RINGS WITH A LOCAL COHOMOLOGY THEOREM AND APPLICATIONS TO COHOMOLOGY RINGS OF GROUPS.

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Abstract. Cohomology rings of various classes of groups have curious duality properties expressed in terms of their local cohomology [2, 3, 12, 4, 5, 6]. We formulate a purely algebraic form of this duality, and investigate its consequences. It is obvious that a Cohen-Macaulay ring of this sort is automatically Gorenstein, and that its Hilbert series therefore satisfies a functional equation, and our main result is a generalization of this to rings with depth one less than their dimension: this proves a conjecture of Benson and Greenlees [4]

1. Introduction

It has recently emerged that the rings of coefficients of equivariant cohomology theories very often have remarkable duality properties. It is the purpose of the present paper to formulate the duality purely algebraically in a particularly favourable case, and to investigate its ring theoretic implications. We give a little background in Appendix A, but readers finding the definition of interest as commutative algebra in its own right may ignore the topology.

Before we describe the duality properties, we need some terminology. Rings arising as coefficients of cohomology theories are $\mathbb{Z}$-graded, and in this paper all elements and ideals are required to be homogeneous. These rings are also graded-commutative in the sense that $rs = (-1)^{\deg(r)\deg(s)}sr$ for all elements r, s. Graded-commutative rings are very close to being commutative, and we want to apply the techniques of commutative algebra to them. Since left and right ideals coincide, the notions of a Noetherian ring and a prime ideal behaves as in the commutative case. The formula $mr := (-1)^{\deg(r)\deg(m)}rm$ allows one to consider left modules as right modules, and we henceforth restrict ourselves to left modules. If R is of characteristic 2 then R is itself commutative; in general the subring R^{ev} of even degree elements is commutative, and the inclusion induces a bijection of primes. Readers uncomfortable with graded-commutative rings should note that our constructions may be made using only the structure of a module over the commutative subring R^{ev}.

For the rest of the paper, R will be a Noetherian graded-commutative local ring of dimension r, with maximal ideal $\mathfrak{m}$ and residue field k. We also suppose R is connected in the sense that it is zero in negative degrees and $R^0 = k$.

To state the duality property we use Grothendieck's local cohomology functors $H^*_{\mathfrak{m}}(\cdot)$. Since R is Noetherian, these calculate the right derived functors of the $\mathfrak{m}$-power torsion functor on graded R-modules $M \longmapsto \Gamma_{\mathfrak{m}}M = \{x \in M \mid \mathfrak{m}^s x = 0 \text{ for } s >> 0\}$:

$$H^*_{\mathfrak{m}}(M) = R^*\Gamma_{\mathfrak{m}}M.$$

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Notice that since M is graded, the local cohomology module $H^i_{\mathfrak{m}}(M)$ is a graded module, and we write $H^{i,j}_{\mathfrak{m}}(M)$ for the degree j part. Readers uncomfortable with graded-commutative rings may interpret $\mathfrak{m}$ as the maximal ideal of R^{ev} throughout.

Local cohomology detects depth in the sense that $\mathrm{depth}(M) = \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$, so that R is Cohen-Macaulay if and only if $H^*_{\mathfrak{m}}(R)$ is concentrated in degree r; in this case R is then Gorenstein if and only if $H^r_{\mathfrak{m}}(R)$ is isomorphic to $DR = \mathrm{Hom}_k(R, k)$ up to a shift of degrees.

Remark 1.1. *(Grading conventions).* Since we are much concerned with modules non-zero in both positive and negative degrees, it is essential to be clear about grading conventions. All grading will be cohomological (upper indexing), and R is concentrated in degrees 0 and above. Other constructions are graded in the standard way. Thus $(DR)^n = \mathrm{Hom}^n_k(R, k) = \mathrm{Hom}_k(R^{-n}, k)$, and DR is concentrated in degrees 0 and below. In general $\mathrm{Hom}^n_k(M, N) = \prod_i \mathrm{Hom}_k(M^i, N^{i+n})$.

We also use topological notation to denote shifts in degrees. Thus $(\Sigma M)^n = M^{n-1}$, and we refer to ΣM as the (cohomological) suspension of M; the alternative notation $\Sigma M = M(1)$ is also widely used.

We are now equipped to define the class of rings we wish to study.

Definition 1.2. We say that R *has a local cohomology theorem with shift* v (or that R is an LCT^v ring) if there is a spectral sequence

$$E_2^{s,t} = H^{s,t}_{\mathfrak{m}}(R) \Longrightarrow \Sigma^v DR$$

with differentials

$$d_u : E_u^{s,t} \longrightarrow E_u^{s+u,t-u+1},$$

and so that $d_u : E_u^{s,*} \longrightarrow E_u^{s+u,*}$ is a map of R modules.

The interest in LCT^v rings arises from a number of examples supplied by the cohomology of groups.

Example 1.3. (a) If $R = H^*(G; k)$ for a finite group G, then R admits a local cohomology theorem with shift 0 [12].
(b) If $R = H^*(BG; k)$ for a compact Lie group of dimension w, and if the adjoint representation is orientable over k then it admits a local cohomology theorem with shift $-w$ [4].
(c) If $R = H^*(G; k)$ for a k-orientable virtual Poincaré duality group G of virtual dimension v, then it admits a local cohomology theorem with shift v [5].
(d) If $R = H^*(G; k)$ for a p-adic Lie group G of dimension v then it admits a local cohomology theorem with shift v [6].

The existence of an LCT^v structure is a form of duality, since the E_2 term is covariant in R, whilst the spectral sequence converges to DR, which is contravariant. For example, if R is an LCT^v ring and Cohen-Macaulay, then the spectral sequence collapses to give an isomorphism $H^r_{\mathfrak{m}}(R) = \Sigma^{v-r} DR$, showing that R is also Gorenstein. Conversely, it is immediate that any Gorenstein ring is an LCT^v ring for some v.

Our main result, Theorem 5.4, describes the consequences of an LCT^v structure on a ring which is almost Cohen-Macaulay in the sense that its depth is one less than its dimension. In particular we prove a conjecture of Benson and the first author for the rings of Example 1.3 (b) above, by giving a pair of functional equations 6.2 for the Hilbert series of any almost Cohen-Macaulay LCT^v ring. Benson and Carlson [3] had previously proved the analogous result for cohomology rings of finite groups as in Example 1.3 (a), by using the particular features of the definition of the cohomology ring.

We show in Section 7 that Grothendieck's method of dual localization shows that the localization of an LCT^v ring at a prime of dimension d is an LCT^{v-d} ring. Accordingly, our results for r-dimensional LCT^v rings of depth $r-1$ or r allow us to deduce 7.4 that any LCT^v ring R is generically Gorenstein, and also very well behaved in codimension 1.

Readers particularly interested in the applications to cohomology rings may not be familiar with standard methods of local cohomology, so we have therefore explained various well-known methods from commutative algebra at some length, particularly in Section 2.

Contents

2. Implications of local duality

In this section we record a number of consequences of local duality that are central to the analysis. These are all well known [18], but some readers may appreciate the simplicity of the proofs in our case.

Let R be an r-dimensional connected graded-commutative local ring with maximal ideal $\mathfrak{m}$ and residue field k. Throughout this section M will be a finitely generated R-module, although our main interest is in the case $M = R$.

By Noether normalization we may choose a polynomial subring $\tilde{R} \subseteq R$ over which R is finite; for definiteness we suppose the generators of $\tilde{R}$ are in even degrees $a_1, a_2, \ldots, a_r$, and we let $a = a_1 + a_2 + \cdots + a_r$.

Remark 2.1. In what follows the primary objects are R-modules. However, it is sometimes convenient to establish various properties by considering the underlying $\tilde{R}$-module, and we pause to clarify this.

(1) The Matlis duality functor is defined on an R-module M by $DM = \mathrm{Hom}_R(M, E(k))$. Note that $E(k) = \mathrm{Hom}_R(R, k)$, and hence we have $DM = \mathrm{Hom}_R(M, E(k)) = \mathrm{Hom}_k(M, k) = \mathrm{Hom}_{\tilde{R}}(M, \tilde{E}(k))$.

(2) Throughout we shall be discussing local cohomology relative to the maximal ideal of the ambient ring. Since $\sqrt{\tilde{\mathfrak{m}}R} = \mathfrak{m}$, where $\tilde{\mathfrak{m}}$ is the maximal ideal of $\tilde{R}$, for every R-module M we

have $H^i_{\tilde{\mathfrak{m}}}(M) = H^i_{\mathfrak{m}}(M)$; it should therefore cause no confusion if we always write $H^*_{\mathfrak{m}}$ for local cohomology functors for the maximal ideal.

(3) It is now clear that the depth of an R-module, the dimension of its support and the dimension of its associated primes can be established by considering it as an $\tilde{R}$-module.

(4) If M is a finitely generated R-module, it is a finitely generated $\tilde{R}$-module, so $H^i_{\mathfrak{m}}(M)$ is an Artinian $\tilde{R}$-module. □

The principal ingredient in our analysis is local duality [17]. We shall use the graded version (suitable forms are immediate from [13, 3.8] or [7, 3.6.19]), which states that for any finitely generated R-module M,

$$DH^i_{\mathfrak{m}}(M) = \Sigma^a \mathrm{Ext}^{r-i}_{\tilde{R}}(M, \tilde{R}).$$

Working with the dual of local cohomology allows us to measure the significance of local cohomology modules by their dimension. Since $DH^i_{\mathfrak{m}}(M)$ is finitely generated, its dimension is equal to that of its support.

Lemma 2.2.

$$\dim(DH^i_{\mathfrak{m}}(M)) \leq i$$

Proof: For a prime $\wp$ of dimension d let $\tilde{\wp}$ be the inverse image of $\wp$ in $\tilde{R}$, also of dimension d. Since $\tilde{R}_{\wp}$ is a regular local ring of dimension $r-d$ we see

$$\mathrm{Ext}^{r-i}_{\tilde{R}}(M, \tilde{R})_{\tilde{\wp}} = \mathrm{Ext}^{r-i}_{\tilde{R}_{\wp}}(M_{\wp}, \tilde{R}_{\wp}) = 0$$

for $r-i > r-d$. □

The top local cohomology module is particularly well behaved and plays a special role: its dual is the *canonical module* $\Omega = DH^r_{\mathfrak{m}}(R)$ of R.

Lemma 2.3. *All associated primes of Ω are of dimension r.*

Proof: There is an exact sequence $0 \longrightarrow \tilde{F} \longrightarrow R \longrightarrow Q \longrightarrow 0$ of $\tilde{R}$-modules where $\tilde{F}$ is a free $\tilde{R}$-module of finite rank and Q has non-zero annihilator. Accordingly $\Omega = DH^r_{\mathfrak{m}}(R) = \mathrm{Hom}_{\tilde{R}}(R, \tilde{R})$ is a submodule of $\mathrm{Hom}_{\tilde{R}}(\tilde{F}, \tilde{R}) = \tilde{F}$. □

The other useful fact [18] about Ω is that it satisfies Serre's Condition S_2 (its localization at $\wp$ has depth ≥ 2 if $\mathrm{ht}(\wp) \geq 2$ and depth 1 if $\mathrm{ht}(\wp) = 1$). We will have occasion to include a proof of this in 5.2 below.

An immediate corollary of local duality allows us to discuss Hilbert series.

Corollary 2.4. *If M is a finitely generated R-module then for each i the local cohomology module $H^i_{\mathfrak{m}}(M)$ is finite in each degree.*

Proof: It suffices to observe $DH^i_{\mathfrak{m}}(M) = \mathrm{Ext}^{r-i}_{\tilde{R}}(M, \tilde{R})$ is a finitely generated $\tilde{R}$-module. □

Although local cohomology modules are all supported at $\mathfrak{m}$, we can use duality to give a useful way of localizing local cohomology modules. Grothendieck's dual localization functor

$$L_{\wp} : R\text{-mod} \longrightarrow R_{\wp}\text{-mod}$$

for a prime $\wp$ is defined by

$$L_\wp M = D((DM)_\wp).$$

Here the innermost D is duality for R and the outer one is duality for $R_\wp$. Note that if $\wp$ is graded and the localization and duality are both interpreted in the graded sense, $L_\wp$ takes graded modules to graded modules, and Matlis duality works as usual [7, 3.6.16].

Lemma 2.5. *If the prime $\wp$ has dimension d and M is a finitely generated R-module then*

$$L_\wp H^i_{\mathfrak{m}}(M) = H^{i-d}_{\mathfrak{m}}(M_\wp)$$

Proof: Note that $\tilde{R}_\wp$ is a regular local ring of dimension $r-d$, and then apply local duality:

$$(DH^i_{\mathfrak{m}}(M))_\wp = \Sigma^a \mathrm{Ext}^{r-i}_{\tilde{R}}(M, \tilde{R})_{\tilde{\wp}} = \Sigma^a \mathrm{Ext}^{r-i}_{\tilde{R}_\wp}(M_\wp, \tilde{R}_\wp) = DH^{(r-d)-(i-d)}_{\mathfrak{m}}(M_\wp)$$

□

Reasonable behaviour in a module is reflected in the small size of its lower local cohomology.

Lemma 2.6. *If M is an R-module of dimension n with no associated primes of dimension $< n$ then* $\dim DH^d_{\mathfrak{m}}(M) \leq d-1$ *for* $d < n$.

Proof: Suppose $d < n$ and $\wp$ is a prime of dimension d. By hypothesis, $M_\wp$ is of depth ≥ 1. Therefore $DH^d_{\mathfrak{m}}(M)_\wp = DH^0_{\mathfrak{m}}(M_\wp) = 0$ and $\wp$ is not in the support of $DH^d_{\mathfrak{m}}(M)$. □

3. The LCT approximation map.

We now begin the investigation of LCT^v rings as defined in 1.2. Thus we suppose R is a graded connected local ring and there is a spectral sequence

$$E_2^{s,t} = H^{s,t}_{\mathfrak{m}}(R) \Longrightarrow \Sigma^v DR.$$

Note immediately that we have an edge homomorphism

$$\Sigma^r H^r_{\mathfrak{m}}(R) \longrightarrow \Sigma^v DR$$

with dual

$$\alpha : R \longrightarrow \Sigma^{v-r}\Omega.$$

We call α the *LCT-approximation*, and it is of central importance.

Lemma 3.1.

$$\ker(\alpha) = \{x \in R \mid \dim(Rx) \leq r-1\}$$

Proof: The spectral sequence shows that the kernel has a finite filtration by subquotients of $DH^0_{\mathfrak{m}}(R), DH^1_{\mathfrak{m}}(R), \ldots, DH^{r-1}_{\mathfrak{m}}(R)$. It therefore consists of elements generating a submodule of dimension $\leq r-1$ by 2.2. Conversely, since every associated prime of Ω is r-dimensional by 2.3, every element generating a submodule of dimension $\leq r-1$ lies in the kernel. □

It follows that α is injective if R is unmixed (i.e. if all associated primes of R have dimension r).

We may make a weaker statement about $\mathrm{cok}(\alpha)$.

Lemma 3.2. $\mathrm{cok}(\alpha)$ *is of dimension* $\leq r-2$.

Proof: The spectral sequence shows that the cokernel has a finite filtration by subquotients of $DH^0_{\mathfrak{m}}(R), DH^1_{\mathfrak{m}}(R), \ldots, DH^{r-2}_{\mathfrak{m}}(R)$. It therefore has dimension $\leq r-2$ by 2.2. □

There is a very convenient way to package a refinement of these observations, and we turn to it in Section 7. However in the meanwhile we give a criterion for α to be an isomorphism.

4. When is R quasi-Gorenstein?

We say that R is *quasi-Gorenstein* if the canonical module $\Omega = DH^r_{\mathfrak{m}}(R)$ is isomorphic to a suspension of R. Thus an r-dimensional ring is Gorenstein if and only if it is quasi-Gorenstein and of depth r.

Lemma 4.1. *Suppose R is equidimensional and unmixed. The following three conditions on a prime $\wp$ in R of dimension d are equivalent.*

1. $\mathrm{ht}(\wp) \geq 2$ *and* $\mathrm{depth}(R_\wp) = 1$.
2. *There is a regular $a \in R$ such that $\wp$ is an embedded prime of* (a).
3. *$\wp$ is a minimal prime of* $\mathrm{Supp}(DH^{d+1}_{\mathfrak{m}}(R))$

Proof: (1) $\iff$ (2): If an element $a \in \wp$ is a non-zero divisor it is regular on $R_\wp$. If $R_\wp$ is of depth 1, the regular sequence a cannot be extended, and hence $R/(a)$ is of depth 0. The argument is reversible. The conditions on height and embeddedness correspond.

(1) $\iff$ (3): The requirement $\mathrm{depth}(R_\wp) = 1$ is equivalent to $H^1_\wp(R) \neq 0$. But $H^1_\wp(R) = D(DH^{d+1}_{\mathfrak{m}}(R)_\wp)$ and hence $DH^{d+1}_{\mathfrak{m}}(R)_\wp \neq 0$, or equivalently, $\wp$ is in the support of $DH^{d+1}_{\mathfrak{m}}(R)$. It is minimal in this support since the fact that R is equidimensional and unmixed implies that $DH^{d+1}_{\mathfrak{m}}(R)$ is supported in dimension $(d+1)-1$ by 2.6. □

Remark 4.2. An R-module M is S_2 in the sense of Serre if $\mathrm{depth}(M_\wp) \geq \min\{2, \dim(R_\wp)\}$ for every prime $\wp$ of R. It follows that if R is unmixed, then R is S_2 if and only if no prime satisfies Condition (1) of Lemma 4.1.

Corollary 4.3. *If R is equidimensional and unmixed, then the set of primes $\wp$ satisfying the equivalent conditions of Lemma 4.1 is finite.*

Proof: For each i, the set of minimal primes of $\mathrm{Supp}(DH^i_{\mathfrak{m}}(R))$ is finite because $DH^i_{\mathfrak{m}}(R)$ is finitely generated. □

Theorem 4.4. *Suppose R is equidimensional and unmixed, and that it satisfies a local cohomology theorem with shift v. The LCT-approximation map $\alpha : R \longrightarrow \Sigma^{v-r}\Omega$ is an isomorphism if and only if R is S_2.*

Proof: We have remarked that R is S_2 if and only if it satisfies the equivalent conditions of the lemma.

It is known that Ω is S_2 [18] (or see 5.2 below), hence if α is an isomorphism, R is S_2.

For the converse we show that if α is not an isomorphism then R is not S_2. Since R is equidimensional and unmixed, the canonical map is injective, and the cokernel has dimension

$\leq r-2$ by 3.2. Now if α is not an isomorphism $\Omega/R \neq 0$. Let $a \in R$ be a regular element such that $a\Omega \subseteq R$. Thus

$$aR \subseteq a\Omega \subseteq R,$$

and hence $a\Omega/aR \subseteq R/aR$. But R/aR has equi-dimension $\dim(R)-1$, while $a\Omega/aR \cong \Omega/R$ has dimension $\leq r-2$ by 3.2. If $\wp$ is a minimal prime of $a\Omega/aR$, then $\wp$ is associated to R/aR, and $\wp$ is an embedded associated prime of (a), so that by 4.1 R is not S_2. □

5. Interaction with Grothendieck's spectral sequence.

We begin by recalling Grothendieck's spectral sequence from [18]. We shall see that its form is quite different to that of the local cohomology theorem. Firstly, it is contravariant in R, both at the E_2 and in the target. Secondly, each entry $E_2^{p,q}$ is itself an R module. Its construction is quite formal and applies to any ring.

Proposition 5.1. *There is a spectral sequence*

$$E_2^{p,q} = H^p_{\mathfrak{m}}(DH^{r-q}_{\mathfrak{m}}(R)) \Longrightarrow DR,$$

where the target is concentrated in total degree r. The differentials $d_u : E_u^{p,q} \longrightarrow E_u^{p+u,q-u+1}$ are maps of R-modules.

Proof: Given a finitely generated module M we have $\Gamma_{\mathfrak{m}}\mathrm{Hom}_{\tilde{R}}(M,N) = \mathrm{Hom}_{\tilde{R}}(M,\Gamma_{\mathfrak{m}}N)$. Furthermore, $\Gamma_{\mathfrak{m}}$ preserves injectives and $\mathrm{Hom}_{\tilde{R}}(M,\cdot)$ takes injectives to $\Gamma_{\mathfrak{m}}$-acyclic modules [18]. Hence we obtain two composite functor spectral sequences converging to the same cohomology

$$E_2^{p,q} = H^p_{\mathfrak{m}}(\mathrm{Ext}^q_{\tilde{R}}(M,N)) \Longrightarrow R^{p+q}(\Gamma_{\mathfrak{m}}\mathrm{Hom}_{\tilde{R}}(M,\cdot))(N) \Longleftarrow \mathrm{Ext}^p_{\tilde{R}}(M,H^q_{\mathfrak{m}}N) = E_2^{p,q}.$$

In particular if N is Cohen-Macaulay, the second spectral sequence collapses. We are interested in the special case $M=R$ and $N=\tilde{R}$.

Next observe that $H^r_{\mathfrak{m}}(\tilde{R}) = D\Sigma^a\tilde{R}$, where a is the sum of the degrees of the polynomial generators of $\tilde{R}$. This is injective, and there is an isomorphism $\mathrm{Hom}_{\tilde{R}}(R,D\tilde{R}) \cong DR$, so we find the spectral sequence takes the form

$$E_2^{p,q} = H^p_{\mathfrak{m}}(DH^{r-q}_{\mathfrak{m}}(R)) \Longrightarrow DR,$$

where the target is concentrated in total degree r. □

It is useful preparation to recall the following application.

Lemma 5.2. [18] *The canonical module Ω satisfies Serre's Condition S_2.*

Proof: Consider the above spectral sequence for the localized ring $R_\wp$ where $\wp$ is a prime of height h. The entries $E_2^{0,0}$ and $E_2^{1,0}$ necessarily survive to E_∞. If $h \geq 1$ this shows $E_2^{0,0}=0$, and if $h \geq 2$ we also have $E_2^{1,0}=0$. Thus Ω is S_2. □

Case 0: R is Cohen-Macaulay.

Lemma 5.3. *If R is Cohen-Macaulay, then Ω is of depth r and $H^r_{\mathfrak{m}}(\Omega)=DR$. If in addition R is an LCT^v-ring then $DR \cong \Sigma^{r-v}D\Omega$ and hence*

$$H^r_{\mathfrak{m}}(\Omega) \cong \Sigma^{r-v}D\Omega.$$

Proof: The first statement is immediate from the fact that Grothendieck's spectral sequence collapses. The next isomorphism is the LCT-approximation. □

Case 1: R is almost Cohen-Macaulay.

It is the analysis of this case which gives us our main theorem. One innovation of our approach is that we deduce module theoretic consequences, rather than simply conditions on Hilbert series. To state the condition we recall that a module M of dimension d is said to be Gorenstein if it is finitely generated R-module of depth d and $H^d_{\mathfrak{m}}(M)$ is a suspension of DM.

We also use the abbreviation $\Lambda = DH^{r-1}_{\mathfrak{m}}R$.

Theorem 5.4. *If R is almost Cohen-Macaulay and satisfies a local cohomology theorem with shift v then*
(i) Ω is of depth r and $H^r_{\mathfrak{m}}(\Omega) \cong \Sigma^{r-v}D\Omega$.
(ii) Λ is of depth $r-1$ and $H^{r-1}_{\mathfrak{m}}(\Lambda) \cong \Sigma^{r-1-v}D\Lambda$.
Thus Ω is Gorenstein of dimension r and Λ is Gorenstein of dimension $r-1$.

Proof: First consider Grothendieck's spectral sequence: it is concentrated on the two rows $q=0$ and $q=1$. Since it converges to DR concentrated in total degree r, we conclude that for $p \leq r-1$ the d_2 differential gives an isomorphism

$$H^{p-2}_{\mathfrak{m}}(\Lambda) \cong H^p_{\mathfrak{m}}(\Omega),$$

and there is an exact sequence

$$0 \longrightarrow H^{r-2}_{\mathfrak{m}}\Lambda \xrightarrow{d_2} H^r_{\mathfrak{m}}\Omega \longrightarrow \Sigma^{-a}DR \longrightarrow H^{r-1}_{\mathfrak{m}}\Lambda \longrightarrow 0.$$

On the other hand, if R has a local cohomology theorem we have the exact sequence

$$0 \longrightarrow \Sigma^{v-r+1}\Lambda \longrightarrow R \longrightarrow \Sigma^{v-r}\Omega \longrightarrow 0.$$

Because R has depth $r-1$, and Λ has dimension $\leq r-1$, applying local cohomology gives the isomorphisms

$$H^{p-1}_{\mathfrak{m}}\Omega \cong \Sigma H^p_{\mathfrak{m}}\Lambda$$

for $p \leq r-2$, the exact sequence

$$0 \longrightarrow H^{r-2}_{\mathfrak{m}}(\Sigma^{v-r}\Omega) \longrightarrow H^{r-1}_{\mathfrak{m}}(\Sigma^{v-r+1}\Lambda) \longrightarrow H^{r-1}_{\mathfrak{m}}(R) \longrightarrow H^{r-1}_{\mathfrak{m}}(\Sigma^{-r}\Omega) \longrightarrow 0,$$

and the isomorphism

$$H^r_{\mathfrak{m}}(R) \cong \Sigma^{v-r}H^r_{\mathfrak{m}}(\Omega).$$

Combining the first two displayed isomorphisms we obtain

$$H^p_{\mathfrak{m}}(\Omega) \cong \Sigma^{-1}H^{p-3}_{\mathfrak{m}}(\Omega) \text{ and } H^{p-1}_{\mathfrak{m}}(\Lambda) \cong \Sigma^{-1}H^{p-4}_{\mathfrak{m}}(\Lambda)$$

for $p \leq r-1$. Hence Ω is of depth r and Λ is of depth $r-1$ as required. The four term exact sequence now becomes the isomorphism of Part (ii). □

Examples of Aoyama [1] show that the depth d of Ω for an almost Cohen-Macaulay ring may take any value $2 \leq d \leq r$, so the theorem is a significant restriction.

6. Hilbert functions.

If M is a graded k-module of finite dimension in each degree, we may consider its Hilbert function (or Poincaré series). We view this as an element of the Grothendieck group of $\mathbb{Z}$ -graded k-modules, and accordingly write $[M]$ for it. We view this as a doubly infinite power series in t so that $[\Sigma^s M] = t^s[M]$. If M is a finitely generated R-module then $[M]$ may be viewed as the expansion of a rational function $[M]_{\mathrm{rf}}(t)$ about $t = 1$, but we retain separate notation for the Hilbert function and the associated rational function.

Once again we consider a polynomial subring $\tilde{R}$ of R. Notice that if $\tilde{R}$ has generators in degrees $a_1, a_2, \ldots a_r$, the Poincaré series $[\tilde{R}]_{\mathrm{rf}}(t) = 1/(1-t^{a_1})(1-t^{a_2})\cdots(1-t^{a_r})$. This gives the equality $[\tilde{R}]_{\mathrm{rf}}(1/t) = (-1)^r t^a [\tilde{R}]_{\mathrm{rf}}(t)$ of rational functions, where $a = a_1 + a_2 + \cdots + a_r$. Note also that $DH^r_{\mathfrak{m}}(\tilde{R}) = \Sigma^a \tilde{R}$.

Case 0: R is Cohen-Macaulay. Any ring with local cohomology theorem which is Cohen-Macaulay is automatically Gorenstein. Stanley has shown [24] that any graded Gorenstein ring satisfies a functional equation. In the presence of a local cohomology theorem, we may give a more direct proof, which also prepares the way for our result in the almost Cohen-Macaulay case.

Proposition 6.1. *If R is a Cohen-Macaulay ring of dimension r with a local cohomology theorem then its Hilbert rational function satisfies the functional equation*

$$[R]_{\mathrm{rf}}(1/t) = t^{-v}(-t)^r [R]_{\mathrm{rf}}(t).$$

Proof: In the Cohen-Macaulay case $R \cong \tilde{R} \otimes F_0$ for some finite graded k-module F_0. This gives the two equations

1. $[R] = [\tilde{R}][F_0]$
2. $[DH^r_{\mathfrak{m}}(R)] = [DH^r_{\mathfrak{m}}(\tilde{R})][F_0^\vee] = t^a[\tilde{R}][F_0^\vee]$,

where $F_0^\vee$ is the k-dual of F_0. If in addition R has a local cohomology theorem with suspension v then $\Sigma^{v-r} DH^r_{\mathfrak{m}}(R) = R$, and this allows us to combine the above statements to give

$$[\tilde{R}][F_0] = t^{v-r+a}[\tilde{R}][F_0^\vee].$$

In short, $[F_0]$ is self-dual up to suspension in the sense that

$$[F_0] = t^{v-r+a}[F_0^\vee].$$

In terms of rational functions this states

$$[R]_{\mathrm{rf}}(t) = [\tilde{R}]_{\mathrm{rf}}(t)[F_0]_{\mathrm{rf}}(t) = t^{v-r+a}[\tilde{R}]_{\mathrm{rf}}(t)[F_0^\vee]_{\mathrm{rf}}(t) = (-1)^r t^{v-r}[R]_{\mathrm{rf}}(1/t)$$

as required. □

Case 1: R is almost Cohen-Macaulay. Benson and Carlson [3] have shown that the cohomology ring of a finite group satisfies a pair of functional equations, and it has been conjectured by Benson and Greenlees [4] that this also holds for cohomology rings of classifying spaces of compact Lie groups (although a sign in [4] is incorrect for odd dimensional rings). We prove an arbitrary almost Cohen-Macaulay ring satisfying a local cohomology theorem satisfies such functional equations.

Theorem 6.2. *Suppose R is of depth $r-1$ and dimension r, and that it satisfies a local cohomology theorem with shift v. Then the Hilbert function of R satisfies the following pair of functional equations:*

$$[R]_{\mathrm{rf}}(1/t) - t^{-v}(-t)^r[R]_{\mathrm{rf}}(t) = (-1)^{r-1}(1+t)[DH^{r-1}_{\mathfrak{m}}(R)]_{\mathrm{rf}}(t)$$

and

$$[DH^{r-1}_{\mathfrak{m}}(R)]_{\mathrm{rf}}(1/t) = t^{-v}(-t)^{r-1}[DH^{r-1}_{\mathfrak{m}}(R)]_{\mathrm{rf}}(t),$$

Proof: In the almost Cohen-Macaulay case we have a short exact sequence

$$0 \longrightarrow \tilde{R}\otimes F_1 \longrightarrow \tilde{R}\otimes F_0 \longrightarrow R \longrightarrow 0$$

for suitable finite graded k-modules F_1 and F_0. Taking local cohomology and dualizing, this gives the exact sequence

$$0 \longleftarrow DH^{r-1}_{\mathfrak{m}}(R) \longleftarrow \Sigma^a\tilde{R}\otimes F_1^\vee \longleftarrow \Sigma^a\tilde{R}\otimes F_0^\vee \longleftarrow DH^r_{\mathfrak{m}}(R) \longleftarrow 0.$$

In terms of Grothendieck groups these give two equations

1. $[R] = [\tilde{R}]([F_0]-[F_1])$
2. $[DH^r_{\mathfrak{m}}(R)] = t^a[\tilde{R}]([F_0^\vee]-[F_1^\vee]) + [DH^{r-1}_{\mathfrak{m}}(R)]$

If R has a local cohomology theorem with suspension v, then we obtain a short exact sequence

$$0 \longrightarrow \Sigma^{v-r+1}DH^{r-1}_{\mathfrak{m}}(R) \longrightarrow R \longrightarrow \Sigma^{v-r}DH^r_{\mathfrak{m}}(R) \longrightarrow 0.$$

In terms of Hilbert functions this gives

$$[R] = t^{v-r}[DH^r_{\mathfrak{m}}(R)] + t^{v-r+1}[DH^{r-1}_{\mathfrak{m}}(R)].$$

This allows us to combine the Equations 1 and 2 to give

$$\begin{aligned}[\tilde{R}]([F_0]-[F_1]) &= t^{v-r}\left(t^a[\tilde{R}]([F_0^\vee]-[F_1^\vee]) + [DH^{r-1}_{\mathfrak{m}}(R)]\right) + t^{v-r+1}[DH^{r-1}_{\mathfrak{m}}(R)] \\ &= t^{v+a-r}[\tilde{R}]([F_0^\vee]-[F_1^\vee]) + t^{v-r}(1+t)[DH^{r-1}_{\mathfrak{m}}(R)]\end{aligned}$$

In terms of rational functions this gives the first of the conjectured equations

$$[R]_{\mathrm{rf}}(t) - (-1)^r t^{v-r}[R]_{\mathrm{rf}}(1/t) = t^{v-r}(1+t)[DH^{r-1}_{\mathfrak{m}}(R)]_{\mathrm{rf}}(t).$$

We have seen in 5.4 that $\Lambda = DH^{r-1}_{\mathfrak{m}}(R)$ enjoys the Gorenstein duality property.

The module Λ is of dimension $\leq r-1$ and we have $\Lambda \cong \Sigma^{r-1-v}DH^{r-1}_{\mathfrak{m}}(\Lambda)$. It is therefore of dimension $r-1$ if it is nonzero. As before Λ is a finitely generated module over the $(r-1)$-dimensional ring $R/ann(\Lambda)$, and hence by Noether normalization it is a finitely generated module over an $(r-1)$-dimensional polynomial ring with generators in degrees $b_1, b_2, \ldots, b_{r-1}$. Since it is Cohen-Macaulay, it is a free module, so we conclude as before that

$$[\Lambda]_{\mathrm{rf}}(t) = (-1)^{r-1}t^{v-(r-1)}[\Lambda]_{\mathrm{rf}}(1/t).$$

□

7. Localization of the spectral sequence.

So far we have concentrated on what can be said about the whole ring, at the expense of adding hypotheses. In this section, we note that for any ring with a local cohomology theorem, its localizations at primes of height 0 and 1 are very well behaved.

Lemma 7.1. *If R has a local cohomology theorem with suspension v and $\wp$ is a prime of dimension d then $R_\wp$ admits a local cohomology theorem with shift $v-d$.*

Proof: Apply the exact functor $L_\wp$ to the spectral sequence. We obtain a new spectral sequence with

$$L_\wp E_2^{s,*} = L_\wp H^s_{\mathfrak{m}}(R) = H^{s-d}_{\mathfrak{m}}(R_\wp) \Longrightarrow \Sigma^v L_\wp DR.$$

Now we calculate $L_\wp DR = D((DDR)_\wp) = D(R_\wp)$. Finally note that it remains only to regrade the spectral sequence with $\overline{E}_u^{s-d,*} = L_\wp E_u^{s,*}$. □

One advantage is that if if we choose a prime of dimension r or $r-1$ we obtain a collapsed spectral sequence. We first examine the 0 and 1 dimensional cases to which we are reduced by localization, and then return to deduce conclusions for arbitrary rings.

Proposition 7.2. *If R is zero dimensional with a local cohomology theorem of shift v then*

$$R = H^0_{\mathfrak{m}}(R) \cong \Sigma^v DR,$$

and R is a Gorenstein ring with dualizing shift v. □

Proposition 7.3. *If R is one dimensional with a local cohomology theorem of shift v then $H^0_{\mathfrak{m}}(R)$ is Gorenstein with dualizing shift v:*

$$DH^0_{\mathfrak{m}}(R) = \Sigma^{-v} H^0_{\mathfrak{m}}(R)$$

The dualizing module Ω is also Gorenstein: it has the property that $H^0_{\mathfrak{m}}(\Omega) = 0$ as usual and

$$H^1_{\mathfrak{m}}(\Omega) \cong \Sigma^{1-v} D\Omega.$$

Proof: If R is a one dimensional ring the spectral sequence amounts to a short exact sequence

$$0 \longrightarrow \Sigma H^1_{\mathfrak{m}}(R) \longrightarrow \Sigma^v DR \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow 0$$

or dualizing, to a sequence

$$0 \longleftarrow \Sigma^{-1} DH^1_{\mathfrak{m}}(R) \longleftarrow \Sigma^{-v} R \longleftarrow DH^0_{\mathfrak{m}}(R) \longleftarrow 0.$$

and $DH^0_{\mathfrak{m}}(R)$ has only $\mathfrak{m}$ associated to it. Now consider the six term exact sequence of local cohomology modules: it splits into two isomorphisms. Indeed, by 2.3, $\Omega = DH^1_{\mathfrak{m}}(R)$ only has one dimensional associated primes, so that $H^0_{\mathfrak{m}} DH^1_{\mathfrak{m}} R = 0$ and hence

$$\Sigma^{-v} H^0_{\mathfrak{m}}(R) \cong H^0_{\mathfrak{m}}(DH^0_{\mathfrak{m}}(R)) = DH^0_{\mathfrak{m}}(R),$$

where the equality arises since $DH^0_{\mathfrak{m}}(R)$ has only $\mathfrak{m}$ associated to it. This states that $H^0_{\mathfrak{m}}(R)$ is self-dual with dualizing dimension v. Since $DH^0_{\mathfrak{m}}(R)$ is 0-dimensional, $H^1_{\mathfrak{m}}(DH^0_{\mathfrak{m}}(R)) = 0$ and so

$$\Sigma^{-1} H^1_{\mathfrak{m}}(\Omega) = \Sigma^{-1} H^1_{\mathfrak{m}}(DH^1_{\mathfrak{m}}(R)) \cong \Sigma^{-v} H^1_{\mathfrak{m}}(R) = \Sigma^{-v} D\Omega.$$

□

This leads to the following general conclusion about the good behaviour of a general ring R with local cohomology theorem in codimension 0 and 1.

Corollary 7.4. *If R is an r dimensional ring with local cohomology theorem of shift v then*
(1) R is generically Gorenstein (ie its localization at any minimal prime is Gorenstein).
(2) R is almost Gorenstein in codimension 1 in the sense that R has the following behaviour localized at a prime $\wp$ of height 1. If $\wp$ is of dimension d, let $\Omega' = DH^{d+1}_{\mathfrak{m}}(R)$ and $\Lambda' = DH^{d-1}_{\mathfrak{m}}(R)$ (so that $\Omega' = \Omega$ and $\Lambda' = \Lambda$ if $d = r-1$). There is a a short exact sequence

$$0 \longrightarrow \Sigma^{v-d}\Lambda'_{\wp} \longrightarrow R_{\wp} \longrightarrow \Sigma^{v-d+1}\Omega'_{\wp} \longrightarrow 0$$

where both $\Omega'_{\wp} = DH^{d+1}_{\mathfrak{m}}(R)_{\wp}$ and both $\Lambda'_{\wp} = DH^{d-1}_{\mathfrak{m}}(R)_{\wp}$ are Gorenstein. More precisely

$$\Sigma^{v-d}\Lambda'_{\wp} \cong D\Lambda'_{\wp},$$

and Ω' has the property that $DH^d_{\mathfrak{m}}(\Omega')_{\wp} = 0$ and

$$\Sigma^{v-d-1}\Omega'_{\wp} = DH^{d+1}_{\mathfrak{m}}(\Omega')_{\wp}.$$

Proof: Part (1) is clear, since the localized spectral sequence gives

$$L_{\wp}H^d_{\mathfrak{m}}(R) \cong \Sigma^{v-d}L_{\wp}DR.$$

For Part (2) we give more details. The one dimensional case gives the short exact sequence

$$0 \longrightarrow L_{\wp}\Sigma H^{d+1}_{\mathfrak{m}}(R) \longrightarrow \Sigma^{v-d}DR_{\wp} \longrightarrow L_{\wp}H^d_{\mathfrak{m}}(R) \longrightarrow 0$$

or equivalently, a short exact sequence

$$0 \longleftarrow \Sigma^{-1}\Omega'_{\wp} \longleftarrow \Sigma^{d-v}R_{\wp} \longleftarrow \Lambda'_{\wp} \longleftarrow 0.$$

Considering the six term local cohomology exact sequence we obtain two isomorphisms: since $DH^{d+1}_{\mathfrak{m}}(R)_{\wp} \cong DH^1_{\mathfrak{m}}(R_{\wp})$ only has associated primes of dimension 1 it has zero $H^0_{\mathfrak{m}}$, and since $DH^d_{\mathfrak{m}}(R)_{\wp} = DH^0_{\mathfrak{m}}(R_{\wp})$ is zero dimensional it has zero $H^1_{\mathfrak{m}}$. Thus we find

$$\Sigma^{d-v}L_{\wp}H^d_{\mathfrak{m}}(R) = \Sigma^{d-v}H^0_{\mathfrak{m}}(R_{\wp}) \cong H^0_{\mathfrak{m}}(DH^d_{\mathfrak{m}}(R)_{\wp}) = DH^d_{\mathfrak{m}}(R)_{\wp},$$

or dualizing

$$\Sigma^{v-d}\Lambda'_{\wp} = \Sigma^{v-d}DH^d_{\mathfrak{m}}(R)_{\wp} \cong D(DH^d_{\mathfrak{m}}(R)_{\wp}) = D\Lambda'_{\wp}.$$

Similarly,

$$\Sigma^{d-v}L_{\wp}H^{d+1}_{\mathfrak{m}}(R) = \Sigma^{d-v}H^1_{\mathfrak{m}}(R_{\wp}) \cong \Sigma^{-1}H^1_{\mathfrak{m}}(DH^{d+1}_{\mathfrak{m}}(R)_{\wp}) = \Sigma^{-1}L_{\wp}H^{d+1}_{\mathfrak{m}}(DH^{d+1}_{\mathfrak{m}}(R)),$$

or dualizing

$$\Sigma^{v-d}\Omega'_{\wp} = \Sigma^{v-d}DH^{d+1}_{\mathfrak{m}}(R)_{\wp} \cong \Sigma DH^{d+1}_{\mathfrak{m}}(DH^{d+1}_{\mathfrak{m}}(R))_{\wp} = \Sigma DH^{d+1}_{\mathfrak{m}}(\Omega')_{\wp}.$$

□

8. Minimal associated primes of dual local cohomology.

In this section we formalize the geometric content of our results, and in the special case of mod p cohomology of groups, we use a theorem of Quillen to relate this in turn to the group theory.

We have been concerned with the defect of local rings, defined by $\mathrm{def}(R) = \dim(R) - \mathrm{depth}(R)$, and we now consider the defect stratification of $X = \mathrm{Spec}(R)$. Thus, we let

$$X_i = \{\wp \in \mathrm{Spec}(R) \mid \mathrm{def}(R_\wp) \geq i\}$$

Evidently this gives a chain of inclusions

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_\delta \supseteq X_{\delta+1} = \emptyset.$$

We write $X_i' = X_i \setminus X_{i+1}$ for the ith pure stratum. Thus $X_0' = X_0 \setminus X_1$ is the Cohen-Macaulay locus (which we have shown is also the Gorenstein locus) and $X_1' = X_1 \setminus X_2$ is the almost Cohen-Macaulay locus (which we have shown is the almost Gorenstein locus).

Now suppose $\wp$ is a prime of height h and dimension d, so that $r \geq h+d$. Because $DH^i_{\mathfrak{m}}(R)_\wp = DH^{i-d}_{\mathfrak{m}}(R_\wp)$ we see that if $\wp \in \mathrm{Supp}(DH^i_{\mathfrak{m}}(R))$ then $\wp \in X_{h+d-i}$, and conversely, if $\wp \in X'_{h+d-i}$ then $\wp \in \mathrm{Supp}(DH^i_{\mathfrak{m}}(R))$. This shows the defect only ever decreases under localization and hence the chain terminates with $\delta = \mathrm{def}(R)$.

For the remainder of the discussion we assume that R is equidimensional, so that $h + d = r$ for all primes. Thus

$$X_i = \mathrm{Supp}(DH^{r-i}_{\mathfrak{m}}(R)) \cup \mathrm{Supp}(DH^{r-i-1}_{\mathfrak{m}}(R)) \cup \ldots \cup \mathrm{Supp}(DH^{r-\delta}_{\mathfrak{m}}(R)).$$

Corollary 8.1. *If R is equidimensional and $\mathfrak{a}_j = ann(DH^j_{\mathfrak{m}}(R)) = ann(H^j_{\mathfrak{m}}(R))$ then*

$$X_i = V(\mathfrak{a}_{r-i}) \cup V(\mathfrak{a}_{r-i-1}) \cup \ldots \cup V(\mathfrak{a}_{r-\delta}).$$

In particular X_i is a closed set. □

From 2.2 the minimal primes over $\mathfrak{a}_j$ have dimension $\leq j$, and hence X_i is of dimension $\leq r-i$.

We now focus on the case when R is the mod p cohomology of a classifying space of a compact Lie group or virtual duality group. Quillen [21, 22] has given a description of the variety of R, and in particular its minimal primes. We briefly recall Quillen's stratification of the variety. For an elementary abelian group A and an algebraically closed field K of characteristic p, let $X_A(K) = A \otimes K$. This defines a variety X_A, and we take $X_A^+ = X_A \setminus \bigcup_{B \subset A} X_B$. Restriction defines a map

$$\coprod_A X_A \longrightarrow X_G$$

where the coproduct is over elementary abelian subgroups of G and X_G is the variety of $H^*(BG)$. Quillen shows that

$$X_G = \coprod_{(A)} X^+_{G,A},$$

where $X^+_{G,A}$ is the image of X_A, and the coproduct is now over conjugacy classes of elementary abelian subgroups. Furthermore $X^+_{G,A} = X^+_A/\overline{W_G(A)}$ where $\overline{W}_G(A) = N_G(A)/C_G(A)$.

The following lemma is useful in understanding Quillen's filtration.

Lemma 8.2. *For any prime $\wp$, let $m(\wp)$ denote the set of minimal primes contained in $\wp$. The set $m(\wp)$ depends only on the Quillen pure stratum $X^+_{G,A}$ containing $\wp$.*

Proof: Closed Quillen strata $X_{G,A}$ and $X_{G,B}$ intersect in unions of strata $X_{G,C}$ for subgroups C of conjugates of A and B. □

The significant feature for us is that the mod p Steenrod algebra $\mathcal{A}_p$ acts on R. Quillen has shown [22, 12.1] that any prime invariant under the Steenrod operations P^i for $i \geq 0$ is equal to the prime $\wp(E)$ obtained by pullback from the minimal prime of $H^*(BE)$ for an elementary abelian subgroup under the restriction map $R = H^*(BG) \longrightarrow H^*(BE)$. This shows that any P^*-invariant prime determines a conjugacy class of elementary abelian subgroups, and therefore has a group theoretic counterpart.

Proposition 8.3. *The minimal primes over $\mathfrak{a}_i$ are invariant under the Steenrod algebra, and hence define a union of Quillen closed strata, namely the one defined by the set*

$$\mathcal{E}_i = \{E \mid \wp(E) \text{ is minimal over } \mathfrak{a}_i\}$$

of elementary abelian subgroups of G closed under conjugacy.

Proof: First note that $H^i_{\mathfrak{m}}(R)$ is an $\mathcal{A}_p$-module: to see this, observe that $\mathcal{A}_p$ acts via differential operators, and apply the result of [19] that $H^i_{\mathfrak{m}}(R)$ is a D-module. It follows that its annihilator $\mathfrak{a}_i$ is $\mathcal{A}_p$-invariant, and the first clause is immediate from [23, 11.2.3]. The rest follows from the description of the Quillen stratification. □

This shows that the defect stratification is subordinate to the Quillen stratification, and in particular the minimal primes of X_i are Quillen strata. This generalizes Duflot's theorem [8] that the associated primes of R are Quillen strata. Indeed, an associated prime of R of height h is a minimal prime of $H^{d-h}_{\mathfrak{m}}(R)$, by local duality.

Appendix A. Topological background

Although equivariant topology plays no role at all in this paper except to provide examples which show the theory is not vacuous, it still seems worth providing a little background. For simplicity, suppose G is a finite group. There are certainly interesting adaptions to the cases of compact Lie groups, discrete virtual duality groups, and also to p-adic analytic groups, and we comment below on how they affect the discussion. There are various interesting examples of cohomology theories $F^*(\cdot)$. Three to bear in mind are (1) ordinary cohomology with coefficients in a field k, (2) K-theory and (3) bordism. These may be applied to the classifying space BG, and we consider the ring $R = F^*(BG)$, and the R-module $F_*(BG)$. It is natural to expect a universal coefficient theorem

$$\mathrm{Ext}^{s,t}_{F^*}(F_*(BG), F_*) \Longrightarrow F^*(BG).$$

Note in particular that both the E_2 term and the target are contravariant in BG. In case (1) the universal coefficient theorem collapses to state $D(F_*(BG)) = F^*(BG)$, and since the cohomology is finite in each degree, this gives $F_*(BG) = DR$. It is natural to expect more generally that the universal coefficient theorem arises from an equivalence $\mathbf{R}\mathrm{D}(F_*(BG)) \simeq R$, in some derived category, where $D(\cdot) = \mathrm{Hom}_{F_*}(\cdot, F_*)$: this will be given a precise meaning below.

So far these are formal properties which are to be expected in any theory. However in many examples (such as the three listed above [12, 11, 16]) a much more interesting relationship arises out of good behaviour of an associated equivariant theory: there is a spectral sequence

$$H_I^{s,t}(F^*(BG)) \Longrightarrow F_*(BG),$$

where $I = \ker(F^*(BG) \longrightarrow F^*)$. Note that this too connects $F^*(BG)$ with $F_*(BG)$ but its E_2 term is contravariant in G whilst it converges to $F_*(BG)$, which is covariant. A derived category form of this would be an equivalence

$$\mathbf{R}\Gamma_I(R) \simeq F_*(BG).$$

Combining the local cohomology theorem with the universal coefficient theorem we obtain a form of duality for the ring $R = F^*(BG)$:

$$\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I(R) \simeq R$$

in some derived category of R-modules.

In the discussion so far we were not obliged to discuss equivariant cohomology theories at all. However, the natural statements are in terms of the equivariant cohomology $F_G^*(EG \times X)$ and homology $F_G^*(EG \times X)$. For cohomology the change of groups isomorphism $F_G^*(EG \times X) \cong F^*(EG \times_G X)$ is straightforward, and generalizes directly to the various classes of infinite group. However, in homology the isomorphism $F_*^G(EG \times X) \cong F_*(EG \times_G X)$ involves a transfer and therefore a suspension or twisting. For example if G is a compact Lie group

$$F_*^G(EG \times X, EG \times *) \cong F_*(EG \times_G ((X, *) \times (D(ad(G)), S(ad(G)))$$

where $(D(ad(G)), S(ad(G)))$ is the pair consisting of the unit disc and unit ball of the adjoint representation. The effect of this is to give a statement like

$$\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I(R) \simeq \Sigma^{-v} R$$

where Σ^{-v} is some invertible functor on the derived category. It would be interesting to investigate the implications of such an algebraic statement along the lines of the present paper.

We finish by giving a context in which the heuristic derived category statements are true. For this we need to work with highly structured ring and module spectra, and more precisely, in the category of S-algebras of Elmendorf-Kriz-Mandell-May [9]. We thus suppose that the representing spectrum F is an S-algebra, and use the highly structured inflation of Elmendorf-May [10] to view it as an equivariant S-algebra. To emphasize the algebraic content, we write $\mathbf{R}$Hom for function spectra. The articles [14, 15] provide an introduction to some of the constructions used here.

Proposition A.1. *Suppose F is a complex orientable S-algebra, and G is a compact Lie group. If $F^*(BG)$ is Noetherian and $I = \ker(F^*(BG) \longrightarrow F^*)$ then there is an equivalence*

$$\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I \circ \mathbf{R}\mathrm{Hom}(EG_+, F) \simeq \mathbf{R}\mathrm{Hom}(EG_+, F)$$

of equivariant $\mathbf{R}\mathrm{Hom}(EG_+, F)$*-modules, where* $\mathbf{R}\mathrm{D}(M) = \mathbf{R}\mathrm{Hom}_F(M, F)$.

Remark A.2. The equivariant homotopy of the right hand side is $R = F^*(BG_+)$. There are various spectral sequences for calculating the equivariant homotopy of the left hand side, their initial terms are related to $\mathrm{Ext}_{F_*}^{*,*}(H_{\mathfrak{m}}^{*,*}(R), F_*)$.

Proof: By [9, IV.4] the universal coefficient theorem is realized by an equivalence

$$\mathbf{R}\mathrm{Hom}_F(F \wedge BG_+, F) \simeq \mathbf{R}\mathrm{Hom}(BG_+, F)$$

of non-equivariant F-modules. We need to work with G-spectra so we note that the universal coefficient equivalence is obtained by passage to Lewis-May fixed points from the equivalence

$$\mathbf{R}\mathrm{Hom}_F(F \wedge EG_+, F) \simeq \mathbf{R}\mathrm{Hom}(EG_+, F)$$

of equivariant $\mathbf{R}\mathrm{Hom}(EG_+, F)$-modules.

Now the local cohomology theorem (see [15] for a proof in this context) is an equivalence

$$F \wedge EG_+ \simeq \mathbf{R}\Gamma_I \circ \mathbf{R}\mathrm{Hom}(EG_+, F),$$

and the result follows by combining the two. □

It may be worth comparing this statement with a more elementary one, which can be viewed as a generalization of Anderson duality. If R is any S-algebra, with R^* Noetherian and I is an ideal of R^* we may form the R-module $\mathbf{R}\Gamma_I R$. There is a spectral sequence

$$H_I^{*,*}(R^*) \Rightarrow \pi_*(\mathbf{R}\Gamma_I R)$$

for calculating its homotopy. Given an R-module E we may consider the duality functor $\mathbf{R}\mathrm{D}(M) = \mathbf{R}\mathrm{Hom}_R(M, E)$, and the composite $\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I(R)$. A particularly familiar case occurs when (R^*, I) is local and E is the injective envelope of the residue field. If R^* is Gorenstein, $\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I(R)$ has homotopy $\Sigma^{-v}R^*$ for some v, and hence $\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I(R) \simeq \Sigma^v R$. More generally we remark that if M^* is an R^*-module of projective dimension ≤ 1 there is an R module M with $\pi_* M = M^*$, and M is unique up to equivalence. Thus if R^* is Cohen-Macaulay and $DH_I^r(R^*)$ is of projective dimension 1, we can again identify the R-module $\mathbf{R}\mathrm{D} \circ \mathbf{R}\Gamma_I R$. Usually this will not be a suspension of R, but rather a wedge of several. Thus A.1 states that $\mathbf{R}\mathrm{Hom}(EG_+, F)$ behaves very much like a Gorenstein ring.

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