STABLE SPLITTINGS FOR CLASSIFYING SPACES OF ALTERNATING, SPECIAL ORTHOGONAL AND SPECIAL UNITARY GROUPS

by

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Abstract

Let $G(n)$ denote either the symmetric group $\Sigma(n)$, the orthogonal group $O(n)$ or the unitary group $U(n)$ and let $SG(n)$ denote either the alternating group $A(n)$, the special orthogonal group $SO(n)$ or the special unitary group $SU(n)$. The classifying spaces $BG(n)$ are known to split stably as $BG(n) \simeq \bigvee_{\ell=1}^{n} BG(\ell)/BG(\ell-1)$. We consider the case of $BSG(n)$ and prove that, after localizing at any prime p, there are similar although somewhat coarser splittings. E.g. we get a stable 2-local splitting $BA(4n) \simeq BA(4n)/BA(4n-2) \vee \bigvee_{\ell=1}^{n-1} BA(4\ell+2)/BA(4\ell-2)$. A crucial ingredient in our proof is a careful study, for finite p- groups P, of the morphism sets $\mathrm{mor}_{\mathcal{A}}(P, SG(n))$ in the "Burnside category" $\mathcal{A}$, and in particular the effect of transfers on these sets.

Introduction

Stable splittings of classifying spaces have received a lot of attention in the last decade. A good introductory exposition is given in [P]. Perhaps the first known splittings were those for the symmetric groups, $B\Sigma(n) \simeq \bigvee_{\ell=1}^{n} B\Sigma(\ell)/B\Sigma(\ell-1)$, which can be obtained via a geometric version of Dold's proof [D] of Nakaoka's [Na] decomposition theorem for the homology of symmetric groups (cf. [KP]). (As usual here and in the following quotients like $B\Sigma(\ell)/B\Sigma(\ell-1)$ are to be interpreted as cofibres of the maps $B\Sigma(\ell-1) \to B\Sigma(\ell)$ induced by the inclusion $\Sigma(\ell-1) \to \Sigma(\ell)$.) In [MP] Mitchell and Priddy show how to extend these

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methods and they produce stable splittings $BG(n) \simeq \bigvee_{\ell=1}^{n} BG(\ell)/BG(\ell-1)$ with $G(\ell) = O(\ell)$, $U(\ell)$, $Sp(\ell)$ or $GL(\ell, \mathbb{F}_q)$. In the last case one has to invert p if $q = p^k$.

In this paper we prove the following

THEOREM 0. *For each natural number n there are p–local stable splittings of the form*

$$BSG(n) \underset{(p)}{\simeq} \bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell-\delta(\ell))$$

in each of the following cases:

(0.1.) $p = 2$, $SG(n) = A(n)$, the alternating group on n letters, and

$$\delta(\ell) = \begin{cases} 4 & \text{if } \ell \equiv 2 \bmod 4 \\ 0 & \text{if } \ell \not\equiv 2 \bmod 4 \text{ and } \ell \neq n \\ k & \text{if } \ell \equiv 2 + k \bmod 4,\ k \in \{1,2,3\} \text{ and } \ell = n \end{cases}$$

(0.2.) $p > 2$, $SG(n) = A(n)$, and

$$\delta(\ell) = \begin{cases} p & \text{if } \ell \equiv 2 \bmod p \\ 0 & \text{if } \ell \not\equiv 2 \bmod p \text{ and } \ell \neq n, \\ k & \text{if } \ell \equiv 2 + k \bmod p,\ k \in \{1,2,\dots,p-1\} \text{ and } \ell = n \end{cases}$$

(0.3.) $p = 2$, $SG(n) = SO(n)$, the special orthogonal group of real $n \times n$–matrices, and

$$\delta(\ell) = \begin{cases} 2 & \text{if } \ell \equiv 1 \bmod 2\ , \\ 0 & \text{if } \ell \equiv 0 \bmod 2 \text{ and } \ell \neq n, \\ 1 & \text{if } \ell \equiv 0 \bmod 2 \text{ and } \ell = n \end{cases}$$

(0.4.) p any prime, $SG(n) = SU(n)$, the special unitary group of complex $n \times n$–matrices, and

$$\delta(\ell) = \begin{cases} 2 & \text{if } \ell \equiv 1 \bmod p\ , \\ 0 & \text{if } \ell \equiv 0 \bmod p \text{ and } \ell \neq n, \\ 1 & \text{if } \ell \not\equiv 0,1 \bmod p;\ \text{or } \ell \equiv 0 \bmod p \text{ and } \ell = n \end{cases}$$

(By convention $SG(m)$ will be the trivial group if $m \leq 0$.)

EXAMPLES

1) $BA(4n) \simeq BA(4n)/BA(4n-2) \vee \bigvee_{\ell=1}^{n-1} BA(4\ell+2)/BA(4\ell-2)$, after localization at 2.

2) $BSO(2n+1) \simeq \bigvee_{\ell=1}^{n} BSO(2\ell+1)/BSO(2\ell-1)$, after localization at 2. After inverting 2, Snaith [Sn] and Mitchell and Priddy [MP] constructed also stable splittings $BSO(2n+1) \simeq \bigvee_{\ell=1}^{n} BSO(2\ell+1)/BSO(2\ell-1)$. Is there a global splitting of this form?

REMARKS

a) The proof of Theorem 0 will actually show that the natural “filtration” of $BSG(n)$ by the $BSG(\ell)$ with $\delta(\ell) \neq 0$ splits (stably and after localization) and hence we may include the limiting case $n = \infty$ in Theorem 0.

b) The splittings of Theorem 0 are coarser than those for $\Sigma(n)$, $O(n)$ and $U(n)$, where $\delta(\ell)$ was always 1. The following observations show that this is to be expected, at least for $A(n)$ and $SO(n)$.

   The restriction maps $H^*(BA(4n+2);\mathbb{F}_2) \to H^*(BA(4n);\mathbb{F}_2)$ and $H^*(BA(pn+2);\mathbb{F}_p) \to H^*(BA(pn);\mathbb{F}_p)$ are not epi if $n \geq 1$ (cf. Remark 4.2.(b) below).

   The restriction map $H^*(BSO(2n+1);\mathbb{F}_2) \to H^*(BSO(2n);\mathbb{F}_2)$ is not split epi as a homomorphism of modules over the Steenrod algebra (for this consider the action of Sq^1 on the Stiefel–Whitney class w_{2n}).

c) It can be shown (cf. [B]) that the inclusion $A(n) \to A(n+1)$ induces an isomorphism in $\bmod p$ cohomology if $n \not\equiv 1, 3 \bmod 4$ and $p = 2$, or $n \not\equiv 1, p-1 \bmod p$ and $p > 2$. Then $BA(n)$ and $BA(n+1)$ are stably homotopy equivalent at p and hence there are other choices of $\delta(\ell)$ which still describe the same splittings.

d) If $n \not\equiv 0, 1 \bmod p$, $p > 2$, then our splitting of $BA(n)$ is an easy consequence of the splitting of $B\Sigma(n)$ and the fact that the inclusion $A(n) \to \Sigma(n)$ induces a p–local stable equivalence in this case (cf. [B]).

The paper is organized as follows. In Section 1 we construct the splitting maps $h : BSG(n) \to \bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell - \delta(\ell))$ by a modification of the construction in the case of $BG(n)$. Our proof of Theorem 0 then depends on studying the composition of h with (suitable stable) maps $BP \to BSG(n)$ where P runs through the class of finite p–groups. The splitting map itself is a Section 2 we will describe a convenient algebraic set up for transfer calculations (cf. [AGM], [Ma], [Ni] and [HLS]) and state an algebraic analogue (Theorem 2.7) of Theorem 0 in this set up. In Section 3 we show how this leads to a proof of Theorem 0 and in Section 4 we give a proof of Theorem 2.7.

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1. Construction of the splitting maps

In the following all spaces and maps are to be considered in the stable category of CW–spectra. We denote X with a disjoint base point by X^+. Stably X^+ and $X \vee S^0$ are equivalent. Let $(G(n), SG(n))$ be either $(\Sigma(n), A(n))$ or $(O(n), SO(n))$ or $(U(n), SU(n))$, and let i_n denote the canonical inclusion $SG(n) \to G(n)$ and Bi_n the induced map $BSG(n)^+ \to BG(n)^+$.

Let det : $G(n) \to G(\epsilon)$ denote either the signum or the determinant map, i.e. $\epsilon = 2$ in case $G(n) = \Sigma(n)$ and $\epsilon = 1$ in the other two cases. Then there is another canonical inclusion $\widetilde{i} : G(n) \times G(\epsilon) \to G(n+\epsilon)$ given by juxtaposition and the homomorphism

$$G(n) \xrightarrow{(id, \det^{-1})} G(n) \times G(\epsilon) \xrightarrow{\widetilde{i}} G(n+\epsilon)$$

has image in $SG(n+\epsilon)$. We denote the resulting homomorphism $G(n) \to SG(n+\epsilon)$ and the corresponding stable map by $d_{n+\epsilon}$ resp. $Bd_{n+\epsilon}$.

Furthermore we write $tr_{n,m}$ for the transfer belonging to the inclusion $G(n) \times G(m) \to G(n+m)$ given again by juxtaposition; $tr_{n,m}$ is a stable map $BG(n+m)^+ \to (BG(n) \times BG(m))^+$. Finally, the projection map $G(n) \times G(m) \to G(n)$ resp. $(BG(n) \times BG(m))^+ \to BG(n)^+$ will be denoted by $\pi_{n,m}$ resp. $B\pi_{n,m}$ or simply by π resp. $B\pi$.

We will now define the components of a splitting map

$$h : BSG(n) \to \bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell - \delta(\ell))$$

(with $\delta(\ell)$ as in Theorem 0) as the composition of a map $h_{n,\ell} : BSG(n)^+ \to BSG(\ell)^+$ with the inclusion $BSG(n) \to BSG(n)^+$ and the canonical projection q_ℓ of $BSG(\ell)^+$ to the quotient $BSG(\ell)^+/BSG(\ell-\delta(\ell))^+ = BSG(\ell)/BSG(\ell - \delta(\ell))$. The map $h_{n,\ell}$ is defined as

— identity if $n = \ell$,

— as the following composition if $\epsilon < \ell < n$.

$$\begin{aligned} &BSG(n)^+ \xrightarrow{Bi_n} BG(n)^+ \xrightarrow{tr_{\ell-\epsilon,n+\epsilon-\ell}} (BG(\ell-\epsilon) \times BG(n+\epsilon-\ell))^+ \\ &\xrightarrow{B\pi} BG(\ell-\epsilon)^+ \xrightarrow{Bd_\ell} BSG(\ell)^+. \end{aligned}$$

— any map if $\ell \leq \epsilon$. (Then $SG(\ell)$ and the corresponding wedge summand are trivial.)

It will be convenient to consider the map

$$h^+ : BSG(n)^+ \to \left(\bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell - \delta(\ell)) \right) \vee S^0$$

with components $q_\ell h_{n,\ell}$, $1 \le \ell \le n$, and $BSG(n)^+ \to B\{e\}^+ \simeq S^0$, induced by the trivial homomorphism $SG(n) \to \{e\}$. Clearly, h^+ is a p–local equivalence if and only if h is one.

2. $A(P, G)$

We will study h^+ by composing it with maps $BP^+ \to BSG(n)^+$ of the form $BP^+ \xrightarrow{tr} BQ^+ \xrightarrow{B\varphi} BSG(n)^+$, where Q is a subgroup of P, $tr = tr_P^Q$ the transfer associated to the inclusion $Q \subset P$ and $\varphi : Q \to SG(n)$ a group homomorphism.

In this section we will describe the appropriate algebraic set up for dealing with such compositions (cf. [AGM], [Ma], [Ni] and [HLS]). The crucial input is the definition of a transfer homomorphism which is modelled after the double coset formula.

2.1. For a fixed finite group P and a compact Lie group G let $A(P, G)$ be the free abelian group with basis the equivalence classes of pairs (Q, φ) consisting of a subgroup Q of P and a homomorphisms $\varphi : Q \to G$. Here the pairs (Q, φ) and (Q', φ') are called equivalent if and only if the groups $\Delta\varphi = \{(q, \varphi q) \mid q \in Q\}$ and $\Delta\varphi' = \{(q', \varphi q') \mid q' \in Q'\}$ are conjugate as subgroups of $P \times G$. The equivalence class of (Q, φ) will be denoted by $[Q, \varphi]$ (or, more precisely, by $[Q, \varphi]_{(P,G)}$). It is easy to see, say by a minor modification of the argument given in [Q2, Lemma 6.3], that the set of these equivalence classes is finite. Therefore, $G \mapsto A(P, G)$ defines a functor from the category $\mathcal{G}_{cL}$ of compact Lie groups to the category $\mathcal{A}b_f$ of finitely generated free abelian groups. Furthermore, for a closed subgroup H of G, there is a transfer homomorphism $\tau = \tau_G^H : A(P, G) \to A(P, H)$ which is defined on a basis element $[Q, \varphi]$ as follows.

Q acts via φ on the homogeneous space G/H, and the double coset space $Q\backslash G/H$ may be decomposed as a disjoint union of "orbit type manifold components" M_i ([F]). More precisely, if Q' runs through a set of representatives of conjugacy classes of subgroups of Q and if G/H is broken up into the disjoint union of the subspaces $(G/H)_{(Q')}$, consisting of those points in G/H whose Q–orbits are isomorphic to Q/Q', then the M_i are the connected components of the corresponding orbit spaces $Q\backslash(G/H)_{(Q')}$. Let $\chi^\#(M_i)$ be the "internal Euler characteristic" of M_i, i.e. $\chi^\#(M_i) = \chi(\overline{M}_i) - \chi(\overline{M}_i - M_i)$ with $\overline{M}_i$ denoting the closure of M_i in $Q\backslash G/H$. Finally choose a representative $g_i \in G$ of any element in M_i so that the isotropy subgroup of g_iH is Q_i. Then we define

$$\tau_G^H([Q, \varphi]) = \sum_i \chi^\#(M_i)[Q_i, g_i^{-1}\varphi g_i]. \tag{2.2.}$$

(Note that $g_i^{-1}\varphi g_i$ maps Q_i into H !) It is easy to see that the right hand side depends neither on the particular choice of g_i (e.g. because the space of equivalence classes is finite, in particular discrete) nor on the chosen representative of the equivalence class $[Q,\varphi]$.

We remark that for G a finite group $A(P,Q)$ agrees with the set of morphisms from P to G in the "Burnside category" as defined in [AGM] .

2.3. This formula for the transfer is fairly complicated. We will now focus on the "leading term" of the transfer and show that it has a more manageable description.

Subconjugation defines a partial order on the set of conjugacy classes (Q) of subgroups of P. From this partial order we get a filtration of $A(P,G)$ by defining $F_{(Q)}A(P,G)$ as the subgroup generated by the classes $[Q',\varphi]$ with $(Q')\leq(Q)$. We let $\overline{F}_{(Q)}A(P,Q)$ denote the quotient of $F_{(Q)}A(P,Q)$ by the subgroups of lower filtration. Obviously $G\to\overline{F}_{(Q)}A(P,G)$ is also a functor from $\mathcal{G}_{cL}$ to $\mathcal{A}b_f$ and (2.2) shows that the transfer for $A(P,?)$ induces one for $\overline{F}_{(Q)}A(P,?)$ which will be denoted by $\overline{\tau}$.

Next we will give a different description of $\overline{F}_{(Q)}$. Let $\mathrm{Rep}(Q,G)$ denote the set of G–conjugacy classes of homomorphisms from Q to G. The class of a homomorphism φ will be denoted by $[\varphi]$ (or more precisely by $[\varphi]_G$). The normalizer $N_P(Q)$ of Q in P acts on $\mathrm{Rep}(Q,G)$ via conjugation. We denote the quotient of $\mathrm{Rep}(Q,G)$ with respect to this action by $\mathrm{Rep}_P(Q,G)$.

Then the map $\mathrm{Rep}(Q,G)\to A(P,G)$, $[\varphi]\mapsto[Q,\varphi]$ induces a natural isomorphism between $\mathbb{Z}[\mathrm{Rep}_P(Q,G)]$ and $\overline{F}_{(Q)}A(P,G)$. (Here and in the following the free abelian group on a set S will be denoted by $\mathbb{Z}[S]$). Thus we get a transfer $\overline{\tau}$ on $\mathbb{Z}[\mathrm{Rep}_P(Q,?)]$ which we will now describe.

For homomorphisms $\varphi:Q\to G$ and $\psi:Q\to H$ let $(G/H)^{\varphi}_{\psi}=\{gH\mid[g^{-1}\varphi g]_H=[\psi]_H\}$ consist of those cosets gH which conjugate $[\varphi]_G$ into $[\psi]_H$ and let $[\varphi:\psi]$ denote the Euler characteristic of $(G/H)^{\varphi}_{\psi}$. Define $\tau=\tau_G^H:\mathbb{Z}[\mathrm{Rep}(Q,G)]\to\mathbb{Z}[\mathrm{Rep}(Q,H)]$ via

$$(2.4.)\qquad [\varphi]\longmapsto\sum_{[\psi]\in\mathrm{Rep}(Q,H)}[\varphi:\psi][\psi].$$

(cf. [HLS]). Clearly, τ_G^H induces a homomorphism $\tau_G'^H:\mathbb{Z}[\mathrm{Rep}_P(Q,G)]\to\mathbb{Z}[\mathrm{Rep}_P(Q,H)]$.

PROPOSITION 2.5. *$\overline{\tau}_G^H$ and $\tau_G'^H$ agree.*

Proof: Consider $\overline{\tau}_G^H$. By (2.2) we only have to consider the contributions coming from the components of the fixed points $(G/H)^Q$. These components are closed, hence $\chi^{\#}(M_i)$ and $\chi(M_i)$ agree. Now the proposition follows from the observation that $(G/H)^{\varphi}_{\psi} = \coprod_{[g_i^{-1}\varphi g_i]=[\psi]} M_i$.

$\square$

The following Lemma will be useful in Section 4 for computing the coefficients $[\varphi : \psi]$ in concrete cases. The proof is straightforward and left to the reader.

LEMMA 2.6. *Suppose $(G/H)^{\varphi}_{\psi}$ is not empty and let $g_0 \in G$ be any element satisfying $g_0^{-1}\varphi g_0 = \psi$. Then the map*

$$C_G(\psi)/C_G(\psi) \cap H \longrightarrow (G/H)^{\varphi}_{\psi}$$

induced by

$$g \longmapsto g_0 g H$$

is a homeomorphism. (Here $C_G(\psi)$ denotes the centralizer of the image of ψ in G, i.e. $C_G(\psi) = \{g \in G \mid g\psi(q) = \psi(q)g$ for all $q \in Q\}$.)

$\square$

We will finish this section by stating the algebraic analogue of Theorem 0. For this let P be a finite p–group and p, $SG(\ell)$ and $\delta(\ell)$ be as in Theorem 0.

We will see in 4.1 that for such P the homomorphism $A(P, SG(\ell - \delta(\ell)) \to A(P, SG(\ell))$ induced by inclusion is mono. In Theorem 2.7 below we will identify $A(P, SG(\ell - \delta(\ell))$ with its image in $A(P, SG(\ell))$.

As in Section 1 we introduce homomorphisms $h_{n,\ell} : A(P, SG(n)) \to A(P, SG(\ell))$. For $\ell = n$ we define $h_{n,n} = id$. For $0 < \ell \leq \epsilon$ we let $h_{n,\ell}$ be trivial, and for $\epsilon < \ell < n$ we define $h_{n,\ell}$ as composition $d_\ell \pi \tau_{\ell-\epsilon,n+\epsilon-\ell} i_n$ where d_ℓ, π and i_n are as in Section 1 and $\tau_{\ell-\epsilon,n+\epsilon-\ell}$ denotes the transfer $\tau^{G(\ell-\epsilon)\times G(n+\epsilon-\ell)}_{G(n)}$.

THEOREM 2.7. *For each natural number n and each finite p–group P the homomorphism*

$$h^+ : A(P, SG(n)) \to \left(\bigoplus_{\ell=1}^{n} A(P, SG(\ell))/A(P, SG(\ell - \delta(\ell))) \right) \oplus A(P, \{e\}) ,$$

with components induced by $h_{n,\ell}$ respectively the trivial homomorphism $P \to \{e\}$, becomes an isomorphism after tensoring with $\mathbb{Z}/p$.

The proof of Theorem 2.7 will be given in Section 4. In the following the target of h^+ will be denoted by $C(P, SG(n))$, and the component $A(P, SG(n)) \to A(P, \{e\}) = A(P, SG(0))$ by $h_{n,0}$.

3. Proof of Theorem 0

The following folklore proposition gives the justification for Section 2. As there P will be a finite group, G a compact Lie group and H a closed subgroup of G.

PROPOSITION 3.1. *There are homomorphisms $\lambda = \lambda_{P,G}$ from $A(P,G)$ to the group $\{BP^+, BG^+\}$ of homotopy classes of stable maps from BP^+ to BG^+, defined by $\lambda_{P,G}[Q,\varphi] = B\varphi \circ tr_P^Q : BP^+ \to BQ^+ \to BG^+$.*

These homomorphisms are natural in P and Q and commute with transfers, i.e. $tr_G^H \circ \lambda_{P,G} = \lambda_{P,H} \circ \tau_G^H$.

Proof: Inner automorphisms induce selfmaps on BP^+ and BG^+ which are homotopic to the identity maps; hence $\lambda_{P,Q}$ is well defined. Naturality is trivial and compatibility with the transfer follows from the double coset formula [F]. (For the existence of transfers $BG^+ \to BH^+$ in the compact Lie group case we refer to [C].)

□

We remark that Lewis, May and McClure [LMM] have shown that $\lambda_{P,Q}$ induces an isomorphism between a suitable completion of $A(P,G)$ and $\{BP^+, BG^+\}$ if G is finite. However, we will not need this in the sequel.

We begin with the proof of Theorem 0, assuming Theorem 2.7. For the remainder of this section let p, $SG(\ell)$ and $\delta(\ell)$ be as in Theorem 0.

3.2. If P_ℓ is any finite p–group and $\varphi_\ell : P_\ell \to SG(\ell)$, $0 \le \ell \le n$, are any homomorphisms (recall that $SG(0) = \{e\}$ by convention), then Theorem 2.7 implies that there are elements $x_\ell \in A(P_\ell, SG(n))$ and $x'_\ell \in C(P_\ell, SG(n))$ such that

$$\overline{[P_\ell, \varphi_\ell]} = h^+(x_\ell) + px'_\ell,$$

where $\overline{[P_\ell, \varphi_\ell]}$ denotes the class of $[P_\ell, \varphi_\ell] \in A(P_\ell, SG(\ell))$ in $C(P_\ell, SG(n))$. Then Proposition 3.1 gives a diagram

$$\text{(3.3.)} \qquad \begin{tikzcd} BSG(n)^+ \arrow[r, "h^+"] & \bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell-\delta(\ell)) \vee S^0 \\ \bigvee_{\ell=0}^{n} \lambda(x_\ell) & \\ & \bigvee_{\ell=0}^{n} BP_\ell^+ \arrow[u, "\bigvee_{\ell=0}^{n} q_\ell \circ B\varphi_\ell"'] \end{tikzcd}$$

which commutes after passing to $\bmod p$–cohomology. Now Lemma 3.4 below implies easily that h^+ induces an isomorphism in $\bmod p$–cohomology and Theorem 0 follows.

LEMMA 3.4.

a) *The* $\bmod p$ *cohomology of* $BSG(n)^+$ *and* $\bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell-\delta(\ell)) \vee S^0$ *are of finite type and isomorphic as graded vector spaces.*

b) *There are finite* p*–groups* P_ℓ *and homomorphisms* φ_ℓ *such that* $\bigvee_{\ell=0}^{n} q_\ell \circ B\varphi_\ell$ *induces a monomorphism in* $\bmod p$*–cohomology.*

Proof: We claim that $q_\ell : BSG(\ell) \to BSG(\ell)/BSG(\ell-\delta(\ell))$ induces a monomorphism in reduced $\bmod p$–cohomology or, equivalently, that the map $BSG(\ell-\delta(\ell)) \xrightarrow{Bi} BSG(\ell)$ with i denoting the inclusion $SG(\ell-\delta(\ell)) \to SG(\ell)$ induces an epimorphism in $\bmod p$–cohomology. This is clear for $SO(\ell)$ and $SU(\ell)$ and will be proved below for $A(\ell)$. Now part a) follows immediately.

For b) we let P_ℓ be either a p–Sylow subgroup of $SG(\ell)$, if $SG(\ell) = A(\ell)$, or the maximal p–torus of $SG(\ell)$, if $SG(\ell) = SO(\ell)$ or $SU(\ell)$, consisting of all diagonal matrices of order p and determinant 1. For φ_ℓ we take the inclusion of P_ℓ into $SG(\ell)$. Part b) follows from the well–known fact that these φ_ℓ induce monomorphisms in $\bmod p$–cohomology.

$\square$

It remains to prove the following

LEMMA 3.5. *For* $m \leq \ell$ *the restriction map* $H^*(BA(\ell);\mathbb{F}_p) \to H^*(BA(m);\mathbb{F}_p)$ *is onto provided that* $m \not\equiv 0,1 \bmod 4$ *if* $p=2$ *resp.* $m \not\equiv 0,1 \bmod p$ *if* $p>2$.

Proof: If we replace $A(\ell)$ and $A(m)$ by $\Sigma(\ell)$ and $\Sigma(m)$ then the restriction map is onto for all $m \leq \ell$ by Nakaoka [Na]. Therefore it suffices to show that the restriction map $H^*(B\Sigma(m);\mathbb{F}_p) \to H^*(BA(m);\mathbb{F}_p)$ is onto if $m \not\equiv 0,1 \bmod 4$ resp. $m \not\equiv 0,1 \bmod p$.

For odd primes a p–Sylow subgroup of $A(m)$ is also one of $\Sigma(m)$ and an easy argument (essentially the same as in the proof of Lemma 4.1 below) with stable elements [CE] shows that $H^*(B\Sigma(m);\mathbb{F}_p) \to H^*(BA(m);\mathbb{F}_p)$ is an isomorphism if $m \not\equiv 0,1 \bmod p$. For another proof we refer to [B].

For $p=2$ the Gysin sequence of the S^0–bundle $BA(m) \to B\Sigma(m)$ tells us that it is enough to show that multiplication with the $\bmod 2$ Euler class $e : H^*(B\Sigma(m);\mathbb{F}_2) \to H^*(B\Sigma(m);\mathbb{F}_2)$ is mono. Now $H^*(B\Sigma(m);\mathbb{F}_2)$ is detected by the $\bmod 2$–cohomology of its maximal elementary abelian 2–groups E [Q1] and hence it suffices to show that e restricts nontrivially to each $H^*(BE;\mathbb{F}_2)$, E maximal. Finally, $e \in H^1(B\Sigma(m);\mathbb{F}_2)$ corresponds to the signum map $\Sigma(m) \to \mathbb{Z}/2$, and for $m \not\equiv 0,1 \bmod 4$ each maximal E contains a single transposition (cf. [Mu,p.346f]), hence e restricts nontrivially to $H^*(BE;\mathbb{F}_2)$.

$\square$

4. Proof of Theorem 2.7

We start by showing that the inclusion $j = j_{m,n} : SG(m) \to SG(n)$ induces an injection $j = j_{m,n} : A(P, SG(m)) \to A(P, SG(n))$, at least for those m with $\delta(m) \neq 0$. Then we will identify $A(P, SG(m))$ with its image and show that the filtration of $A(P, SG(n))$ by the subspaces $A(P, SG(m))$ (for those m with $\delta(m) \neq 0$) splits via h^+ after tensoring with $\mathbb{Z}/p$ (cf. Lemma 4.4 and Proposition 4.5 below). This will prove Theorem 2.7 and gives, via the argument of Section 3, the strengthening of Theorem 0 mentioned in Remark a) of the introduction.

Throughout this section P will denote a finite p–group, Q a subgroup of P and p, $SG(m)$ and $\delta(m)$ are as in Theorem 0.

LEMMA 4.1 *The inclusion* $SG(m) \xrightarrow{j} SG(n)$ *induces injections* $\mathrm{Rep}(Q, SG(m)) \xrightarrow{j} \mathrm{Rep}(Q, SG(n))$, $\mathrm{Rep}_P(Q, SG(m)) \xrightarrow{j} \mathrm{Rep}_P(Q, SG(n))$ *and* $A(P, SG(m)) \xrightarrow{j} A(P, SG(n))$ *provided* $\delta(m) \neq 0$ *(or* $m = 0$*).*

<u>Proof:</u> The cases $m = 0$ and $m = n$ are trivial, so we may assume $n > m > 0$. It suffices to prove the statement for $\mathrm{Rep}(Q, ?)$. The other cases are immediate consequences.

It is clear that the inclusions from $G(m)$ to $G(n)$ induce injections because elements in $\mathrm{Rep}(Q, G(m))$ correspond to isomorphism classes of Q–sets of cardinality m (if $G(m) = \Sigma(m)$) resp. m–dimensional real or complex representations of Q (if $G(m) = O(m)$ or $U(m)$) and inclusion corresponds to adding a trivial Q–set of cardinality $n - m$ resp. a trivial representation of dimension $n - m$.

Therefore it suffices to show that the inclusion $i_m : SG(m) \to G(m)$ induces an injection if $\delta(m) \neq 0$. For this suppose that φ_1 and φ_2 represent elements in $\mathrm{Rep}(Q, SG(m))$ and that there is an element $g \in G(m) \setminus SG(m)$ such that $i_m\varphi_1 = g(i_m\varphi_2)g^{-1}$.

If $SG(m) = A(m)$ we think of $i_m\varphi_k$, $k = 1, 2$, as Q–set of cardinality m. Then $m < n$ and $\delta(m) \neq 0$ imply that $m \equiv 2 \bmod 4$ (if $p = 2$) or $m \equiv 2 \bmod p$ (if $p > 2$), hence these Q–sets contain either two fixed points or an orbit of length 2. Hence there is a transposition τ which commutes with $i_m\varphi_1$ and then we have $i_m\varphi_1 = (\tau g)(i_m\varphi_2)(\tau g)^{-1}$ with $\tau g \in A(m)$, i.e. φ_1 and φ_2 represent the same element in $\mathrm{Rep}(Q, A(m))$.

For $SG(m) = SO(m)$, $m < n$ and $\delta(m) \neq 0$, we have that m is odd and then the matrix $(-id)$ commutes with $i_m\varphi_1$ and has determinant -1. If $SG(m) = SU(m)$, then $\lambda \cdot id$ commutes with $i_\ell\varphi_1$ for each complex number λ and we can choose λ such that $\det(\lambda \cdot id) = \det(g)^{-1}$. In both cases we can continue as in the case of $A(m)$ and conclude that φ_1 and φ_2 agree in $\mathrm{Rep}(Q, SG(m))$.

$\square$

REMARKS 4.2.

a) The proof shows that the restriction $\delta(m) \neq 0$ in 4.1 is actually unnecessary if $SG(m) = SU(m)$.

b) On the other hand one can show that there are even elementary abelian p–groups Q for which the map $\mathrm{Rep}(Q, A(m)) \to \mathrm{Rep}(Q, \Sigma(m))$ is not injective if $m > 0$, $m \equiv 0 \bmod 4$, $p = 2$, or $m > 0$, $m \equiv 0 \bmod p$, p odd.

   This and the proof of 4.1 imply that for such m and all $\ell \geq m+2$ the map $\mathrm{Rep}(Q, A(m)) \to \mathrm{Rep}(Q, A(\ell))$ is not injective and therefore (cf. [HLS, Section 4.2]) the restriction map $H^*(BA(\ell); \mathbb{F}_p) \to H^*(BA(m); \mathbb{F}_p)$ is not even an F - epimorphism in the sense of [Q2], in particular not an epimorphism (cf. Remark b) after Theorem 0).

4.3. Now we introduce filtrations on $A(P, SG(n))$ and $C(P, SG(n))$ by defining

$$E_m A(P, SG(n)) = \mathrm{Im}(A(P, SG(m))) \xrightarrow{j} A(P, SG(n))$$

and

$$E_m C(P, SG(n)) = \left(\bigoplus_{\ell=1}^{m} A(P, SG(\ell))/A(P, SG(\ell - \delta(\ell))) \right) \oplus A(P, SG(0)).$$

By 4.1 we may identify $E_m A(P, SG(n))$ with $A(P, SG(m))$ whenever $\delta(m) \neq 0$. Furthermore, in the definition of $E_m C$, we have used 4.1 in order to identify $A(P, SG(\ell - \delta(\ell)))$ with $\mathrm{Im}(A(P, SG(\ell - \delta(\ell))) \xrightarrow{j} A(P, SG(\ell)))$ for all ℓ. (By convention we take E_m equal to $A(P, SG(0))$ in both filtrations whenever $m \leq 0$.)

The maps $A(P, SG(m)) \xrightarrow{j} A(P, SG(n))$ map basis elements $[Q, \varphi]$ to basis elements $[Q, j\varphi]$ and hence $j \otimes \mathbb{Z}/p$ is mono whenever j is mono. Therefore Theorem 2.7 is clearly implied by the following two results.

LEMMA 4.4. *The homomorphism $h^+ : A(P, SG(n)) \to C(P, SG(n))$ maps $E_m A(P, SG(n))$ to $E_m C(P, SG(n))$ for all $m \leq n$.*

PROPOSITION 4.5. *The map $h^+ \otimes \mathbb{Z}/p$ induces isomorphisms on E_0 and also on filtration quotients $E_m \otimes \mathbb{Z}_p / E_{m-\delta(m)} \otimes \mathbb{Z}/p$ for all m with $0 < m \leq n$ and $\delta(m) \neq 0$.*

In the proofs of 4.4 and 4.5 we will interpret homomorphisms $Q \xrightarrow{\rho} G(m)$ as Q–sets of cardinality m (if $G(m) = \Sigma(m)$) resp. as real or complex Q–representations of dimension m (if $G(m) = O(m)$ or $U(m)$) and we will write $\rho \cong \sum_\lambda c_\lambda \lambda$ with nonnegative numbers c_λ and λ running through a set of representatives of the isomorphism classes of irreducible Q–sets resp. Q–representations. If we denote the cardinality of λ resp. the dimension of λ by $|\lambda|$, then clearly

$\sum\limits_{\lambda} c_\lambda|\lambda| = m$. The trivial Q–set resp. representation will be denoted by $\lambda = 1$. We will abbreviate $\sum\limits_{\lambda\neq 1} c_\lambda|\lambda|$ by $\|\rho\|$.

4.6. Proof of Lemma 4.4: The case $m = 0$ is trivial. For the other cases it suffices to show that the homomorphisms $h_{n,\ell} : A(P, SG(n)) \to A(P, SG(\ell))$ (cf. Section 2) map $E_m A(P, SG(n))$ into $A(P, SG(m)$ (considered as subspace of $A(P, SG(\ell))$ via 4.1) for all $0 \leq m < \ell \leq n$. This is trivial for $\ell = n$ and for $\ell \leq \epsilon$.

In the other cases we get from (2.2) that $h_{n,\ell}$ is given on a class $[Q, \varphi]$ by a linear combination of the form

$$h_{n,\ell}[Q, \varphi] = \sum_i \chi_i [Q_i, d_\ell \pi \psi_i].$$

Here the χ_i are suitable integers, the Q_i suitable subgroups of Q and there are elements $g_i \in G(n)$ such that $\psi_i := g_i^{-1}(i_n\varphi)g_i$ maps Q_i to $G(\ell-\epsilon)\times G(n+\epsilon-\ell)$. We interpret ψ_i as a pair of Q_i–sets resp. Q_i–representations and write $\psi_i \cong (\sum\limits_{\lambda} a^i_\lambda \lambda, \sum\limits_{\lambda} b^i_\lambda \lambda)$.

If $[Q, \varphi]$ is in $E_m A(P, SG(n))$, then $\sum\limits_{\lambda\neq 1}(a^i_\lambda + b^i_\lambda)|\lambda| \leq m$. It is clear that $[Q_i, d_\ell \pi\psi_i]$ is in $E_{k+\epsilon} A(P, SG(\ell))$, if $\|\pi\psi_i\| = \sum\limits_{\lambda\neq 1} a^i_\lambda|\lambda| \leq k$, so we may concentrate on the case that $m - \epsilon < \|\pi\psi_i\| \leq m$. But then we find $b^i_\lambda = 0$ for all $\lambda \neq 1$ (note that $|\lambda| \geq \epsilon$ for all $\lambda \neq 1$) and therefore $\pi\psi_i$ factors through $SG(m)$. Finally, d_ℓ maps $SG(m)$ clearly to itself and this finishes the proof.

$\square$

4.7. For the proof of Proposition 4.5 we need some preparations. First we observe that the statement about E_0 is trivial.

Then we define filtrations $\widetilde{E}_m$ on $\mathbb{Z}[\mathrm{Rep}(Q, SG(n))]$ as in 4.3, i.e.

$$\widetilde{E}_m \mathbb{Z}[\mathrm{Rep}(Q, SG(n))] = \mathrm{Im}(\mathbb{Z}[\mathrm{Rep}(Q, SG(m))] \to \mathbb{Z}[\mathrm{Rep}(Q, SG(n))]) \ .$$

The proof of 4.4 shows also that the maps

$$h_{n,\ell} : \mathbb{Z}[\mathrm{Rep}(Q, SG(n))] \to \mathbb{Z}[\mathrm{Rep}(Q, SG(\ell))]$$

preserve $\widetilde{E}_m$ for all $0 \leq m \leq \ell \leq n$. (Of course, the definition of these maps is as in section 2.)

Now fix ℓ with $0 < \ell \leq n$ and $\delta(\ell) \neq 0$. For the proof of 4.5 it suffices now by 4.4 and 2.3–2.5 to show that $h_{n,\ell}$ induces an isomorphism on the filtration quotient $\widetilde{E}_\ell \otimes \mathbb{Z}/p / \widetilde{E}_{\ell-\delta(\ell)} \otimes \mathbb{Z}/p$ for all subgroups Q of P. We will show that

$h_{n,\ell}$ induces an isomorphism on filtration quotients $\widetilde{E}_m \otimes \mathbb{Z}/p/\widetilde{E}_{m-1} \otimes \mathbb{Z}/p$ for $\ell - \delta(\ell) < m \leq \ell$ and this is clearly enough.

Again the cases $\ell = n$ or $\ell \leq \epsilon$ are trivial, so we will assume $\epsilon < \ell < n$ from now on.

In these cases we get for a basis element $[\varphi] \in \mathrm{Rep}(Q, SG(n))$ by 2.4

$$h_{n,\ell}[\varphi] = \sum_{[\psi]} [i_n\varphi : \psi][d_\ell \pi\psi]$$

with $[\psi]$ running through $\mathrm{Rep}(Q, G(\ell - \epsilon) \times G(n + \epsilon - \ell))$. Now we write $i_n\varphi \cong \sum c_\lambda \lambda$, $\psi \cong (\sum a_\lambda \lambda, \sum b_\lambda \lambda)$. Then $[i_n\varphi : \psi] = 0$ unless $a_\lambda + b_\lambda = c_\lambda$ for all λ. In particular, if we write $\|\varphi\|$ instead of $\|i_n\varphi\|$ we have either $\|\varphi\| = \|\pi\psi\|$ or $\|\pi\psi\| \leq \|\varphi\| - \epsilon$ for such ψ. If $\|\varphi\| = \|\pi\psi\|$ then this $[\psi]$ is unique and we will also denote it by $[\psi_0]$.

Together with the fact that d_ℓ maps terms $[\pi\psi]$ with $\|\pi\psi\| \leq k$ to $\widetilde{E}_{k+\epsilon}$ (cf. proof of 4.4) this implies

$$h_{n,\ell}[\varphi] \equiv [i_n\varphi : \psi_0][d_\ell\pi\psi_0] + \sum_{\substack{[\psi] \\ \|\pi\psi\| = \|\varphi\| - \epsilon}} [i_n\varphi : \psi][d_\ell\pi\psi] \mod \widetilde{E}_{\|\varphi\|-1}. \tag{4.8}$$

(By convention the term involving ψ_0 will be dropped if $\|\varphi\| \neq \|\pi\psi\|$ for all $[\psi]$.)

It will be enough to show that $h_{n,\ell}[\varphi] \equiv C[\varphi] \mod \widetilde{E}_{\|\varphi\|-1}$ for all φ with $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ where C is an integer which is not divisible by p and $[\varphi]$ on the right hand side is regarded as element in $\mathrm{Rep}(Q, SG(\ell))$ (cf. 4.1).

In order to evaluate (4.8) we need the following Lemma. But first note that for those $[\psi]$ with $a_\lambda + b_\lambda = c_\lambda$ and $\|\pi\psi\| = \|\varphi\| - \epsilon$ there is a unique $\lambda' \neq 1$ with $\|\lambda'\| = \epsilon$ and $a_{\lambda'} = c_{\lambda'} - 1$, $a_\lambda = c_\lambda$ for all other $\lambda \neq 1$.

LEMMA 4.9. *For $[\varphi] \in \mathrm{Rep}(Q, SG(n))$ and $[\psi] \in \mathrm{Rep}(Q, G(\ell - \epsilon) \times G(n + \epsilon - \ell))$ with $\delta(\ell) \neq 0$ and $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ we get*

a) If $\|\pi\psi\| = \|\varphi\|$ or $\|\pi\psi\| = \|\varphi\| - \epsilon$, and $a_\lambda + b_\lambda = c_\lambda$ for all λ, then $[d_\ell\pi\psi] = [\varphi]$. (Here φ is considered as element in $\mathrm{Rep}(Q, SG(\ell))$ *by using 4.1.)*

b) If $\|\pi\psi\| = \|\varphi\|$ and $a_\lambda + b_\lambda = c_\lambda$ for all λ (i.e. $\psi = \psi_0$), then

$$[i_n\varphi : \psi] = \chi\left(\frac{G(n - \|\varphi\|)}{G(\ell - \|\varphi\| - \epsilon) \times G(n + \epsilon - \ell)}\right).$$

c) If $\|\pi\psi\| = \|\varphi\| - \epsilon$, and if $a_\lambda + b_\lambda = c_\lambda$ for all λ and λ' is as above, then

$$[i_n\varphi : \psi] = \chi\left(\frac{G(c_{\lambda'})}{G(c_{\lambda'} - 1) \times G(1)} \times \frac{G(n - \|\varphi\|)}{G(\ell - \|\varphi\|) \times G(n - \ell)}\right).$$

We postpone the proof of 4.9 and give now the proof of 4.5.

4.10. Proof of Proposition 4.5: We evaluate (4.8) in the different cases separately.

Case $SG(n) = A(n)$ and p odd: We write $\ell = ps + 2$ (recall that we discuss the case $\delta(\ell) \neq 0$). Now $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ implies $\|\varphi\| = ps$. Furthermore there are no terms with $\|\pi\psi\| = \|\varphi\| - \epsilon$ because $\|\pi\psi\|$ is divisible by p. Therefore 4.9 implies

$$h_{n,\ell}[\varphi] \equiv \binom{n - \|\varphi\|}{n + 2 - \ell} [\varphi] \equiv [\varphi] \operatorname{mod} \widetilde{E}_{\|\varphi\|-1}.$$

Case $SG(n) = A(n)$ and $p = 2$: We write $\ell = 4s + 2$ (Recall that $\delta(\ell) \neq 0$). Now $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ implies $\|\varphi\| = 4s + 2$ or $\|\varphi\| = 4s$.

If $\|\varphi\| = 4s + 2$ then there are only terms with $\|\pi\psi\| = \|\varphi\| - \epsilon$ and we get from 4.9

$$h_{n,\ell}[\varphi] \equiv \sum_{\|\lambda\|=2} c_\lambda \binom{n - \|\varphi\|}{n - \ell} [\varphi] \;\operatorname{mod} \widetilde{E}_{\|\varphi\|-1}.$$

Again the binomial coefficient is 1 and $\|\varphi\| = 4s + 2$ implies that $\sum\limits_{\|\lambda\|=2} c_\lambda$ is odd.

If $\|\varphi\| = 4s$ we find

$$h_{n,\ell}[\varphi] \equiv \left\{ \binom{n - \|\varphi\|}{n + 2 - \ell} + \sum_{\|\lambda\|=2} c_\lambda \binom{n - \|\varphi\|}{n - \ell} \right\} [\varphi] \;\operatorname{mod} \widetilde{E}_{\|\varphi\|-1}.$$

Because of $\|\varphi\| = 4s$ we get $\sum\limits_{\|\lambda\|=2} c_\lambda \equiv 0 \operatorname{mod} 2$ and $\binom{n - \|\varphi\|}{n + 2 - \ell} = \binom{n - \|\varphi\|}{n - \|\varphi\|}$ $= 1$ and we are done again.

The cases $SG(n) = SO(n)$ or $SU(n)$: If $\|\varphi\| = \ell$, then there are only summands with $\|\pi\psi\| = \|\varphi\| - \epsilon$ and 4.9 gives

$$h_{n,\ell}[\varphi] \equiv \sum_{\substack{\|\lambda\|=1 \\ \lambda \neq 1}} \chi(G(c_\lambda)/G(c_\lambda - 1) \times G(1))[\varphi] \;\operatorname{mod} \widetilde{E}_{\|\varphi\|-1}. \tag{4.11.}$$

The space $G(c_\lambda)/G(c_\lambda - 1) \times G(1)$ may be identified with the projective space $KP^{c_\lambda - 1}$ where K denotes $\mathbb{R}$ in the case of $SO(n)$ and $\mathbb{C}$ in the case of $SU(n)$. For the Euler characteristic we get $\chi(\mathbb{R}P^{c_\lambda - 1}) \equiv c_\lambda \operatorname{mod} 2$ and $\chi(\mathbb{C}P^{c_\lambda - 1}) = c_\lambda$. Therefore we find that the coefficient of $[\varphi]$ in (4.11) is $\sum\limits_{\substack{\|\lambda\|=1 \\ \lambda \neq 1}} c_\lambda$ for $SU(n)$ and congruent to $\sum\limits_{\substack{\|\lambda\|=1 \\ \lambda \neq 1}} c_\lambda \operatorname{mod} 2$ for $SO(n)$.

Now Q is a p–group and the dimension of an irreducible Q–representation is either 1 or divisible by p (cf. [Se, Chap. 8.1] for the case of complex representations;

the case of real representations and $p=2$ is easily reduced to the complex case.). Therefore we find $\sum_{\substack{\|\lambda\|=1\\ \lambda\neq 1}} c_\lambda \equiv \|\varphi\| \equiv \ell \bmod p$, and hence the coefficient of $[\varphi]$ is not divisible by p (we have assumed that $\delta(\ell)\neq 0$ and $\ell<n$).

It remains to consider the case $\|\varphi\|\neq \ell$. Then $\ell-\delta(\ell)<\|\varphi\|<\ell$ implies $\|\varphi\|=\ell-1$ and $\ell\equiv 1 \bmod p$. Here 4.9 yields

$$h_{n,\ell}[\varphi] \equiv [\varphi] + \sum_{\substack{\|\lambda\|=1\\ \lambda\neq 1}} \chi(KP^{c_\lambda-1})\cdot\chi(KP^{n-\ell})[\varphi] \mod \widetilde{E}_{\|\varphi\|-1}.$$

The coefficients in the sum are equal to $c_\lambda\cdot(n-\ell+1)$, at least $\bmod p$, and as above we see $\sum_{\substack{\|\lambda\|=1\\ \lambda\neq 1}} c_\lambda \equiv \|\varphi\| \equiv \ell-1\equiv 0 \bmod p$. Therefore $h_{n,\ell}[\varphi]\equiv C[\varphi] \bmod \widetilde{E}_{\|\varphi\|-1}$ with $C\equiv 1 \bmod p$.

$\square$

4.12. It remains to give the

Proof of 4.9: Via 4.1 we consider $[\varphi]$ as element in $\mathrm{Rep}(Q, SG(\ell))$. If $\|\pi\psi\|=\|\varphi\|$ and $a_\lambda+b_\lambda=c_\lambda$ for all λ (i.e. $\psi=\psi_0$) then $\pi\psi$ factors through $SG(\ell-\epsilon)$ and hence $i_\ell d_\ell$ adds only trivial Q–orbits resp. representations to $\pi\psi$. Therefore $i_\ell[d_\ell\pi\psi]=i_\ell[\varphi]$ where i_ℓ denotes the homomorphism induced by the inclusion $SG(\ell)\to G(\ell)$.

If $\|\pi\psi\|=\|\varphi\|-\epsilon$ and $a_\lambda+b_\lambda=c_\lambda$ for all λ, then $\pi\psi$ does not factor through $SG(\ell-\epsilon)$ and $i_\ell d_\ell$ adds the Q–set resp. Q–representation corresponding to the homomorphism $Q\xrightarrow{\pi\psi}G(\ell-\epsilon)\xrightarrow{(\det)^{-1}}G(\epsilon)$. It is easy to see that this corresponds just to the unique λ' with $\lambda'\neq 1$, $\|\lambda'\|=\epsilon$, $a_{\lambda'}=c_{\lambda'}-1$ and $a_\lambda=b_\lambda$ for all other $\lambda\neq 1$. Therefore $i_\ell[d_\ell\pi\psi]=i_\ell[\varphi]$.

Now i_ℓ is mono if $\delta(\ell)\neq 0$ (cf. proof of 4.1) and hence a) follows.

b) and c): The proofs of b) and c) are similar and are just an application of Lemma 2.6. If we abbreviate $G(n)$ by G, $G(\ell-\epsilon)\times G(n+\epsilon-\ell)$ by H and write as before $i_n\varphi\cong\sum c_\lambda\lambda$, $\psi\cong(\sum a_\lambda\lambda,\sum b_\lambda\lambda)$, then $(G/H)^{i_n\varphi}_\psi$ is nonempty if and only if $a_\lambda+b_\lambda=c_\lambda$ for all λ.

Furthermore, it is not hard to see that

$$C_G(\psi)\cong\prod_\lambda \mathrm{Aut}_Q(c_\lambda\lambda)$$

with $\mathrm{Aut}_Q(c_\lambda\lambda)\subset G(c_\lambda|\lambda|)$ denoting the automorphism group of the Q–set respectively Q–representation $c_\lambda\lambda$; similarly we have

$$C_G(\psi)\cap H\cong\prod_\lambda \mathrm{Aut}_Q(a_\lambda\lambda)\times\mathrm{Aut}_Q(b_\lambda\lambda).$$

Hence $(G/H)_\psi^{i_n\varphi}$ is homeomorphic to $\prod_\lambda \mathrm{Aut}_Q(c_\lambda\lambda)/\mathrm{Aut}_Q(a_\lambda\lambda)\times\mathrm{Aut}_Q(b_\lambda\lambda)$ and b) and c) follow easily.

For example, in case c) we have $c_\lambda = a_\lambda$ for all λ except for $\lambda=\lambda'$ as above and $\lambda=1$. Now $\mathrm{Aut}_Q(c_\lambda\lambda)$ is isomorphic to $G(c_\lambda)$ if $|\lambda|=1$, respectively to the wreath product $\mathrm{Aut}_Q(\lambda)\wr G(c_\lambda)$ if $G(n)=\Sigma(n)$ and λ is any irreducible Q–set. Hence we get in this case

$$\begin{aligned}&C_G(\psi)/C_G(\psi)\cap H\cong\\&\cong (G(c_{\lambda'})/G(c_{\lambda'}-1)\times G(1))\times(G(n-\|\varphi\|)/G(\ell-\|\varphi\|)\times G(n-\ell))\,.\end{aligned}$$

$\square$

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