RATIONAL S^1-EQUIVARIANT STABLE HOMOTOPY THEORY.

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Part III

Applications.

CHAPTER 14

Introduction to Part III.

14.1. General Outline

The material in Part III is a collection of applications of the general theory developed in Parts I and II. Accordingly, the chapters are completely independent of each other. Since most of the work has already been incorporated in the general framework, the sections are rather short, and uncluttered by technicalities.

We begin with Chapter 15, which consists of five independent sections answering the main questions motivating this study. Perhaps the most obvious problem of all, in the light of Haeberly's example, is to understand the behaviour of the Atiyah-Hirzebruch spectral sequence. We show that it collapses at E_2 for all $\mathcal{F}$-spaces if and only if $\pi^{\mathbb{T}}_*(K \wedge E\mathcal{F}_+)$ is injective. In general, the differentials encode the Adams short exact sequence, but there are differentials of arbitrary length. For arbitrary spaces, the main message is that the Atiyah-Hirzebruch spectral sequence is not a natural tool. Alternatively, we can always use a cellular decomposition of the domain to try to understand maps. In our case, the graded orbit category can be made quite explicit, and we can therefore do homological algebra over the category of additive functors on this category. We may then construct a spectral sequence, whose E_2 term is calculable and given by homological algebra over the graded orbit category. It is also obviously convergent, but it does not seem a practical tool in general, because it is usually a half-plane spectral sequence. The moral is that we should not decompose spectra by Eilenberg-MacLane spectra or by cells, but rather by injectives in the standard model.

Another basic construction of spectra is by taking the suspension spectrum of a space. Equivariantly it is known that suspension spectra have certain special properties, such as tom Dieck splitting, which are not enjoyed by all spectra. In particular, the inclusion of the H-fixed points can be regarded as a map of $\mathbb{T}$-spaces, and this has implications for the model. However, since our model has no record of purely unstable information, we cannot hope for a precise characterization of suspension spectra.

The special case of K-theory is interesting because it has Bott periodicity, and hence Euler classes of its own. The representing spectrum also turns out to be formal in the torsion model, so that the K-theory of any spectrum depends only on its homology in the torsion model. On the other hand, our analysis is really only the beginning of a study melding the formalisms of the present work with the geometric information of K-theory: in particular it would be interesting to understand the Chern character in more detail and to relate our results to the work of Brylinski and collaborators [3].

After our analysis of function spectra, the rational Segal conjecture is little more than an elementary example: in the torsion model $DE\mathbb{T}_+$ is formal, and represented by the natural map

$$t^{\mathcal{F}}_* \otimes t^{\mathcal{F}}_* \longrightarrow t^{\mathcal{F}}_* \longrightarrow t^{\mathcal{F}}_*/\mathcal{O}_{\mathcal{F}} = \Sigma^2\mathbb{I} \longrightarrow \mathbb{Q}[c, c^{-1}]/\mathbb{Q}[c].$$

We also present a more naive approach by way of comparison.

In Chapter 16 we consider the well known $\mathbb{T}$-equivariant cohomology theory given by cyclic cohomology. This is very simple rationally; more generally rational Tate spectra are also rather simple, but we may make certain intriguing algebraic connections. Finally we are able to identify the *integral* Tate spectrum $t(K\mathbb{Z})$ of integral complex K-theory $K\mathbb{Z}$. This is of interest because $t(K\mathbb{Z})$ is known to be H-equivariantly rational for all finite subgroups H. In fact, writing $K\mathbb{Z}$ for integral complex K-theory for emphasis, we identify the spectra $t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F}$ and $t(K\mathbb{Z}) \wedge \Sigma E\mathcal{F}_+$ together with the map of which $t(K\mathbb{Z})$ is the fibre. The first is obtained from K-theory with suitable coefficients by inflating and smashing with $\tilde{E}\mathcal{F}$, and the second is rational, and we can identify it as an injective Euler-torsion $\mathcal{O}_{\mathcal{F}}$-module.

The final Chapter 17 is more substantial. We turn to examples gaining their importance from algebraic K-theory. Bökstedt, Hsiang and Madsen define the topological cyclic cohomology of a ring or a space [2]. It is obtained by performing various constructions on the topologicial Hochschild homology spectrum and can be used as a close approximation to the completed algebraic K-theory of suitable rings. Hesselholt and Madsen [17] identify the structure required of a $\mathbb{T}$-spectrum X for one to be able to construct $TC(X)$. Spectra X with this structure are called cyclotomic spectra. The motivation for the notion of a cyclotomic spectrum comes from the free loop space $\Lambda X = map(\mathbb{T}, X)$ on a $\mathbb{T}$-fixed space X. This has the property that if we take K-fixed points we obtain the $\mathbb{T}/K$-space $map(\mathbb{T}/K, X)$, and if we identify the circle $\mathbb{T}$ with the circle $\mathbb{T}/K$ by the $|K|$th root isomorphism we recover ΛX. For spectra, one also needs to worry about the indexing universe, but a cyclotomic spectrum is basically one whose geometric fixed point spectrum $\Phi^K X$, regarded as a $\mathbb{T}$-spectrum, is the original $\mathbb{T}$-spectrum X. After the suspension spectrum of a free loop space, the principal example comes from the topological Hochschild homology of $THH(F)$ of a functor F with smash products. Given such a cyclotomic spectrum X, one may construct the topological cyclic spectrum $TC(X)$ of Bökstedt-Hsiang-Madsen [2], which is a non-equivariant spectrum. An intermediate construction of some interest is the $\mathbb{T}$-spectrum $TR(X)$. Although these constructions are principally of interest profinitely, it is instructive to identify the cyclotomic spectra in our model and follow the constructions through. We identify the rational cyclotomic spectra in the torsion model: they are the spectra X so that the function $[N] : \mathcal{F} \longrightarrow \text{torsion}\mathbb{Q}[c] - modules$ modelling $E\mathcal{F}_+ \wedge X$ is constant, and so that the structure map $t^{\mathcal{F}}_* \otimes V \longrightarrow \Sigma N$ commutes with any translation of the finite subgroups. It therefore factors through $t^{\mathcal{F}}_* \otimes V \longrightarrow t^{\mathcal{F}}_*/\mathcal{O}_{\mathcal{F}} \otimes V$ and the map $\mathcal{O}_{\mathcal{F}}/\mathcal{O}_{\mathcal{F}} \otimes V \longrightarrow \Sigma N$ is a direct sum of copies of $\mathbb{Q}[c, c^{-1}]/\mathbb{Q}[c] \otimes V \longrightarrow \Sigma[N](1)$. Furthermore, we may recover Goodwillie's theorem [7] that for any cyclotomic spectrum X

we have $TC(X) = X^{h\mathbb{T}}$: topological cyclic cohomology coincides with cyclic cohomology in the rational setting.

14.2. Prospects and problems.

The main theoretical problem is to show that the equivalence of the category of $\mathbb{T}$-spectra and the derived category of the standard model can be obtained from a chain of equivalences arising from adjoint pairs of functors on underlying Quillen model categories. This would inevitably be linked with a better understanding of the meaning of the standard model.

One of the most interesting prospective applications is that of understanding rational $\mathbb{T}$-equivariant elliptic cohomology. Constructions have recently been given by Grojnowski [8] and by Ginzburg-Kapranov-Vasserot [6], and the cohomology of any $\mathbb{T}$-space is a sheaf over an elliptic curve. One can ask if these theories are represented. This would involve considering sheaves of $\mathbb{T}$-spectra over an elliptic curve, and it would seem a sensible first step to consider sheaves of objects of $\mathcal{A}$.

Both in this case and that of K-theory, there is the task of relating the general model to the geometry of the cohomology theory: in practice this will involve concentration on the Chern character, and comparison with the work of Brylinski [3]. There are a number of other classes of spectra which we do not understand as well as we should like, such as suspension spectra, free loop spaces, THH, ...

In the present work we have concentrated entirely on the circle group $\mathbb{T}$. Although it is unlikely to be possible to give so complete a picture as we have done for $\mathbb{T}$-spectra, we hope to consider other small groups in due course. The continuous quaternion and dihedral groups are prime candidates, both by virtue of their simplicity and the prospects for applications. However, consideration of the case of Mackey functors [11] shows that it is necessary to replace the underlying algebra of $\mathcal{O}_{\mathcal{F}}$-modules by that of sheaves over spaces of subgroups, since the topology on the space of subgroups can no longer be ignored. For groups of rank greater than 1, the injective dimension of the relevant algebraic categories will be greater than 1. There is therefore no prospect of obtaining splittings for formal reasons, and models must be based on a more complete geometric understanding than we have used here.

CHAPTER 15

Classical miscellany.

The sections in this chapter are independent of each other; each answers a natural question about rational $\mathbb{T}$-spectra.

In Section 15.1 we give a complete analysis of the behaviour of the Atiyah-Hirzebruch spectral sequence for $\mathcal{F}$-spectra, generalising the study in Section 1.4. Section 15.2 sets up a calculable spectral sequence for calculating maps between $\mathbb{T}$-spectra from a cellular decomposition, based on the graded orbit category. Section 15.3 shows how the existence of tom Dieck splitting makes the models of suspension spectra very special. Section 15.4 finally returns to complex $\mathbb{T}$-equivariant K-theory, and identifies its place in the torsion model, showing that it is formal. Finally, Section 15.5 identifies the functional dual $DE\mathbb{T}_+$, giving the rational analogue of the geometric equivariant Segal conjecture.

15.1. The collapse of the Atiyah-Hirzebruch spectral sequence.

The purpose of this section is to analyse the Atiyah-Hirzebruch spectral sequence for $\mathcal{F}$-spectra.

We suppose given an arbitrary $\mathbb{T}$-equivariant cohomology theory $K^*_{\mathbb{T}}(\cdot)$, and consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{s,t} = H^s_{\mathbb{T}}(X; \underline{K}^t_{\mathbb{T}}) \Longrightarrow K^*_{\mathbb{T}}(X).$$

This may be constructed either by using the skeletal filtration of X or, in complete generality, by using the Postnikov filtration of K. It is conditionally convergent if X is bounded below.

First, let us suppose that $K = E\mathbb{T}_+^{(2n)}$; we observe that, if X is free, the only relevant part of the Mackey functor $\underline{K}^*_{\mathbb{T}}$ is the nonequivariant homotopy of K. Thus the spectral sequence is concentrated on the lines $q = 1$ and $q = 2n + 2$. There are therefore no differentials except

$$d_{2n+2} : H^p(X/\mathbb{T}) = H^p_{\mathbb{T}}(X; \underline{K}^{2n+2}_{\mathbb{T}}) \longrightarrow H^{p+2n+2}_{\mathbb{T}}(X; \underline{K}^1_{\mathbb{T}}) = H^{p+2n+2}(X/\mathbb{T}).$$

If we take the special case $X = E\mathbb{T}_+^{(2m)}$, we see that the differential must be given by multiplication by c^{n+1} in order to give the correct answer, as calculated by the Adams short exact sequence. In particular, the differential is non-zero if and only if $m > n$.

THEOREM 15.1.1. If X is an $\mathcal{F}$-spectrum then the Atiyah-Hirzebruch spectral collapses at E_2 if $\pi_*^{\mathbb{T}}(K)$ is injective over $\mathcal{O}_{\mathcal{F}}$. Conversely, if the Atiyah-Hirzebruch Spectral sequence collapses at E_2 for all $\mathcal{F}$-spectra X, then $\pi_*^{\mathbb{T}}(K)$ is injective.

More precisely, any differential d_{2i+1} is zero, and the nonzero differentials d_{2n+2} are all explained by the above example, in a sense to be made precise in the proof.

Proof: The first observation is that if X is an $\mathcal{F}$-spectrum then $K_{\mathbb{T}}^*(X) = [X, K]_{\mathbb{T}}^* = [X, K \wedge E\mathcal{F}_+]_{\mathbb{T}}^*$. Since the spectral sequence is natural, we may suppose that both X and $K \wedge E\mathcal{F}_+$ have homotopy in even degrees. We argue that if $\pi_*^{\mathbb{T}}(K)$ is injective then the E_2 term is entirely in even total degrees, and hence the spectral sequence collapses.

First note that for any $\mathcal{F}$-spectrum T,

$$K_{\mathbb{T}}^*(T) = \mathrm{Hom}(\pi_*^{\mathbb{T}}(T), \pi_*^{\mathbb{T}}(K \wedge E\mathcal{F}_+)).$$

Thus, taking $T = \sigma_H^0$, we see that, for any finite subgroup H, the H-equivariant basic homotopy groups are purely in odd degrees, since $\pi_*^{\mathbb{T}}(\sigma_H^0) = \Sigma\mathbb{Q}$ is odd. Thus the part of the graded Mackey functor $\underline{K}_{\mathbb{T}}^*$ over $\mathcal{F}$ is entirely in odd degrees. On the other hand, since X is an $\mathcal{F}$-spectrum, $[X, HM]_{\mathbb{T}}^* = [X, HM \wedge E\mathcal{F}_+]_{\mathbb{T}}^*$, and $HM \wedge E\mathcal{F}_+$ is a wedge of copies of $E\langle H\rangle$, with one factor for each basis element of $V(H) = e_H M(H)$. Thus, if we let $\mathbb{I} \otimes M = \bigoplus_H \mathbb{I}(H) \otimes V(H)$ we have

$$H_{\mathbb{T}}^*(X; M) = \mathrm{Hom}(\pi_*^{\mathbb{T}}(X), \Sigma\mathbb{I} \otimes M),$$

which is entirely in odd degrees. Thus $E_2^{*,*} = H_{\mathbb{T}}^*(X; \underline{K}_{\mathbb{T}}^*)$ is in even total degrees as claimed.

Now suppose that the Atiyah-Hirzebruch spectral sequence does not collapse, and that $x \in E_r^{p,q}$ supports a non-zero differential $d_r(x) = y \neq 0$. We shall prove that $r = 2n+2$ for some n, and that the differential is explained by naturality and the differentials described above.

We may pick a representative $x' \in E_1^{p,q} = [X^{(p)}/X^{(p-1)}, K]_{\mathbb{T}}^{p+q}$ for x. This shows that x' is supported on a map $\sigma_H^p \longrightarrow X/X^{(p-1)} \longrightarrow \Sigma^{p+q}K$. Replacing X by $X/X^{(p-1)}$ and suspending, we may assume that X is (-1)-connected and $p = 0$. We may therefore replace K by its connective cover K_0^∞ without changing the fate of x in the spectral sequence. Now, letting K_m^n denote the Postnikov section of K with non-zero homotopy groups in degree i with $m \leq i \leq n$, we consider the Postnikov tower of K:

$$\begin{array}{ccccccc}
 & & K_{r-1}^{r-1} & \longrightarrow & K_0^{r-1} & \longrightarrow & \Sigma K_r^r \\
 & & & & \downarrow & & \\
 & & K_{r-2}^{r-2} & \longrightarrow & K_0^{r-2} & \longrightarrow & \Sigma K_{r-1}^{r-1} \\
 & & & & \downarrow & & \\
 & & K_2^2 & \longrightarrow & K_0^2 & \longrightarrow & \Sigma K_3^3 \\
 & & & & \downarrow & & \\
 & & K_1^1 & \longrightarrow & K_0^1 & \longrightarrow & \Sigma K_2^2 \\
 & & & & \downarrow & & \\
X & - & \overset{x}{-} & \longrightarrow & K_0^0 & \longrightarrow & \Sigma K_1^1
\end{array}$$

By hypothesis, $x : X \longrightarrow K_0^0$ lifts to $x^{(r)} : X \longrightarrow K_0^{r-2}$ so that the composite $x^{(r)} : X \longrightarrow K_0^{r-2} \longrightarrow \Sigma K_{r-1}^{r-1}$ is essential and represents y. Since X is an $\mathcal{F}$-spectrum, the behaviour is unaltered if the diagram is smashed with $E\mathcal{F}_+$. Since $HM \wedge E\mathcal{F}_+$ is injective, all maps at the E_2-term are detected by their d-invariant, and it is thus appropriate to examine the effect of taking homotopy of the above diagram smashed with $E\mathcal{F}_+$. The basic observation is that,

$$\pi^{\mathbb{T}}_*(HM \wedge E\mathcal{F}_+) = \Sigma\mathbb{I} \otimes M,$$

which is in odd degrees. Furthermore, x is detected by $\pi^{\mathbb{T}}_1$, and we need only look at the odd graded part. This immediately shows that all odd differentials are zero, so that $r = 2n+2$ for some n.

Next, we note that the maps

$$\pi^{\mathbb{T}}_*(K_0^{2s+1} \wedge E\mathcal{F}_+) \longrightarrow \pi^{\mathbb{T}}_*(K_0^{2s} \wedge E\mathcal{F}_+)$$

are injective in odd degrees, whilst the maps

$$\pi^{\mathbb{T}}_*(K_0^{2s+2} \wedge E\mathcal{F}_+) \longrightarrow \pi^{\mathbb{T}}_*(K_0^{2s+1} \wedge E\mathcal{F}_+)$$

are surjective in odd degrees. Furthermore, the image of the composite consists of elements divisible by c. We thus find the diagram

$$\begin{array}{ccccc} & & \pi^{\mathbb{T}}_*(K_0^{2n} \wedge E\mathcal{F}_+) & \longrightarrow & \pi^{\mathbb{T}}_*(\Sigma K^{2n+1}_{2n+1} \wedge E\mathcal{F}_+) = \Sigma^{2n+3}\mathbb{I} \otimes \underline{K}^{2n+1}_{\mathbb{T}} \\ & \nearrow & \downarrow & & \\ \pi^{\mathbb{T}}_*(X) & \longrightarrow & \pi^{\mathbb{T}}_*(K_0^0 \wedge E\mathcal{F}_+) = \Sigma\mathbb{I} \otimes \underline{K}^0_{\mathbb{T}} & & , \end{array}$$

and we know that some element $\tilde{z}$ of $\pi^{\mathbb{T}}_{2n+3}(X)$ maps to $z \in \pi^{\mathbb{T}}_{2n+3}(K_0^{2n} \wedge E\mathcal{F}_+)$, and $c^{n+1}z$ detects x, whilst the image of z in $\pi^{\mathbb{T}}_{2n+3}(\Sigma K^{2n+1}_{2n+1} \wedge E\mathcal{F}_+)$ detects y. We therefore find a map

$$\pi^{\mathbb{T}}_*(E\langle H\rangle^{(2n+2)}) = \Sigma\mathbb{I}_0^{2n+2} \longrightarrow \pi^{\mathbb{T}}_*(X)$$

with $\tilde{z}$ as the image of the top class, and $\tilde{x}$ as the image of the bottom class. By the Adams short exact sequence, this is realised by a map $E\langle H\rangle^{(2n+2)} \longrightarrow X$. □

15.2. Orbit category resolutions.

Integrally one expects the cellular decomposition to be unhelpful in global calculations because one does not know the stable homotopy groups of spheres. Rationally, everything is much simpler. For finite groups, cells are Eilenberg-MacLane spectra, and hence the cellular decomposition is simply another way of viewing the complete splitting [14]. In the present $\mathbb{T}$-equivariant context, cells are not all Eilenberg-MacLane spectra, but one may understand the entire graded category of natural or basic cells. We shall concentrate on the graded category $h\mathcal{SB}_*$ of basic cells (i.e. the full subcategory of the graded stable category with the basic cells as objects).

One thus views the entire homotopy functor $\underline{\pi}^{\mathbb{T}}_*(X)$ of X as a module over the graded category $h\mathcal{SB}_*$. By the Yoneda lemma, the case when X is a cell plays the role of a free object, and a resolution is form of cellular approximation. We understand maps of degree 0 from the discussion of Mackey functors presented in Appendix A. Referring to 2.1.4, we see that composition in $h\mathcal{SB}_*$ is usually zero for dimensional reasons. In fact, there are no maps of degree 2 between any pair of objects, and the only case with maps of degree more than 1 is $[\sigma^0_{\mathbb{T}}, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_* = \mathbb{Q} \oplus \Sigma(\mathbb{Q}\mathcal{F}[c^{-1}])$. For maps of degree 1 we have $[\sigma^0_{\mathbb{T}}, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_1 = \mathbb{Q}\mathcal{F}$, $[\sigma^0_{\mathbb{T}}, \sigma^0_H]^{\mathbb{T}}_1 = \mathbb{Q}$, $[\sigma^0_H, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_1 = 0$ and $[\sigma^0_H, \sigma^0_H]^{\mathbb{T}}_1 = \mathbb{Q}$; for maps of degree 0 we have $[\sigma^0_{\mathbb{T}}, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_0 = \mathbb{Q}$, $[\sigma^0_{\mathbb{T}}, \sigma^0_H]^{\mathbb{T}}_0 = 0$, $[\sigma^0_H, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_0 = \mathbb{Q}$ and $[\sigma^0_H, \sigma^0_H]^{\mathbb{T}}_0 = \mathbb{Q}$. It therefore remains to deal with the composite of a degree 1 morphism (which must be of form $x^{\mathbb{T}}_{\mathbb{T}} : \sigma^1_{\mathbb{T}} \longrightarrow \sigma^0_{\mathbb{T}}$, $x^H_H : \sigma^1_H \longrightarrow \sigma^0_H$ or a multiple of the transfer $tr^{\mathbb{T}}_H : \sigma^1_{\mathbb{T}} \longrightarrow \sigma^0_H$) and a degree 0 morphism (which is either a multiple of the relevant identity or a multiple of the projection $pr^H_{\mathbb{T}} : \sigma^0_H \longrightarrow \sigma^0_{\mathbb{T}}$). We now deal with these remaining cases.

LEMMA 15.2.1. The composites of the x's and y's are as follows.
(i) $x^{\mathbb{T}}_{\mathbb{T}} pr^H_{\mathbb{T}} = 0$ and $pr^H_{\mathbb{T}} x^H_H = 0$,
(ii) $tr^{\mathbb{T}}_H pr^H_{\mathbb{T}} \neq 0$ and $pr^H_{\mathbb{T}} tr^{\mathbb{T}}_H$ corresponds to the inclusion of the Hth factor in $\mathbb{Q}\mathcal{F} = [\sigma^0_{\mathbb{T}}, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_1$.

Proof: Part (i) is clear since the composites lie in the zero group. The first fact in Part (ii) follows from the explicit geometric construction of the transfer. The second fact in Part (ii) follows by construction of the isomorphism in tom Dieck splitting [18, V.11]. □

It is thus natural to take $\tau_H = tr^{\mathbb{T}}_H pr^H_{\mathbb{T}}$ as the basic generator of $[\sigma^0_H, \sigma^0_H]^{\mathbb{T}}_1$ and $\delta_H = pr^H_{\mathbb{T}} tr^{\mathbb{T}}_H$ as a standard basis element of $[\sigma^0_{\mathbb{T}}, \sigma^0_{\mathbb{T}}]^{\mathbb{T}}_1$.

Now, suppose given any $\mathbb{T}$-spectrum X, we consider $[\cdot, X]^{\mathbb{T}}_*$ as a contravariant functor on the graded orbit category. As such, we may form a projective resolution, and we may realise it. In fact we may construct a map $P_0 \longrightarrow X$, which is surjective in graded equivariant homotopy for all subgroups of $\mathbb{T}$, and in which P_0 is a wedge of cells. Now let X_1 be the cofibre of this, and iterate to form the diagram

$$\begin{array}{ccccccccc} X & = & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & P_0 & & P_1 & & P_2 & & \end{array}$$

By construction, all the maps $X_s \longrightarrow X_{s+1}$ are zero in H-equivariant homotopy for all subgroups H, and so $\underset{\rightarrow}{\mathrm{holim}}_s\, X_s$ is contractible by the Whitehead theorem.

It is convenient to form the dual diagram with $X^{p-1} = fibre(X \longrightarrow X_p)$, so that $X^{-1} = *$ and $X \simeq \underset{\rightarrow}{\mathrm{holim}}\, X^p$:

$$\begin{array}{ccccccc} * = X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{-1} P_0 & & \Sigma^{-1} P_1 & & \Sigma^{-1} P_2 & & \end{array}$$

Replacing the maps by inclusions, we view this as a filtration of X with subquotients $X^p/X^{p-1} = P_p$. We may now construct a spectral sequence by applying $[\cdot, Y]^*_{\mathbb{T}}$ to the diagram. It has $E^{p,q}_1 = [P_p, Y]^{p+q}_{\mathbb{T}}$, and $D^{p,q}_1 = [X^p, Y]^{p+q}_{\mathbb{T}}$. The spectral sequence lies in the right half-plane, and the differentials are cohomological, so that $d_r : E^{p,q}_r \longrightarrow E^{p+r,q-r+1}_r$, and it is evidently conditionally convergent. Finally, we see by construction that the sequence

$$\cdots \longrightarrow \underline{\pi}^{\mathbb{T}}_*(\Sigma^{-2} P_2) \longrightarrow \underline{\pi}^{\mathbb{T}}_*(\Sigma^{-1} P_1) \longrightarrow \underline{\pi}^{\mathbb{T}}_*(P_0) \longrightarrow \underline{\pi}^{\mathbb{T}}_*(X) \longrightarrow 0$$

is exact. Hence we may identify the E_2 term as an Ext group, and the spectral sequence takes the form

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{h\mathcal{SB}_*}(\underline{\pi}^{\mathbb{T}}_*(X), \underline{\pi}^{\mathbb{T}}_*(Y)) \Longrightarrow [X,Y]^{p+q}_{\mathbb{T}}.$$

The fact that the category of Mackey functors is one dimensional gives a vanishing line if X is bounded below. Indeed, since cells are also (-1)-connected we may ensure P_0 is (-1)-connected, and hence that X_1 is 0-connected. This formality proves that if the resolution is dimensionally minimal, X_p is $(p-1)$-connected. However, if we use *basic* cells and the fact that the category of Mackey functors is of projective dimension 1, we may ensure that X_2 is 2-connected. By iteration we see that X_{2s} and P_{2s} are $(3s-1)$-connected; similarly X_{2s+1} and P_{2s+1} are $3s$-connected. Thus the map $X^{2s-1} \longrightarrow X$ is $(3s-2)$-connected, and the map $X^{2s} \longrightarrow X$ is $(3s-1)$-connected. Unfortunately this only seems to be useful if the homotopy of the spectrum Y is bounded *above*. Thus if $Y^q_H = 0$ for $q \leq -1$ and all H, then the nonzero entries are in the first quadrant and lie on or above the line $q = (p-1)/2$.

15.3. Suspension spectra.

In this section we suppose given a based $\mathbb{T}$-space Z, and we identify the place of its suspension spectrum in our classification. We follow our usual convention of omitting notation for the suspension spectrum functor, and using the notation $\Psi^{\mathbb{T}}$ for Lewis-May fixed point functor.

Our basic tool is tom Dieck splitting, which states that the Lewis-May fixed points of the suspension spectrum of Z is

$$\Psi^{\mathbb{T}} Z = Z \vee \bigvee_K E\mathbb{T}/K_+ \wedge_{\mathbb{T}/K} \Sigma Z^K;$$

furthermore this is natural, and applies to stable retracts of spaces. The crucial simplification for spaces is that there is a map $Z^{\mathbb{T}} \longrightarrow Z$ of $\mathbb{T}$-spectra, and hence a diagram

$$\begin{array}{ccc} \tilde{E}\mathcal{F} \wedge Z^{\mathbb{T}} & \longrightarrow & \Sigma E\mathcal{F}_+ \wedge Z^{\mathbb{T}} \\ \simeq\downarrow & & \downarrow \\ \tilde{E}\mathcal{F} \wedge Z & \longrightarrow & \Sigma E\mathcal{F}_+ \wedge Z. \end{array}$$

The structure map $\tilde{E}\mathcal{F} \wedge Z \longrightarrow \Sigma E\mathcal{F}_+ \wedge Z$ thus factors through the corresponding map for $Z^{\mathbb{T}}$, which we understand completely, since $Z^{\mathbb{T}}$ is rationally a wedge of spheres.

On the other hand, by naturality of tom Dieck splitting, we find

$$\pi^{\mathbb{T}}_*(Z \wedge E\langle H\rangle) = \pi_*(E\mathbb{T}/H_+ \wedge_{\mathbb{T}/H} \Sigma Z^H) = H_*(E\mathbb{T}/H_+ \wedge_{\mathbb{T}/H} \Sigma Z^H),$$

which we may certainly regard as computable.

Summary 15.3.1. The algebraic model of the suspension spectrum of a space Z is

$$t^{\mathcal{F}}_* \otimes H_*(Z^{\mathbb{T}}) \longrightarrow \Sigma^2 \mathbb{I} \otimes H_*(Z^{\mathbb{T}}) = \Sigma^2 \bigoplus_H H_*(E\mathbb{T}/H_+ \wedge_{\mathbb{T}/H} Z^{\mathbb{T}}) \longrightarrow \Sigma^2 \bigoplus_H H_*(E\mathbb{T}/H_+ \wedge_{\mathbb{T}/H} Z^H).$$

The first map in the diplayed composite necessarily has zero e-invariant, and is simply induced by the quotient $t_*^{\mathcal{F}} \longrightarrow \Sigma^2\mathbb{I}$. However the second map is induced by the inclusion $X^{\mathbb{T}} \longrightarrow X$, and may have non-zero d and e invariant. Again we may be satisfied that the d-invariant is given by a homology calculation. For the e invariant, since the above discussion applies to retracts of spaces, we may assume that $X^{\mathbb{T}}$ and $E\mathcal{F}_+ \wedge X$ are of pure parity, and then identify the e invariant with the Borel homology of the cofibres $X^H/X^{\mathbb{T}}$ as in [1].

There remains the question of whether this characterizes suspension spectra. More precisely, we cannot distinguish $\mathbb{T}$-fixed suspensions so that we are asking if every model of the above sort is the model of a spectrum $\Sigma^n\Sigma^\infty X$ for some $\mathbb{T}$-space X and some integer n. We do not have available the option of identifying the suspension spectrum functor, since there is no algebraic model of rational $\mathbb{T}$-spaces. One might hope to realize finitely generated examples by explicit construction, but one would expect a certain amount of suspension to be necessary to achieve stability in each case. To obtain a global realization one would need a bound on these suspensions; since the model contains no data relevant to the achievment of stability, there is no ready way to do this.

15.4. K-theory revisited.

Let us consider the structure of the $\mathbb{T}$-spectrum K representing rational equivariant K-theory. We know that $K_*^H = R(H)[\beta, \beta^{-1}]$ for all subgroups H, where $R(H)$ is the rationalization of the complex representation ring and $\beta \in K_{-2}$ is the Bott element. Now $R = R(\mathbb{T}) = \mathbb{Q}[z, z^{-1}]$, and $R(H) = \mathbb{Q}[z]/(z^n - 1)$ when H is of order n. The restriction maps are implicit in the notation here, and the induction maps to $\mathbb{T}$ are zero (holomorphic induction maps are not part of the structure). Now, by Bott periodicity we have K-theory Euler classes $c(H)$ of degree -2 for each finite subgroup H, obtained by applying the Thom isomorphism to the image of $e(V(H))$ in K-theory. We may apply Bott periodicity to obtain the usual K-theory Euler class $\lambda(H) \in R(\mathbb{T})$. In other words if H is of order n we have

$$\lambda(H) = 1 - z^n \text{ and } c(H) = \beta\lambda(H).$$

Furthermore

$$\lambda(H) = \prod_{d|n} \Phi_d,$$

where Φ_d is the dth cyclotomic polynomial. Let S be the multiplicative set generated by the Euler classes $1 - z^n$, and T be generated by the cyclotomic polynomials Φ_d; in practice the geometry localizes so as to invert S, which is algebraically the same as inverting T, and the latter is easier to understand. Let $F = S^{-1}R = T^{-1}R$, so that

$$\pi_*^{\mathbb{T}}(K \wedge \tilde{E}\mathcal{F}) \cong F[\beta, \beta^{-1}] \text{ and } \pi_*^{\mathbb{T}}(K \wedge \Sigma E\mathcal{F}_+) \cong (F/R)[\beta, \beta^{-1}].$$

Since the cyclotomic polynomials are coprime, an element of F/R can be written uniquely as a sum of terms $f_d(z)/\Phi_d(z)^n$ for $n \geq 1$ where $f_d(z) \in R$ is not divisible by $\Phi_d(z)$. Hence

$$F/R = \bigoplus_n R[1/\Phi_n]/R.$$

We should relate this to our geometric decompositions.

LEMMA 15.4.1. If $K(H)$ denotes the part of K in $\mathbb{T}$−Spec/H as usual then

$$\pi^{\mathbb{T}}_*(K(H)) = (R[1/\Phi_n]/R)[\beta, \beta^{-1}]$$

and c_H acts as multiplication by $\beta\Phi_n$ times an automorphism.

REMARK 15.4.2. Our map c_H is defined up to a non-zero rational number, whilst the K-theory Euler class is defined absolutely. The multiple is therefore well defined up to a non-zero scalar, but its exact value is not relevant to us at present. Crabb considers studies this value in greater detail [5].

Proof: The only part requiring proof is that the action of c_H is as claimed. We shall show that c_H acts as $c(V(H))$ times an automorphism on $K \wedge E\langle H\rangle$. To do this, we let $V = V(H)$, and unravel definitions.

We have a K-theory Thom isomorphism $t : K \wedge S^V \xrightarrow{\simeq} K \wedge S^{|V|}$, and the $\mathcal{F}$-spectrum Thom isomorphism $\tau : S^V \wedge E\langle H\rangle \xrightarrow{\simeq} S^{|V|} \wedge E\langle H\rangle$; we need to know they are compatible in the sense that the composites

$$K \wedge S^0 \wedge E\langle H\rangle \longrightarrow K \wedge S^V \wedge E\langle H\rangle \longrightarrow K \wedge S^{|V|} \wedge E\langle H\rangle$$

are equal. Now, we observe that both are maps of K-module spectra, and hence it is sufficient to show both induce the same map $S^0 \wedge E\langle H\rangle \longrightarrow K \wedge S^{|V|} \wedge E\langle H\rangle$. Maps of this form are classified by $K^{|V|}_{\mathbb{T}}(E\langle H\rangle) \cong \lim_{\leftarrow n} K^{|V|}(E\langle H\rangle^{(2n)})$. Indeed the maps are classified by what they induce in homotopy:

$$[E\langle H\rangle \wedge K, S^{|V|} \wedge E\langle H\rangle \wedge K]^{K,\mathbb{T}}_0 \xrightarrow{\cong} \mathrm{Hom}_R(R/\Phi_n^\infty, R/\Phi_n^\infty).$$

We know the K-theory Thom isomorphism gives multiplication by Φ_n. We may express the action of c_H in its Φ_n-adic expansion as multiplication by $x_0 + x_1\Phi_n + x_2\Phi_n^2 + \cdots$, where $x_i \in R$. It suffices to prove that the $\mathcal{F}$-spectrum Euler class is multiplication by $\lambda\Phi_n$ mod Φ_n^2 for a non-zero scalar λ, i.e. that $x_0 = 0$ and $x_1 = \lambda$.

First, we know that the map $\sigma^0_H \longrightarrow \sigma^0_H \wedge K$ induces the permutation module map $\mathbb{Q} = e_H A(H) \longrightarrow e_H R(H) = R/\Phi_n$. Now, we need to understand something of the map $E\langle H\rangle^{(2n)} \longrightarrow K \wedge E\langle H\rangle^{(2n)}$, and we can infer enough by considering the diagram

$$\begin{array}{ccccc}
0 & & \mathbb{Q} & & \mathbb{Q} \\
\| & & \| & & \| \\
\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k-2)}) & \longrightarrow & \pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k)}) & \xrightarrow{\cong} & \pi^{\mathbb{T}}_{2k+1}(\sigma^{2k}_H) \\
\downarrow & & \downarrow & & \downarrow \\
\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k-2)} \wedge K) & \longrightarrow & \pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k)} \wedge K) & \longrightarrow & \pi^{\mathbb{T}}_{2k+1}(\sigma^{2k}_H \wedge K) \\
\| & & \| & & \| \\
R/\Phi_n^k & & R/\Phi_n^{k+1} & & R/\Phi_n.
\end{array}$$

This shows that the image g_k of the generator of $\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k)})$ in $\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k)}) = R/\Phi_n^{k+1}$ is λ_k modulo Φ_n^k for a non-zero rational number λ_k.

Now consider the diagram

$$\begin{array}{ccc}
\mathbb{Q} & & \mathbb{Q} \\
\| & & \| \\
\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle^{(2k)}) & \xrightarrow{\cong} & \pi^{\mathbb{T}}_{2k+1}(\Sigma^{kV}E\langle H\rangle^{(0)}) \\
\cong\downarrow & & \downarrow\cong \\
\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle) & \xrightarrow{c_H^k} & \pi^{\mathbb{T}}_{2k+1}(\Sigma^{kV}E\langle H\rangle) \\
\downarrow & & \downarrow \\
\pi^{\mathbb{T}}_{2k+1}(E\langle H\rangle\wedge K) & \longrightarrow & \pi^{\mathbb{T}}_{2k+1}(\Sigma^{kV}E\langle H\rangle\wedge K) \\
\| & & \| \\
R/\Phi_n^\infty & & R/\Phi_n^\infty.
\end{array}$$

We see that c_H^k takes the image of $g_k \in R/\Phi_n^{k+1}$ to the image of $1 \in R/\Phi_n$. Now g_k is mapped to λ_k/Φ_n^{k+1} modulo elements annihilated by Φ_n^k. We conclude from the case $k=1$ that $x_1 = \lambda_1$ as required. The general case shows that $\lambda_k = \lambda_1^k$. □

Corollary 15.4.3. The $\mathbb{Q}[c_H]$-module $\pi^{\mathbb{T}}_*(K(H))$ is injective. Indeed, if L_n^i is the space of Laurent polynomials in z with poles of order at most i at a primitive nth root of unity, then multiplication by Φ_n gives an isomorphism $L_n^{i+1}/L_n^i \longrightarrow L_n^i/L_n^{i-1}$. Thus $\pi^{\mathbb{T}}_*(K(H)) \cong I(H)\otimes(L_n^1/L_n^0)[\beta,\beta^{-1}]$, and hence

$$K(H) \simeq E\langle H\rangle \wedge S^0[\,(L_n^1/L_n^0)[\beta,\beta^{-1}]\,]. \quad \square$$

Accordingly K is characterized by $t_*^{\mathcal{F}}\otimes\pi^{\mathbb{T}}_*(K\wedge\tilde{E}\mathcal{F})$, $\pi^{\mathbb{T}}_*(K\wedge\Sigma E\mathcal{F}_+)$ and the homomorphism between them. To make sense of the following statement, note that $(F/R)[\beta,\beta^{-1}]$ is a module over $\mathcal{O}_{\mathcal{F}}$, since it is $\mathcal{F}$-finite; more explicitly, if $x \in R[1/\Phi_n]/R$ and H is of order n, then $c_H x = \Phi_n x\beta$ as mentioned above. Of course, multiplication by Φ_n^{-1} is not defined on F/R, but it makes sense for F.

Theorem 15.4.4. The $\mathbb{T}$-spectrum K is the unique $\mathbb{T}$-spectrum for which $t_*^{\mathcal{F}}\otimes\pi^{\mathbb{T}}_*(K\wedge\tilde{E}\mathcal{F}) \longrightarrow \pi^{\mathbb{T}}_*(K\wedge\Sigma E\mathcal{F}_+)$ is the map

$$\hat{q}_K : t_*^{\mathcal{F}}\otimes F[\beta,\beta^{-1}] \longrightarrow (F/R)[\beta,\beta^{-1}]$$

described as follows. For $x\in\mathcal{O}_{\mathcal{F}}$ of degree $-2k$ we have $\hat{q}_K(x\otimes f\beta^l) = x\overline{f}\beta^{l+k}$, and $\hat{q}_K(c_H^{-k}\otimes f\beta^l) = \overline{\Phi_n^{-k}f}\beta^{l-k}$.

Proof: The value of $\hat{q}_K(x\otimes f\beta^l)$ is immediate from our method of calculating $\pi^{\mathbb{T}}_*(K\wedge\Sigma E\mathcal{F}_+)$.

For $\hat{q}_K(c_H^{-k}\otimes f\beta^l)$ we apply 6.1.2, using the compatibility statement in 15.4.1 to relate it to our present naming of elements. Consider the diagram

$$\begin{array}{ccccccc}
K & \xleftarrow{1\wedge e} & K\wedge S^{-kV(H)} & \xrightarrow{\simeq} & K\wedge S^{-2k} & \xrightarrow{\simeq} & K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K\wedge\tilde{E}\mathcal{F} & \xleftarrow{\simeq} & K\wedge\tilde{E}\mathcal{F}\wedge S^{-kV(H)} & \xrightarrow{\simeq} & K\wedge\tilde{E}\mathcal{F}\wedge S^{-2k} & \xrightarrow{\simeq} & K\wedge\tilde{E}\mathcal{F} \\
\downarrow & & r\downarrow & & \downarrow & & \downarrow \\
K\wedge\Sigma E\mathcal{F}_+ & \longleftarrow & K\wedge\Sigma E\mathcal{F}_+\wedge S^{-kV(H)} & \xrightarrow{\simeq} & K\wedge\Sigma E\mathcal{F}_+\wedge S^{-2k} & \xrightarrow{\simeq} & K\wedge\Sigma E\mathcal{F}_+.
\end{array}$$

The first horizontal is induced by $e : S^{-kV(H)} \longrightarrow S^0$, the second is the K-theory Thom class, and the third is multiplication by the integer Bott class. By 6.1.2 the relevant map is induced by r.

Applying $\pi_*^{\mathbb{T}}$ we see by definition that the composite from right to left in the first row is multiplication by $\lambda(kV(H))$, and hence this is also true in the second row. Since the bottom right hand vertical induces projection, we identify the second vertical r in the lower ladder as stated. □

It is tempting to rewrite the description of $\hat{q}_K$ above as $\hat{q}_K(x \otimes f\beta^l) = \overline{xf}\beta^{l+k}$ for all $x \in t_*^{\mathcal{F}}$, but this makes no sense, since F is not an $\mathcal{O}_{\mathcal{F}}$-module. This suggests we should perform some algebraic completion to F. This can be achieved geometrically by replacing K with $F(E\mathcal{F}_+, K)$. This is a reasonable thing to do since the fibre of the completion map is the $\mathcal{F}$-contractible spectrum $F(\tilde{E}\mathcal{F}, K)$, which is determined by its homotopy groups.

15.5. The geometric equivariant rational Segal conjecture for $\mathbb{T}$.

In this section we aim to analyse the functional dual $DE\mathbb{T}_+ = F(E\mathbb{T}_+, S^0)$; the title is something of a misnomer since neither Segal nor anyone else has made a conjecture about $DE\mathbb{T}_+$. It completes the description given in [9, 12] of the integral functional dual. It would be possible to give an entirely algebraic treatment since we have a model for function spectra, but we shall first present a more direct treatment so as to avoid relying on the more complicated bits of algebra.

Since $E\mathbb{T}_+ = E\langle 1\rangle$, we should really discuss the more general question of identifying $DE\langle H\rangle$. Indeed, we should also consider $DE\mathcal{F}_+ \simeq D(\bigvee_H E\langle H\rangle) \simeq \prod_H DE\langle H\rangle$. Since the initial stages of the analysis are easier to understand for $E\mathcal{F}_+$, we begin with that.

We already know $E\mathcal{F}_+ \wedge DE\mathcal{F}_+ \simeq E\mathcal{F}_+$, and that $\pi_*^{\mathbb{T}}(DE\mathcal{F}_+) = t_*^{\mathcal{F}}$; accordingly we have the cofibre sequence

$$DE\mathcal{F}_+ \longrightarrow \tilde{E}\mathcal{F} \wedge S^0[t_*^{\mathcal{F}}] \stackrel{a}{\longrightarrow} \Sigma E\mathcal{F}_+,$$

where a induces projection onto the positive dimensional part. In the algebraic model $DE\mathcal{F}_+$ is thus described by the element $\hat{a} \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(t_*^{\mathcal{F}} \otimes t_*^{\mathcal{F}}, \Sigma^2\mathbb{I})$ given by smashing a with $DE\mathcal{F}_+$ and looking in $\pi_*^{\mathbb{T}}$. For definiteness we emphasize that the second copy of $t_*^{\mathcal{F}}$ is the new one.

LEMMA 15.5.1. The map $\hat{a} : t_*^{\mathcal{F}} \otimes t_*^{\mathcal{F}} \longrightarrow \Sigma^2\mathbb{I}$ is given by $\hat{a}(x \otimes y) = a_*(xy)$.

Note that $\hat{a}$ realizes Tate duality between negative and positive parts of $t_*^{\mathcal{F}}$.

Proof: Given an $\mathcal{O}_{\mathcal{F}}$-map $\theta : t_*^{\mathcal{F}} \longrightarrow M$, so that $\theta(1) = m$ the value of θ on $\mathcal{O}_{\mathcal{F}} \subseteq t_*^{\mathcal{F}}$ follows, and if the components of m are uniquely divisible by the relevant c_H then θ is determined. The result follows provided x is of positive degree. The real content of the lemma is that the formula is valid also when $a_*(x) = 0$.

To begin with we note how a nontrivial map $\tilde{E}\mathcal{F} \longrightarrow \Sigma^{2k+1}E\mathcal{F}_+$ is detected. As motivation, we observe that the obvious example is the quotient of $\tilde{E}\mathcal{F} = S^{\infty V(\mathcal{F})}$ by $S^{kV(\mathcal{F})}$. If it were possible to simply desuspend by smashing with a putative spectrum $S^{(-k-1)V(\mathcal{F})}$, then the map would be detected in homotopy. Since the spectrum $S^{V(\mathcal{F})}$ is not invertible we must be slightly less direct by concentrating on a single subgroup H at a time, and using

$\sigma^{kV(H)}$ instead of $S^{kV(\mathcal{F})}$. This part of the analysis is given in the proof of 15.5.2 below. □

The following identifies $DE\langle H\rangle$, and the special case $H = 1$ gives $DE\mathbb{T}_+$.

PROPOSITION 15.5.2. There is a cofibre sequence

$$DE\langle H\rangle \longrightarrow \tilde{E}\mathcal{F} \wedge S^0[\,\mathbb{Q}[c_H, c_H^{-1}]\,] \stackrel{b}{\longrightarrow} \Sigma E\langle H\rangle$$

and the extension is determined by the fact that

$$\hat{b} \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(t_*^{\mathcal{F}} \otimes \mathbb{Q}[c_H, c_H^{-1}], \Sigma^2\mathbb{I}(H)) = \mathrm{Hom}_{\mathbb{Q}[c_H]}(\mathbb{Q}[c_H, c_H^{-1}] \otimes \mathbb{Q}[c_H, c_H^{-1}], \Sigma^2\mathbb{I}(H))$$

is given by $\hat{b}(x \otimes y) = b_*(xy)$, which represents a perfect duality of $\mathbb{Q}[c_H, c_H^{-1}]$

Proof: We smash the standard cofibre sequence $E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}$ with $DE\langle H\rangle$; the terms are identified with those in the statement by the following lemma.

LEMMA 15.5.3. (i)There is an equivalence

$$E\mathcal{F}_+ \wedge DE\langle H\rangle \simeq E\langle H\rangle.$$

(ii)There is an isomorphism

$$\pi_*^{\mathbb{T}}(\Phi^{\mathbb{T}} DE\langle H\rangle) \cong \mathbb{Q}[c_H, c_H^{-1}].$$

Remark: Part (i) of 15.5.3 with $H = 1$ corrects statements in the rational analysis of [9]. More precisely, the space $E\mathcal{F}_+$ should be replaced by EG_+ in 1.6, Theorem B, 4.8, and on the right hand side of 4.5 and page 359 line -3. The correction is discussed in more detail in [12].

Proof of 15.5.3: (i) The equivalence $E\mathcal{F}_+ \wedge DE\langle H\rangle \simeq E\langle H\rangle$ follows since $E\mathcal{F}_+ \wedge DE\langle H\rangle$ is a retract of $E\mathcal{F}_+ \wedge DE\mathcal{F}_+ \simeq E\mathcal{F}_+$. Indeed $E\mathcal{F}_+ \simeq \bigvee_H E\langle H\rangle$, and the idempotent for all subgroups $K \neq H$ annihilates $DE\langle H\rangle$. Hence $E\mathcal{F}_+ \wedge DE\langle H\rangle$ is a retract of $E\langle H\rangle$, and by homotopy groups it is an equivalence.
(ii) The identification of $\Phi^{\mathbb{T}} DE\langle H\rangle$ is immediate from 2.4.1. □

It remains to show that $\hat{b}(x\otimes y) = b_*(xy)$; this follows when x is of positive degree exactly as in 15.5.1. Now suppose $x = c_H^{-k}$ for $k \geq 0$; the verification that $\hat{b}(c_H^{-k} \otimes y) = b_*(c_H^{-k} y)$ in this case will also complete the proof of 15.5.1.

We take the cofibre sequence in the statement and smash it with $\sigma^{nV(H)}$. Since $\sigma^{nV(H)}$ is formed from S^0 and various basic cells with isotropy H, we have $\sigma^{nV(H)} \wedge \tilde{E}\mathcal{F} \simeq \tilde{E}\mathcal{F}$; by the Thom isomorphism $E\langle H\rangle \wedge \sigma^{nV(H)} \simeq E\langle H\rangle \wedge S^{2n}$, and also

$$\sigma^{nV(H)} \wedge DE\langle H\rangle \simeq D(E\langle H\rangle \wedge \sigma^{-nV(H)}) \simeq D(E\langle H\rangle \wedge S^{-2n}).$$

Thus the cofibre sequence becomes

$$S^{2n} \wedge DE\langle H\rangle \longrightarrow \tilde{E}\mathcal{F} \wedge S^0[\,\mathbb{Q}[c_H, c_H^{-1}]\,] \stackrel{1\wedge b}{\longrightarrow} \Sigma^{2n+1} E\langle H\rangle.$$

Because the final term is $2n$-connected and the first has $\pi_*^{\mathbb{T}}$ only in degrees $\leq 2n$ it follows that $\pi_*^{\mathbb{T}}(1 \wedge b)$ is surjective for all n. Taking $n \leq -k-1$ establishes the required formula for $\hat{b}$. □

Now let us turn to the general question of what can be said about $DX = F(X, S^0)$ if we already understand X in the standard model. Thus we suppose that X has model $B = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$, and we let $Q \longrightarrow t_*^{\mathcal{F}} \otimes H$ denote the torsion model of DX.

First we recall that S^0 has model $(\mathcal{O}_{\mathcal{F}} \longrightarrow t_*^{\mathcal{F}})$, this has torsion part $\Sigma\mathbb{I}$, and the natural fibrant model is as the fibre of the map $e(\mathbb{Q}) \longrightarrow f(\Sigma^2\mathbb{I})$. We have seen in Section 9.3 that if S is the torsion part of X, the torsion part of DX is

$$R\Gamma_f \ \mathrm{Hom}(S, \Sigma\mathbb{I}).$$

The vertex is described by the fibre sequence

$$H \longrightarrow V^* \longrightarrow \mathcal{E}^{-1}\mathrm{Hom}(t_*^{\mathcal{F}} \otimes V, \Sigma^2\mathbb{I}).$$

However the real aim here is to give a description of the complete model. We show that the obvious cofibre sequence $DX \longrightarrow DX \wedge \tilde{E}\mathcal{F} \longrightarrow DX \wedge \Sigma E\mathcal{F}_+$ is also natural from the algebraic point of view.

LEMMA 15.5.4. If $B = (N \longrightarrow t_*^{\mathcal{F}} \otimes V)$, then the functional dual of B is described by

$$R\hat{\Gamma} \ R\mathbf{Hom}(B, S^0) = fibre\left(e(V^*) \longrightarrow R\hat{\Gamma} \ f(\mathrm{Hom}(N, \Sigma^2\mathbb{I}))\right).$$

Proof: We use the injective resolution $S^0 \longrightarrow e(\mathbb{Q}) \longrightarrow f(\Sigma^2\mathbb{I})$, and deduce that

$$R\mathbf{Hom}(B, S^0) = fibre\left(\mathbf{Hom}(B, e(\mathbb{Q})) \longrightarrow \mathbf{Hom}(B, \Sigma^2 f(\mathbb{I}))\right).$$

Now $\mathbf{Hom}(B, e(\mathbb{Q}))$ is the injective $e(V^*)$, whilst $\mathbf{Hom}(B, \Sigma^2 f(\mathbb{I})) = f(\mathrm{Hom}(N, \Sigma^2\mathbb{I}))$. Applying $\hat{\Gamma}$ we obtain the result. □

We can say a little more about the term $R\hat{\Gamma} \ f(\mathrm{Hom}(N, \Sigma^2\mathbb{I}))$ in the description For an arbitrary $\mathcal{O}_{\mathcal{F}}$-module M we proved in Example 8.5.8 that there is a fibre sequence $R\hat{\Gamma} f(M) \longrightarrow e(\mathcal{E}^{-1}M) \longrightarrow f(\mathcal{E}^{-1}M/M)$.

CHAPTER 16

Cyclic and Tate cohomology.

This is a short chapter, but fits naturally between its neighbours. The first section identifies rational cyclic cohomology, the second gives an algebraic description of the Tate construction on rational spectra, and the third gives a description of the Tate spectrum of integral complex equivariant K-theory.

16.1. Cyclic cohomology.

In this section we consider periodic cyclic cohomology. We begin by observing that the rationalisation of the integral cyclic cohomology is the cyclic cohomology of the rationals, so that the two possible interpretations coincide.

It was proved in [14] that the representing spectrum for cyclic cohomology with coefficients in an abelian group A is the Tate spectrum $t(HA) = F(E\mathbb{T}_+, HA) \wedge \tilde{E}\mathbb{T}$. The following lemma is special to bounded cohomology theories.

LEMMA 16.1.1. For any Mackey functor A the rationalization of $t(HA)$ is $t(H(A \otimes \mathbb{Q}))$.

Proof: The essential point is that $F(E\mathbb{T}_+, HA) = \operatorname{holim}_{\leftarrow n} F(E\mathbb{T}_+^{(n)}, HA)$, and that the maps induced in $[X, \cdot]^{\mathbb{T}}$ by those of the inverse system are ultimately isomorphisms for each finite X. This inverse limit therefore commutes with direct limit under degree zero selfmaps of HA.

Thus

$$\begin{aligned} t(HA) \wedge S^0\mathbb{Q} &= \operatorname{holim}_{\rightarrow m} (\ F(E\mathbb{T}_+, HA) \wedge \tilde{E}\mathbb{T}, m!\ ) \\ &\simeq F(E\mathbb{T}_+, \operatorname{holim}_{\rightarrow m} (HA, m!)) \wedge \tilde{E}\mathbb{T} \\ &\simeq F(E\mathbb{T}_+, HA \otimes \mathbb{Q}) \wedge \tilde{E}\mathbb{T} = t(H(A \otimes \mathbb{Q})). \quad \square \end{aligned}$$

Henceforth we suppose A is rational.

LEMMA 16.1.2. Provided A is rational, $t(HA)$ is $\mathcal{F}$-contractible. We therefore have an equivalence

$$t(HA) \simeq \tilde{E}\mathcal{F} \wedge S^0[\pi_*^{\mathbb{T}}(t(HA))].$$

Proof: Indeed $t(HA)|_H = t(HA|_H)$, and the rational Tate cohomology of any finite group is 0. □

For any abelian group A we have $\pi^{\mathbb{T}}_*(t(HA)) = \hat{H}C_* \otimes A = A[c, c^{-1}]$.

16.2. Rational Tate spectra.

In this section we discuss the Tate construction of [14], which generalizes the periodic cyclic cohomology discussed in the previous section. Recall that the Tate construction on a $\mathbb{T}$-spectrum is defined by $t(X) = F(E\mathbb{T}_+, X) \wedge \tilde{E}\mathbb{T}$. This simplifies considerably in the rational case, and it seems worth giving a complete description of the Tate construction in the category of rational $\mathbb{T}$-spectra.

We begin with the warning that if X is integral, the map $t(X) \longrightarrow t(X \wedge S^0\mathbb{Q})$ need not be a rational equivalence, so that Lemma 16.1.1 above is special to suitably bounded theories like HA. An example is given by complex K-theory, since $t(K\mathbb{Z})|_H$ is non-trivial and rational for all non-trivial finite subgroups H [10, 14, 15]; the following lemma shows this is false for $t(K\mathbb{Q})$.

We revert to our global assumption that all spectra are rational.

LEMMA 16.2.1. The natural map

$$t(X) = F(E\mathbb{T}_+, X) \wedge \tilde{E}\mathbb{T} \longrightarrow F(E\mathbb{T}_+, X) \wedge \tilde{E}\mathcal{F}$$

is an equivariant equivalence. Thus $t(X)$ is an $\mathcal{F}$-contractible spectrum determined by its homotopy groups.

Proof: We give two proofs. Firstly, the Tate construction commutes with restriction: $t(X)|_H = t(X|_H)$. The lemma follows from the fact that the Tate construction is trivial on rational spectra for finite groups. One way of seeing this is to use the fact that if H is finite and $e \in A(H)$ is the idempotent with support 1 then $EH_+ = eS^0$ and $\tilde{E}H = (1-e)S^0$.

For the second proof, we compare the cofibre sequence $E\mathbb{T}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathbb{T}$ with $E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}$. We see that the lemma is equivalent to showing that the natural map $f : F(E\mathbb{T}_+, X) \wedge E\mathbb{T}_+ \longrightarrow F(E\mathbb{T}_+, X) \wedge E\mathcal{F}_+$ is an equivalence. However, the cofibre of f is a wedge of terms $F(E\mathbb{T}_+, X) \wedge E\langle H \rangle$ with $H \neq 1$; this is contractible, as one sees from the fact that $F(E\mathbb{T}_+, X) \wedge \sigma^0_H \simeq *$ by using cofibre sequences and passing to direct limits. □

PROPOSITION 16.2.2. If X is a rational $\mathbb{T}$-spectrum with associated module $M = \pi^{\mathbb{T}}_*(X \wedge E\mathbb{T}_+)$ over $\mathbb{Q}[c_1]$, then $t(X)$ is the $\mathcal{F}$-contractible spectrum with homotopy groups

$$\hat{H}^0_{(c_1)}(M) \oplus \Sigma \hat{H}^{-1}_{(c_1)}(M)$$

where $\hat{H}^*_{(c_1)}$ denotes local Tate cohomology in the sense of [10].

REMARK 16.2.3. There are two methods for calculating the local Tate cohomology; since (c_1) is principal both are very simple. The second description simplifies further because M is torsion. In fact, the local Tate cohomology is only non-zero in codegrees 0 and -1, and for these cases we have

$$\hat{H}^{-i}_{(c_1)}(M) = (L^{(c_1)}_i M)[1/c_1] = \lim_{\leftarrow}{}^i(M, c_1),$$

where $L^{(c_1)}_*$ denotes the left derived functors of completion at (c_1). Note that this is only likely to be equal to (c_1)-adic completion when M is finitely generated. However, since M is torsion, when it is finitely generated it is already complete; the Tate cohomology therefore vanishes, as we know it must for geometric reasons.

Proof: Observe $F(E\mathbb{T}_+, X) \simeq F(E\mathbb{T}_+, X \wedge E\mathbb{T}_+)$, so that if $M = \pi^{\mathbb{T}}_*(X \wedge E\mathbb{T}_+)$, there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}(\Sigma^2\mathbb{I}, M) \longrightarrow [E\mathbb{T}_+, X]^{\mathbb{T}}_* \longrightarrow \mathrm{Hom}(\Sigma\mathbb{I}, M) \longrightarrow 0.$$

This is precisely parallel to the algebraic situation. We may split X into even and odd parts, and thus the exact sequence splits. Therefore $F(E\mathbb{T}_+, X)$ is modelled by the complex $\mathrm{Hom}(PK(c_1), M)$ where $PK(c_1)$ is a complex of projectives approximating the stable Koszul complex $\mathbb{Q}[c_1] \longrightarrow \mathbb{Q}[c_1, c_1^{-1}]$; the homology of this complex calculates the left derived functors of c_1-completion [13]. In particular, when X is even, $[E\mathbb{T}_+, X]^{\mathbb{T}}_*$ is $L^{(c_1)}_0 M$ in even degrees and $\Sigma L^{(c_1)}_1 M$ in odd degrees.

Now conclude that there is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(\Sigma^2\mathbb{I}, M)[1/c] \longrightarrow t(X)^{\mathbb{T}}_* \longrightarrow \mathrm{Hom}(\Sigma\mathbb{I}, M)[1/c] \longrightarrow 0.$$

Therefore, if $TT_{(c_1)}(M)$ is the complex of the second avatar in the notation of [10], $t(X)$ is modelled by the corresponding torsion free model, $e(TT_{(c_1)}(M))$. Thus, if X is even, $t(X)^{\mathbb{T}}_*$ is $\hat{H}^0_{(c_1)}(M)$ in even degrees and $\Sigma\hat{H}^{-1}_{(c_1)}(M)$ in odd degrees. $\square$

16.3. The integral $\mathbb{T}$-equivariant Tate spectrum for complex K-theory.

In this section we apply the general theory to identify the Tate spectrum of complex equivariant K-theory $K\mathbb{Z}$ *integrally.* However, we note that $t(K\mathbb{Z})$ is not rational, and its rationalization is not $t(K\mathbb{Q})$, so this is not an application of the previous section.

Before we state the theorem, recall that the representation ring $R(\mathbb{T}) = \mathbb{Z}[z, z^{-1}]$, and that the Euler class of the representation z^n is $1-z^n$. In particular, we let $\chi = 1-z$ and find $R(\mathbb{T})^{\wedge}_{(\chi)} = \mathbb{Z}[[\chi]]$; indeed, $z = 1-\chi$ is invertible in $\mathbb{Z}[\chi]/(\chi^n)$, so that $\mathbb{Z}[\chi] \longrightarrow \mathbb{Z}[\chi, z^{-1}] = \mathbb{Z}[z, z^{-1}]$ induces an isomorphism of (χ)-completions. We write $\mathbb{Z}((\chi))$ for the localization $\mathbb{Z}[[\chi]][\chi^{-1}]$, and S for the mulitiplicative set generated by the Euler classes. Note that if $n \geq 2$ the Euler class $1-z^n$ is a multiple of χ. However, although the multiplier is a unit in $\mathbb{Q}((\chi))$, it is not a unit in $\mathbb{Z}((\chi)) \otimes \mathbb{Q}$.

THEOREM 16.3.1. The Tate spectrum $t(K\mathbb{Z})$ is $\mathcal{F}$-equivalent to a rational spectrum, and is thus determined by the homotopy type of $\Phi^{\mathbb{T}}K\mathbb{Z}$ and its rational type. There is an equivalence of $K\mathbb{Z}$-module spectra

$$\Phi^{\mathbb{T}}K\mathbb{Z} \simeq KS^{-1}\mathbb{Z}((\chi)).$$

The rational spectrum $t(K\mathbb{Z}) \wedge S^0\mathbb{Q}$ is classified in the torsion model by

$$t_*^{\mathcal{F}} \otimes S^{-1}\mathbb{Z}((\chi))[\beta, \beta^{-1}] \longrightarrow S^{-1}\mathbb{Z}((\chi))/\mathbb{Z}((\chi))[\beta, \beta^{-1}] = \bigoplus_{H \neq 1} \mathbb{Z}((\chi))/\Phi_{|H|}^{\infty}[\beta, \beta^{-1}],$$

where Φ_n is the nth cyclotomic polynomial, and the structure map is described as in 15.4.4.

Proof: First note that $t(K\mathbb{Z}) = F(E\mathbb{T}_+, K\mathbb{Z}) \wedge \tilde{E}\mathbb{T}$, so that we may calculate its coefficient ring from the Atiyah-Segal completion theorem. First, equivariant K-theory has coefficients $R(\mathbb{T})[\beta, \beta^{-1}]$, with $R(\mathbb{T}) = \mathbb{Z}[z, z^{-1}]$, and the K-theory Euler class of z^n is $1 - z^n$. By the Atiyah-Segal completion theorem, $\pi_*^{\mathbb{T}}(F(E\mathbb{T}_+, K\mathbb{Z})) = R(\mathbb{T})^{\wedge}_{(\chi)}[\beta, \beta^{-1}]$.

Consider the cofibre sequence

$$t(K\mathbb{Z}) \longrightarrow t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F} \longrightarrow t(K\mathbb{Z}) \wedge \Sigma E\mathcal{F}_+.$$

We shall identify $t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F}$, $t(K\mathbb{Z}) \wedge \Sigma E\mathcal{F}_+$, and the map between them in turn.

Firstly, since χ is an Euler class,

$$\pi_*^{\mathbb{T}}(t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F}) = S^{-1}\mathbb{Z}((\chi))[\beta, \beta^{-1}],$$

where S is the multiplicative set generated by $1 - z^n$ for $n \geq 1$. Now $S^{-1}\mathbb{Z}((\chi))$ is flat over $\mathbb{Z}$, and hence the coefficients of $\Phi^{\mathbb{T}}t(K\mathbb{Z})$ are the same as those of K-theory with coefficients in $S^{-1}\mathbb{Z}((\chi))$.

LEMMA 16.3.2. There is an equivalence

$$\Phi^{\mathbb{T}}K\mathbb{Z} \simeq KS^{-1}\mathbb{Z}((\chi))$$

of non-equivariant $K\mathbb{Z}$-module spectra.

Proof: First note that $\Phi^{\mathbb{T}}t(K\mathbb{Z})$ is a module over $K\mathbb{Z}$. Now let $MS^{-1}\mathbb{Z}((\chi))$ be a non-equivariant Moore spectrum, and construct a map $f : MS^{-1}\mathbb{Z}((\chi)) \longrightarrow t(K\mathbb{Z})$ inducing an isomorphism in $\pi_0^{\mathbb{T}}$. Now form the composite

$$KS^{-1}\mathbb{Z}((\chi)) = K \wedge MS^{-1}\mathbb{Z}((\chi)) \longrightarrow K \wedge \Phi^{\mathbb{T}}t(K\mathbb{Z}) \longrightarrow \Phi^{\mathbb{T}}t(K\mathbb{Z})$$

in which the first map is obtained from f by applying $K \wedge \Phi^{\mathbb{T}}(\cdot)$ to f, and the second uses the module structure. By construction this induces an isomorphism in homotopy, and is therefore an equivalence. $\square$

Next, we claim that $t(K\mathbb{Z}) \wedge E\mathcal{F}_+$ is rational. This is immediate from the fact that, $t(K\mathbb{Z})|_H$ is rational for all finite subgroups H [10, 14, 15]. Indeed, we know that $t(K\mathbb{Z}) \wedge \mathbb{T}/H_+$ is induced from $t(K\mathbb{Z})|_H$. Rationally $E\mathcal{F}_+ \simeq \bigvee_H E\langle H \rangle$, so $t(K\mathbb{Z}) \wedge E\mathcal{F}_+$ has a corresponding splitting. The summand for $E\langle 1 \rangle = E\mathbb{T}_+$ is trivial, so we may choose $H \neq 1$ and consider $t(K\mathbb{Z}) \wedge \Sigma E\langle H \rangle$. From the identification of c_H, we find it has homotopy groups $\mathbb{Z}((\chi))/\Phi_n^{\infty}$ in each even degree, and in particular it is injective.

Finally, the map $t(K\mathbb{Z})\tilde{E}\mathcal{F} \longrightarrow t(K\mathbb{Z}) \wedge \Sigma E\mathcal{F}_+$ factors through the rationalization $t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F} \longrightarrow t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F} \wedge S^0\mathbb{Q}$, and the resulting map $t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F} \wedge S^0\mathbb{Q} \longrightarrow t(K\mathbb{Z}) \wedge \Sigma E\mathcal{F}_+$ is classified by its d-invariant since the codomain is injective. The map from $t^{\mathcal{F}}_* \otimes S^{-1}\mathbb{Z}((\chi))[\beta, \beta^{-1}]$ is the analogue of that in Theorem 15.4.4. $\square$

CHAPTER 17

Cyclotomic spectra and topological cyclic cohomology.

In this chapter, we study various $\mathbb{T}$-spectra arising from algebraic K-theory. Various constructions are used to define suitable targets for trace maps from algebraic K-theory, and the most sophisticated takes Bökstedt's Topological Hochschild homology of a ring, and forms the associated topological cyclic spectrum in the sense of Bökstedt-Hsiang-Madsen [2]. Madsen has recently given a very helpful general survey [20].

The topological cyclic construction can be applied to any $\mathbb{T}$-spectrum with appropriate extra structure, and we begin in Section 17.1 by identifying the extra structure involved in specifying such a 'cyclotomic' spectrum. In the following section, we illustrate this by considering the basic examples: free loop spaces on a $\mathbb{T}$-fixed space, and topological Hochschild homology of a functor with smash products. Finally, in Section 17.3 we analyse the topological cyclic construction on rational cyclotomic spectra.

17.1. Cyclotomic spectra.

We must begin by recalling the definition of a cyclotomic spectrum. The basic idea is that it is a spectrum X with the property analogous to that of the free loop space ΛZ, on a $\mathbb{T}$-fixed based space Z, namely that for any finite subgroup K the fixed point set $(\Lambda Z)^K$ is equivalent to the original space ΛZ. The analogue should be that any fixed point spectrum $\Phi^K X$ is equivalent to X again. Of course $\Phi^K X$ is really a $\mathbb{T}/K$-spectrum, so we must begin by explaining exactly how we interpret it as a $\mathbb{T}$-spectrum indexed on the original universe. In addition, we want to avoid redundant structure, so we simply require that the resulting equivalences are transitive.

We wish to consider the group $\mathbb{T}$ and all its quotients $\overline{\mathbb{T}} = \mathbb{T}/K$ by finite subgroups K. We want transitive systems of structure, so we first let $\rho = \rho_K : \mathbb{T} \xrightarrow{\cong} \overline{\mathbb{T}}$ be the isomorphism given by taking the $|K|$th root. If we index our $\mathbb{T}$-spectra on a complete universe $\mathcal{U}$, we index our $\overline{\mathbb{T}}$-spectra on the complete universe $\mathcal{U}^K$. However we want these universes to be comparable, so we say that a complete $\mathbb{T}$-universe $\mathcal{U}$ is *cyclotomic* if it is provided with isomorphisms $\mathcal{U} \xrightarrow{\cong} \rho_K^* \mathcal{U}^K$. Identifying $\mathbb{T}$ and $\overline{\mathbb{T}}$ via ρ_K, this also specifies isomorphisms $\mathcal{U}^L \longrightarrow \rho_{K/L}^* (\mathcal{U}^L)^{K/L}$. We require that these are transitive in the sense that if $L \subseteq K$ then the composite

$$\mathcal{U} \longrightarrow \rho_L^* \mathcal{U}^L \longrightarrow \rho_L^* (\rho_{K/L}^* \mathcal{U}^L)^{K/L} = \rho_K^* \mathcal{U}^K$$

is the isomorphism for K. One such cyclotomic universe is the direct sum $\mathcal{U} = \oplus_{n\in\mathbb{Z}}\mathcal{U}_n$ where $\mathcal{U}_n = \bigoplus_{i\in\mathbb{N}} z^n$ with the isomorphisms suggested by the indexing.

Suppose then that X is a $\mathbb{T}$-spectrum indexed on the cyclotomic universe $\mathcal{U}$. Thus, for any finite subgroup K, $\Phi^K X$ is a $\overline{\mathbb{T}}$-spectrum indexed on $\mathcal{U}^K$; by pullback along the isomorphism ρ_K we obtain a $\mathbb{T}$-spectrum $\rho_K^*\Phi^K X$ indexed on $\rho_K^*\mathcal{U}^K$, which may be viewed as a $\mathbb{T}$-spectrum $\rho_K^!\Phi^K X$ indexed on $\mathcal{U}$ by using the cyclotomic structure of the universe. A *cyclotomic* structure on X consists of a transitive system of $\mathbb{T}$-equivalences

$$r_K : \rho_K^!\Phi^K X \xrightarrow{\simeq} X.$$

By transitivity, the essential pieces of the structure come from the cases that K is of prime order.

Although this structure is really designed to capture profinite information, there is enough residue rationally to make it worthwhile identifying the cyclotomic objects in the algebraic model of rational $\mathbb{T}$-spectra. The essential idea is that in a cyclotomic spectrum all finite subgroups behave in an analogous way, differing only in the multiplicity with which information occurs. There is no significant constraint on total fixed points $\Phi^{\mathbb{T}}X$. The first step of our analysis was to split $\mathcal{F}$-spectra into the parts over different subgroups, so it is easy to describe the cyclotomic structure in these terms. A spectrum X is cyclotomic if we have specified equivalences $X(C_1) \simeq X(C_2) \simeq X(C_3) \simeq \ldots$. This uniformity itself imposes constraints on the assembly map of a $\mathbb{T}$-spectrum.

Let us now describe the algebraic model for cyclotomic spectra more precisely. It is useful to bear in mind the torsion model rather than the standard model.

Definition 17.1.1. The ring of cyclotomic operations is the polynomial ring $\mathbb{Q}[c_0]$ on a single generator c_0 of degree -2. The standard injective $\mathbb{I}_0$ is defined by the exact sequence $0 \longrightarrow \mathbb{Q}[c_0] \longrightarrow \mathbb{Q}[c_0, c_0^{-1}] \longrightarrow \Sigma^2\mathbb{I}_0 \longrightarrow 0$. The cyclotomic torsion category $\mathcal{C}_t$ has objects $(\Sigma^2\mathbb{I}_0 \otimes V \longrightarrow T_0)$ where V is a graded vector space and T_0 is a torsion $\mathbb{Q}[c_0]$-module. The morphisms are given by commutative squares as usual. $\square$

Lemma 17.1.2. *The category $\mathcal{C}_t$ is abelian and 2 dimensional. Hence we may form the derived category $D\mathcal{C}_t$.*

Proof: The proof is precisely analogous to that for the torsion model category $\mathcal{A}_t$. $\square$

Again, it is convenient to have a 1-dimensional model; the analogue of the standard model is considerably simplified in the present context.

Definition 17.1.3. The standard cyclotomic category $\mathcal{C}$ has objects $\mathbb{Q}[c_0]$-maps $N_0 \longrightarrow \Sigma^2\mathbb{I}_0 \otimes V$ with N_0 a torsion module. The morphisms are given by commutative squares as usual.

Lemma 17.1.4. *The cyclotomic category $\mathcal{C}$ is abelian and 1-dimensional. Hence we may form the derived category $D\mathcal{C}$. Furthermore passage to fibre $dg\mathcal{C} \longrightarrow dg\mathcal{C}_t$ and passage to cofibre $dg\mathcal{C} \longrightarrow dg\mathcal{C}_t$ induce inverse equivalences of derived categories, so that $D\mathcal{C} \simeq D\mathcal{C}_t$.*

Proof: The proof is similar to the case of the standard model, but with the simplification that the cofibre functor arrives in the correct category before passing to homology. □

Now define a functor

$$\Delta : \mathcal{C}_t \longrightarrow \mathcal{A}_t$$

as follows. For an object we define $\Delta(\Sigma^2 \mathbb{I}_0 \otimes V \stackrel{s_0}{\longrightarrow} T_0)$ to be the composite

$$(t_*^{\mathcal{F}} \otimes V \longrightarrow \Sigma^2 \mathbb{I} \otimes V = \bigoplus_H \Sigma^2 \mathbb{I}_0 \otimes V_0 \stackrel{\bigoplus_H s_0}{\longrightarrow} \bigoplus_H T_0).$$

Here the first map is induced by the quotient $t_*^{\mathcal{F}} \longrightarrow t_*^{\mathcal{F}}/\mathcal{O}_{\mathcal{F}} = \Sigma^2 \mathbb{I}$, and the second is the direct sum of countably many copies of s_0 made into a $\mathcal{O}_{\mathcal{F}}$-module in the obvious way.

The functor is obviously exact and hence induces a functor

$$\Delta : D\mathcal{C}_t \longrightarrow D\mathcal{A}_t.$$

We may now state a precise theorem.

Theorem 17.1.5. A $\mathbb{T}$-spectrum admits the structure of a cyclotomic spectrum if and only if it corresponds to an object of $D\mathcal{A}_t$ equivalent to one in the image of Δ.

Note that the condition in the theorem gives a rather satisfactory characterization of cyclotomic spectra. It essentially says that a cyclotomic spectrum is one that has two properties. Firstly, the structure map factors through that for its geometric fixed point spectrum (as happens for suspension spectra) and secondly, that all finite subgroups behave alike.

If a spectrum admits a cyclotomic structure then a structure is imposed by choosing particular equivalences between the idempotent pieces of the torsion part of the model. Note that for a spectrum X with torsion model $t_*^{\mathcal{F}} \otimes V \longrightarrow T$ admitting a cyclotomic structure the corresponding cyclotomic spectrum is simply $\Sigma^2 \mathbb{I}_0 \otimes V \longrightarrow T_0$ where $T_0 = e_1 T = \pi_*^{\mathbb{T}}(E\mathbb{T}_+ \wedge X)$ and the map is obtained by factoring s through the projection and applying the idempotent e_1.

Proof: We have explained how to put a cyclotomic structure on an object in the image of Δ. Any imprecision will be eliminated in the course of the proof in the reverse direction.

Suppose then that X is a cyclotomic spectrum with cyclotomic structure maps $r_K : \rho_K^! \Phi^K X \stackrel{\simeq}{\longrightarrow} X$ as required. We already know from Section 10.2 the effect of passage to geometric fixed points. Indeed, by Theorem 10.2.6, if M is the model of X then eM is the model for $\Phi^K X$ where e is the idempotent supported on the subgroups containing K. Here $\overline{\mathcal{O}}_{\overline{\mathcal{F}}}$ is identified with $e\mathcal{O}_{\mathcal{F}}$ by letting a subgroup $\overline{H}$ of $\overline{\mathbb{T}}$ correspond to its inverse image in $\mathbb{T}$. The effect of $\rho_K^!$ simply results from identifying subgroups of $\mathbb{T}$ with those of the same order in $\overline{\mathbb{T}}$.

Define $n_K : \overline{\mathcal{F}} \longrightarrow \mathcal{F}$ by letting $n(\overline{H})$ be the subgroup of $\mathbb{T}$ with the same order as $\overline{H}$, and consider the induced ring isomorphism $n_K^* : \mathcal{O}_{\mathcal{F}} \stackrel{\cong}{\longrightarrow} \overline{\mathcal{O}}_{\overline{\mathcal{F}}}$.

LEMMA 17.1.6. The functor $\rho^!_K : \overline{\mathbb{T}}\text{−Spec} \longrightarrow \mathbb{T}\text{−Spec}$ corresponds to pullback along n_K^* in the usual sense that the diagram

$$\begin{array}{ccc} \overline{\mathbb{T}}\text{−Spec} & \xrightarrow{\rho^!_K} & \mathbb{T}\text{−Spec} \\ \simeq\downarrow & & \downarrow\simeq \\ D\overline{\mathcal{A}} & \xrightarrow{n^K_\#} & D\mathcal{A} \end{array}$$

commutes up to natural isomorphism, and similarly for torsion models. □

It is then clear (for example by using the cyclotomic structure for K itself) that the part $e_K M$ of the model over any subgroup K will be the same as the piece $e_1 M$ over the trivial subgroup. This also forces the map to factor as specified.

If M is an object of the torsion model with zero differential, we see that the structure map must be zero on any element of form $1 \otimes v$; otherwise it would have nonzero image in $e_H T$ for some H, and hence for all finite subgroups H. This contradicts $\mathcal{F}$-finiteness of T. Since this argument passes to an injective resolution, it applies to all differential graded objects. □

17.2. Free loop spaces.

For a based space Z, we intend to identify the place of the free loop space ΛZ in the present scheme. In particular we may consider $K^*_{\mathbb{T}}(\Lambda Z)$, which is a conjectural approximation to $Ell^*(Z)$.

We restrict attention to the case that $Z = \Sigma Y$ is a suspension. Here, Carlsson and Cohen [4] prove the splitting

$$E\mathbb{T}_+ \wedge_{\mathbb{T}} \Lambda\Sigma Y = \bigvee_n (EC_n)_+ \wedge_{C_n} Y^{\wedge n}.$$

Hence, rationally we have

$$\pi^{\mathbb{T}}_*(E\mathbb{T}_+ \wedge \Lambda\Sigma Y) = \Sigma \bigoplus_n \{H_*(Y)^{\otimes n}\}^{C_n}$$

with trivial $H^*(B\mathbb{T})$ action. This leaves us to describe a map

$$\Sigma^2 \mathbb{I}_0 \otimes H_*(\Sigma Y) \longrightarrow \Sigma^2 \bigoplus_n \{H_*(Y)\}^{C_n}.$$

This necessarily has zero d-invariant. Indeed, this is obvious if H_*Y has even parity. In the general case we see that the map is induced by $E\mathbb{T}_+ \wedge_{\mathbb{T}} \Sigma Y \longrightarrow E\mathbb{T}_+ \wedge_{\mathbb{T}} \Lambda\Sigma Y$; by duality it is sufficient to consider cohomology, and the domain has torsion free cohomology whilst the codomain has torsion cohomology. Now, exactly as in the case of suspension spectra, the e invariant is the element of $\mathrm{Ext}(\Sigma^4\mathbb{I}_0 \otimes H_*(Y), \bigoplus_n \Sigma^2 H_*(Y))$ corresponding to the extension obtained by applying homology to

$$E\mathbb{T}_+ \wedge_{\mathbb{T}} \Lambda\Sigma Y \longrightarrow E\mathbb{T}_+ \wedge_{\mathbb{T}} (\Lambda\Sigma Y)/\Sigma Y \longrightarrow E\mathbb{T}_+ \wedge_{\mathbb{T}} \Sigma^2 Y.$$

Since $\mathbb{Q}[c_1]$ acts trivially on $H_*(Y)$, one might hope the extension is always obtained by tensoring a universal extension

$$0 \longrightarrow \bigoplus_n \mathbb{Q} \longrightarrow E \longrightarrow \Sigma^2\mathbb{I}_0 \longrightarrow 0$$

with $\Sigma^2 H_*(Y)$.

There is another important example of cyclotomic spectra.

Example 17.2.1. **Topological Hochschild Homology:**

Suppose that F is a functor with smash products in the sense of Bökstedt. One may define a cyclotomic spectrum $THH(F)$, which comes with a spectral sequence

$$HH_*(F(S^0)_*) \Longrightarrow THH(F)_*$$

for calculating its homotopy groups. One may then hope to calculate $\pi_*^{\mathbb{T}}(E\mathbb{T}_+ \wedge_{\mathbb{T}} THH(F))$ using the skeletal filtration of $E\mathbb{T}_+$.

It is always the case that $\Phi^{\mathbb{T}} THH(F) \simeq S^0$, and so the structure map of the cyclotomic spectrum $THH(F)$ takes the form

$$\Sigma^2\mathbb{I}_0 \longrightarrow \Sigma^2 THH(F)_*^{h\mathbb{T}}.$$

By definition, we always have a map from the identity functor to F and hence a cyclotomic map $S^0 = THH(I) \longrightarrow THH(F)$. Since this is an equivalence of geometric fixed points, and the structure map for S^0 has zero d-invariant we deduce that the structure map for an arbitrary functor F has zero d-invariant. It would be interesting to understand its e-invariant more precisely.

One case of particular interest is when the FSP arises from a ring R. In this case $\pi_*^{\mathbb{T}}(E\mathbb{T}_+ \wedge THH(R)) = \Sigma HC_*(R)$, which can be calculated by the algebraic Loday-Quillen double complex. It remains to identify the torsion model structure map, but we can obtain information by naturality from the unit $\mathbb{Q} \longrightarrow R$. In fact we have the diagram

$$\begin{array}{ccc}
\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ & \longrightarrow & \Sigma E\mathcal{F}_+ \\
\simeq\downarrow & & \downarrow\simeq \\
\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \wedge THH(\mathbb{Q}) & \longrightarrow & \Sigma E\mathcal{F}_+ \wedge THH(\mathbb{Q}) \\
\simeq\downarrow & & \downarrow \\
\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+ \wedge THH(R) & \longrightarrow & \Sigma E\mathcal{F}_+ \wedge THH(R)
\end{array}$$

in which we understand the top row precisely as the structure map of the sphere. Thus we only need to understand the algebraic map $\Sigma^2\mathbb{I}_0 = \Sigma^2 HC_*(\mathbb{Q}) \longrightarrow \Sigma^2 HC_*(R)$. This is in fact either zero or injective: this follows from the Tate spectral sequence by naturality. Indeed $t(THH(R))$ is a module over $t(THH(\mathbb{Q}))$, and hence over the ring spectrum $t(H\mathbb{Q})$, whose coefficients are $\mathbb{Q}[c_0, c_0^{-1}]$: thus the behaviour is completely determined by the image of the unit. If R is augmented, then of course the map is injective.

This special case is not too far from the general case, because any rational FSP arises from a simplicial ring, and $\pi_*^{\mathbb{T}}(E\mathbb{T}_+ \wedge THH(R_\bullet))$ can be calculated algebraically, since there is a $\mathbb{T}$-map $THH(R_\bullet) \longrightarrow HH(R_\bullet)$ which is a non-equviariant rational equivalence. $\square$

17.3. Topological cyclic cohomology of cyclotomic spectra.

In the first instance, the topological cyclic cohomology of a ring is designed to be the target of a refined trace map from the algebraic K-theory. Hesselholt and Madsen have shown that the cyclotomic trace is very close to being an isomorphism in many cases [17]. The construction of the topological cyclic cohomology in this case begins with the topological Hoschschild homology, and the definition of a cyclotomic spectrum abstracts precisely what is required to make the construction.

Goodwillie has identified the topolical cyclic cohomology of the topological Hoschschild homology of a rational functor with smash products [7], and we generalize this to an arbitrary cyclotomic spectrum. This is not a deep result, but it demonstrates the character of the topological cyclic cohomology and illustrates the adequacy of the present theory. The author is grateful to L. Hesselholt for many helpful discussions.

We must begin by describing the construction. For any $\mathbb{T}$-spectrum X, if $L \subseteq K$ we have an inclusion of the Lewis-May fixed points $F_L^K : \Psi^K X \longrightarrow \Psi^L X$; the letter F is chosen because it corresponds to the Frobenius map in algebraic K-theory. To avoid confusion, the reader should ignore for the duration of the present section the fact that F_L^K induces the restriction map from K-equivariant to L-equivariant homotopy groups. The cyclotomic structure supplies a second set of maps $R_L^K : \Psi^K X \longrightarrow \Psi^L X$ defined as follows. First we let L^* be the subgroup of K with order $|K/L|$. Now consider the inclusion $X \longrightarrow X \wedge \tilde{E}[\not\supseteq L^*]$; applying Lewis-May L^*-fixed points we obtain a map $\Psi^{L^*} X \longrightarrow \Phi^{L^*} X$. Applying $\rho^!_{L^*}$ and the cyclotomic structure we obtain

$$\rho^!_{L^*} \Psi^{L^*} X \longrightarrow \rho^!_{L^*} \Phi^{L^*} X \xrightarrow{r_{L^*}} X;$$

finally we apply L-fixed points and obtain the required map

$$\rho^!_{K/L} \Psi^K X = \rho^!_{K/L} \Psi^{K/L^*} \Psi^{L^*} X = \Psi^L(\rho^!_{L^*} \Psi^{L^*} X) \longrightarrow \Psi^L X.$$

Again, the letter R is chosen because the induced map is the restriction map in algebraic K-theory.

To simplify notation, we index F and R simply by the order of the quotient K/L. Thus we find $F_1 = R_1 = 1$, $F_r F_s = F_{rs}$ and $R_r R_s = R_{rs}$. It turns out that the Frobenius and restriction maps also commute.

The most familiar version of the topological cyclic cohomology construction is simply to take the the homotopy inverse limit of the system of non-equivariant fixed point spectra under the restriction and Frobenius maps:

$$TC'(X) = \underset{\leftarrow}{\text{holim}}(\underset{\leftarrow}{\text{holim}}(\Psi^K X, R)), F) = \underset{\leftarrow}{\text{holim}}(\underset{\leftarrow}{\text{holim}}(\Psi^K X, F), R).$$

It may help later motivation to view this as the homotopy fixed point object of an 'action of a category'. It turns out that the intermediate object

$$TR'(X) = \underset{\leftarrow}{\text{holim}}_K (\Psi^K X, R),$$

has significance of its own, so we prefer the first description $TC'(X) = \underset{\leftarrow}{\text{holim}}(TR'(X), F)$. Furthermore, we note that the above construction shows that the map $R_L^K : \rho^!_K \Psi^K X \longrightarrow$

$\rho^!_L \Psi^L X$ is a map of $\mathbb{T}$-spectra so

$$TR(X) = \operatorname{holim}_{\leftarrow K} (\rho^!_K \Psi^K X, R)$$

is a $\mathbb{T}$-spectrum with underlying spectrum $TR'(X)$. However, we warn that the identification of all terms with $TR(X)$ means that the Frobenius maps are not maps of $\mathbb{T}$-spectra. We shall identify the relevant equivariance below.

For non-profinite work, Goodwillie points out that the diagram given by the restriction and Frobenius maps should be augmented by adding in the circle action; we may now think of an action by a topological category. Since R also commutes with the Frobenius, passing to limits under R, we obtain a diagram with a copy of $TR(X)$ for each finite subgroup, and Frobenius maps relating them; the quotient category acts on $TR(X)$. For the present we view all objects as the same and hence we think of having an action of the monoid M occurring in a split exact sequence $1 \longrightarrow \mathbb{T} \longrightarrow M \longrightarrow \mathbb{Z}_{>0} \longrightarrow 1$; in fact if $w, z \in \mathbb{T}$ then $(wF_r)(zF_s) = wz^r F_{rs}$. This leads to the definition

$$TC(X) = TR(X)^{hM} \simeq (TR(X)^{h\mathbb{T}})^{hF}.$$

The following result may simply be regarded as evidence that the definition is a reasonable one: rationally, the topological cyclic construction is a complicated way of doing something familiar.

Theorem 17.3.1. *(Goodwillie)* For any rational cyclotomic spectrum X we have an equivalence of rational spectra

$$TC(X) \simeq X^{h\mathbb{T}},$$

so that the topological cyclic cohomology agrees with the Borel cohomology.

Goodwillie proves this in the case that $X = THH(F)$ for a rational functor F with smash products [7, 14.2].

Proof: The first step is to note that homotopy fixed points commute with homotopy inverse limits, and that the homotopy fixed point spectrum of a non-equivariantly contractible spectrum is contractible. Thus

$$\begin{aligned} TR(X)^{h\mathbb{T}} &= (\operatorname{holim}_{\leftarrow K} \rho^!_K \Psi^K X)^{h\mathbb{T}} \\ &= \operatorname{holim}_{\leftarrow K} ((\rho^!_K \Psi^K X)^{h\mathbb{T}}) \\ &= \operatorname{holim}_{\leftarrow K} ((E\mathbb{T}_+ \wedge \rho^!_K \Psi^K X)^{h\mathbb{T}}) \end{aligned}$$

This shows that it is really only necessary to understand $X(1) = E\mathbb{T}_+ \wedge X$. Of course the end result is simply a non-equivariant rational spectrum, so it is only necessary to calculate homotopy groups.

The main result of Part I is that the $\mathbb{T}$-free spectrum $E\mathbb{T}_+ \wedge \rho^! \Psi^K X$ is determined by its homotopy groups as modules over $\mathbb{Q}[c_1]$.

Lemma 17.3.2. If X is a cyclotomic spectrum with $\pi^{\mathbb{T}}_*(E\mathbb{T}_+ \wedge X) = T_0$ then

$$\pi^{\mathbb{T}}_*(E\mathbb{T}_+ \wedge \rho^!_K \Psi^K X) = \bigoplus_{L \subseteq K} T_0$$

and hence

$$E\mathbb{T}_+ \wedge \rho^!_K \Psi^K X = \bigvee_{L \subseteq K} X(1).$$

Proof: This is immediate from 17.1.6 together with our exact identification of Lewis-May fixed points in Theorem 12.2.2, (or more directly from 12.3.2). □

The relevant inverse system thus has Kth term given as a wedge of copies of the spectrum $X(1)$ indexed by the subgroups of K. It will perhaps be clearest if we think of this as the set of functions from the finite set $[\subseteq K]$ of subgroups of K to $X(1)$. The advantage is that it permits a helpful notation for maps: any function $f : A \longrightarrow B$ of finite sets induces $f^* : X(1)^B \longrightarrow X(1)^A$. If f is an inclusion the map f^* is simply projection.

LEMMA 17.3.3. When $L \subseteq K$, the restriction map R^K_L induces the projection corresponding to the function $\lambda^K_L : [\subseteq L] \longrightarrow [\subseteq K]$ defined by requiring $\lambda^K_L(H)$ to have order $|H| \cdot |K/L|$, in the sense that the diagram

$$\begin{array}{ccc} E\mathbb{T}_+ \wedge \rho^!_K \Psi^K X & \stackrel{1\wedge R^K_L}{\longrightarrow} & E\mathbb{T}_+ \wedge \rho^!_L \Psi^L X \\ \simeq\downarrow & & \downarrow\simeq \\ \bigvee_{H\subseteq K} X(1) & \stackrel{(\lambda^K_L)^*}{\longrightarrow} & \bigvee_{H\subseteq L} X(1) \end{array}$$

commutes.

Proof: Recall that L^* denotes the subgroup of K with order $|K/L|$. The effect of the map $\Psi^{L^*} X \longrightarrow \Psi^{L^*}(X \wedge \tilde{E}[\not\subseteq L^*]) = \Phi^{L^*} X$ follows from our account of the Lewis-May fixed points. Now we just need to rename subgroups using $\rho^!_{L^*}$ 17.1.6, and apply L-fixed points as described in 12.3.2. □

COROLLARY 17.3.4. We have an equivalence

$$TR(X)^{h\mathbb{T}} \simeq \prod_H X(1)^{h\mathbb{T}}. \quad \square$$

Notice that the above argument could also be used to identify the $\mathbb{T}$-spectrum $TR(X)$ exactly in the algebraic model. Indeed the maps $\rho^!_K \Psi^K X \longrightarrow \rho^!_L \Psi^L X$ are all identified exactly, and we can form the homotopy inverse limit in the algebraic model. However, since inverse limits do not preserve $\mathcal{F}$-free objects, the answer is not very attractive. Our present purpose requires much less; indeed, since $TR(X)^{h\mathbb{T}}$ is just a rational spectrum, and it remains only to understand the action of F on homotopy groups.

LEMMA 17.3.5. When $L \subseteq K$ the Frobenius map F^K_L induces the projection corresponding to the inclusion $i^K_L : [\subseteq L] \longrightarrow [\subseteq K]$ in the sense that the diagram

$$\begin{array}{ccc} E\mathbb{T}_+ \wedge \Psi^K X & \stackrel{1\wedge F^K_L}{\longrightarrow} & E\mathbb{T}_+ \wedge \Psi^L X \\ \simeq\downarrow & & \downarrow\simeq \\ \bigvee_{H\subseteq K} X(1) & \stackrel{(i^K_L)^*}{\longrightarrow} & \bigvee_{H\subseteq L} X(1) \end{array}$$

commutes.

Proof: The first necessity is to understand the statement. We begin with a map $\Psi^K X \longrightarrow \Psi^L X$, which we may view as a map of $\mathbb{T}$ spectra indexed on $\mathcal{U}^K$. Once we have smashed with $E\mathbb{T}_+$ the universe may be replaced by a complete one, and we obtain a map in the category for which we have a model.

To understand the map we factor $\Psi^K X \longrightarrow \Psi^L X$ as $\Psi^K X \longrightarrow \mathrm{inf}\Psi^K X \longrightarrow \Psi^L X$. If we view these as maps of $\mathbb{T}/L$ spectra, the second map is the counit of the K/L fixed point adjunction, completely understood by 12.2.2 and the contents of Section 11.3. The first map has the property that it is a nonequivariant equivalence. The result now follows from our description of the adjunction. $\square$

It remains only to index the terms so that the relevant structure is visible, and to verify that the circle action does not get in the way.

We begin by replacing subgroups by their orders, and defining a category as follows. The object set $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ consists of integer points in the strictly positive orthant. There are morphisms $(\phi_s, \rho_t) : (m, n) \longrightarrow (ms, nt)$ for $s, t \in \mathbb{Z}_{>0}$. Next, consider the diagram D of divisors defined by $D(m, n) = \{d \mid d \text{ divides } mn\}$ and $\phi_s : D(m, n) \longrightarrow D(ms, n)$ is inclusion, $\rho_t : D(m, n) \longrightarrow D(m, nt)$ is the multiplication by t. Finally, for an object Y we consider the contravariant functor Y^D defined by taking functions from D into Y; thus on objects, $Y^D(m, n) = Y^{D(m,n)}$. The connection with the restriction and Frobenius diagram is immediate from 17.3.5 and 17.3.3. The maps will be clearest if we replace $D(m, n)$ by the set of rational numbers i/j where i divides m and j divides n. It is easy to check that these fractions are in bijective correspondence to divisors of mn: if d divides mn the relevant fraction is d/n. With this indexing, both R and F simply drop irrelevant coordinates. Now, passing to limits under restriction maps we obtain $Y^D(m) := \lim_{\leftarrow n} Y^D(m, n)$, which simply consists of sequences $(y_{i/j})$ with i dividing m. The map $F_s : Y^D(ms) \longrightarrow Y^D(m)$ again simply drops coordinates with numerator dividing ms but not m. In other words, if we now identify $Y^D(m)$ with $Y^D(1)$ by dividing the coordinate indexes by m we find F_s is the shift map specified by mutiplying indices by s and ignoring fractions with an integer numerator bigger than 1. The system consists of surjections, so $\lim^1_{\leftarrow m} (Y^D(1), F) = 0$, and evidently the only compatible families are those with all coordinates equal: $\lim_{\leftarrow m} (Y^D(1), F) = Y$. This description suggests that we should have a means for discussing subgroups of $\mathbb{T}$ with fractional orders, which suggests we should be considering the solenoid $\mathbb{S} := \lim_{\leftarrow}(\mathbb{T}, \psi_s)$ which is the inverse limit of copies of the circle under the power maps ψ_s .

We must now check that the fact we have taken homotopy $\mathbb{T}$-fixed points between the R and F stages does not invalidate the above procedure. The time has come to be precise about the equivariance of the Frobenius maps. First, note that although we have the behaviour $F_s z = z^s F_s$ in the monoid M, so that F_s is identified with the map $\psi_s : E\mathbb{T}_+ \longrightarrow \psi_s^* E\mathbb{T}_+$, we expect the reverse type of behaviour for the objects acted upon.

Lemma 17.3.6. The Frobenius map induces a map of $\mathbb{T}$-spectra along ψ_s, in the sense that $F_s : \psi_s^* TR(X) \longrightarrow TR(X)$ is a map of naive $\mathbb{T}$-spectra.

Proof: We must remember that the map F_s arose from the inclusions $\Psi^K X \longrightarrow \Psi^L X$, which is a map of $\mathbb{T}$-spectra. However, when we have applied $\rho^!$ in the appropriate way, we must insert the power map ψ_s to retrieve the equivariance. □

The relevant map $TR(X)^{h\mathbb{T}} \longrightarrow TR(X)^{h\mathbb{T}}$ is then obtained by passage to fixed points from

$$F(\psi_s, F_s) : \psi_s^* F(E\mathbb{T}_+, TR(X)) = F(\psi_s^* E\mathbb{T}_+, \psi_s^* TR(X)) \longrightarrow F(E\mathbb{T}_+, TR(X)).$$

The relevant untwisting result is as follows.

LEMMA 17.3.7. The sth power map $\psi_s : E\mathbb{T}_+ \longrightarrow \psi_s^* E\mathbb{T}_+$ is a stable rational equivalence. □

Let $Y = \prod_n X(1)$, and consider the map $F_s : \psi_s^* Y \longrightarrow Y$ of $\mathbb{T}$-spectra. the commutative diagram

$$\begin{array}{ccc} F(E\mathbb{T}_+, \psi_s^* Y) & \xrightarrow{F(1,F_s)} & F(E\mathbb{T}_+, Y) \\ F(\psi_s, 1) \uparrow\simeq & & \uparrow= \\ F(\psi_s^* E\mathbb{T}_+, \psi_s^* Y) & \xrightarrow{F(\psi_s,F_s)} & F(E\mathbb{T}_+, Y) \end{array}$$

has an equivalence in its left hand vertical. Hence we can untwist the action on $\prod_n X(1)^{h\mathbb{T}}$.

COROLLARY 17.3.8. Rationally, we may identify the system of copies of $TR(X)^{h\mathbb{T}}$ under the Frobenius map with $\prod_{n>0} X(1)^{h\mathbb{T}}$ and with the Frobenius F_s acting via multiplication by s shifts. □

The theorem now follows. □