EQUIVARIANT FORMS OF CONNECTIVE K THEORY

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Abstract. We construct a C_p-equivariant form of connective K-theory with several good properties.

1. Introduction

In the non-equivariant world, connective K-theory is an extremely useful cohomology theory. It is not much harder to calculate than periodic K-theory, but can be a lot more powerful. However, there does not appear to be any abstract explanation for the importance of the theory. To qualify, such an explanation would tell me how to construct (or at least recognize) an equivariant version of the theory.

I still do not understand the abstract nature of the theory, but I do have a list of desirable properties for an equivariant form of connective K-theory. This paper arises since none of the constructions I previously knew had all of these properties. Initially, I suspected the properties were inconsistent, but in fact they almost determine the construction. The resulting cohomology theory needs further investigation: it may be expected to improve certain existing results proved with periodic equivariant K-theory and connective K-theory of the Borel construction. At present the construction is restricted to groups of prime order. Any treatment for arbitrary finite groups will have to use a more geometric construction.

Let G be a group of order p.

Properties 1.1. Consider the following properties of a G-spectrum E.

1. E is a split ring spectrum, and nonequivariantly ku.
2. $E[v^{-1}] \simeq K$ (equivariantly) where v is the degree 2 Bott element arising from the split structure.
3. E is complex orientable (equivariantly).
4. E^G_* is concentrated in even degrees.
5. E^*_G is a Noetherian ring.

Property 1 simply states that E is an equivariant form of connective K-theory, and is therefore not negotiable. The other properties are in decreasing order of importance. Note that complex orientability is enough to ensure that a completion theorem holds; provided E^*_G is Noetherian this states in particular that

$$E^*(BG_+) = (E^*_G)^\wedge_I$$

where $I = ker(E^*_G \longrightarrow E^* = ku^*)$ is the augmentation ideal.

The purpose of this note is to prove the following theorem.

Theorem 1.2. *There is a G-spectrum ku with all five of the properties listed in 1.1. It has coefficient ring*

$$ku^*_G = R(G)[v, y]/(vy = \chi(\alpha), y\rho)$$

where α is the natural representation of G, $\chi(\alpha) = 1 - \alpha$ is its K-theory Euler class and $\rho = 1 + \alpha + \cdots + \alpha^{p-1}$ is the regular representation. The Bott element v is in degree 2, and the element y is in degree -2.

Remark 1.3. Note that the coefficient ring is $R(G)$ in each positive even degree, and J^n in degree $-2n$. This is the Rees ring of $R(G)$ with respect to the augmentation ideal J [10], and the associated projective scheme is that of the blowup of $\mathrm{Spec}(R(G))$ at the subscheme defined by J. It is tempting to hope that for any finite group Γ there is a Γ-equivariant form of ku whose coefficient ring is the Rees ring of $R(\Gamma)$ with respect to J. However such an equivariant form cannot have all of the properties listed in 1.1, since these imply the completion theorem holds. This is inconsistent with known values of $ku^*(B\Gamma_+)$, since

$$Rees(R(\Gamma), J)^\wedge_J = Rees(R(\Gamma)^\wedge_J, J^\wedge_J)$$

is in even degrees, whilst $ku^*(B\Gamma_+)$ is often nonzero in odd degrees (for example if Γ is elementary abelian of rank ≥ 3) [12, 8]. □

The rest of the paper is layed out as follows. We begin by discussing some known equivariant forms of ku, and showing they do not have the required properties. Next, we summarize facts about periodic K-theory. This ensures we have a good example in mind, allows us to introduce notation, and is an input to the verification of Property 2. We then spend two sections showing how much of the construction is forced by complex stability and the comparison with periodic K-theory; this is logically unnecessary, but I believe it makes the construction more compelling. Finally we turn to the construction and properties of a good equivariant form of ku.

I am grateful to R.R.Bruner for his instruction on connective K-theory and his explanation of various calculations, and to J.P.May, N.P.Strickland, P.R.Turner and J.Wolbert for useful conversations.

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2. Examples

We summarize three known ways to construct equivariant versions of ku. We show that none of them have all five properties listed in 1.1, thereby establishing the need for the subsequent sections in which an equivariant form with these properties is constructed.

Example 2.1. *The connective cover $K\langle 0\rangle$ of equivariant K-theory.*

This spectrum is constructed in the usual way as the fibre of the map $K \longrightarrow K(-\infty, -1]$ killing π_n and π_n^G for all $n \geq 0$. The resulting spectrum $K\langle 0\rangle$ certainly has Properties 1, 2 and 4. We shall see in the Section 4 that no connective theory can be complex stable. Also, since $K\langle 0\rangle$ is bounded below, the geometric fixed point spectrum $\Phi^G K\langle 0\rangle$ is contractible. □

Example 2.2. *May's spectrum MU-$\mathit{inf}_1^G(ku)$ [11], obtained as an inflation of MU-modules.*

Formal manipulations show that since ku is an MU-algebra up to homotopy, the MU-induced spectrum is a split MU-algebra up to homotopy and hence a split ring spectrum up to homotopy. In particular it is a module over equviariant MU and hence complex orientable. It thus has Properties 1 and 3. We shall show that it also has Property 4 but not Property 2. Its geometric fixed points are enormous:

$$\Phi^G(MU\text{-}\mathrm{inf}_1^G(ku)) = (\Phi^G MU) \wedge_{MU} ku = ku \wedge (BU_+[z, z^{-1}])^{\wedge(p-1)},$$

and hence this is different from the previous example. □

Example 2.3. *The inflation $\mathrm{inf}_1^G(ku)$ obtained by change of universe from ku.*

This spectrum is commonly written $i_* ku$ if i denotes the inclusion of the fixed points in a chosen complete G-universe.

It is no surprise that such a crude construction is very unsuccessful. The result is connective and therefore not complex stable (see Section 4), so does not have Property 3. It does not have Property 2 either, since $\mathrm{inf}_1^G(ku)[1/v] = \mathrm{inf}_1^G(K)$; the fact that $\mathrm{inf}_1^G(K) \neq K$ is well known, and one argument can be obtained by adapting the following calculation of $\pi_*^G(\mathrm{inf}_1^G(ku))$.

Consider homotopy groups and see that Property 4 also fails. To calculate $\pi_*^G(\mathrm{inf}_1^G(ku))$ we use the long exact sequence arising from

$$EG_+ \wedge \mathrm{inf}_1^G(ku) \longrightarrow \mathrm{inf}_1^G(ku) \longrightarrow \tilde{E}\mathcal{P} \wedge \mathrm{inf}_1^G(ku).$$

The first term has homotopy groups $ku_*(BG_+)$, and the last term has homotopy groups $\mathbb{Z}[v]$. By considering the map in degree zero we see that last map is surjective, so that $\pi_*^G(\mathrm{inf}_1^G(ku)) = \mathbb{Z}[v] \oplus ku_*(BG_+)$. It is well known (and discussed in Section 5 below) that $ku_*(BG_+)$ is non-zero and torsion in odd degrees.

It may be worth comparing this to the first example. Since $ku = S^0 \cup e^2 \cup \ldots$, the inflation $\mathrm{inf}_1^G(ku)$ has a similar cell decomposition, and its zeroth homotopy is $A(G)$. Indeed, from the fact that periodic K-theory is split, we have a map $\mathrm{inf}_1^G(ku) \longrightarrow K\langle 0\rangle$. In degree 0 it is the permutation representation homomorphism $A(G) \longrightarrow R(G)$. When $p = 2$ this is an isomorphism, and since it is a map of $A(G)[v]$ modules, we find that the homotopy groups of the fibre are the odd degree (torsion) part of $ku_*(BG_+)$. □

3. Periodic K-theory and associated notation.

Our basic tool is the Tate pullback square [7]. In the case of periodic K-theory it reads

$$\begin{array}{ccc} K & \longrightarrow & K \wedge \tilde{E}\mathcal{P} \\ \downarrow & \ulcorner & \downarrow \\ F(EG_+, K) & \longrightarrow & t(K) \end{array}$$

This method is useful because we have simpler descriptions of the two spectra on the right, and $F(EG_+, K)$ only depends on non-equivariant K-theory.

We need some more notation. Recall that $K^G_0 = R(G)$ and $K^G_* = R(G)[v, v^{-1}]$, where v is the Bott element of degree 2. Associated to the representation ring there are a number of other algebraic objects. Most familiar perhaps is the augmentation ideal $J = \ker(R(G) \longrightarrow \mathbb{Z})$. The regular representation ρ generates the ideal of J-power torsion elements, which is additively isomorphic to $\mathbb{Z}$; the quotient ring $\overline{R} = R(G)/(\rho)$ will be important to us. A source of some confusion is that J is isomorphic to $\overline{R}$ as an $R(G)$-module, however neither the exact sequence $0 \longrightarrow (\rho) \longrightarrow R(G) \longrightarrow \overline{R} \longrightarrow 0$ nor the exact sequence $0 \longrightarrow J \longrightarrow R(G) \longrightarrow \mathbb{Z} \longrightarrow 0$ are split over $R(G)$. The effect of J-completion is trivial on $\mathbb{Z}$ and (ρ), and p-completion on J and $\overline{R}$; the sequences still do not split after completion.

For the group of order p we let α denote a faithful one dimensional representation, so that $R(G) = \mathbb{Z}[\alpha]/(\alpha^p - 1)$, $\rho = 1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1}$, and $\overline{R}$ may be identified with the ring generated by a primitive p-th root of unity in the complex numbers. The Euler class $\chi = \chi(\alpha) = 1 - \alpha$ also plays a role. Note that, for any i, $\chi(\alpha^i)$ is a multiple of $\chi(\alpha)$, and vice versa if i is prime to p. Thus (χ) is the ideal generated by any non-zero Euler class.

Lemma 3.1. *(i) The geometric fixed points of periodic K-theory are given by*

$$\Phi^G K \simeq K\overline{R}[1/p]$$

(ii) The Tate spectrum of periodic K-theory is given by

$$t(K) \simeq \tilde{E}\mathcal{P} \wedge \prod_{2n} \Sigma^{2n} H\overline{R}^{\wedge}_p[1/p].$$

Proof: (i) To calculate the coefficients of $\Phi^G X$ for any complex stable X we have

$$\pi_*(\Phi^G X) = \pi^G_*(\tilde{E}\mathcal{P} \wedge X) = \pi^G_*(S^{\infty\alpha} \wedge X) = \pi^G_*(X)[1/\chi(\alpha)].$$

Since $\chi(\alpha) = 1-\alpha$, a short calculation shows that the coefficient ring of $\Phi^G K$ is $\overline{R}[1/p]$ in each even degree. Furthermore $\Phi^G K$ is a non-equivariant K-algebra up to homotopy; since the coefficients are free over $\mathbb{Z}[1/p]$ the result follows.
(ii) This is a special case of [6, A.5] or [7, 13.1]. □

For periodic K-theory, both v and χ act isomorphically, so it is important to be clear that we use the following convention.

Convention 3.2. Multiplication by v is used to relate notation for homotopy groups in different dimensions.

When we come to the analogous situation for ku, multiplication by v will not be an isomorphism, whereas multiplication by χ will be. The correct action of v will then need to be chosen by reference to the action of χ. This is recorded in the following lemma.

Lemma 3.3. *Multiplication by $\chi = \chi(\alpha) = 1 - \alpha$ on $\overline{R}$ has the property that its $(p-1)$st power is p times an isomorphism.*

Proof: The module $\overline{R}$ has basis $1, \alpha, \ldots, \alpha^{p-2}$. If we write out the matrix of $\chi(\alpha) = 1 - \alpha$ with respect to this basis we see that it has determinant p, and it is easy to see that $(1-\alpha)^{p-1}$ is divisible by p. □

4. Complex stability

In this section we assume that E is complex stable and non-equivariantly bounded below (for example if Properties 1 and 3 of 1.1 hold). This tells us rather a lot about the coefficients of E and its geometric fixed point spectrum.

The complex stability condition states that for complex representations V there is a Thom isomorphism

$$E^G_*(S^V \wedge X) \cong E^G_*(S^{|V|} \wedge X)$$

(where $|V|$ is the trivial representation of the same dimension as V). A great deal of information comes from complex stability in conjunction with the fact that E is nonequivariantly bounded below.

Thus we may choose a faithful one dimensional complex representation V and consider its cell structure in the following two cofibre sequences

$$G_+ \stackrel{1-R_g}{\longrightarrow} G_+ \longrightarrow C$$

and

$$C \longrightarrow S^0 \longrightarrow S^V,$$

where g is a generator of G and R_g denotes right multiplication by g.

The first cofibre sequence lets us calculate the homology of C.

Lemma 4.1. *There is an isomorphism*

$$E^G_*(C) \cong ku_* \oplus \Sigma ku_*$$

of ku_*-modules.

Proof: Since G acts trivially on the nonequivariant homotopy ku_* we see $(1-R_g)_*$ is the zero map of E^G_*. □

Now the inclusion $S^0 \longrightarrow S^V$ together with complex stability induces multiplication by the Euler class

$$\chi(V) : E^G_k \longrightarrow E^G_{k-2}.$$

Thus the second cofibre sequence together with our calculation for C gives the following powerful restriction.

Corollary 4.2. *For $k \leq -1$ the Euler class is an isomorphism*

$$\chi(V) : E^G_k \xrightarrow{\cong} E^G_{k-2}. \quad \square$$

It is thus convenient temporarily to ignore Convention 3.2, and let $P = E^G_{-2}$ and $Q = E^G_{-1}$ denote the typical even and odd groups in the negative homotopy of E^G_*. This gives the homotopy groups of the geometric fixed point spectrum.

Corollary 4.3. *The homotopy groups of the geometric fixed point spectrum of E are $\Phi^G E_{2k} = P$ and $\Phi^G E_{2k+1} = Q$. Furthermore the action of v (as an endomorphism of P or Q) is independent of degree.* $\square$

The sequence in positive degrees takes the form

...

$$\begin{array}{c}
\mathbb{Z} \longrightarrow E^G_3 \xrightarrow{\chi} E^G_1 \longrightarrow \\
\mathbb{Z} \longrightarrow E^G_2 \xrightarrow{\chi} E^G_0 \longrightarrow \\
\mathbb{Z} \longrightarrow E^G_1 \xrightarrow{\chi} E^G_{-1} \longrightarrow \\
\mathbb{Z} \longrightarrow E^G_0 \xrightarrow{\chi} E^G_{-2} \longrightarrow 0
\end{array}$$

.

If E^G_* is all in even degrees then we obtain

$$0 \longrightarrow ku_* \longrightarrow E^G_* \xrightarrow{\chi} \Sigma^2 E^G_* \longrightarrow \Sigma^2 ku_* \longrightarrow 0$$

and $E^G_* = ku_* \oplus \chi(V) E^G_*$. Now let

$$I_{2k} = \mathrm{im}(E^G_{2k} \xrightarrow{\chi^{k+1}} E^G_{-2} = P),$$

and

$$K_{2k} = \ker(E^G_{2k} \xrightarrow{\chi^{k+1}} E^G_{-2} = P),$$

so that $I_0 = P$ and $K_0 = \mathbb{Z}$. Now compare two exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{2k+2} & \longrightarrow & E^G_{2k+2} & \longrightarrow & I_{2k+2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{2k} & \longrightarrow & E^G_{2k} & \longrightarrow & I_{2k} & \longrightarrow & 0.
\end{array}$$

The right hand vertical is injective, and the central vertical has kernel and cokernel $\mathbb{Z}$. We conclude
(i) I_{2k}/I_{2k+2} is a cyclic group, and
(ii) $K_{2k+2} = \mathbb{Z} \oplus K'_{2k}$, where $K'_{2k} = K_{2k}$ or $K'_{2k} \oplus \mathbb{Z} = K_{2k}$ according to whether the cyclic group in (i) is infinite or finite. In any case we see by induction that K_{2k+2} is a free abelian group with rank either that of K_{2k} or one more. Furthermore, if $P \otimes \mathbb{Q}$ is finite dimensional, the cyclic group in (i) is eventually finite and so the rank of the kernel K_{2k} is eventually constant.

5. The complete part

If E is split with underlying non-equivariant spectrum ku, we have the pullback square

$$\begin{array}{ccc} E & \longrightarrow & E \wedge \tilde{E}\mathcal{P} \\ \downarrow & \lrcorner & \downarrow \\ F(EG_+, E) & \longrightarrow & t(E) \\ \| & & \| \\ F(EG_+, ku) & & t(ku) \end{array}$$

The point here is the two equalities at the bottom: we have written ku because the spectra are independent of which equivariant form of ku has been used. For definiteness, and to emphasize the logic of the construction, we should use $\inf_1^G ku$. In this section we make the groups concerned explicit. In the present context it seems most appropriate to give the descriptions as abelian groups, with v action. If lu is the principal Adams summand, we have

$$ku = lu \vee \Sigma^2 lu \vee \ldots \vee \Sigma^{2(p-2)} lu$$

and

$$lu^* = \mathbb{Z}[v^{p-1}].$$

Furthermore

$$\pi_n^G(F(EG_+, lu)) = lu^{-n}(BG_+) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_p^\wedge & \text{if } n \geq 0 \text{ is a multiple of } 2(p-1) \\ \mathbb{Z}_p^\wedge & \text{for other even } n \\ 0 & \text{otherwise} \end{cases}$$

Here v^{p-1} acts isomorphically on the $\mathbb{Z}$ factor, and also on the $\mathbb{Z}_p^\wedge$ factor when the codomain is in degree 0 or more. In lower even degrees v^{p-1} acts as multiplication by p.

$$\pi_n^G(t(lu)) = t(lu)_n^G = \begin{cases} \mathbb{Z}_p^\wedge & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

In all even degrees v^{p-1} acts as multiplication by p.

The fibre of $F(EG_+, lu) \longrightarrow t(lu)$ is $lu \wedge EG_+$. Its homotopy groups can be deduced from the long exact sequence and the fact that it is connective:

$$\pi_n^G(lu \wedge EG_+) = lu_n(BG_+) = \begin{cases} \mathbb{Z} & \text{if } n \geq 0 \text{ is a multiple of } 2(p-1) \\ \mathbb{Z}/p^{j+1} & \text{if } n = 2j(p-1) + 2s + 1 \text{ with } 0 \leq s < p-1 \\ 0 & \text{otherwise} \end{cases}$$

By taking suitable wedges of suspensions of lu we obtain the corresponding calculations for ku. For comparison with periodic K-theory we note that as abelian groups $R(G)_J^\wedge = \mathbb{Z} \oplus (\mathbb{Z}_p^\wedge)^{p-1}$. However the formal group descriptions $K^*(BG_+) = \mathbb{Z}[v, v^{-1}][[y]]/([p](y))$ and $ku^*(BG_+) = \mathbb{Z}[v][[y]]/([p](y))$ (where $[p](y) = (1-(1-vy)^p)/v$), show that in degrees $n \geq 0$, the map $ku^{-n}(BG_+) \longrightarrow K^{-n}(BG_+)$ is isomorphic.

$$\pi_n^G(F(EG_+, ku)) = ku^{-n}(BG_+) = \begin{cases} R(G)_J^\wedge & \text{if } n \geq 0 \text{ is even} \\ \overline{R}_p^\wedge & \text{if } n < 0 \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Here v acts isomorphically on the $\mathbb{Z}$ factor, and also on the $\overline{R}_p^\wedge$ factor when the codomain is in degree 0 or more. In lower even degrees the action is more complicated: we are therefore violating Convention 3.2 in writing $\overline{R}_p^\wedge$ for the negative dimensional homotopy groups.

$$\pi_n^G(t(ku)) = t(ku)_n^G = \begin{cases} \overline{R}_p^\wedge & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

In all even degrees v^{p-1} acts as p times an isomorphism.

Lemma 5.1. *We may choose a basis so that v acts on $t(ku)_*^G$ as multiplication by χ. This may be represented in accordance with Convention 3.2 by*

$$t(ku)_{2n}^G = \chi^{-n}\overline{R}_p^\wedge. \quad \square$$

The fibre of $F(EG_+, ku) \longrightarrow t(ku)$ is $ku \wedge EG_+$. Its homotopy group $\pi_n^G(ku \wedge EG_+)$ is as follows:

$$ku_n(BG_+) = \begin{cases} \mathbb{Z} & \text{if } n \geq 0 \text{ is even} \\ (\mathbb{Z}/p^{j+1})^{s+1} \oplus (\mathbb{Z}/p^j)^{p-s-2} & \text{if } n = 2j(p-1) + 2s + 1 \text{ with } 0 \leq s < p-1 \\ 0 & \text{otherwise} \end{cases}$$

In other words, in odd positive degrees $ku_n(BG_+)$ is the sum of $p-1$ cyclic groups, and as n increases, they take turns to increase in order by a factor of p.

The Mayer-Vietoris sequence associated to the pullback square becomes a short exact sequence if we assume (Property 4) that homotopy is concentrated in even degrees:

$$0 \longrightarrow E_{2k}^G \longrightarrow P \oplus ku^{-2k}(BG_+) \longrightarrow t(ku)_{2k}^G \longrightarrow 0.$$

In negative degrees it tells us (as we learnt before in Section 4) that $E_{2k}^G \cong P$. If $k \geq 0$ it becomes

$$0 \longrightarrow E_{2k}^G \longrightarrow P \oplus R(G)_J^\wedge \longrightarrow \overline{R}_p^\wedge \longrightarrow 0.$$

6. Relation to periodic K-theory

Assuming Property 2, we have a map $i : E \longrightarrow K$ which becomes an equivalence when v is inverted. Assuming complex stability, in negative even degrees it is $i_{-2k} : P = E_{-2k}^G \longrightarrow K_{-2k}^G = R(G)$. This map is not zero since if we pass to the limit as $k \longrightarrow \infty$ (using $\chi(\alpha)$) we reach

$$i_{-\infty} : P = \pi_{-2k}(\Phi^G E) \longrightarrow \pi_{-2k}(\Phi^G K) = R(G)[1/\chi(\alpha)] = \overline{R}[1/p].$$

If we now invert v, we obtain an equivalence (since this is the same as first inverting v and then $\chi(\alpha)$). Hence i_{-2k} maps P onto a non-trivial subgroup of $R(G)$ with rank $p-1$. This is necessarily $\mathbb{Z}$-projective, and hence $P = P' \oplus \overline{R}$ additively, where $\overline{R} = R(G)/(\rho)$ as above. This shows that there is essentially a unique minimal equivariant form of connective K-theory with Properties 1.1.

Furthermore, we may deduce the action of v on $\overline{R}$ by using Lemma 3.3, bearing in mind that in Section 4 multiplication by χ was used to identify the various copies of $\overline{R}$ in different dimensions. In other words if we normalize so that in degree 0 the image of P in $R(G)[1/\chi]$ is $\overline{R} \subset \overline{R}[1/p]$ the image in degree $-2k$ is $\chi^k\overline{R}$. This then follows Convention 3.2

7. The solenoidal ku-module.

The purpose of this section is to construct and study a candidate for the geometric fixed point spectrum of equivariant connective K-theory.

After our algebraic calculations, and especially Section 6, we see that its coefficients should be the $\mathbb{Z}[v]$-algebra $\overline{R}[y, y^{-1}, v]/(v = \chi y)$: we have also met this as the $R(G)[v]$-submodule of $\overline{R}[1/p][v, v^{-1}]$ generated by the elements $\chi^{-n}v^n$. Since v acts in the solenoidal fashion by multiplication with χy we call this module $\overline{R}Sol$, and emphasize that it is additively $\overline{R}$ in each even degree. Since $v - \chi y$ is regular, $\overline{R}Sol$ admits a presentation

$$0 \longrightarrow \overline{R}[y, y^{-1}, v] \stackrel{v-\chi y}{\longrightarrow} \overline{R}[y, y^{-1}, v] \longrightarrow \overline{R}Sol \longrightarrow 0$$

by free $\mathbb{Z}[v]$-modules. Now ku is an E_∞-ring spectrum by infinite loop space theory, and hence an algebra over the sphere spectrum in the sense of [4]. Working in the homotopy category of ku-modules [4] we may therefore define a ku-module $ku\overline{R}Sol$ by copying this:

$$ku\overline{R}[y, y^{-1}] \stackrel{v-\chi y}{\longrightarrow} ku\overline{R}[y, y^{-1}] \longrightarrow ku\overline{R}Sol.$$

We begin with the more elementary properties.

Lemma 7.1. *(i) The homotopy groups of $ku\overline{R}Sol$ are*

$$\pi^G_*(ku\overline{R}Sol) \cong \overline{R}Sol$$

as $ku_* = \mathbb{Z}[v]$ modules, and thus $\overline{R}$ in each even degree, and 0 in each odd degree.
(ii) Inverting p we have an equivalence

$$ku\overline{R}Sol[1/p] \simeq K\overline{R}[1/p]$$

Proof: Part (i) is immediate from the definition and the fact that $v - \chi y$ is a regular element. Part (ii) comes from the fact that χ is an isomorphism of $\overline{R}[1/p]$. □

The defining cofibre sequence for $ku\overline{R}Sol$ gives the following calculation.

Lemma 7.2. *Given a ku-module M we have a short exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{ku_*}(ku\overline{R}Sol_*, \Sigma^{-1}M_*) \longrightarrow [ku\overline{R}Sol, M]_{ku} \longrightarrow \mathrm{Hom}_{ku_*}(ku\overline{R}Sol_*, M_*) \longrightarrow 0. \quad \square$$

Corollary 7.3. *The ku-module $ku\overline{R}Sol$ is determined up to equivalence by its homotopy as a ku_*-module.* □

The critical property for us is that $ku\overline{R}Sol^\wedge_p$ splits as a product of Eilenberg-MacLane spectra. From coefficients alone one sees this is not a splitting of ku-modules. This contrasts with the fact that, neither $ku\overline{R}Sol$, nor even $ku\overline{R}Sol[1/p] \simeq K\overline{R}[1/p]$ splits as a product of Eilenberg-MacLane spectra.

To obtain a splitting we need maps to Eilenberg-MacLane spectra. Fortunately, the $\mathbb{Z}[v]$-module $(H\mathbb{Z}^\wedge_p)^*(ku)$ is known and determined by its effect in homotopy.

Lemma 7.4. [1, III.16.5] *For $r \geq 0$ define a numerical function by $m(r) = \prod_p p^{[r/p-1]}$.*
(i) The homology of ku is additively $\mathbb{Z}$ in each even degree ≥ 0, and as a $\mathbb{Z}[v]$-module $H\mathbb{Z}_*(ku) \subseteq H\mathbb{Q}_*(ku) = \mathbb{Q}[v]$ it is generated by $v^r/m(r)$.
(ii) The cohomology $H\mathbb{Z}^*(ku)$ is additively $\mathbb{Z}$ in each even degree. Furthermore, the degree map

$$H\mathbb{Z}^*(ku) = [ku, H\mathbb{Z}]^* \longrightarrow \mathrm{Hom}(ku_*, H\mathbb{Z}_*)$$

is injective, and the image of a generator in cohomological degree $2r$ is multiplication by $\pm m(r)$ as a map $\mathbb{Z} = ku_{2r} \longrightarrow (H\mathbb{Z})_0 = \mathbb{Z}$. As a submodule of $H\mathbb{Q}^*(ku) = \mathbb{Q}[v]^* = \mathbb{Q}\{v_0, v_1, v_2, \ldots\}$ (with $vv_{r+1} = v_r, v_{-1} = 0$) it is additively generated by the elements $m(r)v_r$ for $r \geq 0$. □

The splitting arises even after localizing at p.

Proposition 7.5. *The p-local spectrum $ku\overline{R}Sol_{(p)}$ splits as a wedge of Eilenberg-MacLane spectra:*

$$ku\overline{R}Sol_{(p)} \simeq \prod_n \Sigma^{2n} H\overline{R}_{(p)}.$$

Proof: It suffices to construct maps

$$\lambda_{2n} : ku\overline{R}Sol_{(p)} \longrightarrow \Sigma^{2n} H\overline{R}_{(p)}$$

which induce an isomorphism in π_{2n}. We use multiplication by powers of y to identify each even homotopy group of $ku\overline{R}Sol_{(p)}$ with $\overline{R}_{(p)}$. If the map λ_{2n} exists, the composite λ_{2n}^{2m}

$$\Sigma^{2m} ku\overline{R}_{(p)} \xrightarrow{y^{-m}} ku\overline{R}_{(p)}[y, y^{-1}] \longrightarrow ku\overline{R}Sol_{(p)} \longrightarrow \Sigma^{2n} H\overline{R}_{(p)}$$

must be multiplication by χ^{m-n} provided $m \geq n$.

We reverse this deduction to give a construction of λ_{2n}. By 7.4 the map

$$\pi_{2n} : [\Sigma^{2m} ku\overline{R}_{(p)}, \Sigma^{2n} H\overline{R}_{(p)}] \longrightarrow \mathrm{Hom}(\overline{R}_{(p)}, \overline{R}_{(p)}) = \overline{R}_{(p)}$$

is injective and has image consisting of multiples of $p^{[(m-n)/p-1]}$. Since χ^{m-n} is divisible by $p^{[(m-n)/p-1]}$, the map λ_{2n}^{2m} exists as required. Now assemble the λ_{2n}^{2m} into a map

$$\lambda'_{2n} : ku\overline{R}_{(p)}[y, y^{-1}] \longrightarrow \Sigma^{2n} H\overline{R}_{(p)}.$$

This has the property that the composite

$$ku\overline{R}_{(p)}[y, y^{-1}] \xrightarrow{v-\chi y} ku\overline{R}_{(p)}[y, y^{-1}] \longrightarrow \Sigma^{2n} H\overline{R}_{(p)}$$

is trivial in homotopy, and hence trivial by 7.4. Hence λ'_{2n} extends to a map λ_{2n}. The extension is unique since $H^*(ku)$ is in even degrees. □

Remark 7.6. (i) Because of the fact that $t(ku) = F(EG_+, ku) \wedge \tilde{E}\mathcal{P}$ is p-local, the calculation 7.4 is also behind the splitting of $t(ku)$ into Eilenberg-MacLane spectra.
(iii) It is therefore an arithmetic coincidence that has led to $t(ku)^G$ splitting for the group of order p. For a general group Γ it is natural to expect the proper Tate spectrum $t_{\mathcal{P}}(ku)^\Gamma$ to be like $ku\overline{R}Sol$ in character. Similarly if Γ is the cyclic group of order p^2 it should not be hard to identify the Γ-spectrum $t(ku)$ exactly, using the equivariant form of ku constructed here.

8. Construction of equivariant connective K theory

In this section we construct a G-spectrum $\mathbb{E}$, which is the best available equivariant form of connective K-theory. Ultimately this should be called simply ku, but, until its properties are justified, we use the more neutral name $\mathbb{E}$. We shall see in Sections 9 to 11 that it has Properties 1.1 as described in Section 1.

For the construction, we use the pullback square

$$\begin{array}{ccl} \mathbb{E} & \longrightarrow & \tilde{E}\mathcal{P} \wedge \mathbb{E} \simeq \tilde{E}\mathcal{P} \wedge \Phi^G\mathbb{E} \\ \downarrow & \lrcorner & \downarrow \\ F(EG_+, \mathbb{E}) & \longrightarrow & t(\mathbb{E}) \\ \| & & \| \\ F(EG_+, ku) & & t(ku) \end{array}$$

We already know the spectra at the bottom: indeed, we have written ku because the spectra are independent of which equivariant form of ku has been used. For definiteness, and to emphasize the logic of the construction we should use $\inf_1^G ku$. The homotopy groups concerned were described in the Section 5; geometrically, by [3] we know that $t(ku)$ is a wedge of suspensions of $\tilde{E}\mathcal{P} \wedge H\overline{R}_p^\wedge$, one in each even degree. In view of our Convention 3.2, the action by v should give comparison between different degrees, so we should display this more visibly. Note that we have a comparison map $t(ku) \longrightarrow t(K)$, and we may build this into the notation by writing HM for an Eilenberg-MacLane spectrum where $M \subseteq \overline{R}_p^\wedge[1/p]$. Now, as $R(G)$-modules $\chi^k\overline{R}_p^\wedge \cong \overline{R}_p^\wedge$, so abstractly $H(\chi^k\overline{R}_p^\wedge) \simeq H\overline{R}_p^\wedge$. Thus we write

$$t(ku) \simeq \tilde{E}\mathcal{P} \wedge \prod_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge),$$

thereby implicitly giving a map to

$$t(K) \simeq \tilde{E}\mathcal{P} \wedge \prod_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge[1/p]).$$

To complete the construction, we need only specify $\Phi^G\mathbb{E}$ and give a map to $t(ku)$. If we are to define $\mathbb{E}$ so as to ensure $\Phi^G\mathbb{E} = ku\overline{R}Sol$, the right hand vertical in the Tate pullback square will need to be a map

$$\tilde{E}\mathcal{P} \wedge ku\overline{R}Sol \simeq \tilde{E}\mathcal{P} \wedge \mathbb{E} \longrightarrow t(\mathbb{E}) \simeq \tilde{E}\mathcal{P} \wedge \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge).$$

It is equivalent to get a non-equivariant map

$$\lambda : ku\overline{R}Sol \longrightarrow \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge).$$

This must be done so as to be compatible with the known map for periodic K-theory. In other words we must obtain a square

$$\begin{array}{ccl} ku\overline{R}Sol & \longrightarrow & \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge) \\ \downarrow & & \downarrow \\ K\overline{R}[1/p] & \longrightarrow & \bigvee_n \Sigma^{2n} H(\overline{R}_p^\wedge[1/p]) \end{array}$$

which in π_{2n} is the pullback square of inclusions

$$\begin{array}{ccc} \chi^{-n}\overline{R} & \longrightarrow & \chi^{-n}\overline{R}^{\wedge}_p \\ \downarrow & & \downarrow \\ \overline{R}[1/p] & \longrightarrow & \overline{R}^{\wedge}_p[1/p]. \end{array}$$

We may take λ to be the completion map followed by the splitting of 7.5.

Definition 8.1. We define the principal equivariant form of ku to be the G spectrum $\mathbb{E}$ given by the pullback square

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol \\ \downarrow & \lrcorner & \downarrow 1 \wedge \lambda \\ F(EG_+, ku) & \longrightarrow & t(ku) \end{array}$$

9. The first properties of equivariant connective K theory

In this section we begin to investigate the properties of the spectrum $\mathbb{E}$ defined in 8.1. We calculate its coefficients, show that it is split and that it is well related to periodic K-theory.

Lemma 9.1. *The spectrum $\mathbb{E}$ is split with underlying non-equivariant spectrum ku.*

Proof: The spectrum $\mathbb{E}$ is certainly non-equivariantly ku since the two right hand entries in the Tate pullback square are non-equivariantly contractible.

To show that $\mathbb{E}$ is split, we must construct a map $s : \inf_1^G(ku) \longrightarrow \mathbb{E}$. Indeed, we have a natural map $s' : \inf_1^G(ku) \longrightarrow F(EG_+, \inf_1^G(ku)) = F(EG_+, ku)$, and we choose the map $s'' : \inf_1^G(ku) \longrightarrow \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol$ corresponding to the unit map $ku \longrightarrow ku\overline{R}Sol$ of non-equivariant spectra.

By construction, the composites of s' and s'' into $t(ku)$ are both induce the inclusion of $\mathbb{Z}[v]$ in homotopy. Since homotopy detects such maps by 7.4, s' and s'' are compatible and hence combine to give a map s as required. Furthermore this specifies s uniquely: there are no maps of odd degree $\inf_1^G(ku) \longrightarrow t(ku)$ because $(H\mathbb{Z}_p^{\wedge})^*(ku)$ is zero in odd degrees by 7.4. The map s is a non-equivariant equivalence because s' is by construction. $\square$

Proposition 9.2. *The spectrum $\mathbb{E}$ has Property 2:*

$$\mathbb{E}[1/v] = K.$$

Proof: To obtain a comparison map we need to compare the defining pullback for $\mathbb{E}$ with the pullback for periodic K-theory.

We now need only form

$$\begin{array}{ccccc} F(EG_+, ku) & \longrightarrow & t(ku) \simeq \tilde{E}\mathcal{P} \wedge \prod_{2n} \Sigma^{2n} H(\chi^{-n}\overline{R}^{\wedge}_p) & \stackrel{\lambda}{\longleftarrow} & \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol \\ \downarrow & & \downarrow & & \downarrow \\ F(EG_+, K) & \longrightarrow & t(K) \simeq \tilde{E}\mathcal{P} \wedge \prod_{2n} \Sigma^{2n} H\overline{R}^{\wedge}_p[1/p] & \longleftarrow & \tilde{E}\mathcal{P} \wedge K\overline{R}[1/p] \end{array}$$

Here the left hand square exists and commutes by the definition of Tate cohomology. We have chosen $\mathbb{E}$ so that the right hand vertical comes from the natural map $ku\overline{R}Sol \longrightarrow ku\overline{R}Sol[1/p] \simeq K\overline{R}[1/p]$, where the last equivalence was proved in 7.1. The resulting right hand square induces a commutative square of homotopy groups by construction; since the codomain is rational, this ensures the square itself is homotopy commutative.

Finally, we need to know the resulting map $\mathbb{E} \longrightarrow K$ becomes an equivalence when v is inverted. Of course v is already invertible in the second row, so it is sufficient (though not necessary) to show that if we invert v in the first row we obtain the second. Since v^{p-1} is p times an isomorphism, this is holds in the middle and on the right. Finally we claim $F(EG_+, ku)[1/v] = F(EG_+, ku[1/v])$. This is a calculation not a formality, but we have already reported the relevant facts in Section 5 □

We calculate the coefficient group of $\mathbb{E}$, giving it a ring structure as a subring of K_G^*. In the next section we show this agrees with the product arising from the product we construct on $\mathbb{E}$.

Proposition 9.3. *The coefficient ring of the principal equivariant form of connective K-theory is*

$$\mathbb{E}_G^* = R(G)[v, y]/(vy = \chi, y\rho),$$

where v is of degree 2, y is of degree -2, ρ is the regular representation and $\chi = 1 - \alpha$.

Proof: We first observe that the map $\mathbb{E} \longrightarrow K$ induces an injective map on coefficients. Indeed, we consider the induced map of Mayer-Vietoris sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}_G^* & \longrightarrow & ku^*(BG_+) \oplus \overline{R}Sol & \longrightarrow & \overline{R}Sol_p^\wedge & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_G^* & \longrightarrow & K^*(BG_+) \oplus \overline{R}Sol[1/p] & \longrightarrow & \overline{R}Sol_p^\wedge[1/p] & \longrightarrow & 0. \end{array}$$

Note that it suffices to observe that $ku^*(BG_+) \longrightarrow K^*(BG_+)$ and $ku\overline{R}Sol_* \longrightarrow K\overline{R}[1/p]_*$ are injective: this was shown in Section 5 and Lemma 7.1.

This lets us describe $\mathbb{E}_G^*$ as a subring of $K_G^* = R(G)[v, v^{-1}]$. In degrees 0 and above $ku^*(BG_+) \longrightarrow K^*(BG_+)$ is an isomorphism, and any element of the kernel of $R(G)_J^\wedge \oplus \overline{R}[1/p] \longrightarrow R(G)_J^\wedge[1/p]$ necessarily has second component in $\overline{R}$. Since $t(ku) \longrightarrow t(K)$ is also injective in homotopy we see $\mathbb{E}_G^* \longrightarrow K_G^*$ is an isomorphism in positive degrees.

In negative degrees the maps $ku^*(BG_+) \longrightarrow t(ku)_*^G$ and $ku\overline{R}Sol_* \longrightarrow t(ku)_*^G$ are both injective, so the coefficient group $\mathbb{E}_{-2n}^G$ is their intersection

$$\mathbb{E}_{-2n}^G = \chi^n \overline{R}_p^\wedge \cap \chi^n \overline{R} = \chi^n \overline{R}. \quad \square$$

Remark 9.4. A useful consequence of this calculation is that if L is a free module over $\mathbb{E}_G^*$ on a single generator of degree d then $x \in L$ is a generator if and only if (i) x is a generator of $L[1/v]$ and (ii) x is in degree d.

10. Multiplicative properties of equivariant connective K-theory.

In this section we show that $\mathbb{E}$ is a commutative ring spectrum and that the map $\mathbb{E} \longrightarrow K$ is a map of ring spectra.

Proposition 10.1. *The spectrum $\mathbb{E}$ is a split ring spectrum with underlying non-equivariant spectrum ku.*

Proof: We need to check that $\mathbb{E}$ is a ring spectrum, and that the splitting is a map of ring spectra. The trick here is to recognize that the difficulties at and away from p are different.

We can treat the problems separately and then assemble them using the arithmetic Hasse pullback square

$$\begin{array}{ccc} \mathbb{E} & \longrightarrow & \mathbb{E}[1/p] \\ \downarrow & \ulcorner & \downarrow 1 \wedge \lambda \\ \mathbb{E}_p^{\wedge} & \longrightarrow & \mathbb{E}_p^{\wedge}[1/p] \end{array}$$

The main point is that this gives an exact sequence for calculating $[\mathbb{E}^{\wedge i}, \mathbb{E}]^G$.

Lemma 10.2. *There is an exact sequence*

$$0 \longrightarrow [\mathbb{E}^{\wedge i}, \mathbb{E}]^G \longrightarrow [\mathbb{E}^{\wedge i}, \mathbb{E}[1/p]]^G \oplus [\mathbb{E}^{\wedge i}, \mathbb{E}_p^{\wedge}]^G \longrightarrow [\mathbb{E}^{\wedge i}, \mathbb{E}_p^{\wedge}[1/p]]^G$$

Proof: It remains to verify that $[\mathbb{E}^{\wedge i}, \mathbb{E}_p^{\wedge}[1/p]]^G_*$ is concentrated in even degrees.

Note first that by 9.3, the group $\mathbb{E}_G^*$ consists of finitely generated abelian groups in even degrees, so the coefficients of $\mathbb{E}_p^{\wedge}[1/p]$ are finite dimensional vector spaces over $\mathbb{Q}_p$ in even degrees. Since rational G-spectra split as products of Eilenberg-MacLane spectra by [7, Appendix A], both $\mathbb{E}^{\wedge i} \otimes \mathbb{Q}$ and $\mathbb{E}_p^{\wedge}[1/p]$ are wedges of rational Eilenberg-MacLane spectra in even degrees. The result follows since the category of rational G-Mackey functors has global dimension 0. □

It thus suffices to construct products $\mu_p^{\wedge}$ on $\mathbb{E}_p^{\wedge}$ and $\mu[1/p]$ on $\mathbb{E}[1/p]$ that are suitably compatible. Indeed this gives us

$$\mathbb{E} \wedge \mathbb{E} \longrightarrow \mathbb{E}_p^{\wedge} \wedge \mathbb{E}_p^{\wedge} \xrightarrow{\mu_p^{\wedge}} \mathbb{E}_p^{\wedge}$$

and similarly for $\mu[1/p]$, which we must show agree when composed with the natural map into the rational spectrum $\mathbb{E}_p^{\wedge}[1/p]$. Thus compatibility may be verified in homotopy.

First consider the p-completion. By 7.5, the map $\lambda : \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol \longrightarrow t(ku)$ becomes an equivalence on p-completion, so that $\mathbb{E}_p^{\wedge} \simeq F(EG_+, ku_p^{\wedge})$. Since non-equivariant ku is a commutative and associative ring spectrum the same is true for $\mathbb{E}_p^{\wedge}$. Indeed, if we use the highly structured inflation of Elmendorf-May [5], we can construct $\mathbb{E}_p^{\wedge}$ as a ku-algebra.

For $\mathbb{E}[1/p]$ we may adopt a more naive approach. The analogue of the following lemma is false without inverting p.

Lemma 10.3. *For $s \geq 0$ the group $[\mathbb{E}^{\wedge s}[1/p], \mathbb{E}[1/p]]_G$ is the kernel of*

$$[ku^{\wedge s}[1/p], ku[1/p]] \times [ku\overline{R}Sol^{\wedge s}[1/p], ku\overline{R}Sol[1/p]] \longrightarrow \prod_n [ku\overline{R}Sol^{\wedge s}[1/p], \Sigma^{2n} H\overline{R}_p^{\wedge}[1/p]]$$

Proof: This follows from the Mayer-Vietoris sequence of the defining pullback square. We need to calculate maps into the three corners. First note that for any G-spectra X and Y we have $[X[1/p], Y[1/p]]^G = [X[1/p], Y]^G$. This avoids difficulties in using the smash function adjunction. We may now calculate

$$\begin{array}{rcl} [\mathbb{E}^{\wedge s}[1/p], F(EG_+, ku)]_G^* & = & [\mathbb{E}^{\wedge s} \wedge EG_+[1/p], ku]_G^* \\ & = & [F(EG_+, ku)^{\wedge s} \wedge EG_+[1/p], ku]_G^* \\ & = & [ku^{\wedge s} \wedge BG_+[1/p], ku] \end{array}$$

$$\begin{array}{rcl} [\mathbb{E}^{\wedge s}[1/p], \mathbb{E} \wedge \tilde{E}\mathcal{P}]_G^* & = & [\Phi^G \mathbb{E}^{\wedge s}[1/p], ku\overline{R}Sol]^* \\ & = & [ku\overline{R}Sol^{\wedge s}[1/p], ku\overline{R}Sol]^* \end{array}$$

and

$$\begin{array}{rcl}[\mathbb{E}^{\wedge s}[1/p], t(\mathbb{E})]^*_G & = & [\Phi^G\mathbb{E}^{\wedge s}[1/p], \prod_n H\Sigma^{2n}\overline{R}^{\wedge}_p]^* \\ & = & [ku\overline{R}Sol^{\wedge s}[1/p], \prod_n H\Sigma^{2n}\overline{R}^{\wedge}_p]^*\end{array}$$

Finally we need to know the relevant odd dimensional part is zero. This follows since $[ku\overline{R}Sol^{\wedge s}[1/p], H\overline{R}^{\wedge}_p[1/p]]$ is zero in odd degrees: indeed, $\overline{R}^{\wedge}_p[1/p]$ is rational, and so it suffices to observe the rational homotopy of $ku\overline{R}Sol^{\wedge s}$ is in even degrees, as follows from the case $s = 1$. □

To define a product $\mu[1/p]$ we must find a compatible pair of products $\mu'[1/p]$ (on $ku[1/p]$) and $\mu''[1/p]$ (on $ku\overline{R}Sol[1/p]$). To obtain $\mu'[1/p]$ we invert p on the usual product $\mu' : ku \wedge ku \longrightarrow ku$. For $\mu''[1/p]$ we take geometric fixed points of the usual product on equivariant K-theory and use the fact that $\Phi^G K \simeq ku\overline{R}Sol[1/p]$. To check these are compatible we need only verify they agree in rational homotopy.

This shows that $\mathbb{E}$ is a commutative ring spectrum. We need to check the splitting map of 9.1 is a ring map. The map s' is a ring map integrally. At p we have only defined the ring structure on $ku\overline{R}Sol$ via its equivalence with $t(ku)$, and away from p, the map s'' is a ring map because of the square

$$\begin{array}{ccc}\mathrm{inf}_1^G(ku)[1/p] & \longrightarrow & ku\overline{R}Sol[1/p] \\ \downarrow & & \downarrow\simeq \\ \mathrm{inf}_1^G(K)[1/p] & \longrightarrow & K\overline{R}[1/p].\end{array} \quad \square$$

Lemma 10.4. *The map $\mathbb{E} \longrightarrow \mathbb{E}[1/v] \simeq K$ constructed above is a map of ring spectra.*

Proof: This follows by comparing pullback squares, since K may be shown to be a ring spectrum by the same method as we used for $\mathbb{E}$. □

11. Complex orientability of equivariant connective K-theory.

Finally we establish that the spectrum $\mathbb{E}$ has a canonical complex orientation.

Proposition 11.1. *The spectrum $\mathbb{E}$ is complex stable.*

Proof: We need to find a Thom class for S^β for all one dimensional representations β. This is trivial if β is G-fixed, and the other spheres are homeomorphic as G-spaces, so it suffices to deal with $\beta = \alpha$. One way to do this is to construct equivalences $E \wedge S^\alpha \simeq E \wedge S^2$. We do this again by using the defining pullback square, so it suffices to find vertical equivalences in the diagram

$$\begin{array}{ccccc}S^2 \wedge F(EG_+, ku) & \longrightarrow & S^2 \wedge t(ku) \simeq S^2 \wedge \tilde{E}\mathcal{P} \wedge \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}^{\wedge}_p) & \longleftarrow & S^2 \wedge \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol \\ a\downarrow & A & b\downarrow & B & c\downarrow \\ S^\alpha \wedge F(EG_+, ku) & \longrightarrow & S^\alpha \wedge t(ku) \simeq S^\alpha \wedge \tilde{E}\mathcal{P} \wedge \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}^{\wedge}_p) & \longleftarrow & S^\alpha \wedge \tilde{E}\mathcal{P} \wedge ku\overline{R}Sol\end{array}$$

so that the squares A and B commute.

For the square B, we note that $S^\alpha \wedge \tilde{E}\mathcal{P} \simeq \tilde{E}\mathcal{P}$, so the idea is to take b and c to be multiplication by χ. For b this amounts to a shift of terms in the wedge of Eilenberg-MacLane spectra, and for c we use 7.2 to deduce there is a unique ku-map $\Phi^G c : \Sigma^2 ku\overline{R}Sol \longrightarrow ku\overline{R}Sol$

inducing multiplication by χ in homotopy. To see the resulting square B is commutative note that the codomain is $\tilde{E}\mathcal{P} \wedge ku\overline{R}Sol_p^\wedge$ as a ku-module. Now, by 7.2 again, ku-maps $\Sigma^2 ku\overline{R}Sol \longrightarrow ku\overline{R}Sol_p^\wedge$ are detected by their effect in homotopy.

It remains to check we may find an equivalence a so as to make the square A commute. Noting that $EG_+ \simeq \underset{\rightarrow}{\mathrm{holim}}_k\, S(k\alpha)_+$, we view the map $F(EG_+, ku) \longrightarrow t(ku) = F(\tilde{E}\mathcal{P}, ku \wedge \Sigma EG_+)$ as

$$\underset{\leftarrow}{\mathrm{holim}}_k\, D(S(k\alpha)_+) \wedge EG_+ \wedge ku \longrightarrow \underset{\leftarrow}{\mathrm{holim}}_k\, \Sigma D(S^{k\alpha}) \wedge EG_+ \wedge ku,$$

so to obtain a map it suffices to use a fixed Thom equivalence $EG_+ \wedge ku \wedge S^2 \simeq EG_+ \wedge ku \wedge S^\alpha$, smash it with the maps $D(S(k\alpha)_+) \longrightarrow D(S^{k\alpha})$ and pass to inverse limits. This gives a map on inverse limits, unique up to maps which are zero on homotopy; we make an arbitrary choice of one. Finally, we need to know that our two descriptions of the map b are consistent. This is the statement that $v = \chi y$, since χ is also the ku-Euler class of α: in other words we need the diagram

$$\begin{array}{ccc}
\underset{\leftarrow}{\mathrm{holim}}_k\, \Sigma^3 BG^{-k\alpha} \wedge ku & \overset{\simeq}{\longleftarrow} & \bigvee_n \Sigma^{2n+2} H(\chi^{-n}\overline{R}_p^\wedge) \\
\chi \downarrow & & \downarrow \chi \\
\underset{\leftarrow}{\mathrm{holim}}_k\, \Sigma BG^{-k\alpha} \wedge ku & \overset{\simeq}{\longleftarrow} & \bigvee_n \Sigma^{2n} H(\chi^{-n}\overline{R}_p^\wedge)
\end{array}$$

to commute. Up to this point we have not needed to know anything more about the horizontal equivalence than their effect in homotopy. We now claim it may be chosen so that the above diagram commutes. Indeed a map

$$\theta^{2i} : \Sigma^{2i} H(\chi^{-i}\overline{R}_p^\wedge) \longrightarrow \bigvee_n \Sigma^{2n+2} H(\chi^{-n}\overline{R}_p^\wedge)$$

will be part of an equivalence provided its component θ^{2i}_{2i} in the ith factor is an isomorphism. Now the composite of θ^{2i}_{2i} and the Thom isomorphism may have a non-zero component in degree $2(i+j)$ for $j \geq 0$; we may choose $\theta^{2(i+j)}_{2i}$ to cancel this, because the Thom isomorphism is an equivalence on the $2(i+j)$th factor. □

To treat complex orientations, recall that there is a cofibre sequence $\mathbb{C}P(V) \longrightarrow \mathbb{C}P(V \oplus \alpha) \longrightarrow S^{V \otimes \alpha^{-1}}$. A complex orientation is an element $x \in E_G^*(\mathbb{C}P(\mathcal{U})_+)$ so that (i) x restricts to zero on $\mathbb{C}P(\epsilon)_+ = *_+$ and (ii) x restricts to a generator on $(\mathbb{C}P(\epsilon \oplus \alpha), \mathbb{C}P(\alpha))$.

Proposition 11.2. *The spectrum $\mathbb{E}$ is complex orientable.*

Proof: Periodic K-theory has a natural orientation, $x_K \in K_G^2(\mathbb{C}P(\mathcal{U})_+)$; the restriction of this to $K^2(\mathbb{C}P(\epsilon \oplus \alpha), \mathbb{C}P(\epsilon)) \cong K_G^2(S^\alpha)$ is the Bott class. It suffices to show that x_K is the image of an element $x \in \mathbb{E}_G^*(\mathbb{C}P(\mathcal{U})_+)$; indeed, since $\mathbb{E}_G^* \longrightarrow K_G^*$ is injective it follows that its restriction to $\mathbb{C}P(\epsilon)_+$ is zero, and by Remark 9.4 it follows that the restriction of $x(\epsilon)$ to $(\mathbb{C}P(\epsilon \oplus \alpha), \mathbb{C}P(\alpha))$ is a generator.

To see that x_K lifts to $\mathbb{E}$ theory we choose a complete flag $\epsilon = V^1 \subseteq V^2 \subseteq V^3 \subseteq \cdots$ in $\mathcal{U}$. We argue by induction that the class $x_K^{V^n}$ on $\mathbb{C}P(V^n)$ lifts to a class x^{V^n} in $\mathbb{E}$ theory. This is immediate by degree for $n = 1$. For the inductive step observe that the cofibre sequence $\mathbb{C}P(V) \longrightarrow \mathbb{C}P(V \oplus \alpha) \longrightarrow S^{V \otimes \alpha^{-1}}$, together with complex stability and concentration in even degrees allows us to deduce that we have a split short exact sequence in both $\mathbb{E}$ theory and K theory:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{E}^*_G(S^{V\otimes\alpha^{-1}}) & \longrightarrow & \mathbb{E}^*_G(\mathbb{C}P(V\oplus\alpha)) & \longrightarrow & \mathbb{E}^*_G(\mathbb{C}P(V)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K^*_G(S^{V\otimes\alpha^{-1}}) & \longrightarrow & K^*_G(\mathbb{C}P(V\oplus\alpha)) & \longrightarrow & K^*_G(\mathbb{C}P(V)) & \longrightarrow & 0. \end{array}$$

From the top exact sequence x^V lifts to a class $\overline{x}^{V\oplus\alpha}$, and because the left hand vertical is surjective in the relevant degree we can add an adjustment term to obtain $x^{V\oplus\alpha}$ mapping to $x_K^{V\oplus\alpha}$. $\square$

12. Highly structured products

The author does not know if $\mathbb{E}$ admits the structure of a commutative S^0-algebra, or that of an algebra over $\mathrm{inf}_1^G ku$. However, we have a partial result in this direction, in the spirit of those of McClure [9]. This result is independent of 10.1.

McClure works in the language of E_∞-ring spectra, so we pause to translate his results into the language of [4]. I am grateful to J.P.May for guidance. Up until now, we have assumed that G-spectra are indexed on a complete G-universe $\mathcal{U}$. We now need to consider naive G-spectra indexed on a G-fixed universe, such as $\mathcal{U}^G$. We have the forgetful functor

$$N : G\text{-spectra} \longrightarrow \text{Naive } G\text{-spectra}.$$

The results of [4] are independent of universe, and apply equally well to naive G-spectra. We refer to the 0-sphere spectrum in the relevant category as S^0; as is usually the way with suspension spectra, this means something different in the categories of non-equivariant spectra, of G-spectra and of naive G-spectra. We shall say that a (genuine) G-spectrum E is a naive S^0-algebra if NE is an S^0-algebra. Since passage to Lewis-May fixed points factors through N, this means that E^G is an S^0-algebra.

By the work of Elmendorf-May [5], since ku is a non-equivariant S^0-algebra, the inflation $\mathrm{inf}_1^G ku$ may be taken to be an equivariant S^0-algebra. Hence $F(EG_+, ku)$ may be taken to be an S^0-algebra. McClure's results state that if E is is an equivariant S^0-algebra then the Tate spectrum $t(E)$ is never an equivariant S^0-algebra (unless it is contractible), but it is a naive S^0-algebra.

Theorem 12.1. *The spectrum $\mathbb{E}$ may be constructed so that $N\mathbb{E}$ is homotopy equivalent to an S^0-algebra, and the map $N\mathbb{E} \longrightarrow NK$ is homotopic to a map of S^0-algebras.*

Proof: We still use the defining pullback square, but this time in the category of naive S^0-algebras.

Lemma 12.2. *The non-equivariant spectrum $ku\overline{R}Sol$ may be constructed as an S^0-algebra.*

Proof: The spectrum may be obtained from $ku\overline{R}Sol[1/p]$, $ku\overline{R}Sol_p^\wedge$ and $ku\overline{R}Sol_p^\wedge[1/p]$ by an arithmetic Hasse square.

Now $ku\overline{R}Sol[1/p] \simeq K\overline{R}[1/p]$ has the structure of a ku-algebra since K may be constructed from ku by Bousfield localization. The spectra $ku\overline{R}Sol_p^\wedge$ and $ku\overline{R}Sol_p^\wedge[1/p]$ are generalized Eilenberg-MacLane spectra and hence S^0-algebras.

Replacing the two maps in the fork by fibrations to ensure the right homotopy type, we construct $ku\overline{R}Sol$ as a pullback of S^0-algebras. □

The map $\lambda : ku\overline{R}Sol \longrightarrow \Pi_n \Sigma H\overline{R}_p^{\wedge}$ may be constructed as a map of S^0-algebras since it is p-completion. It follows from [9] that $\tilde{E}G \wedge ku\overline{R}Sol$ is a naive S^0-algebra, and the map $\tilde{E}G \wedge ku\overline{R}Sol \longrightarrow t(ku)$ is a map of naive S^0-algebras. Furthermore McClure shows $F(EG_+, ku) \longrightarrow t(ku)$ is also a map of naive S^0-algebras. Because $t(ku)$ is a product of Eilenberg-MacLane spectra, the two implicit S^0-algebra structures on it my be taken to agree. Replacing the two maps in the fork of the defining pullback by fibrations to ensure the right homotopy type, we construct $N\mathbb{E}$ as a pullback of S^0-algebras. □

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