On the homotopy groups of 2-cell complexes

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0. The homotopy groups of CW complexes are, in general, much more mysterious than the stable homotopy groups. A notable exception is the case of spheres or cases when the unstable Adams spectral sequence can be utilized. The problem is clearest in the case of a 2-cell complex $S^k \cup e^n$. Very little knowledge of such spaces was known before the work of Cohen, Moore, and Neisendorfer who analyzed the case of the Moore space $S^k \cup_{p^r} e^{k+1}$ for p an odd prime ([CMN]). Their work gave a clear understanding of the kinds of things that can occur, and the depth of their analysis was demonstrated by their determination of the exponent of the homotopy groups of Moore spaces when $p > 2$. Our purpose is to discuss the homotopy groups of two cell complexes in case the attaching map is an arbitrary element in an even stem. In some cases we will have results as strong as those in [CMN].

Throughout this paper all spaces will be localized at a prime $p > 2$. Let $\theta : S^{2n-1} \longrightarrow S^{2m-1}$ and write $P = S^{2m-1} \cup_\theta e^{2n}$, $P^r = S^{r-2n}P$ for $r \geq 2n$, and $\sigma = 2n - 2m + 1$.

In section 1, we deal with quite a general decomposition theorem for spaces of the form $\Omega S(X \cup_\phi e^{2n-1})$. There are two applications

Corollary 1.3. *Suppose $S^{2n+1}\theta \not\simeq *$. Then there is a homotopy equivalence:*

$$\Omega P^{2n+2} \simeq F_{S^2\theta} \times \Omega(P^{4n+3} \wedge \Omega S^{2m+1^+})$$

where $F_{S^2\theta}$ is the fiber of $S^2\theta : S^{2n+1} \longrightarrow S^{2m+1}$.

This generalizes [CMN;1.1]. Another application gives

Corollary 1.4. *Suppose $p > 3$. Let $V(1)_{2n}$ be a space stably homotopy equivalent to the Smith-Toda complex $S^{2n}V(1)$, and let $V(3/4)_{2n}$ be the $2n + 2p - 1$ skeleton of $S^{2n}V(1)$. Then*

$$\Omega V(1)_{2n} \simeq F \times \Omega\big(S^{2n+2p-1}V(1)_{2n} \wedge \Omega SV(3/4)^+_{2n}\big)$$

where F is the fiber of the attaching map of the top cell of $V(1)_{2n}$:

$$S^{2n+2p-1} \longrightarrow V(3/4)_{2n}.$$

This has application to the program in [G6].

Section 2 is preparatory for the case of P^{2n+1}. In this section we establish a Bockstein homomorphism in $H_*(\Omega P^{2n+1}; Z_{(p)})$ corresponding to the homotopy Bockstein in $\pi_*(\quad ; P)$.

In section 3 we prove

Theorem 3.2. *Suppose $\theta = p\varphi$ and $p > 3$. Then $\Omega P^{2n+1} \simeq T \times \Omega W$ where $W = \bigvee_\alpha P^{n_\alpha}$ with $n_\alpha \geq 2m + 2n$ and T belongs to a fibration sequence:*

$$\Omega^2 S^{2n+1} \longrightarrow S^{2m-1} \times \prod_{i \geq 1} S^{2np^i - \sigma}\{p\theta\} \longrightarrow T \longrightarrow \Omega S^{2n+1}.$$

We also prove

Theorem 3.3. *Suppose $S^{2n(p-3)+4m-2}\theta \sim *$. Then there is a fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\ \pi\ } S^{2m-1} \longrightarrow T_\infty \longrightarrow \Omega S^{2n+1}$$

where $\pi\big|_{S^{2n-1}} \sim \theta$ and $E^2\pi \sim \Omega^2 S^2\theta : \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2m+1}$. Furthermore, $\Omega P^{2n+1} \simeq T_\infty \times \Omega W \times \prod_{i\geq 1} (S^{2np^i-\sigma} \times \Omega S^{2np^i - 2\sigma+1})$ with $W \simeq \bigvee_\alpha P^{n_\alpha}$ where $n_\alpha \geq 2n + 2m$.

Finally in section 4 we show that the decomposition in 3.2 does not hold in general - in particular, in case $\theta = \beta_1$.

1. In this section we will give a splitting theorem for ΩSY where $Y \wedge Y$ has a splitting property. This is a property enjoyed by the Spanier-Whitehead duals of ring spectra and applies to two interesting cases: P^{2n+2} and $V(1)^{2n+2p}$ (the $2n+2p$ dimensional version of the Smith-Toda comples $V(1)$).

Definition 1.1. A map $p : Y \longrightarrow S^k$ will be called a co-unit if $p \wedge 1 : Y \wedge Y \longrightarrow S^k \wedge Y$ has a right homotopy inverse q.

Theorem 1.2. *Suppose $S^{2n}(X \cup_\phi e^{2n-1})$ is p-complete and atomic, and the projection*

$$p : X \cup_\phi e^{2n-1} \longrightarrow S^{2n-1}$$

is a co-unit. Then there are fibration sequences:

$$F_{S\phi} \longrightarrow S^{2n-1} \xrightarrow{S\phi} SX$$

$$S^{2n-1} \longrightarrow SX \vee S^{2n}(X \cup_\phi e^{2n-1}) \longrightarrow S(X \cup_\phi e^{2n-1})$$

$$F_{S\phi} \xrightarrow{*} S^{2n}(X \cup_\phi e^{2n-1}) \wedge \Omega SX^+ \xrightarrow{ad} S(X \cup_\phi e^{2n-1})$$

inducing a homotopy equivalence:

$$\Omega S(X \cup_\phi e^{2n-1}) \simeq F \times \Omega\big(S^{2n}(X \cup_\phi e^{2n-1}) \wedge \Omega SX^+\big)$$

where $ad = \bigvee_{i=0}^{\infty} ad_i : S^{2n}(X \cup_\phi e^{2n-1}) \wedge X^{(i)} \longrightarrow S(X \cup_\phi e^{2n-1})$ with $ad_i = [\iota, ad_{i-1}]$ and $ad_0 = [\iota, \iota] \circ Sq$ where q is the right homotopy inverse to $p \wedge 1$.

Corollary 1.3. *Suppose $\theta : S^{2n-1} \longrightarrow S^{2m-1}$ is such that $S^{2n+1}\theta \not\simeq *$. Then there is a homotopy equivalence:*

$$\Omega P^{2n+2} \simeq F_{S^2\theta} \times \Omega(P^{4n+3} \wedge \Omega S^{2m+1^+}).$$

This generalizes the first decomposition theorem in [CMN].

Let us write $V(1)_{2n} = S^{2n} \cup_{p\iota} e^{2n+1} \cup_{\alpha_1} e^{2n+2p-1} \cup_{p\iota} e^{2n+2p}$ for the space whose suspension spectrum is $S^{2n}V(1)$ for $n \geq 1$. Likewise, write $V(3/4)_{2n}$ for the $2n+2p-1$ skeleton of $V(1)_{2n}$. Then if $p > 3$, the projection $V(1)_{2n} \longrightarrow S^{2n+2p}$ is a co-unit, so we get

Corollary 1.4. *Suppose $p > 3$. Then there is a homotopy equivalence:*

$$\Omega V(1)_{2n} \simeq F \times \Omega\big(S^{2n+2p-1}V(1)_{2n} \wedge \Omega SV(3/4)_n^+\big)$$

where F is the fiber of the attaching map:

$$S^{2n+2p-1} \longrightarrow V(3/4)_{2n}.$$

This result has an application of the program developed in [G6].

The proof of these results depends on constructing the fibration sequences in 1.2. To this end we recall the clutching constructions of [G5]. Given a Hurewicz fibration $F \longrightarrow E \xrightarrow{\pi} X$, a subspace $A \subset X$, and a trivialization θ of $\pi\big|_A$, there is a quasi fibering:

$$F \longrightarrow E^\theta \xrightarrow{\pi^\theta} X \cup CA$$

where

$$E^\theta = E \coprod F \times CA/(f, a, 0) \sim \theta(f, a).$$

Here $\theta : F \times A \longrightarrow E$ is a trivialization of $\pi\big|_A$. A classical example when $X = CA$ and $A = F \simeq E$ is the Hopf construction defined by $\theta : A \times A \longrightarrow A$ when A is an H-space (see [S; 1.3, 1.4]).

Localized away from 2, there is a multiplication

$$\mu_n : S^{2n-1} \times S^{2n-1} \longrightarrow S^{2n-1}$$

giving a well known fibration

$$S^{2n-1} \longrightarrow S^{4n-1} \xrightarrow{h_n} S^{2n}.$$

Lemma 1.5. $2h_n \sim [\iota_n, \iota_n]$.

Proof: The multiplication μ_n is the restriction of the multiplication on $\Omega^2 S^{2n+1}$ to S^{2n-1}. Consequently there is a commutative diagram of Hopf constructions:

$$\begin{array}{ccc} S^{4n-1} & \longrightarrow & \Omega^2 S^{2n+1} * \Omega^2 S^{2n+1} \\ \downarrow h_n & & \downarrow h' \\ S^{2n} & \longrightarrow & S\Omega^2 S^{2n+1} \end{array}$$

however h' is the restruction of the classifying space construction for $\Omega^2 S^{2n+1}$, so

$$\Omega^2 S^{2n+1} * \Omega^2 S^{2n+1} \xrightarrow{h'} S\Omega^2 S^{2n+1} \longrightarrow \Omega S^{2n+1}$$

is null homotopic. It follows that

$$S^{4n-1} \xrightarrow{h_n} S^{2n} \longrightarrow \Omega S^{2n+1}$$

is nullhomotopic, so $h_n = k[\iota_n, \iota_n]$. Applying the second Hopf invariant we get $k = 1/2$.

More generally, we may use the map $\phi : S^{2n-1} \longrightarrow X$ to construct:

$$\theta : S^{2n-1} \times S^{2n-1} \longrightarrow X \times S^{2n-1}$$

by $\theta(a, b) = \big(\phi(a), \mu_n(a, b)\big)$. θ is a trivialization of the trivial bundle $X \times S^{2n-1} \longrightarrow X$ pulled back by ϕ, and we get

Proposition 1.6. *There is a quasi fibering:*

$$S^{2n-1} \longrightarrow E^{\theta} \longrightarrow X \cup_{\phi} e^{2n-1}$$

with $E^{\theta} = X \times S^{2n-1} \cup_{\theta} B^{2n} \times S^{2n-1}$ *where* $S^{2n-1} \times S^{2n-1} \subset B^{2n} \times S^{2n-1}$ *is identified with it's image under* $\theta : S^{2n-1} \times S^{2n-1} \longrightarrow X \times S^{2n-1}$.

Moreover this fibering is natural in X and is the Hopf construction in case $X = *$.

Proposition 1.7. *There is a cofibration sequence:*

$$X \to E^{\theta} \xrightarrow{j} S^{2n-1} \wedge (X \cup_{\phi} e^{2n})$$

which is natural in X*.*

Proof: E_{θ} is the push out of the diagram:

$$\begin{array}{ccc} X \times S^{2n-1} & & \\ \uparrow \theta & & \\ S^{2n-1} \times S^{2n-1} & \longrightarrow & B^{2n} \times S^{2n-1}. \end{array}$$

We will show that this is equivalent to another push out diagram. Let $\Gamma : S^{2n-1} \times S^{2n-1} \longrightarrow S^{2n-1} \times S^{2n-1}$ be the homotopy equivalence given by $\Gamma(a,b) = \big(a, \mu_n(a,b)\big)$. Then there is a homotopy commutative diagram:

$$\begin{array}{ccccc} X \times S^{2n-1} & \xleftarrow{F} & S^{2n-1} \times S^{2n-1} & \longrightarrow & B^{2n} \times S^{2n-1} \\ \downarrow = & & \downarrow \Gamma & & \downarrow \pi_2 \\ X \times S^{2n-1} & \xleftarrow{\phi \times 1} & S^{2n-1} \times S^{2n-1} & \xrightarrow{\alpha} & S^{2n-1} \end{array}$$

where α is the second component of Γ^{-1}. It follows that E^{θ} is homotopy equivalent to the push out of the diagram:

$$\begin{array}{ccc} X \times S^{2n-1} & & \\ \uparrow \phi \times 1 & & \\ S^{2n-1} \times S^{2n-1} & \xrightarrow{\alpha} & S^{2n-1}. \end{array}$$

Clearly α has degrees -1 and $+1$ on the axes. Therefore there is a cofibration sequence of push out diagrams:

$$
\begin{array}{ccc}
X \vee S^{2n-1} & & \\
\uparrow \phi\vee 1 & & \\
S^{2n-1} \vee S^{2n-1} & \xrightarrow{-1\vee 1} & S^{2n-1} \\
 & & \downarrow \\
X \times S^{2n-1} & & \\
\uparrow \phi\times 1 & & \\
S^{2n-1} \times S^{2n-1} & \xrightarrow{\alpha} & S^{2n-1} \\
 & & \downarrow \\
X \wedge X^{2n-1} & & \\
\uparrow \phi\wedge 1 & & \\
S^{2n-1} \wedge S^{2n-1} & \longrightarrow & *
\end{array}
$$

This gives a cofibration sequence of the pushouts, which is the conclusion of 1.7.

Proposition 1.8. *Suppose $X = SX'$ and $\phi = s\phi'$ where $\phi' : S^{2n-2} \longrightarrow X'$. Then there is a map $w : S(X' \cup e^{2n-1}) \wedge (X' \cup e^{2n-1}) \longrightarrow E^\theta$ so that:*

$\pi^\theta \circ w \sim [\iota, \iota] : S(X' \cup e^{2n-1}) \wedge (X' \cup e^{2n-1}) \to S(X' \cup e^{2n-1})$ and $S^{2n-1} p \circ j \circ w \sim 2S(p \wedge p)$

Proof: Since the Whitehead product is natural, there is a commutative diagram:

$$
\begin{array}{ccc}
S(X' \cup e^{2n-1}) \wedge (X' \cup e^{2n-1}) & \xrightarrow{[\iota,\iota]} & S(X' \cup e^{2n-1}) \\
\downarrow S(p\wedge p) & & \downarrow Sp \\
S^{4n-1} & \xrightarrow{[\iota,\iota]} & S^{2n}.
\end{array}
$$

On the other hand, there is a pull back diagram:

$$
\begin{array}{ccc}
E^\theta & \xrightarrow{S^{2n-1} p \circ j} & S^{4n-1} \\
\downarrow \pi^\theta & & \downarrow hn \\
S(X' \cup e^{2n-1}) & \xrightarrow{Sp} & S^{2n}
\end{array}
$$

so we may lift the Whitehead product map on $S(X' \cup e^{2n-1})$ through π^θ to E^θ.

Proof of 1.2: Replacing X' with X and ϕ' with ϕ, we show that the cofibration sequence (2.4):

$$SX \to E^\theta \to S^{2n}(X \cup_\phi e^{2n-1})$$

splits and $E^\theta \simeq SX \vee S(X \cup_\phi e^{2n-1})$. To obtain the splitting, we consider the composite

$$\Lambda : S^{2n} \wedge (X \cup_\phi e^{2n-1}) \xrightarrow{Sq} S(X \cup e^{2n-1}) \wedge (X \cup e^{2n-1}) \xrightarrow{w} E^\theta$$

where q is the right inverse to $p \wedge 1$. We have

$$\begin{array}{ccc}
S^{2n} \wedge (X \cup_\phi e^{2n-1}) & \xrightarrow{\Lambda} & E^\theta \\
\Big\downarrow {\scriptstyle 2S^{2n}p} & & {\scriptstyle j}\Big\downarrow \\
S^{4n-1} & \underset{S^{2n}p}{\longleftarrow} & S^{2n}(X \cup_\phi e^{2n-1}).
\end{array}$$

Now since p is a count, $p^* : H^{2n-1}(S^{2n-1}; \mathbf{Z}/p) \to H^{2n-1}(X \cup_\phi e^{2n-1}; \mathbf{Z}/p)$ is a monomorphism; it follows that the self map $j\Lambda : S^{2n}(X \cup e^{2n-1}) \to S^{2n}(X \cup e^{2n-1})$ is not topologically nilpotent. Since $S^{2n}(X \cup e^{2n-1})$ is atomic it follows from the results in [AK] that $j\Lambda$ is an equivalence. Thus we have a homotopy equivalence:

$$SX \vee S^{2n}(X \cup_\phi e^{2n-1}) \xrightarrow{\iota \vee \Lambda} E^\theta.$$

We now construct the diagram of fibrations:

$$\begin{array}{ccccccc}
\Omega S(X \cup_\phi e^{2n-1}) & \longrightarrow & F & \longrightarrow & S^{2n}(X \cup_\phi e^{2n-1}) \wedge \Omega SX^+ & \longrightarrow & S(X \cup_\phi e^{2n-1}) \\
\Big\downarrow {\scriptstyle 1} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow {\scriptstyle 1} \\
\Omega S(X \cup_\phi e^{2n-1}) & \longrightarrow & S^{2n-1} & \longrightarrow & SX \vee S^{2n}(X \cup_\phi e^{2n-1}) & \longrightarrow & S(X \cup_\phi e^{2n-1}) \\
\Big\downarrow {\scriptstyle *} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow {\scriptstyle *} \\
* & \longrightarrow & SX & \longrightarrow & SX & \longrightarrow & *.
\end{array}$$

Clearly there is a right homotopy inverse for $\Omega S(X \cup_\phi e^{2n-1}) \to F$, so $\Omega S(X \cup \phi e^{2n-1}) \simeq F \times \Omega\big(S^{2n}(X \cup_\phi e^{2n-1}) \wedge \Omega SX^+\big)$. Here we use the fibering [G2]

$$\bigvee_{i=0}^{\infty} SB \wedge A^{(i)} \simeq SB \wedge \Omega SA^+ \xrightarrow{f} SA \vee SB \to SA$$

where $f = \vee f^i$ and $SB \wedge A^{(i)} \xrightarrow{f_i} SA \vee SB$ is $ad_i(A)(B)$.

2. In order to decompose ΩSP, we need to construct a Bockstein homomorphism in homology which is compatible with the homotopy Bockstein

$$\beta : \pi_r(X;P) \longrightarrow \pi_{r-\sigma}(X;P)$$

under the Hurewicz homomorphism. Such a homology operation cannot be defined and natural for all spaces.

Definition 2.1. A space X is P-free if there is a homotopy equivalence $X \simeq \bigvee_\alpha P^{n_\alpha}$. A space is stably P-free if it has the stable homotopy type of a P-free space.

Theorem 2.2. *There is a unique Bockstein operation:*

$$\widehat{p} : \widetilde{H}_r(X;\mathbf{Z}/p) \longrightarrow \widetilde{H}_{r-\sigma}(X;\mathbf{Z}/p)$$

defined when X is stably P-free and natural under continuous maps such that

a) $\widehat{\beta} : \widetilde{H}_r(P;\mathbf{Z}/p) \longrightarrow \widetilde{H}_{r-\sigma}(P;\mathbf{Z}/p)$ *is a isomorphism for all r*

b) $\widehat{\beta}s = s\widehat{\beta}$ *where σ is the homology suspension*

c) $\ker\widehat{\beta} = \mathrm{im}\widehat{\beta}$

d) *if $r-\sigma \geq 2n$, there is a commutative diagram:*

$$\begin{array}{ccc} \pi_r(X;P) & \xrightarrow{\beta} & \pi_{r-\sigma}(X;P) \\ \downarrow h & & \downarrow h \\ \widetilde{H}_r(X:\mathbf{Z}/p) & \xrightarrow{\widehat{\beta}} & H_{r-\sigma}(X;\mathbf{Z}/p). \end{array}$$

Proof: It is clear what the intention is. The problem is with naturality, and in particular, to show that the definition does not depend on the choice of an equivalence $X \simeq \bigvee_\alpha P^{n_\alpha}$.

Let $u \in H_{2m-1}(P;\mathbf{Z}_{(p)})$ and $v \in H_{2n}(P;\mathbf{Z}_{(p)})$ be homology generators. Then define $u_k = S^{k-2m+1}u \in H_k(P^{k+\sigma};\mathbf{Z}_{(p)})$ and $v_k = S^{k-2n}v \in H_k(P^k;\mathbf{Z}_{(p)})$. With this we define Hurewicz homomorphisms when $r \geq 2n$:

$$h : \pi_r(X;P) \longrightarrow H_r(X;\mathbf{Z}/p)$$
$$h' : \pi_r(X;P) \longrightarrow H_{r-\sigma}(X;\mathbf{Z}/p)$$

by $h(f) = f_*(v_r)$, $h'(f) = f_*(u_{r-\sigma})$ for $f : P^r \longrightarrow X$ (see [G7]). If $r-\sigma \geq 2n$, $h' = h\beta$ where

$$\beta : \pi_r(X;P) \longrightarrow \pi_{r-\sigma}(X;P)$$

is the homotopy Bockstein.

Lemma 2.3. *Suppose X is P-free. Then*

a) *h is onto*

b) $\ker h \subset \ker h'$.

Proof: Clearly $h : \pi_r(P^{n_\alpha}; P) \longrightarrow \widetilde{H}_r(P^{n_\alpha}; Z/p)$ is onto if $r = n_\alpha$; it is also onto in case $r = n_\alpha - \sigma$ if $r \geq 2n$. Consequently, if X is P-free, a) follows. Now suppose $h(f) = 0$, where $f : P^r \longrightarrow X$. Since P^r is compact, $f(P^r) \subset P^{n_i} \vee \cdots \vee P^{n_k}$. Then

$$h'(f) = \sum a_i u_{n_i - \sigma} + b_i v_{n_i}.$$

If $b_j \neq 0$ for some j, the composite

$$P^r \longrightarrow P^{n_1} \vee \cdots \vee P^{n_k} \longrightarrow S^{n_j}$$

would be non zero in homology and $r - \sigma = n_j$; it follows that $b_j\theta = 0$, so $b_j \equiv 0 (\text{mod } p)$. Now suppose that $a_j \neq 0$ for some j. Then the composite

$$P^r \longrightarrow P^{n_1} \vee \cdots \vee P^{n_k} \longrightarrow P^{n_j}$$

is non zero in homology in dimension $r - \sigma = n_j - \sigma$. However since $h(f) = 0$, the degree on the top cell is divisible by p. It follows that the degree on the bottom cell, a_j, is also divisible by p.

Proof of 2.2. We first deal with the case that X is P-free. In this case Lemma 2.3 implies that we can define $\widehat{\beta}$ such that $h' = \widehat{\beta} h$. Since h is onto, $\widehat{\beta}$ is unique, natural, and commutes with s. Condition d) follows since if $r - \sigma \geq 2n$, $h'(f) = h(\beta f)$. Condition a) is immediate. To show that $\widehat{\beta}^2 = 0$, note that $\widehat{\beta} = 0$ if $r < 2n$. Suppose $u \in \widetilde{H}_r(X; \mathbf{Z}/p)$ and $r - \sigma \geq 2n$. Let $u = h(f)$; then $\widehat{\beta}(u) = h'(f) = h\big(\beta(f)\big)$. Thus $\widehat{\beta}^2(u) = h'\big(\beta(f)\big) = 0$. Since the homology of $\widetilde{H}_*(X; \mathbf{Z}/p)$ under $\widehat{\beta}$ is the direct sum of the homology of $\widetilde{H}_r(P^{n_\alpha}; \mathbf{Z}/p)$, it is zero and c) holds. In case X is stably P-free, we define $\widehat{\beta}$ as the composite:

$$\widetilde{H}_r(X; \mathbf{Z}/p) \xrightarrow[\cong]{s^i} \widetilde{H}_{r+i}(S^i X; \mathbf{Z}/p) \xrightarrow{\widehat{\beta}} \widetilde{H}_{r+i-\sigma}(S^i X; \mathbf{Z}/p) \xleftarrow[\cong]{s^i} \widetilde{H}_{r-\sigma}(X; \mathbf{Z}/p).$$

Clearly this does not depend on i and is compatible under the Hurewicz homomorphism with the homotopy Bockstein. The other conditions are immediate.

Proposition 2.4. *The category of stably P-free spaces is closed under Cartesian and smash products and retracts. If X is P-free, $\Omega S X$ is stably P-free.*

Proof: The only difficult part is to show that a retract of a P-free space is P-free. This follows from

Lemma 2.5. *Suppose for a given space X, h is onto,* $\ker h \subset \ker h'$, *and* $\widehat{\beta} = h'h^{-1}$ *satisfies* $\ker \widehat{\beta} = im\ \widehat{\beta}$. *Then X is P-free.*

Proof: Choose a basis $\{v_{\alpha_1} u_\alpha\}$ for $\widehat{H}_*(X; \mathbf{Z}/p)$ with $\widehat{\beta}(v_\alpha) = u_\alpha$. Choose $f_\alpha : P^{n_\alpha} \longrightarrow X$ with $h(f_\alpha) = v_\alpha$. Define

$$f : \bigvee_\alpha P^{n_\alpha} \longrightarrow X$$

by $f\big|_{P^{n_\alpha}} = f_\alpha$. Then f_* is an isomorphism.

Proposition 2.6. *Suppose X and Y are stably P-free and $u \otimes v \in \widehat{H}_*(X \times Y; \mathbf{Z}/p)$. Then $\widehat{\beta}(u \otimes v) = \widehat{\beta}(u) \otimes v + (-1)^{|u|} u \otimes \widehat{\beta}(v)$.*

Proof: It suffice to prove this formula in case $u \otimes v \in \widetilde{H}_*(X \wedge Y : \mathbf{Z}/p)$. We may assume that X and Y are P-free. In this case h is onto so the formula follows from the corresponding formula in homotopy with coefficients in P.

3. In this section we will discuss the decomposition of ΩSP. Our main tool will be the method of [CMN]. We will use the results of [G7] to obtain an identical splitting to that in [CMN] when $p\big|\theta$. We will also obtain a splitting generalizing [G4] when θ is unstable and discuss the obstructions in the general case.

Recall from [G7] that if G is a group like space there is a Samelson product pairing

$$[\ \ ,\ \ ] : \pi_k(G; P) \otimes \pi_\ell(G; P) \longrightarrow \pi_{k+\ell}(G; P)$$

when $k, \ell \geq 2n$ satisfying the usual conditions for a graded Lie algebra except that for the Jacobi identity we require $p > 3$. Furthermore $\pi_*(G; P)$ is a Lie module over $\pi_*(G)$. There is a Bockstein homomorphism:

$$\beta : \pi_k(X; P) \longrightarrow \pi_{k-\sigma}(X; P)$$

defined when $k - \sigma \geq 2n$ which is a derivation with respect to the Samelson product. Finally, there is a Hurewicz homomorphism

$$h : \pi_k(X; P) \longrightarrow \widetilde{H}_k(X; Z_{(p)})$$

and $h\big([x, y]\big) = \big[h(x), h(y)\big]$, the graded commutator in the ring $H_*(X; \mathbf{Z}_{(p)})$.

The following result is an adaption of the results of [CMN] using the results of section 2 and the above.

Theorem 3.1. *There is a commutative diagram of fibrations:*

$$\begin{array}{ccccccc}
\Omega^2 S^{n+1} & \xrightarrow{\partial} & V & \longrightarrow & T_{2m-1} & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & {\scriptstyle *}\downarrow & & \downarrow \\
* & \longrightarrow & W & \xrightarrow{1} & W & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
\Omega S^{2n+1} & \longrightarrow & F & \xrightarrow{r} & SP & \longrightarrow & S^{2n+1}
\end{array}$$

with the following properties:

(a) Ωf *has a left homotopy inverse, so*

$$\Omega SP \simeq \Omega W \times T_{2m-1}$$
$$\Omega F \simeq \Omega W \times V$$

(b) $H_*(\Omega F; \mathbf{Z}_{(p)}) \simeq T(x_i, 2n(i-1) + 2m - 1, i \geq 1)$ *where* T *is a tensor algebra on the generators* x_i *and* $r_*(x_i) = ad^{i-1}(v)(u)$.

(c) $H_*(V; \mathbf{Z}_{(p)}) \simeq \bigotimes_{i \geq 0} \bigwedge_{(p)}(\tau_i, 2np^i - \sigma) \otimes \bigotimes_{i \geq 1} \mathbf{Z}_{(p)}(\sigma_i, 2np^i - 2\sigma)$

as coalgebras where $\bigwedge_{(p)}$ *and* $\mathbf{Z}_{(p)}$ *denote exterior and polynomial algebras respectively and* τ_i *corresponds to* x_{p^i} *while* σ_i *corresponds to*

$$\frac{1}{2p} \sum_{j=1}^{p^i - 1} \binom{p^i}{j} [x_j, x_{p^i} - j]$$

under the equivalence in (a).

Proof: As in [CMN; section 12], write $L(u, v)$ for the free Lie algebra over $\mathbf{Z}_{(p)}$ generated by u and v with $\dim u = 2m - 1$ and $\dim v = 2n$. Let L_0 be the Lie algebra kernel of the natural projection: $L \to \langle v \rangle$. Then L_0 is the free Lie algebra generated by x_i, $i \geq 1$, and $H_*(\Omega F; \mathbf{Z}_{(p)}) \simeq UL_0$, $H_*(\Omega SP; \mathbf{Z}_{(p)}) \cong UL\langle u, v \rangle$. This is the same as in [CMN] except $\dim x_i = 2n(i-1) + 2m - 1$ in this case while $m = n$ in [CMN]. In [CMN; 12.3] specific free sub-Lie algebras $L^{(k)}$ of $L^{(0)}$ are constructed, and we consider the same sub-Lie algebras regraded as appropriate. Let $L^{(\infty)} = \bigcap L^{(k)}$. Since ΩSP is stably P-free, we introduce Bockstein $\widehat{\beta} : H_r(\Omega SP; \mathbf{Z}/p) \longrightarrow H_{r-\sigma}(\Omega SP; \mathbf{Z}/p)$ from section 2. Clearly $\widehat{\beta}(v) = u$ and $\widehat{\beta}$ is a derivation. Consequently apart from grading, $\left(L\langle u, v \rangle, \widehat{\beta}\right)$ is isomorphic to the differential Lie algebra L in [CMN], and $L^{(\infty)} \otimes \mathbf{Z}/p$ is acyclic. Choose a free basis u_α, v_α for $L^{(\infty)}$ with $\widehat{\beta}(v_\alpha) \equiv u_\alpha (\mathrm{mod}\ p)$. Let $u = h(\mu)$ and

$v = h(\nu)$ with $\mu \in \pi_{2m-1}(\Omega SP)$ and $\nu \in \pi_{2n}(\Omega SP; P)$. Using the Samelson product and the action of $\pi_*(\Omega SP)$ on $\pi_*(\Omega SP; P)$, we see that each element of $L(u,v) \otimes \mathbf{Z}/p$ except possibly $[u,u]$ is in the image of $h : \pi_*(\Omega SP; P) \longrightarrow H_*(\Omega SP; \mathbf{Z}/p)$. Choose $f_\alpha : P^{n_\alpha} \longrightarrow \Omega SP$ so that $h(f_\alpha) = v_\alpha$. Then $(f_\alpha)_*(v_{n_\alpha}) = v_\alpha$ and $(f_\alpha)_*(u_{n_\alpha - \sigma}) \equiv u_\alpha (\text{mod } p)$. Construct $W' = \bigvee_\alpha P^{n_\alpha}$ and $f' : W' \longrightarrow \Omega SP$ with $f'\big|_{P^{n_\alpha}} = f_\alpha$. Then in $(f')_* : H_*(W'; \mathbf{Z}_{(p)}) \longrightarrow H_*(\Omega SP; \mathbf{Z}_{(p)})$ is contained in $L^{(\infty)}$ and with $\mathbf{Z}/p$ coefficients $(f')_*$ is an isomorphism onto $L^{(\infty)} \otimes \mathbf{Z}/p$. It follows that $(f')_*$ is an isomorphism onto $L^{(\infty)}$ with $\mathbf{Z}_{(p)}$ coefficients. Let $W = SW'$ and $f : W \longrightarrow SP$ be the adjoint of f'. Then $(\Omega f)_* : H_*(\Omega W; \mathbf{Z}_{(p)}) \longrightarrow H_*(\Omega SP; \mathbf{Z}_{(p)})$ is the inclusion of $UL^{(\infty)}$ into $UL\langle u, v\rangle$. Since each f_α is an iterated Samelson product and ΩS^{2n+1} is homotopy commutative, f factors through F, and we have constructed the diagram of the theorem. As in the case of [CMN], $S\Omega f$ has a left homotopy inverse, so Ωf does as well and we have proven (a). Part (b) follows exactly as in [CMN] as does (c) where we write σ_i and τ_i in $H_*(V; \mathbf{Z}_{(p)})$ for the images of σ_i, τ_i in $H_*(\Omega F; \mathbf{Z}_{(p)})$.

We now write $S^k\{p\theta\}$ for the fiber of the map $p\theta : S^k \longrightarrow S^{k-\sigma}$.

Theorem 3.2. *Suppose $\theta = p\phi$ and $p > 3$. Then*

$$V \simeq S^{2m-1} \times \prod_{i \geq 1} S^{2np^i - \sigma}\{p\theta\}.$$

Proof: Using the extended ideal structure of $\pi_*(\Omega F; P)$ in $\pi_*(\Omega SP; P)$ we construct classes $\widetilde{\tau}_\iota, \widetilde{\sigma}_\iota$ in $\pi_*(\Omega F; P)$ whose Hurewicz images are τ_i and σ_i respectively. Since $p > 3$, we may use the Jacobi identity and apply [CMN, 4.4] to conclude that $\beta\widetilde{\tau}_i = p\widetilde{\sigma}_\iota$. Now let $\gamma_i : P^{2np^i - 2\sigma}(p\iota) \longrightarrow P^{2np^i - 2\sigma}$ be a map which is degree 1 on the top cell and ϕ on the bottom cell where $P^{2np^i - 2\sigma}(p\iota)$ is the mod p Moore space of that dimension. Now $\beta\widetilde{\tau}_i \circ \gamma_i = p\widetilde{\sigma}_i \circ \gamma_i = \widetilde{\sigma}_\iota \circ \gamma_i \circ p\iota = 0$. Now $\beta\widetilde{\tau}_i$ is the composition of $\widetilde{\tau}_\iota$ with

$$B^{2np^i - \sigma} : P^{2np^i - 2\sigma} \longrightarrow S^{2np^i - 2\sigma} \longrightarrow P^{2np^i - \sigma}$$

so $\widetilde{\tau}_i$ extends over the mapping cone of $B^{2np^i - \sigma} \circ \gamma_i$. This mapping cone is $P^{2np^i - \sigma}(p\theta)$. This extension has homology image generated by σ_i and τ_i. Using the H space structure we extend and consider the composite

$$S^{2np^i - \sigma}\{p\theta\} \longrightarrow \Omega P^{2np^i - \sigma + 1} \longrightarrow \Omega F \longrightarrow V.$$

Clearly the image of this map in homology is $\bigwedge_{(p)}(\tau_i, 2np^i - \sigma) \otimes \mathbf{Z}_{(p)}(\sigma_i, 2np^i - 2\sigma)$. Now multiply these maps together and with the inclusion of S^{2m-1} in ΩF to obtain a homotopy equivalence:

$$S^{2m-1} \times \prod_{i \geq 1} S^{2np^i - \sigma}\{p\theta\} \longrightarrow \Omega F \longrightarrow V.$$

Theorem 3.3. *Suppose* $E^{2n(p-3)+4m-2}\theta \sim *$. Then*

$$V \simeq S^{2m-1} \times \prod_{i\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1}$$

and $T \simeq T_\infty \times \prod_{i\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1}$ where T_∞ fits into a fibration sequence:

$$\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2m-1} \longrightarrow T_\infty \longrightarrow \Omega S^{2n+1}$$

where $\pi\big|_{S^{2n-1}} \sim \theta$ and $E^2 \circ \pi \sim.\ \ \Omega^2 S^2\theta : \Omega^2 S^{2n+1} \longrightarrow \Omega^2 S^{2m+1}$

Proof: The condition on θ implies that $P^k \simeq S^k \vee S^{k-\sigma}$ when $k \geq 2np-2\sigma$. Consequently the classes $\widetilde{\sigma}_i$ and $\widetilde{\tau}_i$ may be constructed and we conclude that $\sigma_i,\ \tau_i \in H_*(\Omega F; \mathbf{Z}_{(p)})$ are spherical. As in the case of 3.2, we easily construct a composite:

$$S^{2m-1} \times \prod_{i\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1} \longrightarrow \Omega F \longrightarrow V$$

which is a homotopy equivalence. Since the composition $V \longrightarrow T \longrightarrow \Omega SP$ is a homology monomorphism and $S\Omega SP$ is a wedge of spheres and copies of P^k with $k < 2np-2\sigma$, there are maps $\alpha_i : S\Omega SP \longrightarrow S^{2np^i-\sigma+1}$, $\beta_i : S\Omega SP \longrightarrow S^{2np^i-2\sigma+1}$ carrying τ_i and σ_i in homology. From this we construct a map

$$\Omega SP \longrightarrow \prod_{i\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1}$$

which is a left inverse to the inclusion

$$\prod_{i\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1} \longrightarrow \Omega F \longrightarrow \Omega SP.$$

It follows that this product factors off of ΩF, ΩSP, V and T. Finally, construct the diagram:

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\pi} & \Omega F & \longrightarrow & \Omega SP & \longrightarrow & \Omega S^{2n+1} \\
\downarrow{\scriptstyle \Omega^2 S^2\theta} & & \downarrow{\scriptstyle \Omega\chi} & & \downarrow & & \downarrow{\scriptstyle \Omega S^2\theta} \\
\Omega^2 S^{2m+1} & \longrightarrow & \Omega^2 S^{2m+1} & \longrightarrow & * & \longrightarrow & \Omega S^{2m+1}
\end{array}$$

$\Omega\chi$ is null homotopic on ΩW and $\prod_{2\geq 1} S^{2np^i-\sigma} \times \Omega S^{2np^i-2\sigma+1}$ since it is an H map and each piece is mapped in by a Samelson product which is null in $\Omega^2 S^{2m+1}$. Consequently, the composite factors though S^{2m-1} which is included in $\Omega^2 S^{2m+1}$ as the bottom cell.

An example of this result in the Toda fibration $S^{2n-1} \longrightarrow \Omega S^{2n} \longrightarrow \Omega S^{2np-1}$ created from the class $w_n \in \pi_{2np-3}(S^{2n-1})$.

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*If $p > 3$ this is equivalent to θ begin stably trivial.

4. The object of this section is to study the space V, and in particular to show that it does not always split as in the case that $\theta = p\phi$. In fact we prove

Proposition 4.1. *The $2np - 2\sigma$ skeleton of V is $S^{2m-1} \cup_{w_m \theta^{p-2}} e^{2np-2\sigma}$ where $w_m : S^{2mp-3} \longrightarrow S^{2m-1}$ is the first element in the kernel of the double suspension.*

Proposition 4.2. *Let $\beta_1 : S^{2m+pq-3} \longrightarrow S^{2m-1}$ for $m \geq p$ be a desuspension of the class β_1 in the stable $pq-2$ stem. Suppose $m \not\equiv 0 (\text{mod } p)$; then $w_m \beta_1^{p-2} \not\simeq *$.*

The proof of 4.1 will require the construction of a generalization of the Toda-Hopf invariant to this situation. Let us recall the relative James construction [G1]. This is a reduced product space construction for the fiber of a pinch map. Consider a fibration:

$$F \to X \cup CA \to SA.$$

Then there is a model $(X, A)_\infty$ consisting of all words in X_∞ with the property that all letters after the first letter are required to lie in A, and a weak homotopy equivalence:

$$\rho_\infty : (X, A)_\infty \to F.$$

There is a natural James-Hopf invariant:

$$(X, A)_\infty \xrightarrow{H_k} (XA^{(k-1)})_\infty$$

which agrees with the classical one in case $A = X$. We now generalize the Toda-Hopf invariant given in [G4]. In [G4], a map ℓ was constructed fitting into a diagram:

$$\begin{array}{ccc} \Omega(X \cup CA) & \xrightarrow{\ell} & \Omega\big((X \cup CA) \wedge A\big)_\infty \\ \uparrow{\scriptstyle \Omega\rho_\infty} & & \uparrow{\scriptstyle \Omega i} \\ \Omega(X, A)_\infty & \xrightarrow{\Omega H_2} & \Omega(X \wedge A)_\infty. \end{array}$$

We filter $(X, A)_\infty$ by $(X, A)_k$ which consists of words of length at most k. Then we have

$$(SX, SA)_{k-1} = (SX, SA)_{k-2} \cup_{\gamma_k} C(S^{k-2} XA^{(k-2)})$$

where $\gamma_k : S^{k-2} XA^{(k-2)} \longrightarrow (SX, SA)_{k-2}$ is natural in (X, A). Combining this with the map ℓ and evaluation yields a map

$$\Omega(SX, SA)_{k-1} \to \Omega\big((SX, SA)_{k-1} \wedge S^{k-2} XA^{(k-2)}\big)_\infty \to \Omega(S^{k-1} X^{(2)} A^{(k-2)})_\infty.$$

Now consider the case that $SX = S^{2m}$, $SA = S^{2n}$, and the inclusion corresponds to $S\theta$. Applying the natural retraction yields

Proposition 4.3. *Localized away from 2 there is a Toda-Hopf invariant:*

$$H'_k : \Omega(S^{2m}, S^{2n})_{k-1} \to \Omega S^{2nk-2\sigma+1}$$

which is natural for maps of pairs (S^{2m-1}, S^{2n-1}) *and is null homotopic when restricted to* $\Omega(S^{2m}, S^{2n})_{k-2}$.

In particular, there is a commutative diagram:

$$\begin{array}{ccc}
\Omega S^{2n}_{k-1} & \xrightarrow{H'_k} & \Omega S^{2mk-1} \\
\downarrow & & \downarrow \Omega\theta^{k-2} \\
\Omega(S^{2m}, S^{2n})_{k-1} & \xrightarrow{H'_k} & \Omega S^{2nk-2\sigma+1} \\
\downarrow & & \downarrow \Omega\theta^{2} \\
\Omega S^{2m}_{k-1} & \xrightarrow{H'_k} & \Omega S^{2nk-1}
\end{array}$$

where the upper and lower Hopf invariants are the Toda-Hopf invariants of [G4].

Proposition 4.4 $H'_p(\widetilde{\sigma}_1) : P^{2np-2\sigma} \longrightarrow \Omega S^{2np-2\sigma+1}$ *is the adjoint of the projection* $j : P^{2np-2\sigma+1} \longrightarrow S^{2np-2\sigma+1}$.

Proof:

$$\sigma_1 = \frac{1}{2p} \sum_{i=1}^{p-1} \binom{p}{i} [\widetilde{x}_i, \widetilde{x}_{p-i}]$$

where $\widetilde{x}_i = ad^{i-1}(\nu)(\mu)$ is a mod P homotopy class with $h(\widetilde{x}_i) = x_i$. Now

$$H_*(\Omega(S^{2m}, S^{2n})_i; \mathbf{Z}_{(p)}) = T(x_1, \dots, x_i),$$

the tensor algebra generated by $x_1, \dots, x_i$. Hence the pair $\big(\Omega(S^{2m}, S^{2n})_\infty, \Omega(S^{2m}, S^{2n})_i\big)$ is $2m + 2in - 1$ connected and $\widetilde{x}_i$ is in the image of the inclusion

$$\pi_{2m+2n(i-1)-1}\big(\Omega(S^{2m}, S^{2n})_i; P\big) \to \pi_{2m+2n(i-1)-1}\big(\Omega(S^{2m}, S^{2n})_\infty; P\big).$$

In particular, if $1 < i < p-1$, $[\widetilde{x}_i, \widetilde{x}_{p-i}]$ is in the image of $\pi_*\big(\Omega(S^{2m}, S^{2n})_{p-2}; P\big)$ and hence $H'_p\big([\widetilde{x}_i, \widetilde{x}_{p-i}]\big) = 0$ in this case. Thus $H'_p(\widetilde{\sigma}_1) = H'_p\big([\widetilde{x}_1, \widetilde{x}_{p-1}]\big)$.

Now consider the homotopy groups of the fibering:

$$\Omega\big((S^{2m}, S^{2n})_{p-2}, S^{2m+2n(p-2)-1}\big)_\infty \to \Omega(S^{2m}, S^{2n})_{p-1} \to \Omega S^{2m+2n(p-2)}$$

$[\widetilde{x}_1, \widetilde{x}_{p-1}]$ factors through the left hand space and projects to $x_1 x_{p-1} + x_{p-1} x_1$ in the homology groups in the middle. It factors through $\Omega\big((S^{2m}, S^{2n})_{p-2}, S^{2m+2n(p-2)-1}\big)_2$ for dimensional reasons. It does not lie in the image of $H_*\big(\Omega S^{2m}, S^{2n})_{p-2}; \mathbf{Z}_{(p)}\big) \simeq T(x_1, \dots, x_{p-2})$.

However the $2np-2\sigma$ skeleton of $\Omega\big((S^{2m},S^{2n})_{p-2},\ S^{2m+2n(p-2)-1}\big)_2$ is contained in $\Omega\big((S^{2m},S^{2n})_{p-2}\cup e^{2np-2\sigma+1}\big)$. This is the first cell that maps nontrivially under the relative Hopf invariant

$$\big((S^{2m},S^{2n})_{p-2},S^{2m+2n(p-2)-1}\big)_2 \xrightarrow{H_2} S^{2m+2n(p-2)-1}\wedge(S^{2m},S^{2n})_{p-2}.$$

It follows from the definition of H'_p that $H'_p\big([\widetilde{x}_1,\widetilde{x}_{p-2}]\big)$ induces an isomorphism in homology in dimension $2np-2\sigma$.

Proof of 4.1: Let $V^{2np-2\sigma}$ be the $2np-2\sigma$ skeleton of V. then $\widetilde{H}_*(V^{2np-2\sigma};\mathbf{Z}_{(p)})$ is freely generated by x_1 and σ_1. Since $\widetilde{\sigma_1}$ factors through $V^{2np-2\sigma}$ and $V^{2np-2\sigma}$ factors through $\Omega(S^{2m},S^{2n})_{p-1}$ for dimensional reasons, we have a composite

$$P^{2np-2\sigma}\xrightarrow{\widetilde{\sigma_1}} V^{2np-2\sigma}\to\Omega(S^{2m},S^{2n})_{p-1}\to\Omega S^{2np-2\sigma+1}$$

which is a homology isomorphism in dimension $2np-2\sigma$. Now consider the diagram:

$$\begin{array}{ccccc}
V^{2np-2\sigma}=S^{2m-1}\cup_g e^{2np-2\sigma} & \longrightarrow & \Omega(S^{2m},S^{2n})_{p-1} & \xrightarrow{H;_p} & \Omega S^{2np-2\sigma+1}\\
\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle\theta^{p-2}}\\
\Omega(S^{2m}\cup_{w_m} e^{2mp-1}) & \longrightarrow & \Omega S^{2m}_{p-1} & \xrightarrow{H'_p} & \Omega S^{2mp-1}
\end{array}$$

where the left hand vertical map exists since ΩS^{2m}_{p-1} is a retract of $\Omega(S^{2m}\cup_{w_m} e^{2mp-2})$. From this we see that $g\sim w_m\theta^{p-2}$.

Proof of 4.2. According to [G3; Theorem 12], if $m\not\equiv 0(\text{mod }p)$, $H'(w_m)=\alpha_1$. Consequently $H'(w_m\beta_1^{p-2})=\alpha_1\beta_1^{p-2}$ which according to [O] is non-zero (stably) and not divisible by p. Hence $w_m\beta_1^{p-2}\neq 0$ in $\pi_*(S^{2m-1})$.

Analogously to 4.1, one can easily derive the following result using the James-Hopf invariant ([G1]) instead of the Toda-Hopf invariant.

Proposition 4.6. *Suppose $p\theta\sim *$. Then there is a map $S^{2m-1}\cup_\xi e^{2np-\sigma}\to V$ which induces a homology monomorphism where*

$$\begin{array}{ccc}
S^{2np-\sigma-1} & \xrightarrow{\xi} & S^{2m-1}\\
\big\downarrow{\scriptstyle\theta^{p-1}} & & \big\downarrow{\scriptstyle E}\\
\Omega^3S^{2mp+1} & \xrightarrow{P} & \Omega\widehat{S}^{2m}.
\end{array}$$

In particular, $H(\xi)=(1-m)\alpha_1\circ\theta^{p-1}:S^{2np-\sigma-1}\to S^{2(m-1)p+1}$ and $\xi\not\sim *$ in case $\theta=\beta_1$ and $m\not\equiv 1(\text{mod }p)$.

Theorem 4.7. *If $m \not\equiv 0, 1 (\mathrm{mod}\ p)$ there is no extension $B : \Omega^2 S^{2m+pq-1} \longrightarrow S^{2m-1}$ of $\beta_1 : S^{2m+pq-3} \longrightarrow S^{2m-1}$.*

Proof: According to [G4], $w_m \beta_1^p = \beta_1 w_{m+p(p-1)-1}$. It follows that if an extension exists, $w_m \beta_1^p = 0$. We will show that this is impossible in case $m \not\equiv 0, 1 (\mathrm{mod} p)$. We will base this on the fact that the Toda Brocket $\{\beta_1^p, \alpha_1, \alpha_1\} \neq 0$ (see [O]). The element w_m is obtained as the composition:

$$S^{2mp-3} \xrightarrow{a_m} S^{2m-1} B^{(m-1)q} \xrightarrow{\lambda} S^{2m-1}.$$

where a_m is the $2m-1$ fold suspension of the attaching map of the cell of dimension $mq-1$ in B^{mq-1} and λ is the Kahn-Priddy map (see [G3]). We examine the composite:

$$S^{2mp+(pq-2)p-3} \xrightarrow{\beta_1^p} S^{2mp-3} \xrightarrow{a_m} S^{2m-1} B^{(m-1)q}$$

stably. It is easily seen to factor through $S^{2m-1} B^{(m-2)q-1}$ and its projection onto the top cell $S^{(m-2)2p+2}$ belongs to the bracket $\{\alpha_1, \alpha_1, \beta_1^p\}$ since

$$P^2 : H^{(m-2)q-1}(B^{mq}; \mathbf{Z}/p) \to H^{mq-1}(B^{mq}; \mathbf{Z}/p)$$

is non-zero when $m \not\equiv 0, 1$. In other words, we have a commutative diagram:

$$\begin{array}{ccc} S^{2mp+(pq-3)p-3} & \xrightarrow{\beta_1^p} & S^{2mp-3} \\ \downarrow & & \downarrow a_m \\ S^{2m-1} B^{(m-2)q-1} & \longrightarrow & S^{2m-1} B^{(m-1)q} \\ \downarrow & & \\ S^{2p(m-2)+2} & & \end{array}$$

where the left hand composite, ϵ', is an element of $\{\alpha_1, \alpha_1, \beta_1^p\}$. This implies that $w_m \beta_1^p$ desuspends to an element $e_m \in \pi_{2mp+(pq-2)p-4}(\widehat{S}^{2(m-2)})$ with $H'(e_m) = \epsilon'$. Such an element is, of course, non-zero, so $w_m \beta_1^p \neq 0$.

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