circ S^k\Omega f$. This establishes the result. □

We now apply this to the map $\sigma\gamma\colon S^2K_n \longrightarrow S^{2n+1}$:

$$\begin{array}{ccc} \Omega S^2 K_m & \xrightarrow{\Omega(\sigma g)} & \Omega S^{2m+1} \\ {\scriptstyle \lambda_p}\downarrow & & \downarrow{\scriptstyle \lambda_p} \\ \Omega^p S^{2p} K_m^{(p)} & \longrightarrow & \Omega^p S^{(2m+1)p} \end{array}$$

$\Omega S^2 K_m \simeq (SK_m)_\infty$ and $J_{p-1}(SK_m)$ maps trivially under H_p and hence under λ_p. Thus the composition:

$$J_{p-1}(SK_m) \longrightarrow (SK_m)_\infty \longrightarrow S^{2m}_\infty \xrightarrow{H_p} S^{2mp}_\infty \longrightarrow \Omega^p S^{(2m+1)p}$$

is null-homotopic. Since $\dim\big(J_{p-1}(SK)\big) = (2mp-1)(p-1) < 2(mp+1)p - 3$, the composite of the first three maps:

$$J_{p-1}(SK_m) \longrightarrow (SK_m)_\infty \longrightarrow S^{2m}_\infty \xrightarrow{H_p} S^{2mp}_\infty$$

is null-homotopic. We have proven the first part of

Corollary A.6. *There is a map $\phi\colon J_{p-1}(SK_m) \longrightarrow \widehat{S}^{2m}$ such that the diagram:*

$$\begin{array}{ccc} J_{p-1}(SK_m) & \xrightarrow{\phi} & \widehat{S}^{2m} \\ \downarrow & & \downarrow \\ (SK_m)_\infty & \xrightarrow{(\sigma\gamma)_\infty} & S^{2m}_\infty \end{array}$$

homotopy commutes.

Furthermore, $\phi\big|_{SK_m} \sim \gamma$

Proof. For the second part, we observe that the composites:

$$SK_m \longrightarrow J_{p-1}(SK_m) \xrightarrow{\phi} \widehat{S}^{2m} \xrightarrow{E} S^{2m}_\infty$$
$$SK_m \xrightarrow{\gamma} \widehat{S}^{2m} \xrightarrow{E} S^{2m}_\infty$$

are homotopic, so the difference $\phi\big|_{SK_m} - \gamma$ factors through the fiber of E. We will alter ϕ so that this difference vanishes. To do this, note that the choice of ϕ can be modified by any element of $[J_{p-1}(SK_m), \Omega^2 S^{2mp+1}]$. But the restriction:

$$[J_{p-1}(SK_m), \Omega^2 S^{2mp+1}] \longrightarrow [SK_m, \Omega^2 S^{2mp+1}]$$

is onto since $SJ_{p-1}(SK_m)$ splits. Therefore an appropriate choice of ϕ yields $\phi\big|_{SK_m} \sim \gamma$. □

□

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