BAR CONSTRUCTIONS AND QUILLEN HOMOLOGY OF MODULES OVER OPERADS

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1. Introduction

There are many interesting situations in which algebraic structure can be naturally described by operads [28, 31, 32, 33]. In many of these, there is a notion of abelianization or stabilization [1, 2, 13, 14, 15, 16, 17, 36, 37, 40, 42] which provides a notion of (derived) homology. In this context, homology is not just a graded collection of abelian groups, but a geometric object like a chain complex or spectrum, and distinct algebraic structures tend to have distinct notions of homology. For commutative algebras this is the cotangent complex appearing in Andre-Quillen homology, and for the empty algebraic structure on spaces this is a chain complex calculating the singular homology of spaces.

In this paper we consider algebraic structures parametrized by operads acting on symmetric sequences of unbounded chain complexes and symmetric spectra [26]; such algebraic structures are called modules over an operad [27, 38], which includes the more familiar algebras over an operad [12, 28, 29, 33, 38] as a special case. Underlying every operad is a symmetric sequence [3, 9, 10, 12, 27, 38, 45, 46], and it turns out that symmetric sequences provide a useful setting for studying the derived homology of algebraic structures. In addition to the classical associative algebra, commutative algebra, and Lie algebra structures, operads can describe in a useful manner various highly structured homotopy versions of these, as naturally appear for example in the algebraic analogs of n-fold loop spaces and infinite loop spaces [2, 13, 17, 21, 23, 28, 29, 31, 32, 33]. In other words, operads arise because they act on many objects.

Even in the case of a simple algebraic structure such as commutative algebras, homology provides interesting invariants; in [34] Miller proves the Sullivan conjecture on maps from classifying spaces, and in his proof derived homology of commutative algebras [15, 16, 36, 37, 39] is a critical ingredient. This suggests that homology, for the larger class of algebraic structures parametrized by an action of an operad, will provide interesting and useful invariants.

Consider any catgory C with all small limits, and with terminal object denoted by $*$. Let C_{ab} be the category of abelian group objects in $(\mathsf{C},\times,*)$ and define

$$\mathsf{C} \underset{U}{\overset{\mathrm{Ab}}{\rightleftarrows}} \mathsf{C}_{\mathsf{ab}}$$

abelianization Ab to be the left adjoint of the forgetful functor U, if it exists. Then if C and C_{ab} are equipped with an appropriate homotopy theoretic structure, homology is the total left derived functor of abelianization; i.e., if $X \in \mathsf{C}$ then *Quillen homology* of X is defined by $\mathsf{QH}(X) := \mathsf{L}\,\mathrm{Ab}(X)$. This notion of homology is interesting in several contexts, including left modules and algebras over augmented operads $\mathcal{O}$ in unbounded chain complexes over a commutative ring k. In this context, the abelianization-forgetful adjunction takes the form of a "change of operads" adjunction

$$\mathsf{Lt}_{\mathcal{O}} \underset{}{\overset{I\circ_{\mathcal{O}}-}{\rightleftarrows}} \mathsf{Lt}_I = \mathsf{SymSeq} = (\mathsf{Lt}_{\mathcal{O}})_{\mathsf{ab}} \qquad \mathsf{Alg}_{\mathcal{O}} \underset{}{\overset{I\circ_{\mathcal{O}}(-)}{\rightleftarrows}} \mathsf{Alg}_I = \mathsf{Ch}_{\mathsf{k}} = (\mathsf{Alg}_{\mathcal{O}})_{\mathsf{ab}}$$

with left adjoints on top, provided that $\mathcal{O}[\mathbf{0}] = *$ and $\mathcal{O}[\mathbf{1}] = \mathsf{k}$; hence in this setting, abelianization is the "indecomposables" functor. Using the framework and corresponding homotopy theory established in [19], we show that the desired Quillen homology functors are well-defined and can be calculated as realization of simplicial bar constructions. The theorem is this.

Theorem 1.1. *Let k be a field of characteristic zero and let Ch_{k} be the category of unbounded chain complexes over k. If $f:\mathcal{O}\longrightarrow I$ is a morphism of operads in Ch_{k} and X is a left $\mathcal{O}$-module (resp. $\mathcal{O}$-algebra), then there is a zig-zag of weak equivalences*

$$I\circ^{\mathsf{L}}_{\mathcal{O}} X \simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad I\circ^{\mathsf{L}}_{\mathcal{O}}(X) \simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))\Big)$$

natural in X. In particular, Quillen homology $\mathsf{QH}(X)\simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$ provided that $\mathcal{O}[\mathbf{0}] = *$ and $\mathcal{O}[\mathbf{1}] = \mathsf{k}$.

The condition in Theorem 1.1 that k is a field of characteristic zero, ensures the appropriate homotopy theoretic structures exist on the category of left $\mathcal{O}$-modules and the category of $\mathcal{O}$-algebras, when $\mathcal{O}$ is an arbitrary operad in chain complexes [19].

When passing from the context of chain complexes to the context of symmetric spectra, abelian group objects appear less meaningful, and the interesting corresponding notion of homology is derived "indecomposables". If X is a left module or algebra over an augmented operad $\mathcal{O}$ in symmetric spectra, there is a "change of operads" adjunction

$$\mathsf{Lt}_{\mathcal{O}} \underset{}{\overset{I\circ_{\mathcal{O}}-}{\rightleftarrows}} \mathsf{Lt}_I = \mathsf{SymSeq} \qquad \mathsf{Alg}_{\mathcal{O}} \underset{}{\overset{I\circ_{\mathcal{O}}(-)}{\rightleftarrows}} \mathsf{Alg}_I = \mathsf{Sp}^{\Sigma}$$

with left adjoints on top. If $\mathcal{O}[\mathbf{0}] = *$ and $\mathcal{O}[\mathbf{1}] = S$, then *Quillen homology* of X is defined by $\mathsf{QH}(X) := I\circ^{\mathsf{L}}_{\mathcal{O}} X$ for left $\mathcal{O}$-modules and by $\mathsf{QH}(X) := I\circ^{\mathsf{L}}_{\mathcal{O}}(X)$ for $\mathcal{O}$-algebras; hence in this setting, Quillen homology is the total left derived functor of "indecomposables". Using the framework and corresponding homotopy theory established in [19, 20], we show that the desired Quillen homology functors are well-defined and can be calculated as realization of simplicial bar constructions, modulo cofibrancy conditions. The theorem is this.

Theorem 1.2. *Let Sp^{Σ} be the category of symmetric spectra. If $f:\mathcal{O}\longrightarrow I$ is a morphism of operads in Sp^{Σ} and X is a left $\mathcal{O}$-module (resp. $\mathcal{O}$-algebra) such that one of the following is true:*

(a) *the simplicial bar construction $\mathrm{Bar}(\mathcal{O},\mathcal{O},X)$ is objectwise cofibrant in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$), or*
(b) *the simplicial bar construction $\mathrm{Bar}(\mathcal{O},\mathcal{O},X^c)$ is objectwise cofibrant in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) for some functorial factorization $\emptyset\longrightarrow X^c\longrightarrow X$ in $\mathsf{Lt}_{\mathcal{O}}$ giving a cofibration followed by a weak equivalence,*

then there is a zig-zag of weak equivalences

$$I \circ^{\mathsf{L}}_{\mathcal{O}} X \simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad I \circ^{\mathsf{L}}_{\mathcal{O}} (X) \simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))\Big)$$

natural in such X. In particular, Quillen homology $\mathsf{QH}(X) \simeq \mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$ for such X provided that $\mathcal{O}[\mathbf{0}] = *$ and $\mathcal{O}[\mathbf{1}] = S$.

Remark 1.3. The conditions in (a) are satisfied if $\mathcal{O}[\mathbf{0}] = *$, $\mathcal{O}$ is cofibrant in SymSeq, and X is cofibrant in SymSeq (resp. Sp^Σ). The conditions in (b) are satisfied if the forgetful functor $\mathsf{Lt}_{\mathcal{O}} \longrightarrow \mathsf{SymSeq}$ (resp. $\mathsf{Alg}_{\mathcal{O}} \longrightarrow \mathsf{Sp}^\Sigma$) preserves cofibrant objects. These cofibrancy conditions in Sp^Σ and SymSeq are with respect to the stable flat positive model structures (Section 3.8).

Working with several model category structures, we give a homotopical proof of Theorems 1.1 and 1.2, once we have proved that certain homotopy colimits in left $\mathcal{O}$-modules and $\mathcal{O}$-algebras can be easily understood. The key result here, which is at the heart of this paper, is showing that the forgetful functor commutes with certain homotopy colimits. The theorem is this.

Theorem 1.4. *Let k be a field of characteristic zero. If $\mathcal{O}$ is an operad in Sp^Σ or Ch_{k} and X is a simplicial left $\mathcal{O}$-module (resp. simplicial $\mathcal{O}$-algebra), then there is a zig-zag of weak equivalences*

$$\Phi \operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \Phi X$$
$$\Big(\textit{resp.}\quad \Phi \operatorname*{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \Phi X\Big)$$

natural in X, with Φ the forgetful functor.

In this paper we develop results for both chain complexes and symmetric spectra, in parallel. It turns out, we can use the techniques developed in [20] in the context of symmetric spectra to compare homotopy categories of modules (resp. algebras) over operads in the context of chain complexes. The theorem is this.

Theorem 1.5. *Let k be a field of characteristic zero. Suppose $\mathcal{O}$ is an operad in Ch_{k} and let $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$) be the category of left $\mathcal{O}$-modules (resp. $\mathcal{O}$-algebras). If $f : \mathcal{O} \longrightarrow \mathcal{O}'$ is a map of operads, then the adjunction*

$$\mathsf{Lt}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'}, \qquad \Big(\textit{resp.}\quad \mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'},\Big)$$

is a Quillen adjunction with left adjoint on top and f^* the forgetful functor. If furthermore, f is an objectwise weak equivalence, then the adjunction is a Quillen equivalence, and hence induces an equivalence on the homotopy categories.

In the last few sections of this paper, we present analogous results for non-Σ operads, operads in chain complexes over a commutative ring, and right modules over operads.

1.1. **Relationship to previous work.** One of the main theorems of Fresse [9] is that for positive chain complexes over a field of characteristic zero, and for left modules and operads which are trivial at zero (e.g., such modules do not specialize to algebras over operads), then under additional conditions, the total left derived "indecomposables" functor is well-defined, and can be calculated as realization of a simplicial bar construction in the underlying category. Theorem 1.1 improves this result to unbounded chain complexes over a field of characteristic zero, to algebras and arbitrary left modules over operads, and also provides a simplified homotopical proof of Fresse's original result.

One of the main theorems of Hinich [22] is that for unbounded chain complexes over a field of characteristic zero, a morphism of operads which is an objectwise weak equivalence induces a Quillen equivalence between categories of algebras over operads. Theorem 1.5 improves this result to the category of left modules over operads.

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2. Symmetric sequences

2.1. Two contexts.

Definition 2.1. Let k be a commutative ring.

- $(\mathsf{Sp}^\Sigma, \wedge, S)$ is the category of symmetric spectra.
- $(\mathsf{Ch}_\mathsf{k}, \otimes, \mathsf{k})$ is the category of unbounded chain complexes over k.

Both are symmetric monoidal closed categories with all small limits and colimits; the null object is denoted by $*$.

Remark 2.2. By *closed* we mean there exists a functor

$$\big(\mathsf{Sp}^\Sigma\big)^{\mathrm{op}} \times \mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma, \qquad (Y,Z) \longmapsto \mathrm{Map}(Y,Z),$$
$$\Big(\text{resp.} \quad \mathsf{Ch}_\mathsf{k}^{\mathrm{op}} \times \mathsf{Ch}_\mathsf{k} \longrightarrow \mathsf{Ch}_\mathsf{k}, \qquad (Y,Z) \longmapsto \mathrm{Map}(Y,Z)\Big)$$

which we call *mapping object*, which fits into isomorphisms

$$\hom(X \wedge Y, Z) \cong \hom(X, \mathrm{Map}(Y,Z))$$
$$\Big(\text{resp.} \quad \hom(X \otimes Y, Z) \cong \hom(X, \mathrm{Map}(Y,Z))\Big)$$

natural in X, Y, Z.

2.2. **Symmetric sequences.** Define the sets $\mathbf{n} := \{1, \dots, n\}$ for each $n \geq 0$, where $\mathbf{0} := \emptyset$ denotes the empty set. Define the totally ordered sets $[n] := \{0, 1, \dots, n\}$ for each $n \geq 0$, and given their natural ordering. If T is a finite set, define $|T|$ to be the number of elements in T.

Definition 2.3. Let k be a commutative ring. Let $n \geq 0$.

- Σ is the category of finite sets and their bijections.

- A *symmetric sequence* in Sp^Σ (resp. Ch_k) is a functor $A:\Sigma^{\mathrm{op}}\longrightarrow\mathsf{Sp}^\Sigma$ (resp. $A:\Sigma^{\mathrm{op}}\longrightarrow\mathsf{Ch}_\mathsf{k}$). SymSeq is the category of symmetric sequences in Sp^Σ (resp. Ch_k) and their natural transformations.
- A symmetric sequence A is *concentrated at n* if $A[\mathbf{r}]=*$ for all $r\neq n$.

2.3. **Symmetric sequences build functors.**

Definition 2.4. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Each $A\in\mathsf{SymSeq}$ determines a corresponding functor defined objectwise by

$$\mathsf{Sp}^\Sigma\longrightarrow\mathsf{Sp}^\Sigma \qquad Z\longmapsto A(Z):=\coprod_{t\geq 0}A[\mathbf{t}]\wedge_{\Sigma_t}Z^{\wedge t}$$

$$\Big(\text{resp.}\quad \mathsf{Ch}_\mathsf{k}\longrightarrow\mathsf{Ch}_\mathsf{k} \qquad Z\longmapsto A(Z):=\coprod_{t\geq 0}A[\mathbf{t}]\otimes_{\Sigma_t}Z^{\otimes t}\Big)$$

If a symmetric sequence A has the extra structure of an operad (Section 3.6), then the corresponding functor $A(-)$ has the extra structure of a monad (or triple), and the assignment $Z\longmapsto A(Z)$ fits into a free-forgetful adjunction. Consider $A,B\in\mathsf{SymSeq}$. In the next section, the tensor product $A\check{\otimes}B\in\mathsf{SymSeq}$ is presented and used to define the circle product $A\circ B\in\mathsf{SymSeq}$, which has the property that $(A\circ B)(Z)\cong A\big(B(Z)\big)$.

3. Monoidal structures on SymSeq

To remain consistent with [20], and to avoid confusion with other tensor products appearing in this paper, we use the following $\check{\otimes}$ notation.

3.1. **Tensor product.**

Definition 3.1. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Let $A_1,\dots,A_t\in\mathsf{SymSeq}$. The *tensor products* $A_1\check{\otimes}\cdots\check{\otimes}A_t\in\mathsf{SymSeq}$ are the left Kan extensions of objectwise smash (resp. objectwise tensor) along coproduct of sets,

$$\begin{array}{ccccc} (\Sigma^{\mathrm{op}})^{\times t} & \xrightarrow{A_1\times\cdots\times A_t} & (\mathsf{Sp}^\Sigma)^{\times t} & \xrightarrow{\wedge} & \mathsf{Sp}^\Sigma \\ \downarrow{\scriptstyle\amalg} & & & & \\ \Sigma^{\mathrm{op}} & & \xrightarrow[\text{left Kan extension}]{A_1\check{\otimes}\cdots\check{\otimes}A_t} & & \mathsf{Sp}^\Sigma \end{array} \qquad \begin{array}{ccccc} (\Sigma^{\mathrm{op}})^{\times t} & \xrightarrow{A_1\times\cdots\times A_t} & (\mathsf{Ch}_\mathsf{k})^{\times t} & \xrightarrow{\otimes} & \mathsf{Ch}_\mathsf{k} \\ \downarrow{\scriptstyle\amalg} & & & & \\ \Sigma^{\mathrm{op}} & & \xrightarrow[\text{left Kan extension}]{A_1\check{\otimes}\cdots\check{\otimes}A_t} & & \mathsf{Ch}_\mathsf{k} \end{array}$$

The following calculations will be useful when working with tensor products.

Proposition 3.2. *Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Let $A_1,\dots,A_t\in\mathsf{SymSeq}$ and $R\in\Sigma$, with $r:=|R|$. There are natural isomorphisms,*

$$\begin{aligned} (A_1\check{\otimes}\cdots\check{\otimes}A_t)[R] &\cong \coprod_{\substack{\pi:R\longrightarrow\mathbf{t}\\ \text{in } \mathsf{Set}}} A_1[\pi^{-1}(1)]\wedge\cdots\wedge A_t[\pi^{-1}(t)],\\ &\cong \coprod_{r_1+\cdots+r_t=r} A_1[\mathbf{r_1}]\wedge\cdots\wedge A_t[\mathbf{r_t}]\underset{\Sigma_{r_1}\times\cdots\times\Sigma_{r_t}}{\cdot}\Sigma_r,\\ \textit{resp.}\quad (A_1\check{\otimes}\cdots\check{\otimes}A_t)[R] &\cong \coprod_{\substack{\pi:R\longrightarrow\mathbf{t}\\ \text{in } \mathsf{Set}}} A_1[\pi^{-1}(1)]\otimes\cdots\otimes A_t[\pi^{-1}(t)],\\ &\cong \coprod_{r_1+\cdots+r_t=r} A_1[\mathbf{r_1}]\otimes\cdots\otimes A_t[\mathbf{r_t}]\underset{\Sigma_{r_1}\times\cdots\times\Sigma_{r_t}}{\cdot}\Sigma_r, \end{aligned}$$

3.2. **Tensor powers.** It will be useful to extend the definition of tensor powers $A^{\check{\otimes} t}$ to situations in which the integers t are replaced by a finite set T.

Definition 3.3. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Let $A \in \mathsf{SymSeq}$ and $R, T \in \Sigma$. The *tensor powers* $A^{\check{\otimes} T} \in \mathsf{SymSeq}$ are defined objectwise by

$$(A^{\check{\otimes}\emptyset})[R] := \coprod_{\substack{\pi: R \longrightarrow \emptyset \\ \text{in } \mathsf{Set}}} S, \qquad (A^{\check{\otimes} T})[R] := \coprod_{\substack{\pi: R \longrightarrow T \\ \text{in } \mathsf{Set}}} \wedge_{t \in T} A[\pi^{-1}(t)] \quad (T \neq \emptyset),$$

$$\Big(\text{resp.} \quad (A^{\check{\otimes}\emptyset})[R] := \coprod_{\substack{\pi: R \longrightarrow \emptyset \\ \text{in } \mathsf{Set}}} \mathsf{k}, \qquad (A^{\check{\otimes} T})[R] := \coprod_{\substack{\pi: R \longrightarrow T \\ \text{in } \mathsf{Set}}} \otimes_{t \in T} A[\pi^{-1}(t)] \quad (T \neq \emptyset).\Big)$$

We will use the abbreviation $A^{\check{\otimes} 0} := A^{\check{\otimes}\emptyset}$. The smash products (resp. tensor products) indexed by T are regarded as unordered [19].

3.3. **Circle product (composition product).**

Definition 3.4. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Let $A, B \in \mathsf{SymSeq}$, $R \in \Sigma$, and define $r := |R|$. The *circle product* (or composition product) $A \circ B \in \mathsf{SymSeq}$ is defined objectwise by the coend

$$(A \circ B)[R] := A \wedge_\Sigma (B^{\check{\otimes} -})[R] \cong \coprod_{t \geq 0} A[\mathbf{t}] \wedge_{\Sigma_t} (B^{\check{\otimes} t})[\mathbf{r}]$$

$$\Big(\text{resp.} \quad (A \circ B)[R] := A \otimes_\Sigma (B^{\check{\otimes} -})[R] \cong \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} (B^{\check{\otimes} t})[\mathbf{r}]\Big).$$

3.4. **Monoidal structures.**

Proposition 3.5. *Consider symmetric sequences in* Sp^Σ *(resp. in* Ch_k*).*

(a) $(\mathsf{SymSeq}, \check{\otimes}, 1)$ *has the structure of a symmetric monoidal closed category with all small limits and colimits. The unit for* $\check{\otimes}$ *denoted "1" is the symmetric sequence concentrated at* 0 *with value* S *(resp.* k*).*

(b) $(\mathsf{SymSeq}, \circ, I)$ *has the structure of a monoidal closed category with all small limits and colimits. The unit for* $\circ$ *denoted "I" is the symmetric sequence concentrated at* 1 *with value* S *(resp.* k*). Circle product is not symmetric.*

3.5. **Symmetric sequences build functors (revisited).**

Definition 3.6. Let $Z \in \mathsf{Sp}^\Sigma$ (resp. $Z \in \mathsf{Ch}_\mathsf{k}$). Define $\hat{Z} \in \mathsf{SymSeq}$ to be the symmetric sequence concentrated at 0 with value Z.

The category Sp^Σ (resp. Ch_k) embeds in SymSeq as the full subcategory of symmetric sequences concentrated at 0, via the functor

$$\mathsf{Sp}^\Sigma \longrightarrow \mathsf{SymSeq} \qquad Z \longmapsto \hat{Z}$$

$$\Big(\text{resp.} \quad \mathsf{Ch}_\mathsf{k} \longrightarrow \mathsf{SymSeq} \qquad Z \longmapsto \hat{Z}\Big)$$

Definition 3.7. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Each $A \in \mathsf{SymSeq}$ determines a corresponding functor defined objectwise by

$$\mathsf{Sp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma \qquad Z \longmapsto A \circ (Z) := \coprod_{t \geq 0} A[\mathbf{t}] \wedge_{\Sigma_t} Z^{\wedge t} \cong (A \circ \hat{Z})[\mathbf{0}]$$

$$\Big(\text{resp.} \quad \mathsf{Ch}_\mathsf{k} \longrightarrow \mathsf{Ch}_\mathsf{k} \qquad Z \longmapsto A \circ (Z) := \coprod_{t \geq 0} A[\mathbf{t}] \otimes_{\Sigma_t} Z^{\otimes t} \cong (A \circ \hat{Z})[\mathbf{0}]\Big)$$

3.6. **Modules and algebras over operads.**

Definition 3.8. An *operad* is a monoid object in $(\mathsf{SymSeq}, \circ, I)$ and a *morphism of operads* is a morphism of monoid objects in $(\mathsf{SymSeq}, \circ, I)$.

Definition 3.9. Let $\mathcal{O}$ be an operad in Sp^Σ (resp. $\mathsf{Ch_k}$).

- A *left $\mathcal{O}$-module* is an object in $(\mathsf{SymSeq}, \circ, I)$ with a left action of $\mathcal{O}$ and a *morphism of left $\mathcal{O}$-modules* is a map which respects the left $\mathcal{O}$-module structure.
- A *right $\mathcal{O}$-module* is an object in $(\mathsf{SymSeq}, \circ, I)$ with a right action of $\mathcal{O}$ and a *morphism of right $\mathcal{O}$-modules* is a map which respects the right $\mathcal{O}$-module structure.
- An *$\mathcal{O}$-algebra* is an object $X \in \mathsf{Sp}^\Sigma$ (resp. $X \in \mathsf{Ch_k}$) with a left $\mathcal{O}$-module structure on $\hat{X}$. Let X and X' be $\mathcal{O}$-algebras. A *morphism of $\mathcal{O}$-algebras* is a map $f : X \longrightarrow X'$ in Sp^Σ (resp. $\mathsf{Ch_k}$) such that $\hat{f} : \hat{X} \longrightarrow \hat{X}'$ is a morphism of left $\mathcal{O}$-modules.
- $\mathsf{Lt}_\mathcal{O}$ is the category of left $\mathcal{O}$-modules and their morphisms.
- $\mathsf{Rt}_\mathcal{O}$ is the category of right $\mathcal{O}$-modules and their morphisms.
- $\mathsf{Alg}_\mathcal{O}$ is the category of $\mathcal{O}$-algebras and their morphisms.

Proposition 3.10. *Let $\mathcal{O}$ be an operad. The category $\mathsf{Alg}_\mathcal{O}$ embeds in $\mathsf{Lt}_\mathcal{O}$ as the full subcategory of left $\mathcal{O}$-modules concentrated at 0, via the functor*

$$\mathsf{Alg}_\mathcal{O} \longrightarrow \mathsf{Lt}_\mathcal{O} \qquad Z \longmapsto \hat{Z}$$

Hence, an $\mathcal{O}$-algebra is the same as a left $\mathcal{O}$-module concentrated at 0.

3.7. **Free-forgetful adjunctions.** It will be useful to summarize the following basic properties of $\mathsf{Alg}_\mathcal{O}$ and $\mathsf{Lt}_\mathcal{O}$. For an operad $\mathcal{O}$, the assignment $Z \longmapsto \mathcal{O} \circ (Z)$ given in Definition 3.7, is the free algebra on the underlying object.

Proposition 3.11. *Let $\mathcal{O}$ be an operad in Sp^Σ (resp. $\mathsf{Ch_k}$).*

(a) *There are adjunctions*

$$\mathsf{SymSeq} \underset{U}{\overset{\mathcal{O}\circ -}{\rightleftarrows}} \mathsf{Lt}_\mathcal{O} \qquad \mathsf{Sp}^\Sigma \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_\mathcal{O} \qquad \Big(\textit{resp.} \quad \mathsf{Ch_k} \underset{U}{\overset{\mathcal{O}\circ(-)}{\rightleftarrows}} \mathsf{Alg}_\mathcal{O}\Big)$$

with left adjoints on top and U the forgetful functor.

(b) *All small colimits exist in $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$, and both reflexive coequalizers and filtered colimits are preserved (and created) by the forgetful functors.*

(c) *All small limits exist in $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$, and are preserved (and created) by the forgetful functors.*

3.8. **Model category structures.** We assume the reader is familiar with model categories. A useful introduction is given in [6]. See also the original articles by Quillen [35, 37], and the more recent [4, 14, 18, 24, 25]. The adjunctions in Proposition 3.11(a) can be used to create model category structures on $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$ [19, 20]. We recall the statements here.

Theorem 3.12. *Let k be a field of characteristic zero. Let $\mathcal{O}$ be an operad in $\mathsf{Ch_k}$. Then $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$ both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the homology isomorphisms and the dimensionwise surjections in $\mathsf{Ch_k}$.*

Theorem 3.13. *Let $\mathcal{O}$ be an operad in Sp^Σ. Then $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$ both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable flat positive fibrations in Sp^Σ.*

We have followed Schwede [43] in using the term *flat* (e.g., stable flat model structure) for what is called S (e.g., stable S-model structure) in [26, 41, 44].

Theorem 3.14. *Let $\mathcal{O}$ be an operad in Sp^Σ. Then $\mathsf{Lt}_\mathcal{O}$ and $\mathsf{Alg}_\mathcal{O}$ both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable positive fibrations in Sp^Σ.*

4. Simplicial objects

We assume the reader is familiar with simplicial objects [7, 11, 14, 18, 47].

Definition 4.1. Let D be a category with all small limits and colimits.

- Δ is the category with objects the totally ordered sets $[n]$ for $n \geq 0$ and morphisms the maps of sets $\xi : [n] \longrightarrow [n']$ which respect the ordering; i.e., such that $k \leq l$ implies $\xi(k) \leq \xi(l)$.
- Δ_+ is the subcategory of Δ with all the objects and morphisms the surjective maps.
- A *simplicial object* in D is a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathsf{D}$. $\mathsf{sD} := \mathsf{D}^{\Delta^{\mathrm{op}}}$ is the category of simplicial objects in D and their natural transformations.
- A *cosimplicial object* in D is a functor $A : \Delta \longrightarrow \mathsf{D}$. $\mathsf{cD} := \mathsf{D}^{\Delta}$ is the category of cosimplicial objects in D and their natural transformations.
- If $X \in \mathsf{sD}$, we will sometimes use the notation $\pi_0 X := \operatorname{colim}\big(X : \Delta^{\mathrm{op}} \longrightarrow \mathsf{D}\big)$.
- $\emptyset$ denotes an initial object in D and $*$ denotes a terminal object in D.

If $X \in \mathsf{sD}$ and $n \geq 0$, we usually use the notation $X_n := X([n])$.

4.1. Model structures on simplicial objects.

Definition 4.2. Let D be a category with all small colimits. If $X \in \mathsf{sD}$ (resp. $X \in \mathsf{D}$) and $K \in \mathsf{sSet}$, then $X \cdot K \in \mathsf{sD}$ is defined objectwise by

$$(X \cdot K)_n := \coprod_{K_n} X_n \qquad \Big(\text{resp.} \quad (X \cdot K)_n := \coprod_{K_n} X\Big)$$

the coproduct in D, indexed over the set K_n, of copies of X_n (resp. X). Let $z \geq 0$ and define the *evaluation* functor $\mathrm{Ev}_z : \mathsf{sD} \longrightarrow \mathsf{D}$ objectwise by $\mathrm{Ev}_z(X) := X_z$.

Theorem 4.3. *Let k be a field of characteristic zero. Let $\mathcal{O}$ be an operad in Sp^Σ or Ch_k. Consider $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$) with any of the model structures in Section 3.8. Then $\mathsf{sLt}_\mathcal{O}$ (resp. $\mathsf{sAlg}_\mathcal{O}$) has a corresponding natural model category structure. The weak equivalences and fibrations in this model structure are inherited in an appropriate sense from the weak equivalences and fibrations in $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$).*

Proof. The model category structure on $\mathsf{sLt}_\mathcal{O}$ is created by the set of adjunctions

$$\mathsf{Lt}_\mathcal{O} \underset{\mathrm{Ev}_z}{\overset{-\cdot\Delta[z]}{\rightleftarrows}} \mathsf{sLt}_\mathcal{O}\ , \quad z \geq 0,$$

with left adjoints on top. Define a map f in $\mathsf{sLt}_{\mathcal{O}}$ to be a weak equivalence (resp. fibration) if $\mathrm{Ev}_z(f)$ is a weak equivalence (resp. fibration) in $\mathsf{Lt}_{\mathcal{O}}$ for every $z \geq 0$. Define a map f in $\mathsf{sLt}_{\mathcal{O}}$ to be a cofibration if it has the left lifting property with respect to all acyclic fibrations in $\mathsf{sLt}_{\mathcal{O}}$. To verify the model category axioms, argue as in the proof of [19, Theorem 12.2]. Since the right adjoints Ev_z commute with filtered colimits, the smallness conditions needed for the (possibly transfinite) small object arguments are satisfied. By construction, the model category is cofibrantly generated.

The model category structure on $\mathsf{sAlg}_{\mathcal{O}}$ is created by the set of adjunctions

$$\mathsf{Alg}_{\mathcal{O}} \underset{\mathrm{Ev}_z}{\overset{-\cdot\Delta[z]}{\rightleftarrows}} \mathsf{sAlg}_{\mathcal{O}}\ , \quad z \geq 0,$$

with left adjoints on top. Argue as in the $\mathsf{sLt}_{\mathcal{O}}$ case. □

4.2. **Homotopy colimit functors.** Here we define certain homotopy colimit functors.

Proposition 4.4. *Let* k *be a field of characteristic zero. Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k}. *The left derived functors*

$$\begin{array}{ccccc} \mathsf{sLt}_{\mathcal{O}} & \xrightarrow{\operatorname{colim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}} & \mathsf{Lt}_{\mathcal{O}} & \longrightarrow & \mathsf{Ho}(\mathsf{Lt}_{\mathcal{O}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{sLt}_{\mathcal{O}}) & & \xrightarrow[\text{left derived functor}]{\operatorname{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}} & & \mathsf{Ho}(\mathsf{Lt}_{\mathcal{O}}) \end{array} \qquad \begin{array}{ccccc} \mathsf{sAlg}_{\mathcal{O}} & \xrightarrow{\operatorname{colim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}} & \mathsf{Alg}_{\mathcal{O}} & \longrightarrow & \mathsf{Ho}(\mathsf{Alg}_{\mathcal{O}}) \\ \downarrow & & & & \\ \mathsf{Ho}(\mathsf{sAlg}_{\mathcal{O}}) & & \xrightarrow[\text{left derived functor}]{\operatorname{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}} & & \mathsf{Ho}(\mathsf{Alg}_{\mathcal{O}}) \end{array}$$

exist.

Proof. It is enough to verify that the adjunction

$$\mathsf{sLt}_{\mathcal{O}} \underset{}{\overset{\operatorname{colim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}} \qquad \Big(\text{resp.} \quad \mathsf{sAlg}_{\mathcal{O}} \underset{}{\overset{\operatorname{colim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}}\Big)$$

with left adjoint on top, is a Quillen pair. Noting that the right adjoint preserves fibrations and acyclic fibrations finishes the proof. □

4.3. **Bar constructions of modules over operads.**

Definition 4.5. Let $\mathcal{O}$ be an operad, $X \in \mathsf{Rt}_{\mathcal{O}}$, and $Y \in \mathsf{Lt}_{\mathcal{O}}$. The *simplicial bar construction* $\mathrm{Bar}(X, \mathcal{O}, Y) \in \mathsf{sSymSeq}$ looks like (showing the face maps only)

$$X \circ Y \underset{\mathrm{id}\circ m}{\overset{m\circ\mathrm{id}}{\leftleftarrows}} X \circ \mathcal{O} \circ Y \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; X \circ \mathcal{O} \circ \mathcal{O} \circ Y \;\overset{\longleftarrow}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longleftarrow}}}\; \cdots$$

and is defined objectwise by $\mathrm{Bar}(X, \mathcal{O}, Y)_k := X \circ \mathcal{O}^{\circ k} \circ Y$ with the obvious face and degeneracy maps. Similarly, let $\mathcal{O}$ be an operad in Sp^{Σ} (resp. Ch_{k}), $X \in \mathsf{Rt}_{\mathcal{O}}$, and $Y \in \mathsf{Alg}_{\mathcal{O}}$. The *simplicial bar construction* $\mathrm{Bar}(X, \mathcal{O}, Y)$ in sSp^{Σ} (resp. $\mathsf{sCh}_{\mathsf{k}}$) is defined objectwise by $\mathrm{Bar}(X, \mathcal{O}, Y)_k := X \circ \mathcal{O}^{\circ k} \circ (Y)$.

Sometimes the simplicial bar construction has a naturally occurring left (or right) simplicial $\mathcal{O}$-module structure.

Remark 4.6. Let $\mathcal{O}$ be an operad, $X \in \mathsf{Rt}_{\mathcal{O}}$, and $Y \in \mathsf{Lt}_{\mathcal{O}}$. Then $\mathrm{Bar}(\mathcal{O}, \mathcal{O}, Y) \in \mathsf{sLt}_{\mathcal{O}}$ and $\mathrm{Bar}(X, \mathcal{O}, \mathcal{O}) \in \mathsf{sRt}_{\mathcal{O}}$.

Theorem 4.7. *Let k be a field of characteristic zero. Let $\mathcal{O}$ be an operad in Sp^Σ or Ch_k and $X \in \mathsf{Lt}_\mathcal{O}$ (resp. $X \in \mathsf{Alg}_\mathcal{O}$). There is a zig-zag of weak equivalences*

$$\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(\mathcal{O},\mathcal{O},X) \simeq X$$

$$\Big(\textit{resp.}\quad \operatorname*{hocolim}^{\mathsf{Alg}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(\mathcal{O},\mathcal{O},X) \simeq X\Big)$$

in $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$), natural in X.

Proof. Consider any $X \in \mathsf{Lt}_\mathcal{O}$ and define $BX := \mathrm{Bar}(\mathcal{O},\mathcal{O},X) \in \mathsf{sLt}_\mathcal{O}$ and $\Delta X := X \cdot \Delta[0] \in \mathsf{sLt}_\mathcal{O}$. By Theorem 1.4 there is a commutative diagram

$$\begin{array}{ccc}
\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} BX & \overset{\simeq}{\text{———}} & \mathrm{R}(BX) \\
\downarrow{\scriptstyle (*)} & & \downarrow{\scriptstyle (**)} \\
\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \Delta X & \overset{\simeq}{\text{———}} & \mathrm{R}(\Delta X)
\end{array}$$

with each row a zig-zag of weak equivalences. We know that $(**)$ is a weak equivalence (Section 8), hence $(*)$ is a weak equivalence. The map $\emptyset \longrightarrow X$ factors functorially $\emptyset \longrightarrow X^c \longrightarrow X$ in $\mathsf{Lt}_\mathcal{O}$ as a cofibration followed by an acyclic fibration. This gives a natural zig-zag of weak equivalences

$$\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} BX \simeq \operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \Delta X \simeq \operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \Delta(X^c) \simeq \operatorname*{colim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} \Delta(X^c) \cong X^c \simeq X$$

in $\mathsf{Lt}_\mathcal{O}$, which finishes the proof. ☐

4.4. **Total left derived "change of operads" functors.** In this section we calculate certain left derived functors as realizations of simplicial bar constructions.

Proposition 4.8. *Let k be a field of characteristic zero. Let $f : \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism of operads in Sp^Σ or Ch_k. The left derived functors*

$$\begin{array}{ccccc}
\mathsf{Lt}_\mathcal{O} & \xrightarrow{\mathcal{O}' \circ_\mathcal{O} -} & \mathsf{Lt}_{\mathcal{O}'} & \longrightarrow & \mathsf{Ho}(\mathsf{Lt}_{\mathcal{O}'}) \\
\downarrow & & & & \\
\mathsf{Ho}(\mathsf{Lt}_\mathcal{O}) & & \xrightarrow[\textit{left derived functor}]{\mathcal{O}' \circ^{\mathsf{L}}_\mathcal{O} -} & & \mathsf{Ho}(\mathsf{Lt}_{\mathcal{O}'})
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{Alg}_\mathcal{O} & \xrightarrow{\mathcal{O}' \circ_\mathcal{O} (-)} & \mathsf{Alg}_{\mathcal{O}'} & \longrightarrow & \mathsf{Ho}(\mathsf{Alg}_{\mathcal{O}'}) \\
\downarrow & & & & \\
\mathsf{Ho}(\mathsf{Alg}_\mathcal{O}) & & \xrightarrow[\textit{left derived functor}]{\mathcal{O}' \circ^{\mathsf{L}}_\mathcal{O} (-)} & & \mathsf{Ho}(\mathsf{Alg}_{\mathcal{O}'})
\end{array}$$

exist.

Proof. It is enough to verify that the adjunction

$$\mathsf{Lt}_\mathcal{O} \underset{f^*}{\overset{\mathcal{O}' \circ_\mathcal{O} -}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'} \qquad \Big(\text{resp.}\quad \mathsf{Alg}_\mathcal{O} \underset{f^*}{\overset{\mathcal{O}' \circ_\mathcal{O} (-)}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'}\Big)$$

with left adjoint on top, is a Quillen pair. Noting that the forgetful functor f^* preserves fibrations and acyclic fibrations finishes the proof. ☐

Theorem 4.9. *Let* k *be a field of characteristic zero. If* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *is a morphism of operads in* Ch_k *and* $X\in\mathsf{Lt}_\mathcal{O}$ *(resp.* $X\in\mathsf{Alg}_\mathcal{O}$*), then there is a zig-zag of weak equivalences*

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}X\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad \mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}(X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))\Big)$$

natural in X.

Proof. Consider the case of $\mathsf{Lt}_\mathcal{O}$. For each $X\in\mathsf{Lt}_\mathcal{O}$, consider the zig-zags of weak equivalences

$$\begin{aligned}\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}X&\simeq\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}\Big(\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}}\mathrm{Bar}(\mathcal{O},\mathcal{O},X)\Big)\simeq\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}\mathrm{Bar}(\mathcal{O},\mathcal{O},X)\big)\\&\simeq\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathcal{O}'\circ_{\mathcal{O}}\mathrm{Bar}(\mathcal{O},\mathcal{O},X)\big)\simeq\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)\big)\\&\simeq\mathrm{R}\big(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X)\big).\end{aligned}$$

To verify these weak equivalences, use Theorem 4.7, Proposition 4.11, and Theorem 6.3. We have used the fact that every object in SymSeq is cofibrant. Argue similarly for the case of $\mathsf{Alg}_\mathcal{O}$. □

Theorem 4.10. *If* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *is a morphism of operads in* Sp^Σ *and* $X\in\mathsf{Lt}_\mathcal{O}$ *(resp.* $X\in\mathsf{Alg}_\mathcal{O}$*) such that one of the following is true:*

(a) *the simplicial bar construction* $\mathrm{Bar}(\mathcal{O},\mathcal{O},X)$ *is objectwise cofibrant in* $\mathsf{Lt}_\mathcal{O}$ *(resp.* $\mathsf{Alg}_\mathcal{O}$*), or*
(b) *the simplicial bar construction* $\mathrm{Bar}(\mathcal{O},\mathcal{O},X^c)$ *is objectwise cofibrant in* $\mathsf{Lt}_\mathcal{O}$ *(resp.* $\mathsf{Alg}_\mathcal{O}$*) for some functorial factorization* $\emptyset\longrightarrow X^c\longrightarrow X$ *in* $\mathsf{Lt}_\mathcal{O}$ *giving a cofibration followed by a weak equivalence,*

then there is a zig-zag of weak equivalences

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}X\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad \mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}(X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))\Big)$$

natural in such X.

Proof. Argue as in the proof of Theorem 4.9. □

Proposition 4.11. *Let* k *be a field of characteristic zero. Let* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *be a morphism of operads in* Sp^Σ *or* Ch_k *and* $X\in\mathsf{sLt}_\mathcal{O}$ *(resp.* $X\in\mathsf{sAlg}_\mathcal{O}$*). There is a zig-zag of weak equivalences*

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}\big(\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}}X\big)\simeq\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}X\big)$$
$$\Big(\textit{resp.}\quad\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}\big(\operatorname*{hocolim}^{\mathsf{Alg}_\mathcal{O}}_{\Delta^{\mathrm{op}}}X\big)\simeq\operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\big(\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}(X)\big)\Big)$$

natural in X.

5. Simplicial objects in Sp^Σ and Ch_k

This section is a first step in comparing realization with certain homotopy colimits. Similar homotopy invariance arguments appear in a variety of contexts [5, Appendix A], [8, Section X.2], [18, Section IV.1], [24, Chapter 18].

5.1. **Model category structures.**

Theorem 5.1. *Let* k *be a commutative ring. Consider symmetric sequences in* $\mathsf{Ch_k}$*. Then* $\mathsf{sCh_k}$ *and* $\mathsf{sSymSeq}$ *have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the homology isomorphisms and the dimensionwise surjections in* $\mathsf{Ch_k}$*.*

Theorem 5.2. *Consider symmetric sequences in* Sp^Σ*. Then* sSp^Σ *and* $\mathsf{sSymSeq}$ *both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable flat fibrations in* Sp^Σ*.*

Theorem 5.3. *Consider symmetric sequences in* Sp^Σ*. Then* sSp^Σ *and* $\mathsf{sSymSeq}$ *both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable fibrations in* Sp^Σ*.*

Proof. The model category structures are created by the set of adjunctions

$$\mathsf{Ch_k} \underset{\mathrm{Ev}_z}{\overset{-\cdot\Delta[z]}{\rightleftarrows}} \mathsf{sCh_k}\ ,\quad z\geq 0,\qquad \mathsf{Sp}^\Sigma \underset{\mathrm{Ev}_z}{\overset{-\cdot\Delta[z]}{\rightleftarrows}} \mathsf{sSp}^\Sigma\ ,\quad z\geq 0,$$

$$\mathsf{SymSeq} \underset{\mathrm{Ev}_z}{\overset{-\cdot\Delta[z]}{\rightleftarrows}} \mathsf{sSymSeq}\ ,\quad z\geq 0.$$

with left adjoints on top. Argue as in the case of Theorem 4.3. □

5.2. **Realization calculates hocolim.**

Theorem 5.4. *Let* k *be a commutative ring. Let* $X\in\mathsf{sSp}^\Sigma$ *(resp.* $X\in\mathsf{sCh_k}$*). There is a zig-zag of weak equivalences*

$$\operatorname*{hocolim}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(X)$$

natural in X*.*

An intermediate step in the proof of Theorem 5.4 is the following homotopy invariance property.

Theorem 5.5. *Let* k *be a commutative ring. If* $f:X\longrightarrow Y$ *in* sSp^Σ *(resp.* $\mathsf{sCh_k}$*) is an objectwise weak equivalence, then* $\mathrm{R}f:\mathrm{R}X\longrightarrow\mathrm{R}Y$ *is a weak equivalence.*

Before proving Theorems 5.5 and 5.4, we establish some notation.

5.3. **Realization.**

Definition 5.6.

- sSet is the category of simplicial sets.
- sSet_* is the category of pointed simplicial sets.

There are adjunctions

$$\mathsf{sSet} \underset{U}{\overset{(-)_+}{\rightleftarrows}} \mathsf{sSet}_* \overset{S\otimes G_0}{\rightleftarrows} \mathsf{Sp}^\Sigma \qquad \mathsf{sSet} \underset{U}{\overset{\mathsf{k}}{\rightleftarrows}} \mathsf{sMod_k} \overset{\mathrm{N}}{\rightleftarrows} \mathsf{Ch}^+_\mathsf{k} \rightleftarrows \mathsf{Ch_k}$$

with left adjoints on top, U the forgetful functor, N the normalization functor appearing in the Dold-Kan correspondence [18, Section III.2], and the right-hand functor on top the natural inclusion of categories. We will denote by $\mathrm{N}\mathsf{k}:\mathsf{sSet}\longrightarrow\mathsf{Ch_k}$ the composition of the left adjoints on the right-hand side.

Remark 5.7. The functor $S{\otimes}G_0$ is left adjoint to "evaluation at 0"; the notation agrees with [20, Section 4.1] and [26, after Definition 2.2.5]. Let $X \in \mathsf{Sp}^\Sigma$ and $K \in \mathsf{sSet}_*$. There are natural isomorphisms $X \wedge K \cong X \wedge (S{\otimes}G_0 K)$ in Sp^Σ.

Remark 5.8. If $X \in \mathsf{sSet}_*$, there are natural isomorphisms $X \times_\Delta \Delta[-] \cong X \wedge_\Delta \Delta[-]_+$.

Definition 5.9. Let $n \geq 0$.

(a) The *realization* functors R and R_n for simplicial symmetric spectra are defined objectwise by the coends

$$\mathrm{R}: \mathsf{sSp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma \qquad X \longmapsto X \wedge_\Delta \Delta[-]_+$$
$$\mathrm{R}_n: \mathsf{sSp}^\Sigma \longrightarrow \mathsf{Sp}^\Sigma \qquad X \longmapsto X \wedge_\Delta \mathrm{sk}_n \Delta[-]_+$$

(b) The *realization* functors R and R_n for simplicial chain complexes are defined objectwise by the coends

$$\mathrm{R}: \mathsf{sCh}_\mathsf{k} \longrightarrow \mathsf{Ch}_\mathsf{k} \qquad X \longmapsto X \otimes_\Delta \mathrm{N}\mathsf{k}\Delta[-]$$
$$\mathrm{R}_n: \mathsf{sCh}_\mathsf{k} \longrightarrow \mathsf{Ch}_\mathsf{k} \qquad X \longmapsto X \otimes_\Delta \mathrm{N}\mathsf{k}\mathrm{sk}_n \Delta[-]$$

Proposition 5.10. *Let* k *be a commutative ring. Let* $n \geq 0$. *The realization functors fit into adjunctions*

$$\mathsf{sSp}^\Sigma \underset{}{\overset{\mathrm{R}}{\rightleftarrows}} \mathsf{Sp}^\Sigma \qquad \mathsf{sSp}^\Sigma \underset{}{\overset{\mathrm{R}_n}{\rightleftarrows}} \mathsf{Sp}^\Sigma$$
$$\mathsf{sCh}_\mathsf{k} \underset{}{\overset{\mathrm{R}}{\rightleftarrows}} \mathsf{Ch}_\mathsf{k} \qquad \mathsf{sCh}_\mathsf{k} \underset{}{\overset{\mathrm{R}_n}{\rightleftarrows}} \mathsf{Ch}_\mathsf{k}$$

with left adjoints on top. Each adjunction is a Quillen pair.

Proof. Consider the case of sSp^Σ (resp. sCh_k). Use the universal property of a coend to verify that the functor given objectwise by

$$\mathrm{Map}(S{\otimes}G_0\Delta[-]_+, Y) \qquad \Big(\text{resp.} \quad \mathrm{Map}(\mathrm{N}\mathsf{k}\Delta[-], Y)\Big)$$

is a right adjoint of R. To check the adjunctions form Quillen pairs, it is enough to verify the right adjoints preserve fibrations and acyclic fibrations; since the model structures on Sp^Σ and Ch_k are monoidal model category structures, this follows by noting that $S{\otimes}G_0\Delta[m]_+$ and $\mathrm{N}\mathsf{k}\Delta[m]$ are cofibrant for each $m \geq 0$. Argue similarly for R_n. □

Proposition 5.11. *Let* $n \geq 0$ *and* $X \in \mathsf{Sp}^\Sigma$ *(resp.* $X \in \mathsf{Ch}_\mathsf{k}$*). There are isomorphisms* $\mathrm{R}(X \cdot \Delta[0]) \cong X$ *and* $\mathrm{R}_n(X \cdot \Delta[0]) \cong X$*, natural in* X.

Proof. This follows from uniqueness of left adjoints (up to isomorphism). □

5.4. **Normalization.**

Definition 5.12. Let $X \in \mathsf{sCh}_\mathsf{k}$ (resp. $X \in \mathsf{sMod}_\mathsf{k}$) and $n \geq 0$. Define the subobject $\mathrm{N}X_n \subseteq X_n$ by

$$\mathrm{N}X_0 := X_0 \qquad \mathrm{N}X_n := \bigcap_{0 \leq i \leq n-1} \ker(d_i) \subseteq X_n \qquad (n \geq 1)$$

Proposition 5.13. *Let* $X \in \mathsf{sCh}_\mathsf{k}$ *(resp.* $X \in \mathsf{sMod}_\mathsf{k}$*). There is a natural isomorphism between* X *and a simplicial object of the form*

$$\mathrm{N}X_0 \leftleftarrows \mathrm{N}X_0 \amalg \mathrm{N}X_1 \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, \mathrm{N}X_0 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_1 \amalg \mathrm{N}X_2 \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots$$

(showing the face maps only) which is given objectwise by isomorphisms

$$X_n \cong \coprod_{\substack{[n]\longrightarrow[k]\\ \text{in } \Delta_+}} \mathrm{N}X_k. \tag{5.14}$$

Proof. This follows from the Dold-Kan correspondence [18, Section III.2] that normalization fits into the following

$$\mathsf{sCh_k} \underset{}{\overset{\mathrm{N}}{\rightleftarrows}} \mathsf{Ch}^+(\mathsf{Ch_k}) \qquad \Big(\text{resp.} \quad \mathsf{sMod_k} \overset{\mathrm{N}}{\rightleftarrows} \mathsf{Ch}^+_{\mathsf{k}}\Big) \tag{5.15}$$

equivalence of categories. □

5.5. **Skeletal filtration of realization.**

Proposition 5.16. *Let $X \in \mathsf{sSp}^\Sigma$ (resp. $X \in \mathsf{sCh_k}$). The realization $\mathrm{R}(X)$ is naturally isomorphic to a filtered colimit of the form*

$$\mathrm{R}(X) \cong \operatorname{colim}\Big(\mathrm{R}_0(X) \longrightarrow \mathrm{R}_1(X) \longrightarrow \mathrm{R}_2(X) \longrightarrow \cdots\Big)$$

Proof. Consider the case of $\mathsf{sCh_k}$. We know that $\Delta[-] \cong \operatorname{colim}_n \mathrm{sk}_n\Delta[-]$ in sSet^Δ. Since the functors $\mathrm{N}\mathsf{k} : \mathsf{sSet}^\Delta \longrightarrow \mathsf{Ch}_{\mathsf{k}}^\Delta$ and $X \otimes_\Delta - : \mathsf{Ch}_{\mathsf{k}}^\Delta \longrightarrow \mathsf{Ch_k}$ preserve colimiting cones, it follows that there are natural isomorphisms

$$\begin{aligned} \mathrm{N}\mathsf{k}\Delta[-] &\cong \operatorname*{colim}_n \mathrm{N}\mathsf{k}\mathrm{sk}_n\Delta[-] \\ X \otimes_\Delta \mathrm{N}\mathsf{k}\Delta[-] &\cong \operatorname*{colim}_n X \otimes_\Delta \mathrm{N}\mathsf{k}\mathrm{sk}_n\Delta[-]. \end{aligned}$$

Consider the case of sSp^Σ. We know that $\Delta[-]_+ \cong \operatorname{colim}_n \mathrm{sk}_n\Delta[-]_+$ in sSet_*^Δ. Since the functors

$$\begin{aligned} S \otimes G_0 &: \mathsf{sSet}_*^\Delta \longrightarrow \big(\mathsf{Sp}^\Sigma\big)^\Delta \\ X \wedge_\Delta - &: \big(\mathsf{Sp}^\Sigma\big)^\Delta \longrightarrow \mathsf{Sp}^\Sigma \end{aligned}$$

preserve colimiting cones, a similar argument finishes the proof. □

Definition 5.17. Let $X \in \mathsf{sSp}^\Sigma$ (resp. $X \in \mathsf{sCh_k}$) and $n \geq 0$. Define the subobject $DX_n \subseteq X_n$ by

$$\begin{aligned} &DX_0 := * \qquad DX_n := \bigcup_{0 \leq i \leq n-1} s_i X_{n-1} \subseteq X_n \qquad (n \geq 1) \\ \Big(\text{resp.} \quad &DX_0 := * \qquad DX_n := \sum_{0 \leq i \leq n-1} s_i X_{n-1} \subseteq X_n \qquad (n \geq 1)\Big) \end{aligned}$$

We refer to DX_n as the *degenerate subobject* of X_n.

Proposition 5.18. *Let $X \in \mathsf{sSp}^\Sigma$ (resp. $X \in \mathsf{sCh_k}$) and $n \geq 1$. There are pushout diagrams*

$$\begin{array}{ccc} (DX_n \wedge \Delta[n]_+) \cup (X_n \wedge \partial\Delta[n]_+) & \longrightarrow & \mathrm{R}_{n-1}(X) \\ \downarrow & & \downarrow \\ X_n \wedge \Delta[n]_+ & \longrightarrow & \mathrm{R}_n(X) \end{array} \tag{5.19}$$

$$\text{resp.}\quad \begin{array}{ccc} (DX_n \otimes \mathrm{N}\mathsf{k}\Delta[n]) + (X_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n]) & \longrightarrow & \mathrm{R}_{n-1}(X) \\ \downarrow & & \downarrow \\ X_n \otimes \mathrm{N}\mathsf{k}\Delta[n] & \longrightarrow & \mathrm{R}_n(X) \end{array} \tag{5.20}$$

in Sp^Σ (resp. $\mathsf{Ch_k}$). The vertical maps in (5.19) *and* (5.20) *are monomorphisms.*

The following will be useful.

Proposition 5.21. *Let $X \in \mathsf{sSp}^\Sigma$ (resp. $X \in \mathsf{sCh_k}$) and $n \geq 1$. There are pushout diagrams*

$$\begin{array}{ccc} DX_n \wedge \partial\Delta[n]_+ & \longrightarrow & X_n \wedge \partial\Delta[n]_+ \\ \downarrow & & \downarrow \\ DX_n \wedge \Delta[n]_+ & \longrightarrow & (DX_n \wedge \Delta[n]_+) \cup (X_n \wedge \partial\Delta[n]_+) \end{array} \tag{5.22}$$

$$\text{resp.}\quad \begin{array}{ccc} DX_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n] & \longrightarrow & X_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n] \\ \downarrow & & \downarrow \\ DX_n \otimes \mathrm{N}\mathsf{k}\Delta[n] & \longrightarrow & (DX_n \otimes \mathrm{N}\mathsf{k}\Delta[n]) + (X_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n]) \end{array} \tag{5.23}$$

in Sp^Σ (resp. in $\mathsf{Ch_k}$). The maps in (5.22) *and* (5.23) *are monomorphisms.*

Proof. In the case of sSp^Σ, the pushout diagrams (5.22) follow from the corresponding pushout diagrams for a bisimplicial set [18, Section IV.1]. In the case of $\mathsf{sCh_k}$, use Proposition 5.13 to reduce to verifying that the diagram

$$\begin{array}{ccc} DX_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n] & = & DX_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n] \\ \downarrow & & \downarrow \\ DX_n \otimes \mathrm{N}\mathsf{k}\Delta[n] & = & DX_n \otimes \mathrm{N}\mathsf{k}\Delta[n] \end{array}$$

is a pushout diagram. ☐

Proof of Proposition 5.18. In the case of sSp^Σ, the pushout diagrams (5.19) follow from the corresponding pushout diagrams for a bisimplicial set [18, Section IV.1]. In the case of $\mathsf{sCh_k}$, use Proposition 5.13 to reduce to verifying that the diagram

$$\begin{array}{ccc} \mathrm{N}X_n \otimes \mathrm{N}\mathsf{k}\partial\Delta[n] & \longrightarrow & \mathrm{R}_{n-1}(X) \\ \downarrow & & \downarrow \\ \mathrm{N}X_n \otimes \mathrm{N}\mathsf{k}\Delta[n] & \longrightarrow & \mathrm{R}_n(X) \end{array} \tag{5.24}$$

is a pushout diagram in $\mathsf{Ch_k}$, which follows from the simplicial identities and the property that $\mathrm{Nk}:\mathsf{sSet}\longrightarrow\mathsf{Ch_k}$ preserves colimiting cones. □

Proposition 5.25. *Let* k *be a commutative ring. If* $f:X\longrightarrow Y$ *in* sSp^Σ *(resp.* $\mathsf{sCh_k}$*) is a monomorphism, then* $\mathrm{R}f:\mathrm{R}X\longrightarrow\mathrm{R}Y$ *is a monomorphism.*

Proof. In the case of sSp^Σ, this follows from the corresponding property for realization of a bisimplicial set [18, Section IV.1]. Consider the case of $\mathsf{sCh_k}$. Use Proposition 5.13 to argue that $\mathrm{N}:\mathsf{sCh_k}\longrightarrow\mathsf{Ch_k}$ preserves monomorphisms; either use the Dold-Kan correspondence (5.15) and note that right adjoints preserve monomorphisms, or use (5.14) and note that monomorphisms are preserved under retracts. To finish the argument, forget differentials and use the pushout diagrams (5.24) to give a particularly simple filtration of $\mathrm{R}f:\mathrm{R}X\longrightarrow\mathrm{R}Y$ in the underlying category of graded k-modules. Since $\mathrm{N}X_n\longrightarrow\mathrm{N}Y_n$ is a monomorphism for each $n\geq 0$, it follows from this filtration that $\mathrm{R}f$ is a monomorphism. □

5.6. **Homotopy invariance of realization.**

Proof of Theorem 5.5. Consider the case of sSp^Σ (resp. $\mathsf{sCh_k}$). Skeletal filtration gives a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{R}_0(X) & \longrightarrow & \mathrm{R}_1(X) & \longrightarrow & \mathrm{R}_2(X) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{R}_0(f)} & & \downarrow{\scriptstyle \mathrm{R}_1(f)} & & \downarrow{\scriptstyle \mathrm{R}_2(f)} & & \\
\mathrm{R}_0(Y) & \longrightarrow & \mathrm{R}_1(Y) & \longrightarrow & \mathrm{R}_2(Y) & \longrightarrow & \cdots
\end{array}$$

We know that $\mathrm{R}_0(f)\cong f_0$ is a weak equivalence. Since the horizontal maps are monomorphisms and we know that

$$\mathrm{R}_n(X)/\mathrm{R}_{n-1}(X)\cong(X_n/D_nX)\wedge(\Delta[n]/\partial\Delta[n])$$
$$\Big(\text{resp.}\quad \mathrm{R}_n(X)/\mathrm{R}_{n-1}(X)\cong(X_n/D_nX)\otimes(\mathrm{Nk}\Delta[n]/\mathrm{Nk}\partial\Delta[n])\Big)$$

it is enough to verify that $Df_n:DX_n\longrightarrow DY_n$ is a weak equivalence for each $n\geq 1$, and Proposition 5.26 finishes the proof. □

Proposition 5.26. *Let* k *be a commutative ring. If* $f:X\longrightarrow Y$ *in* $\mathsf{sCh_k}$ *is an objectwise weak equivalence, then* $Df_n:DX_n\longrightarrow DY_n$ *is a weak equivalence for each* $n\geq 1$*.*

Before proving this, it will be useful to filter the degenerate subobjects.

Definition 5.27. Let $X\in\mathsf{sSp}^\Sigma$ (resp. $X\in\mathsf{sCh_k}$) and $n\geq 1$. For each $0\leq r\leq n-1$, define the subobjects $s_{[r]}X_{n-1}\subseteq X_n$ by

$$s_{[r]}X_{n-1}:=\bigcup_{0\leq i\leq r}s_iX_{n-1}\subseteq X_n,$$
$$\Big(\text{resp.}\quad s_{[r]}X_{n-1}:=\sum_{0\leq i\leq r}s_iX_{n-1}\subseteq X_n\Big)$$

The following proposition is motivated by the corresponding statement for bisimplicial sets [18, Section IV.1].

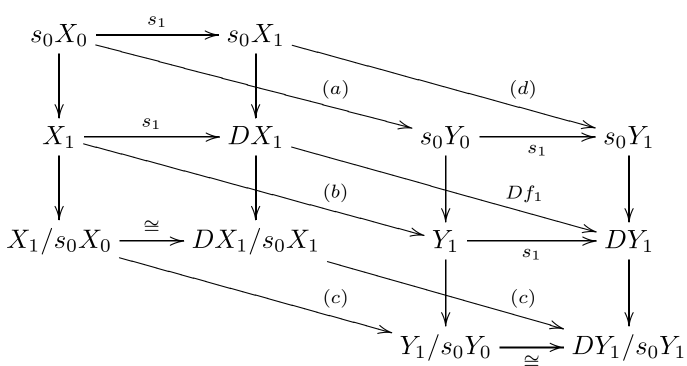
Proposition 5.28. *Let* k *be a commutative ring. Let* $X \in \mathsf{sSp}^\Sigma$ *(resp.* $X \in \mathsf{sCh_k}$*) and* $n \geq 1$*. For each* $0 \leq r \leq n-1$*, the diagram*

$$\begin{array}{ccc} s_{[r]}X_{n-1} & \xrightarrow{s_{r+1}} & s_{[r]}X_n \\ \downarrow{\subseteq} & & \downarrow{\subseteq} \\ X_n & \xrightarrow{s_{r+1}} & s_{[r+1]}X_n \end{array} \tag{5.29}$$

is a pushout diagram. The maps in (5.29) *are monomorphisms.*

Proof. Consider the case of sSp^Σ. This follows from the corresponding pushout diagrams for a bisimplicial set [18, Section IV.1]. Consider the case of $\mathsf{sCh_k}$. This follows from the Proposition 5.13 and the simplicial identities. □

Proof of Proposition 5.26. Consider the case of sSp^Σ (resp. $\mathsf{sCh_k}$). Let $n = 1$. By Proposition 5.28, Df_1 fits into the commutative diagram



Since we know the maps (a) and (b) are weak equivalences, it follows that each map (c) is a weak equivalence. Since we know the map (d) is a weak equivalence, it follows that Df_1 is a weak equivalence. Similarly, use Proposition 5.28 in an induction argument to verify that $Df_n : DX_n \longrightarrow DY_n$ is a weak equivalence for each $n \geq 2$. □

5.7. **Comparing realization and hocolim.**

Proof of Theorem 5.4. Consider any map $X \longrightarrow Y$ in sSp^Σ (resp. $\mathsf{sCh_k}$). Use functorial factorization to obtain a commutative diagram

$$\begin{array}{ccccc} * & \longrightarrow & X^c & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & Y^c & \longrightarrow & Y \end{array}$$

in sSp^Σ (resp. sCh_k) such that each row is a cofibration followed by an acyclic fibration. Hence we get a corresponding commutative diagram

$$\begin{array}{ccccccccc}
\operatorname{hocolim}_{\Delta^{\mathrm{op}}} X & \longleftarrow & \operatorname{hocolim}_{\Delta^{\mathrm{op}}} X^c & \longrightarrow & \operatorname{colim}_{\Delta^{\mathrm{op}}} X^c & \xleftarrow{(*)} & \mathrm{R}(X^c) & \xrightarrow{(**)} & \mathrm{R}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\operatorname{hocolim}_{\Delta^{\mathrm{op}}} Y & \longleftarrow & \operatorname{hocolim}_{\Delta^{\mathrm{op}}} Y^c & \longrightarrow & \operatorname{colim}_{\Delta^{\mathrm{op}}} Y^c & \xleftarrow{(*)} & \mathrm{R}(Y^c) & \xrightarrow{(**)} & \mathrm{R}(Y)
\end{array}$$

such that the rows are maps of weak equivalences; the maps $(*)$ and $(**)$ are weak equivalences by Proposition 5.30 and Theorem 5.5, respectively. □

Proposition 5.30. *Let* k *be a commutative ring. If* $Z \in \mathsf{sSp}^\Sigma$ *(resp.* $Z \in \mathsf{sCh}_\mathsf{k}$*) is cofibrant, then the natural map*

$$\mathrm{R}(Z) \longrightarrow \mathrm{R}\big((\pi_0 Z)\cdot\Delta[0]\big)$$

is a weak equivalence.

Proof. Let $X \longrightarrow Y$ be a generating cofibration in Sp^Σ (resp. Ch_k). Consider the pushout diagram

$$\begin{array}{ccc}
X\cdot\Delta[z] & \longrightarrow & Z_0 \\
\downarrow & & \downarrow \\
Y\cdot\Delta[z] & \longrightarrow & Z_1
\end{array} \tag{5.31}$$

in sSp^Σ (resp. sCh_k) and the natural maps

$$\mathrm{R}(Z_0) \longrightarrow \mathrm{R}\big((\pi_0 Z_0)\cdot\Delta[0]\big) \tag{5.32}$$

$$\mathrm{R}(Z_1) \longrightarrow \mathrm{R}\big((\pi_0 Z_1)\cdot\Delta[0]\big) \tag{5.33}$$

Assume (5.32) is a weak equivalence; let's verify (5.33) is a weak equivalence. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}(Z_0) & \longrightarrow & \mathrm{R}(Z_1) & \longrightarrow & \mathrm{R}\big((Y/X)\cdot\Delta[z]\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{R}\big((\pi_0 Z_0)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((\pi_0 Z_1)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((Y/X)\cdot\Delta[0]\big)
\end{array}$$

The left-hand horizontal maps are monomorphisms, the left-hand vertical map is a weak equivalence by assumption, and the right-hand vertical map is a weak equivalence by Section 8, hence the middle vertical map is a weak equivalence. Consider a sequence

$$Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \cdots$$

of pushouts of maps as in (5.31). Assume Z_0 makes (5.32) a weak equivalence; we want to show that for $Z_\infty := \operatorname{colim}_k Z_k$ the natural map

$$\mathrm{R}(Z_\infty) \longrightarrow \mathrm{R}\big((\pi_0 Z_\infty)\cdot\Delta[0]\big) \tag{5.34}$$

is a weak equivalence. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{R}(Z_0) & \longrightarrow & \mathrm{R}(Z_1) & \longrightarrow & \mathrm{R}(Z_2) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{R}\big((\pi_0 Z_0)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((\pi_0 Z_1)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((\pi_0 Z_2)\cdot\Delta[0]\big) & \longrightarrow & \cdots \end{array}$$

We know that the horizontal maps are monomorphisms and the vertical maps are weak equivalences, hence the induced map (5.34) is a weak equivalence. Noting that every cofibration $*\longrightarrow Z$ in sSp^Σ (resp. sCh_k) is a retract of a (possibly transfinite) composition of pushouts of maps as in (5.31), starting with $Z_0 = *$, finishes the proof. □

6. Simplicial objects in SymSeq

In this section we verify that realization for simplicial symmetric sequences enjoys similar properties.

6.1. **Realization.**

Definition 6.1. Let k be a commutative ring. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). The *realization* functor R is defined objectwise by

$$\mathrm{R}:\mathsf{sSymSeq}\longrightarrow\mathsf{SymSeq} \qquad X\longmapsto \mathrm{R}X \qquad \Big(T\longmapsto R\big(X[T]\big)\Big)$$

Proposition 6.2. *Let k be a commutative ring. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). There is an adjunction*

$$\mathsf{sSymSeq} \underset{}{\overset{\mathrm{R}}{\rightleftarrows}} \mathsf{SymSeq}$$

with left adjoint on top. The adjunction is a Quillen pair.

6.2. **Realization calculates hocolim.**

Theorem 6.3. *Let k be a commutative ring. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). Let $X\in\mathsf{SymSeq}$.*

(a) *There is a zig-zag of weak equivalences $\operatorname{hocolim}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(X)$, natural in X.*
(b) *If $f : X\longrightarrow Y$ in $\mathsf{sSymSeq}$ is a weak equivalence, then $\mathrm{R}f:\mathrm{R}X\longrightarrow\mathrm{R}Y$ is a weak equivalence.*

Proof. Consider part (b). This follows from Definition 6.1 and Theorem 5.5. Consider part (a). Use the argument in the proof of Theorem 5.4, except replace the categories sSp^Σ and sCh_k with the category $\mathsf{sSymSeq}$; the maps $(*)$ and $(**)$ are weak equivalences by Proposition 6.4 and part (b), respectively. □

6.3. **Comparing realization and hocolim.**

Proposition 6.4. *Let k be a commutative ring. Consider symmetric sequences in Sp^Σ (resp. in Ch_k). If $Z\in\mathsf{sSymSeq}$ is cofibrant, then the natural map*

$$\mathrm{R}(Z)\longrightarrow \mathrm{R}\big((\pi_0 Z)\cdot\Delta[0]\big)$$

is a weak equivalence.

Proof. Use the argument in the proof of Proposition 5.30, except replace the categories sSp^Σ and sCh_k with the category $\mathsf{sSymSeq}$. □

7. Simplicial objects in $\mathsf{Lt}_{\mathcal{O}}$ and $\mathsf{Alg}_{\mathcal{O}}$

7.1. Realization calculates hocolim.

Theorem 7.1. *Let* k *be a field of characteristic zero. Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k} *and* $X \in \mathsf{sLt}_{\mathcal{O}}$ *(resp.* $X \in \mathsf{sAlg}_{\mathcal{O}}$*). There is a zig-zag of weak equivalences*

$$\Phi \operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(\Phi X)$$

$$\Big(\textit{resp.}\quad \Phi \operatorname*{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(\Phi X)\Big)$$

natural in X*, with* Φ *the forgetful functor.*

Proof. Consider any $X \in \mathsf{sLt}_{\mathcal{O}}$. The map $\emptyset \longrightarrow X$ factors functorially $\emptyset \longrightarrow X^c \longrightarrow X$ in $\mathsf{sLt}_{\mathcal{O}}$ as a cofibration followed by an acyclic fibration. This gives a diagram

$$\operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \longleftarrow \operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X^c \longrightarrow \operatorname*{colim}_{\Delta^{\mathrm{op}}} X^c \xleftarrow{(*)} \mathrm{R}(X^c) \xrightarrow{(**)} \mathrm{R}(X)$$

of weak equivalences; the maps $(*)$ and $(**)$ are weak equivalences by Proposition 7.15 and Theorem 6.3(b), respectively. □

7.2. Forgetful functor commutes with hocolim.

Proof of Theorem 1.4. Let $X \in \mathsf{sLt}_{\mathcal{O}}$ (resp. $X \in \mathsf{sAlg}_{\mathcal{O}}$). By Theorems 7.1 and 6.3(a), there is a zig-zag of weak equivalences

$$\Phi \operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(\Phi X) \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \Phi X$$

$$\Big(\text{resp.}\quad \Phi \operatorname*{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} X \simeq \mathrm{R}(\Phi X) \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \Phi X\Big)$$

natural in X, with Φ the forgetful functor. □

7.3. Analysis of pushouts in $\mathsf{sLt}_{\mathcal{O}}$ and $\mathsf{sAlg}_{\mathcal{O}}$.

Definition 7.2. Let $(\mathsf{C}, \otimes, \mathsf{k})$ be a monoidal category. If $X, Y \in \mathsf{sC}$ then $X \otimes Y \in \mathsf{sC}$ is defined objectwise by

$$(X \otimes Y)_n := X_n \otimes Y_n.$$

Definition 7.3. Consider symmetric sequences in Sp^{Σ} (resp. Ch_{k}).

- A *symmetric array* in Sp^{Σ} (resp. Ch_{k}) is a symmetric sequence in SymSeq; i.e. a functor $A : \Sigma^{\mathrm{op}} \longrightarrow \mathsf{SymSeq}$.
- $\mathsf{SymArray} := \mathsf{SymSeq}^{\Sigma^{\mathrm{op}}}$ is the category of symmetric arrays in Sp^{Σ} (resp. Ch_{k}) and their natural transformations.

Proposition 7.4. *Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k}*,* $A \in \mathsf{sLt}_{\mathcal{O}}$ *(resp.* $A \in \mathsf{Lt}_{\mathcal{O}}$*), and* $Y \in \mathsf{sSymSeq}$ *(resp.* $Y \in \mathsf{SymSeq}$*). Consider any coproduct in* $\mathsf{sLt}_{\mathcal{O}}$ *(resp.* $\mathsf{Lt}_{\mathcal{O}}$*) of the form*

$$A \amalg (\mathcal{O} \circ Y). \tag{7.5}$$

There exists $\mathcal{O}_A \in \mathsf{sSymArray}$ *(resp.* $\mathcal{O}_A \in \mathsf{SymArray}$*) and natural isomorphisms*

$$A \amalg (\mathcal{O} \circ Y) \cong \coprod_{q \geq 0} \mathcal{O}_A[\mathbf{q}] \check{\otimes}_{\Sigma_q} Y^{\check{\otimes} q}$$

in the underlying category $\mathsf{sSymSeq}$ *(resp.* SymSeq*).*

Proof. The case of $A \in \mathsf{Lt}_{\mathcal{O}}$ and $Y \in \mathsf{SymSeq}$ is given in [19, Proposition 13.8]. If $A \in \mathsf{sLt}_{\mathcal{O}}$ and $Y \in \mathsf{sSymSeq}$, use the colimit construction in [19, Proposition 13.8] to define $\mathcal{O}_A : \Delta^{\mathrm{op}} \longrightarrow \mathsf{SymArray}$ objectwise. □

Proposition 7.6. *Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k}*,* $A \in \mathsf{sLt}_{\mathcal{O}}$*,* $Y \in \mathsf{sSymSeq}$*, and* $t \geq 0$*. There are natural isomorphisms*

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}^{\mathsf{Lt}_{\mathcal{O}}}\Big(A \amalg (\mathcal{O} \circ Y)\Big) \cong \coprod_{q \geq 0} \mathcal{O}_{\pi_0 A}[\mathbf{q}] \check{\otimes}_{\Sigma_q} (\pi_0 Y)^{\check{\otimes} q}, \tag{7.7}$$

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}\Big(\mathcal{O}_A[\mathbf{t}]\Big) \cong \mathcal{O}_{\pi_0 A}[\mathbf{t}], \tag{7.8}$$

in the underlying category SymSeq*.*

Proof. To argue (7.7), use the natural isomorphisms

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}^{\mathsf{Lt}_{\mathcal{O}}}\Big(A \amalg (\mathcal{O} \circ Y)\Big) \cong (\pi_0 A) \amalg \pi_0(\mathcal{O} \circ Y) \cong (\pi_0 A) \amalg \big(\mathcal{O} \circ (\pi_0 Y)\big)$$

in $\mathsf{Lt}_{\mathcal{O}}$ together with Proposition 7.4. To argue (7.8), use the properties of reflexive coequalizers in [19, Section 8.1] together with the colimit construction in [19, Proposition 13.8]. □

Definition 7.9. Let $i : X \longrightarrow Y$ be a morphism in $\mathsf{sSymSeq}$ (resp. SymSeq) and $t \geq 1$. Define $Q_0^t := X^{\check{\otimes} t}$ and $Q_t^t := Y^{\check{\otimes} t}$. For $0 < q < t$ define Q_q^t inductively by the pushout diagrams

$$\begin{array}{ccc}
\Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\check{\otimes}(t-q)} \check{\otimes} Q_{q-1}^q & \xrightarrow{\mathrm{pr}_*} & Q_{q-1}^t \\
\big\downarrow{\scriptstyle i_*} & & \big\downarrow{\scriptstyle i_*} \\
\Sigma_t \cdot_{\Sigma_{t-q} \times \Sigma_q} X^{\check{\otimes}(t-q)} \check{\otimes} Y^{\check{\otimes} q} & \longrightarrow & Q_q^t
\end{array}$$

in $\mathsf{sSymSeq}^{\Sigma_t}$ (resp. $\mathsf{SymSeq}^{\Sigma_t}$).

Proposition 7.10. *Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k}*,* $A \in \mathsf{sLt}_{\mathcal{O}}$ *(resp.* $A \in \mathsf{Lt}_{\mathcal{O}}$*), and* $i : X \longrightarrow Y$ *in* $\mathsf{sSymSeq}$ *(resp. in* SymSeq*). Consider any pushout diagram in* $\mathsf{sLt}_{\mathcal{O}}$ *(resp.* $\mathsf{Lt}_{\mathcal{O}}$*) of the form,*

$$\begin{array}{ccc}
\mathcal{O} \circ X & \xrightarrow{f} & A \\
\big\downarrow{\scriptstyle \mathrm{id} \circ i} & & \big\downarrow{\scriptstyle j} \\
\mathcal{O} \circ Y & \longrightarrow & A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y).
\end{array} \tag{7.11}$$

The pushout in (7.11) *is naturally isomorphic to a filtered colimit of the form*

$$A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y) \cong \operatorname{colim}\Big(A_0 \xrightarrow{j_1} A_1 \xrightarrow{j_2} A_2 \xrightarrow{j_3} \cdots \Big) \tag{7.12}$$

in the underlying category sSymSeq *(resp.* SymSeq*), with* $A_0 := \mathcal{O}_A[\mathbf{0}] \cong A$ *and* A_t *defined inductively by pushout diagrams in* sSymSeq *(resp.* SymSeq*) of the form*

$$\begin{array}{ccc} \mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t} Q^t_{t-1} & \xrightarrow{f_*} & A_{t-1} \\ \downarrow{\scriptstyle \mathrm{id}\check{\otimes}_{\Sigma_t} i_*} & & \downarrow{\scriptstyle j_t} \\ \mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t} Y^{\check{\otimes} t} & \xrightarrow{\xi_t} & A_t \end{array} \tag{7.13}$$

Proof. The case of $A \in \mathsf{Lt}_{\mathcal{O}}$ and $i : X \longrightarrow Y$ in SymSeq is given in [19, Proposition 13.13]. If $A \in \mathsf{sLt}_{\mathcal{O}}$ and $i : X \longrightarrow Y$ in sSymSeq, use Proposition 7.4 together with the argument in the proof of [19, Proposition 13.13] to construct the pushout diagrams (7.13) objectwise. □

Proposition 7.14. *Let* $\mathcal{O}$ *be an operad in* Sp^Σ *or* Ch_k*,* $A \in \mathsf{sLt}_{\mathcal{O}}$*, and* $i : X \longrightarrow Y$ *in* sSymSeq*. Consider any pushout diagram in* $\mathsf{sLt}_{\mathcal{O}}$ *of the form* (7.11)*. Then* $\pi_0(-)$ *commutes with the filtered diagrams in* (7.12)*; i.e., there are natural isomorphisms which make the diagram*

$$\begin{array}{ccccccc} \pi_0(A_0) & \xrightarrow{\pi_0(j_1)} & \pi_0(A_1) & \xrightarrow{\pi_0(j_2)} & \pi_0(A_2) & \xrightarrow{\pi_0(j_3)} & \cdots \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \\ (\pi_0 A)_0 & \xrightarrow{j_1} & (\pi_0 A)_1 & \xrightarrow{j_2} & (\pi_0 A)_2 & \xrightarrow{j_3} & \cdots \end{array}$$

commute.

Proof. Use the properties of reflexive coequalizers in [19, Section 8.1] together with Proposition 7.6 and the arguments in the proof of [19, Proposition 13.13]. □

7.4. **Comparing realization with hocolim.**

Proposition 7.15. *Let* k *be a field of characteristic zero. Let* $\mathcal{O}$ *be an operad in* Sp^Σ *or* Ch_k*. If* $Z \in \mathsf{sLt}_{\mathcal{O}}$ *(resp.* $Z \in \mathsf{sAlg}_{\mathcal{O}}$*) is cofibrant, then the map*

$$\mathrm{R}(\Phi Z) \longrightarrow \mathrm{R}((\pi_0 \Phi Z)\cdot \Delta[0])$$

is a weak equivalence, with Φ *the forgetful functor.*

Proof. Let $X \longrightarrow Y$ be a generating cofibration in SymSeq and consider the pushout diagram

$$\begin{array}{ccc} \mathcal{O}\circ X\cdot \Delta[z] & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ \mathcal{O}\circ Y\cdot \Delta[z] & \longrightarrow & Z_1 \end{array} \tag{7.16}$$

in $\mathsf{sLt}_{\mathcal{O}}$. For each cofibrant $W_\alpha \in \mathsf{SymSeq}$, $l_\alpha \geq 0$, and set $\mathcal{A}$, consider the natural maps

$$\mathrm{R}\Big(Z_0 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha \cdot \Delta[l_\alpha])\Big) \longrightarrow \mathrm{R}\Big(\big((\pi_0 Z_0) \amalg \coprod_{\alpha\in\mathcal{A}} \mathcal{O}\circ W_\alpha\big)\cdot \Delta[0]\Big), \tag{7.17}$$

$$\mathrm{R}\Big(Z_1 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha \cdot \Delta[l_\alpha])\Big) \longrightarrow \mathrm{R}\Big(\big((\pi_0 Z_1) \amalg \coprod_{\alpha\in\mathcal{A}} \mathcal{O}\circ W_\alpha\big)\cdot \Delta[0]\Big), \tag{7.18}$$

and note that the diagram

$$\begin{array}{ccc} \mathcal{O}\circ X\cdot\Delta[z] & \longrightarrow & Z_0 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha]) =: A \\ \downarrow & & \downarrow \\ \mathcal{O}\circ Y\cdot\Delta[z] & \longrightarrow & Z_1 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha]) \cong A_\infty \end{array}$$

is a pushout diagram in $\mathsf{sLt}_{\mathcal{O}}$. Assume (7.17) is a weak equivalence for each cofibrant $W_\alpha \in \mathsf{SymSeq}$, $l_\alpha \geq 0$, and set $\mathcal{A}$; let's verify (7.18) is a weak equivalence for each cofibrant $W_\alpha \in \mathsf{SymSeq}$, $l_\alpha \geq 0$, and set $\mathcal{A}$. Suppose $\mathcal{A}$ is a set, $W_\alpha \in \mathsf{SymSeq}$ cofibrant, and $l_\alpha \geq 0$, for each $\alpha \in \mathcal{A}$. By Proposition 7.10 there are corresponding filtrations together with induced maps ξ_t $(t \geq 1)$ which make the diagram

$$\begin{array}{ccccccc} \mathrm{R}(A_0) & \longrightarrow & \mathrm{R}(A_1) & \longrightarrow & \mathrm{R}(A_2) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \mathrm{R}(\xi_0)} & & \downarrow{\scriptstyle \mathrm{R}(\xi_1)} & & \downarrow{\scriptstyle \mathrm{R}(\xi_2)} & & \\ \mathrm{R}\big((\pi_0 A_0)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((\pi_0 A_1)\cdot\Delta[0]\big) & \longrightarrow & \mathrm{R}\big((\pi_0 A_2)\cdot\Delta[0]\big) & \longrightarrow & \cdots \end{array}$$

in SymSeq commute. Since $\mathrm{R}(-)$ commutes with colimits we get

$$\begin{array}{ccc} \mathrm{colim}_t\, \mathrm{R}(A_t) & \xrightarrow{\cong} & \mathrm{R}(A_\infty) \\ \downarrow & & \downarrow \\ \mathrm{colim}_t\, \mathrm{R}\big((\pi_0 A_t)\cdot\Delta[0]\big) & \xrightarrow{\cong} & \mathrm{R}\big((\pi_0 A_\infty)\cdot\Delta[0]\big). \end{array}$$

By assumption we know $\mathrm{R}(\xi_0)$ is a weak equivalence, and to verify (7.18) is a weak equivalence, it is enough to check that $\mathrm{R}(\xi_t)$ is a weak equivalence for each $t \geq 1$. Since the horizontal maps are monomorphisms and we know that there are natural isomorphisms

$$\mathrm{R}(A_t)/\mathrm{R}(A_{t-1}) \cong \mathrm{R}\big(\mathcal{O}_A[\mathbf{t}]\check{\otimes}_{\Sigma_t}(Y/X\cdot\Delta[z])^{\check{\otimes} t}\big),$$
$$\mathrm{R}\big((\pi_0 A_t)\cdot\Delta[0]\big)/\mathrm{R}\big((\pi_0 A_{t-1})\cdot\Delta[0]\big) \cong \mathrm{R}\big((\mathcal{O}_{\pi_0 A}[\mathbf{t}]\check{\otimes}_{\Sigma_t}(Y/X)^{\check{\otimes} t})\cdot\Delta[0]\big),$$

it is enough to verify that

$$\mathrm{R}\Big(A \amalg \mathcal{O}\circ\big((Y/X)\cdot\Delta[z]\big)\Big) \longrightarrow \mathrm{R}\Big(\big((\pi_0 A)\amalg\mathcal{O}\circ(Y/X)\big)\cdot\Delta[0]\Big)$$

is a weak equivalence. Noting that Y/X is cofibrant finishes the argument that (7.18) is a weak equivalence. Consider a sequence

$$Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \cdots$$

of pushouts of maps as in (7.16). Assume Z_0 makes (7.17) a weak equivalence for each cofibrant $W_\alpha \in \mathsf{SymSeq}$, $l_\alpha \geq 0$, and set $\mathcal{A}$; we want to show that for $Z_\infty := \mathrm{colim}_k\, Z_k$ the natural map

$$\mathrm{R}\Big(Z_\infty \amalg \coprod_{\alpha\in\mathcal{A}}(\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha])\Big)\longrightarrow\mathrm{R}\Big(\big((\pi_0 Z_\infty)\amalg\coprod_{\alpha\in\mathcal{A}}\mathcal{O}\circ W_\alpha\big)\cdot\Delta[0]\Big) \tag{7.19}$$

is a weak equivalence for each cofibrant $W_\alpha \in \mathsf{SymSeq}$, $l_\alpha \geq 0$, and set $\mathcal{A}$. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{R}\Big(Z_0 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha \cdot \Delta[l_\alpha])\Big) & \longrightarrow & \mathrm{R}\Big(Z_1 \amalg \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha \cdot \Delta[l_\alpha])\Big) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathrm{R}\Big(\big((\pi_0 Z_0) \amalg \coprod_{\alpha\in\mathcal{A}} \mathcal{O}\circ W_\alpha\big)\cdot \Delta[0]\Big) & \longrightarrow & \mathrm{R}\Big(\big((\pi_0 Z_1) \amalg \coprod_{\alpha\in\mathcal{A}} \mathcal{O}\circ W_\alpha\big)\cdot \Delta[0]\Big) & \longrightarrow & \cdots
\end{array}$$

in SymSeq. The horizontal maps are monomorphisms and the vertical maps are weak equivalences, hence the induced map (7.19) is a weak equivalence. Noting that every cofibration $\mathcal{O}\circ * \cdot \Delta[0] \longrightarrow Z$ in $\mathsf{sLt}_\mathcal{O}$ is a retract of a (possibly transfinite) composition of pushouts of maps as in (7.16), starting with $Z_0 = \mathcal{O}\circ * \cdot \Delta[0]$, together with Proposition 8.4, finishes the proof. $\square$

8. Simplicial homotopies

Definition 8.1. Let D be a category with all small colimits and consider the left-hand diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \qquad X \cong X\cdot\Delta[0] \underset{\mathrm{id}\cdot d^0}{\overset{\mathrm{id}\cdot d^1}{\rightrightarrows}} X\cdot\Delta[1] \xrightarrow{H} Y \tag{8.2}$$

in sD. A *simplicial homotopy* from f to g is any map $H : X\cdot\Delta[1] \longrightarrow Y$ in sD such that the two diagrams in (8.2) are identical. The map f is *simplicially homotopic* to g if there exists a simplicial homotopy from f to g.

Remark 8.3. This definition of simplicial homotopy agrees with [18, Section I.6] and [30, between Propositions 6.2 and 6.3].

Proposition 8.4. *Let k be a commutative ring. Let $\mathcal{O}$ be an operad in Sp^Σ or Ch_k. Let $\mathcal{A}$ be a set, $W_\alpha \in \mathsf{SymSeq}$, and $l_\alpha \geq 0$, for each $\alpha\in\mathcal{A}$. Consider the maps*

$$\coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha]) \xrightarrow{r} \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[0]) \xrightarrow{s} \coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha])$$

in $\mathsf{sLt}_\mathcal{O}$ induced by the maps $\Delta[l_\alpha] \xrightarrow{r_\alpha} \Delta[0] \xrightarrow{s_\alpha} \Delta[l_\alpha]$ in simplicial sets, such that each map s_α represents the vertex 0. Then the map

$$\mathrm{R}\Big(\coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[l_\alpha])\Big) \xrightarrow{\mathrm{R}(r)} \mathrm{R}\Big(\coprod_{\alpha\in\mathcal{A}} (\mathcal{O}\circ W_\alpha\cdot\Delta[0])\Big)$$

in SymSeq is a weak equivalence.

Proof. For each $\alpha\in\mathcal{A}$, we know that $r_\alpha s_\alpha = \mathrm{id}$ and $s_\alpha r_\alpha$ is simplicially homotopic to the identity map. Hence $rs = \mathrm{id}$ and sr is simplicially homotopic to the identity map. In the case of symmetric spectra, since objectwise weak equivalences are weak equivalences, it follows that $\mathrm{R}(r)$ is a weak equivalence. In the case of chain complexes, since $\mathrm{Tot}^\oplus \mathrm{N}$ takes simplicially homotopic maps to chain homotopic maps, it follows from Proposition 8.5 that $\mathrm{R}(r)$ is a weak equivalence. $\square$

Proposition 8.5. *Let* k *be a commutative ring. If* $X \in \mathsf{sCh_k}$*, then there are isomorphisms*

$$\mathrm{R}(X) = X \otimes_\Delta \mathrm{N}\mathsf{k}\Delta[-] \cong \mathrm{Tot}^\oplus \mathrm{N}(X)$$

natural in X*.*

Proof. If $A, B \in \mathsf{Ch_k}$, define the objectwise tensor $A \dot{\otimes} B \in \mathsf{Ch}(\mathsf{Ch_k})$ such that $A \otimes B = \mathrm{Tot}^\oplus(A \dot{\otimes} B)$. It follows that there are natural isomorphisms,

$$X \otimes_\Delta \mathrm{N}\mathsf{k}\Delta[-] \cong \mathrm{Tot}^\oplus\big(X \dot{\otimes}_\Delta \mathrm{N}\mathsf{k}\Delta[-]\big) \cong \mathrm{Tot}^\oplus\big(\mathrm{N}\mathsf{k}\Delta[-] \dot{\otimes}_\Delta X\big).$$

Arguing as in the proof of Proposition 5.16, verify that

$$\mathrm{N}\mathsf{k}\Delta[-] \dot{\otimes}_\Delta X \cong \operatorname*{colim}_n \big(\mathrm{N}\mathsf{k}\mathrm{sk}_n \Delta[-] \dot{\otimes}_\Delta X\big)$$

and use the pushout diagrams

$$\begin{array}{ccc} \mathrm{N}\mathsf{k}\partial\Delta[n] \dot{\otimes} \mathrm{N}X_n & \longrightarrow & \mathrm{N}\mathsf{k}\mathrm{sk}_{n-1}\Delta[-] \dot{\otimes}_\Delta X \\ \downarrow & & \downarrow \\ \mathrm{N}\mathsf{k}\Delta[n] \dot{\otimes} \mathrm{N}X_n & \longrightarrow & \mathrm{N}\mathsf{k}\mathrm{sk}_n\Delta[-] \dot{\otimes}_\Delta X \end{array}$$

in $\mathsf{Ch}(\mathsf{Ch_k})$ to verify that $\mathrm{N}\mathsf{k}\Delta[-] \dot{\otimes}_\Delta X \cong \mathrm{N}(X)$, which finishes the proof. ☐

9. Modules over Non-Σ Operads

In this section we indicate some of the analogous results for non-Σ operads. These follow from essentially the same arguments given in the previous sections, but using the non-Σ filtrations in [19, Proposition 13.35] instead of the filtrations in [19, Proposition 13.13].

9.1. **Model category structures.** Here we recall the model category structures established in [19, Theorems 1.2 and 1.6] for modules and algebras over non-Σ operads.

Theorem 9.1. *Let* k *be a commutative ring. Let* $\mathcal{O}$ *be a non-*Σ *operad in* $\mathsf{Ch_k}$*. Then* $\mathsf{Lt}_\mathcal{O}$ *and* $\mathsf{Alg}_\mathcal{O}$ *both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the homology isomorphisms and the dimensionwise surjections in* $\mathsf{Ch_k}$*.*

Theorem 9.2. *Let* $\mathcal{O}$ *be a non-*Σ *operad in* Sp^Σ*. Then* $\mathsf{Lt}_\mathcal{O}$ *and* $\mathsf{Alg}_\mathcal{O}$ *both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable flat fibrations in* Sp^Σ*.*

We have followed Schwede [43] in using the term *flat* (e.g., stable flat model structure) for what is called S (e.g., stable S-model structure) in [26, 41, 44].

Theorem 9.3. *Let* $\mathcal{O}$ *be a non-*Σ *operad in* Sp^Σ*. Then* $\mathsf{Lt}_\mathcal{O}$ *and* $\mathsf{Alg}_\mathcal{O}$ *both have natural model category structures. The weak equivalences and fibrations in these model structures are inherited in an appropriate sense from the stable weak equivalences and the stable fibrations in* Sp^Σ*.*

The underlying model category structures for Sp^Σ were established in [26, 44].

9.2. Analogous results for non-Σ operads.

Theorem 9.4. *Let k be a commutative ring. If $f:\mathcal{O}\longrightarrow\mathcal{O}'$ is a morphism of non-Σ operads in Sp^Σ or Ch_k and $X\in\mathsf{Lt}_\mathcal{O}$ (resp. $X\in\mathsf{Alg}_\mathcal{O}$) such that one of the following is true:*

- (a) *the simplicial bar construction $\mathrm{Bar}(\mathcal{O},\mathcal{O},X)$ is objectwise cofibrant in $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$), or*
- (b) *the simplicial bar construction $\mathrm{Bar}(\mathcal{O},\mathcal{O},X^c)$ is objectwise cofibrant in $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$) for some functorial factorization $\emptyset\longrightarrow X^c\longrightarrow X$ in $\mathsf{Lt}_\mathcal{O}$ giving a cofibration followed by a weak equivalence,*

then there is a zig-zag of weak equivalences

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}} X\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad \mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}} (X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))\Big)$$

natural in such X.

Proof. Argue as in the proof of Theorem 1.2. □

Theorem 9.5. *Let k be a field. If $f:\mathcal{O}\longrightarrow\mathcal{O}'$ is a morphism of non-Σ operads in Ch_k and X is an $\mathcal{O}$-algebra, then there is a zig-zag of weak equivalences*

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}} (X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$

natural in X. In particular, Quillen homology $\mathsf{QH}(X)\simeq\mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$ provided that $\mathcal{O}'=I$, $\mathcal{O}[\mathbf{0}]=*$, and $\mathcal{O}[\mathbf{1}]=\mathsf{k}$.

Proof. Since k is a field, it follows that every object in Ch_k is cofibrant. Hence the conditions of Theorem 9.4(a) are satisfied. □

Theorem 9.6. *Let k be a commutative ring. If $\mathcal{O}$ is a non-Σ operad in Sp^Σ or Ch_k and X is a simplicial left $\mathcal{O}$-module (resp. simplicial $\mathcal{O}$-algebra), then there is a zig-zag of weak equivalences*

$$\Phi\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}} X\simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\Phi X$$
$$\Big(\textit{resp.}\quad \Phi\operatorname*{hocolim}^{\mathsf{Alg}_\mathcal{O}}_{\Delta^{\mathrm{op}}} X\simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\Phi X\Big)$$

natural in X, with Φ the forgetful functor.

Proof. Argue as in Section 7 and the proof of Theorem 1.4, together with Proposition 9.10. □

Theorem 9.7. *Let k be a commutative ring. If $\mathcal{O}$ is a non-Σ operad in Sp^Σ or Ch_k and $X\in\mathsf{Lt}_\mathcal{O}$ (resp. $X\in\mathsf{Alg}_\mathcal{O}$), then there is a zig-zag of weak equivalences*

$$\operatorname*{hocolim}^{\mathsf{Lt}_\mathcal{O}}_{\Delta^{\mathrm{op}}}\mathrm{Bar}(\mathcal{O},\mathcal{O},X)\simeq X$$
$$\Big(\textit{resp.}\quad \operatorname*{hocolim}^{\mathsf{Alg}_\mathcal{O}}_{\Delta^{\mathrm{op}}}\mathrm{Bar}(\mathcal{O},\mathcal{O},X)\simeq X\Big)$$

in $\mathsf{Lt}_\mathcal{O}$ (resp. $\mathsf{Alg}_\mathcal{O}$), natural in X.

Proof. Argue as in the proof of Theorem 4.7. □

Theorem 9.8. *Let* k *be a commutative ring. If* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *is a map of non-*Σ *operads in* Sp^Σ *or* Ch_k*, then the adjunction*

$$\mathsf{Lt}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'}, \qquad \Big(\textit{resp.} \quad \mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'},\Big)$$

is a Quillen adjunction with left adjoints on top and f^* *the forgetful functor. If furthermore,* f *is an objectwise weak equivalence, then the adjunction is a Quillen equivalence, and hence induces an equivalence on the homotopy categories.*

Proof. Argue as in the proof of [20, Theorem 1.4 and Proposition 4.41], together with Proposition 9.9. □

In the next proposition, we use notation from [19] for non-Σ operads and their underlying sequences.

Proposition 9.9. *Let* k *be a commutative ring. Consider sequences in* Sp^Σ *or* Ch_k*. If* $Z\in\mathsf{Seq}$ *is cofibrant, then the functor*

$$-\,\hat{\circ}\,Z:\mathsf{Seq}\longrightarrow\mathsf{Seq}$$

preserves weak equivalences.

Proof. Consider the case of Sp^Σ. We know that smashing with a cofibrant symmetric spectrum preserves weak equivalences. Consider the case of Ch_k. We know that tensoring with a cofibrant chain complex preserves weak equivalences. Use [19, Proposition 5.4] together with [19, Theorem 12.4] and the assumption that Z is cofibrant in Seq to finish off the proof. □

Proposition 9.10. *Let* k *be a commutative ring. Consider sequences in* Sp^Σ *or* Ch_k*. If the map* $i:X\longrightarrow Y$ *in* [19, Proposition 13.35] *is a generating cofibration in* Seq*, then* j_t *is a monomorphism for each* $t\geq 1$*.*

Proof. Consider the case of Sp^Σ. This follows from [19, Proposition 13.41]. Consider the case of Ch_k. After forgetting differentials, the map $i:X\longrightarrow Y$ has the form $i:S\longrightarrow S\amalg S'$, and hence (forgetting differentials) $Q^t_{t-1}\longrightarrow Y^{\hat{\otimes}t}$ has a left inverse in Seq for each $t\geq 1$, which finishes the proof. □

10. Operads in chain complexes over a commutative ring

In this section, we indicate some of the analogous results for operads in chain complexes over a commutative ring. The main difficulty here is that an appropriate model category structure on modules and algebras over an arbitrary operad may not exist. On the other hand, sometimes it is possible to establish an appropriate homotopy theory for particularly nice operads. One approach to this is studied in [3]. As indicated in the statements below, it is assumed that a suitable model category structure is available.

10.1. **Model category structures.** Here we recall the model category structure on symmetric sequences [19, Theorem 2.2] that comes up when using the free-forgetful adjunction in Proposition 3.11 to create a model category structure on modules over an operad, if such a model structure exists.

Theorem 10.1. *Let* k *be a commutative ring. Then* SymSeq *has a natural model category structure. The weak equivalences and fibrations in this model structure are inherited in an appropriate sense from the homology isomorphisms and the dimensionwise surjections in* Ch_{k}.

10.2. **Analogous results.**

Theorem 10.2. *Let* k *be a commutative ring. Let* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *be a morphism of operads in* Ch_{k} *and suppose the free-forgetful adjunction creates a model category structure on* $\mathsf{Lt}_{\mathcal{O}}$ *and* $\mathsf{Lt}_{\mathcal{O}'}$ *resp. (*$\mathsf{Alg}_{\mathcal{O}}$ *and* $\mathsf{Alg}_{\mathcal{O}'}$*). If* $X\in\mathsf{Lt}_{\mathcal{O}}$ *(resp.* $X\in\mathsf{Alg}_{\mathcal{O}}$*) such that one of the following is true:*

(a) *the simplicial bar construction* $\mathrm{Bar}(\mathcal{O},\mathcal{O},X)$ *is objectwise cofibrant in* $\mathsf{Lt}_{\mathcal{O}}$ *(resp.* $\mathsf{Alg}_{\mathcal{O}}$*), or*
(b) *the simplicial bar construction* $\mathrm{Bar}(\mathcal{O},\mathcal{O},X^c)$ *is objectwise cofibrant in* $\mathsf{Lt}_{\mathcal{O}}$ *(resp.* $\mathsf{Alg}_{\mathcal{O}}$*) for some functorial factorization* $\emptyset\longrightarrow X^c\longrightarrow X$ *in* $\mathsf{Lt}_{\mathcal{O}}$ *giving a cofibration followed by a weak equivalence,*

then there is a zig-zag of weak equivalences

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}X\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$
$$\Big(\textit{resp.}\quad \mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}(X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))\Big)$$

natural in such X.

Proof. Argue as in the proof of Theorem 1.2. □

Theorem 10.3. *Let* k *be a field. Let* $f:\mathcal{O}\longrightarrow\mathcal{O}'$ *be a morphism of operads in* Ch_{k} *and suppose the free-forgetful adjunction creates a model category structure on* $\mathsf{Alg}_{\mathcal{O}}$ *and* $\mathsf{Alg}_{\mathcal{O}'}$*. Then there is a zig-zag of weak equivalences*

$$\mathcal{O}'\circ^{\mathsf{L}}_{\mathcal{O}}(X)\simeq \mathrm{R}(\mathrm{Bar}(\mathcal{O}',\mathcal{O},X))$$

natural in X*. In particular, Quillen homology* $\mathsf{QH}(X)\simeq\mathrm{R}(\mathrm{Bar}(I,\mathcal{O},X))$ *provided that* $\mathcal{O}'=I$, $\mathcal{O}[\mathbf{0}]=*$, *and* $\mathcal{O}[\mathbf{1}]=\mathsf{k}$.

Proof. Since k is a field, it follows that every object in Ch_{k} is cofibrant. Theorem 10.2(a) finishes the proof. □

Theorem 10.4. *Let* k *be a commutative ring. Let* $\mathcal{O}$ *be an operad in* Ch_{k} *and suppose the free-forgetful adjunction creates a model category structure on* $\mathsf{Lt}_{\mathcal{O}}$ *(resp.* $\mathsf{Alg}_{\mathcal{O}}$*). If* X *is a simplicial left* $\mathcal{O}$*-module (resp. simplicial* $\mathcal{O}$*-algebra), then there is a zig-zag of weak equivalences*

$$\Phi\,\operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}X\simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\Phi X$$
$$\Big(\textit{resp.}\quad \Phi\,\operatorname*{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}X\simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}}\Phi X\Big)$$

natural in X*, with* Φ *the forgetful functor.*

Proof. Argue as in Section 7 and the proof of Theorem 1.4, together with Proposition 10.8. □

Theorem 10.5. *Let k be a commutative ring. Let $\mathcal{O}$ be an operad in Ch_{k} and suppose the free-forgetful adjunction creates a model category structure on $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$). If $X \in \mathsf{Lt}_{\mathcal{O}}$ (resp. $X \in \mathsf{Alg}_{\mathcal{O}}$), then there is a zig-zag of weak equivalences*

$$\operatorname*{hocolim}^{\mathsf{Lt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(\mathcal{O},\mathcal{O},X) \simeq X$$
$$\Big(\textit{resp.}\quad \operatorname*{hocolim}^{\mathsf{Alg}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}} \mathrm{Bar}(\mathcal{O},\mathcal{O},X) \simeq X\Big)$$

in $\mathsf{Lt}_{\mathcal{O}}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$), natural in X.

Proof. Argue as in the proof of Theorem 4.7. □

Theorem 10.6. *Let k be a commutative ring. Let $f : \mathcal{O} \longrightarrow \mathcal{O}'$ be a morphism of operads in Ch_{k} and suppose the free-forgetful adjunction creates a model category structure on $\mathsf{Lt}_{\mathcal{O}}$ and $\mathsf{Lt}_{\mathcal{O}'}$ (resp. $\mathsf{Alg}_{\mathcal{O}}$ and $\mathsf{Alg}_{\mathcal{O}'}$). Then the adjunction*

$$\mathsf{Lt}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Lt}_{\mathcal{O}'}, \qquad \Big(\textit{resp.}\quad \mathsf{Alg}_{\mathcal{O}} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathsf{Alg}_{\mathcal{O}'},\Big)$$

is a Quillen adjunction with left adjoint on top and f^* the forgetful functor. If furthermore, f is an objectwise weak equivalence and both $\mathcal{O}$ and $\mathcal{O}'$ are cofibrant in the underlying category SymSeq, then the adjunction is a Quillen equivalence, and hence induces an equivalence on the homotopy categories.

Proof. Argue as in the proof of [20, Theorem 1.4 and Proposition 4.41], together with Proposition 10.7. □

Proposition 10.7. *Let k be a commutative ring. Consider symmetric sequences in Ch_{k}. If $Z \in \mathsf{SymSeq}$ is cofibrant, then the functor*

$$- \circ Z : \mathsf{SymSeq} \longrightarrow \mathsf{SymSeq}$$

preserves weak equivalences between cofibrant objects.

Proof. This follows from [19, Proposition 12.16]. □

Proposition 10.8. *Let k be a commutative ring. Consider symmetric sequences in Ch_{k}. If the map $i : X \longrightarrow Y$ in Proposition 7.10 is a generating cofibration in SymSeq, then j_t is a monomorphism for each $t \geq 1$.*

Proof. After forgetting differentials, the map $i : X \longrightarrow Y$ has the form $i : S \longrightarrow S \amalg S'$, and hence (forgetting differentials) $Q^t_{t-1} \longrightarrow Y^{\check{\otimes} t}$ has a left inverse in $\mathsf{SymSeq}^{\Sigma_t}$ for each $t \geq 1$, which finishes the proof. □

11. Right modules over an operad

In this section we briefly indicate some of the analogous results for $\mathsf{Rt}_{\mathcal{O}}$. Compared to $\mathsf{Lt}_{\mathcal{O}}$ and $\mathsf{Alg}_{\mathcal{O}}$, the corresponding arguments for $\mathsf{Rt}_{\mathcal{O}}$ are substantially less complicated, since colimits in $\mathsf{Rt}_{\mathcal{O}}$ are calculated in the underlying category [19, Section 8.4].

11.1. **Model category structures.** First we establish appropriate model category structures. Similar model structures are considered in [10].

Theorem 11.1. *Let* k *be a commutative ring. Let* $\mathcal{O}$ *be an operad in* Ch_{k}*. Then* $\mathsf{Rt}_{\mathcal{O}}$ *has a natural model category structure. The weak equivalences and fibrations in this model structure are inherited in an appropriate sense from the homology isomorphisms and the dimensionwise surjections in* Ch_{k}*.*

Proof. Argue as in the proof of [19, Theorem 1.4]. To verify the appropriate $\mathsf{Rt}_{\mathcal{O}}$ version of [19, Proposition 13.4] (but without the cofibrancy assumption), use Proposition 11.4 and note that pushouts in $\mathsf{Rt}_{\mathcal{O}}$ are calculated in the underlying category. Use the property that tensoring with a cofibrant chain complex preserves weak equivalences, and note that the generating (acyclic) cofibrations have cofibrant domains. □

Theorem 11.2. *Let* $\mathcal{O}$ *be an operad in* Sp^{Σ}*. Then* $\mathsf{Rt}_{\mathcal{O}}$ *has a natural model category structure. The weak equivalences and fibrations in this model structure are inherited in an appropriate sense from the stable weak equivalences and the stable flat fibrations in* Sp^{Σ}*.*

Theorem 11.3. *Let* $\mathcal{O}$ *be an operad in* Sp^{Σ}*. Then* $\mathsf{Rt}_{\mathcal{O}}$ *has a natural model category structure. The weak equivalences and fibrations in this model structure are inherited in an appropriate sense from the stable weak equivalences and the stable fibrations in* Sp^{Σ}*.*

Proof. Argue as in the proof of [19, Theorem 1.4]. To verify the appropriate $\mathsf{Rt}_{\mathcal{O}}$ version of [19, Proposition 13.4] (but without the cofibrancy assumption), use Proposition 11.4 and note that pushouts in $\mathsf{Rt}_{\mathcal{O}}$ are calculated in the underlying category. Use the property that smashing with a cofibrant symmetric spectrum in the stable flat model structure preserves weak equivalences, and note that the generating (acyclic) cofibrations have cofibrant domains. □

Proposition 11.4. *Let* k *be a commutative ring. Consider symmetric sequences in* Sp^{Σ} *or* Ch_{k}*. Let* $B \in \mathsf{SymSeq}$*. If* $X \longrightarrow Y$ *in* SymSeq *is a generating (acyclic) cofibration, then* $X \circ B \longrightarrow Y \circ B$ *is a monomorphism.*

Proof. In both cases, this follows from Definition 3.4. In the case of chain complexes, first forget differentials. See the proof of [19, Theorem 12.2] for the particular form of the generating (acyclic) cofibrations. □

11.2. **Analogous results for right modules over operads.**

Theorem 11.5. *Let* k *be a commutative ring. If* $\mathcal{O}$ *is an operad in* Sp^{Σ} *or* Ch_{k} *and* X *is a simplicial right* $\mathcal{O}$*-module, then there is a zig-zag of weak equivalences*

$$\Phi \operatorname*{hocolim}_{\Delta^{\mathrm{op}}}^{\mathsf{Rt}_{\mathcal{O}}} X \simeq \operatorname*{hocolim}_{\Delta^{\mathrm{op}}} \Phi X$$

natural in X*, with* Φ *the forgetful functor.*

Proof. Argue as in the proof of Theorems 1.4 and 7.1, and Proposition 5.30, except replace (5.31) with pushout diagrams of the form

$$\begin{array}{ccc} \Delta[z]\cdot X\circ\mathcal{O} & \longrightarrow & Z_0 \\ \big\downarrow & & \big\downarrow \\ \Delta[z]\cdot Y\circ\mathcal{O} & \longrightarrow & Z_1 \end{array}$$

in $\mathsf{sRt}_{\mathcal{O}}$, with $X\longrightarrow Y$ a generating cofibration in SymSeq, and note that pushouts in $\mathsf{sRt}_{\mathcal{O}}$ are calculated in the underlying category $\mathsf{sSymSeq}$. □

Theorem 11.6. *Let* k *be a commutative ring. Let* $\mathcal{O}$ *be an operad in* Sp^{Σ} *or* Ch_{k} *and* $X\in\mathsf{Rt}_{\mathcal{O}}$*. There is a zig-zag of weak equivalences*

$$\operatorname*{hocolim}^{\mathsf{Rt}_{\mathcal{O}}}_{\Delta^{\mathrm{op}}}\mathrm{Bar}(X,\mathcal{O},\mathcal{O})\simeq X$$

in $\mathsf{Rt}_{\mathcal{O}}$*, natural in* X*.*

Proof. Argue as in the proof of Theorem 4.7. □

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