COMMUTATIVE ALGEBRA OF UNSTABLE K - MODULES, LANNES' T - FUNCTOR AND EQUIVARIANT MOD - P COHOMOLOGY

by

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Abstract

Let p be a fixed prime and let K be an unstable algebra over the mod - p Steenrod algebra A such that K is finitely generated as graded $\mathbb{F}_p$ - algebra. Let $K_{fg}-\mathcal{U}$ denote the abelian category of finitely generated K - modules with a compatible unstable A - module structure. We study various concepts of commutative algebra in this setting. The rôle of the prime ideal spectrum of a commutative ring is here taken by a category $\mathcal{R}(K)$ which, roughly speaking, consists of the A - invariant prime ideals of K together with certain "Galois information"; sheafs will correspond to functors on this category, and the rôle of the sheaf associated to a module will be taken by the components of Lannes' T - functor. We discuss the notions of support, of $\mathfrak{a}$ - torsion modules (for an invariant ideal $\mathfrak{a}$ of K) and of localization away from the Serre subcategory $\mathcal{T}ors(\mathfrak{a})$ of $\mathfrak{a}$ - torsion modules in our setting. We show that the category $K_{fg}-\mathcal{U}$ has enough injectives and use these injectives to study these localizations and their derived functors; they are closely related to the derived functors of the $\mathfrak{a}$ - torsion functor $F_{\mathfrak{a}}$. Our results are formally analogous to Grothendieck's results in the classical situation of modules over a noetherian commutative ring R [Gr].

Important for applications is the case $K = H^*BG$, the mod - p cohomology of a classifying space of a compact Lie group (or a suitable discrete group), and $M = H^*_G X$ where X is a (suitable) G - CW - complex. In these cases the category $\mathcal{R}(K)$ and the functor on $\mathcal{R}(K)$ associated to $H^*_G X$ can be described in terms of group theoretic and geometric data, and our theory yields a far-reaching generalization of a result of Jackowski and McClure [JM] resp. of Dwyer and Wilkerson [DW2]. As a concrete application of our theory we describe the size of the kernel of the restriction map from the unknown mod - 2 cohomology of the S - arithmetic group $GL(n, \mathbb{Z}[1/2])$ to the known cohomology of its subgroup D_n of diagonal matrices.

0. Introduction

Let p be a prime number and K be an unstable algebra over the mod - p Steenrod algebra A [S, SE]. An unstable K - A - module is an unstable A - module together with an A - linear map $K \otimes M \longrightarrow M$ which defines on M the structure of a K - module. We will ususally drop the A from the notation and say that M is unstable K - module. The mod - p cohomology H^*X of a space X is the main source of examples for unstable algebras and examples for unstable H^*X - modules are given by the mod - p cohomology of spaces over X, or more generally of Thom spaces of vector bundles on spaces over X.

We will be mostly concerned with the case that K is finitely generated as an $\mathbb{F}_p$ - algebra, i.e. noetherian, and that M is finitely generated as K - module. We will call such an M an unstable finitely generated K - module and denote the abelian category of such modules by $K_{fg} - \mathcal{U}$ (cf. section 1). Interesting geometric examples are provided by the case of equivariant mod - p cohomology (cf. Theorem 0.2). Our first result reads as follows.

THEOREM 0.1. *Let K be an unstable noetherian algebra. Then the category $K_{fg} - \mathcal{U}$ has enough injectives.*

We will prove this in section 1 by actually constructing enough injectives. If V is an elementary abelian p - group (i.e. $V \cong (\mathbb{Z}/p)^n$ for some natural number n) and if $K = H^*BV$, the mod - p cohomology of the classifying space of V, such a theorem was proved by Lannes and Zarati [LZ2], and in fact, they even determined all indecomposable injectives. We would like to point out that there would not be enough injectives if we worked with modules which are finitely generated using both the K and A - module structure together, as the case $K = \mathbb{F}_p$ (i.e. the case of ordinary unstable A - modules) shows (cf. [LSc]).

Now assume that K is noetherian. We will use these injectives to study localization functors on the category $K_{fg} - \mathcal{U}$ away from suitable subcategories. However, before we will get to this we need to discuss various concepts of commutative algebra in $K_{fg} - \mathcal{U}$.

The rôle of the prime ideal spectrum in the classical case will be taken by an appropriate category $\mathcal{R}(K)$ (cf. 1.3 for a precise definition and section 2 for a discussion of $\mathcal{R}(K)$). Here we just recall that the objects of $\mathcal{R}(K)$ are morphisms of unstable algebras $\varphi : K \longrightarrow H^*BV$ where V is an elementary abelian p - group such that φ makes H^*BV into a finitely generated K - module. (Observe that Rad(Ker φ), the radical of the kernel of φ is a prime ideal which is invariant with respect to the action of the Steenrod reduced power operations and, in fact, all such "invariant" prime ideals are obtained in this way (cf. 2.3).)

For each unstable finitely generated K - module M we have a functor $\overline{M}$ from $\mathcal{R}(K)$ to $K_{fg} - \mathcal{U}$, which sends (V, φ) to the "component" $T_V(M; \varphi)$ where T denotes Lannes' functor (cf. 1.4 for precise definitions). The functor $\overline{M}$ should be considered as the analogue of the sheaf $\widetilde{M}$ which one associates to M in the classical case of modules over a commutative ring. In fact, if one considers $\mathcal{R}(K)^{op}$ as a site equipped with the trivial Grothendieck topology then sheaves on this site are precisely covariant functors on $\mathcal{R}(K)$ (cf. [MM, III.2 and III.4]). We will not go into the details of this analogy but we will point out further analogies when it seems appropriate. For a further justification for this philosophy we refer to the comments after Theorem 0.4. The discussion of support in section 2, in particular 2.10, suggests also to consider $T_V(M; \varphi)$ as an unstable analogue of the appropriate stalk of $\widetilde{M}$, i.e. as an analogue of the classical localization of M at the prime ideal Rad(Ker φ). For earlier work on relations between Lannes' functor and classical localization we refer to work by Dwyer and Wilkerson [DW1].

For the remainder of the introduction we concentrate on the case of equivariant mod - p cohomology. This case is easier to explain and is also particularly interesting because here the category $\mathcal{R}(K)$ can be understood in group theoretic terms and the components of the T - functor are of geometric and group theoretic significance (cf. Theorem 0.2). However, we stress that the main results (Theorem 0.4 and Corollary 0.5 below) are true for unstable finitely generated K - modules over an unstable noetherian algebra K (Theorem 3.9 and Corollary 3.10) and the proof in the case of equivariant cohomology requires the same machinery as in the general case.

To state Theorem 0.2 we introduce some notation. As usual we denote the classifying space of the topological group G by BG, the total space of the universal principal G - bundle over BG by EG and the mod - p cohomology of the Borel construction $EG \times_G X$ by $H_G^* X$. Furthermore, for a fixed prime p, let $\mathcal{A}(G)$ denote the category whose objects are the elementary abelian p - subgroups of G and whose morphism sets consist of those group homomorphisms which are induced by conjugation with an element in G [Q].

THEOREM 0.2. *Fix a prime p. Assume we are in one of the following cases.*

a) G is a compact Lie group and X is a G - CW - complex with finitely many G - cells.

b) G is a discrete group for which there exists a mod - p acyclic G - CW - complex F with finitely many G - cells and finite isotropy groups, and let X be any G - CW - complex, again with finitely many G - cells and with finite isotropy groups (e.g. $X = F$).

c) Let G be a profinite group such that the continuous mod - p cohomology $H_c^* G$ is finitely generated as $\mathbb{F}_p$ - algebra.

I. Then $H^* BG$ is an unstable noetherian algebra and there is a canonical equivalence of categories

$$\mathcal{A}(G) \to \mathcal{R}(H^* BG), \qquad E \mapsto (E, \mathrm{res}_{G,E})$$

with $\mathrm{res}_{G,E}$ denoting the restriction homomorphism $H^* BG \longrightarrow H^* BE$.

II. Furthermore $H_G^* X$ is an unstable finitely generated $H^* BG$ - module and there are isomorphisms

$$T_E(H_G^* X; \mathrm{res}_{G,E}) \cong H^*_{C_G(E)}(X^E)$$

which are natural in $E \in \mathcal{A}(G)$.

(Here $C_G(E)$ is the centralizer of E in G, X^E denotes the E - fixed points of E acting on X, cohomology is with coefficient in $\mathbb{F}_p$, and if G is profinite we assume that X is a point and we read $H_G^* X$ as $H_c^* G$.)

If $X = F$ (in case b)), then the spaces X^E are also mod - p acyclic by Smith theory, and we obtain the following easy but important consequence.

COROLLARY 0.3. *Assume G satisfies the assumptions of Theorem 0.2.b). Then there are isomorphisms*

$$T_E(H^* BG; \mathrm{res}_{G.E}) \cong H^* BC_G(E) \ ,$$

which are natural in E. $\square$

Statement II of Theorem 0.2 and the Corollary are to a large extent due to Lannes. Case (a) is proved in the important and still unpublished preprint [L1] (cf. [L3] for a proof in case G is finite). The cases (b) and (c) are consequences of part (a) (for G finite); (c) was proved in [H2] and (b) will be proved in the appendix of this paper. In the case that G is a group of finite virtual cohomologial dimension (f.v.c.d.) the corollary was also proved in [L1]. The finite generation part of Statement I is well known in case (a); for part (b) we refer again to the appendix. The equivalence of categories is essentially a folk result (cf. [HLS2, I.5.3]).

Interesting classes of groups which are covered by (c) are p - adic analytic groups in the sense of Lazard [Lz], while (b) covers the case of (S) - arithmetic groups [Se], mapping class groups of orientable surfaces [H], outer automorphism groups of free groups [CV] (these are all groups of f.v.c.d.) and word-hyperbolic groups in the sense of Gromov [GH].

Now we turn to localizations. We consider an invariant ideal $\mathfrak{a}$ in H^*BG, i.e. $\mathfrak{a}$ is invariant with respect to the action of the Steenrod reduced power operations. Then the class of unstable finitely generated H^*BG - modules which are annihilated by some power of $\mathfrak{a}$ (we will call such modules unstable $\mathfrak{a}$ - torsion modules) forms a Serre class and we will study localization away from the full subcategory $\mathcal{T}ors(\mathfrak{a})$ of such modules.

These localizations away from $\mathcal{T}ors(\mathfrak{a})$ are closely related to the right derived functors $R^iF_{\mathfrak{a}}$ of the functor which associates to M its largest unstable $\mathfrak{a}$ - torsion submodule $F_{\mathfrak{a}}M$. The ideal $\mathfrak{a}$ determines a subcategory $\mathcal{O}(\mathfrak{a})$ of $\mathcal{A}(G)$, namely the full subcategory of all objects E for which $\mathfrak{a} \not\subset \mathrm{Rad}(\mathrm{Ker}\,\mathrm{res}_{G,E})$, i.e. for which $\mathfrak{a}$ is not contained the radical of the kernel of the restriction map $\mathrm{res}_{G,E} : H^*BG \longrightarrow H^*BE$. The category $\mathcal{O}(\mathfrak{a})$ should be thought of as the analogue of the open complement (in the classical prime ideal spectrum) of the subset $V(\mathfrak{a})$ which is defined by $\mathfrak{a}$.

As usual we denote the inverse limit of a functor F defined on a category $\mathcal{C}$ by $\lim_{\mathcal{C}} F$ and its derived functors by $\lim^i_{\mathcal{C}} F$. Our main result reads now as follows.

THEOREM 0.4. *Let G and X be as in Theorem 0.2 and let $\mathfrak{a}$ be an invariant ideal in H^*BG.*

a) Then there is a natural exact sequence

$$0 \longrightarrow F_{\mathfrak{a}}H^*_GX \longrightarrow H^*_GX \xrightarrow{\rho} \lim_{\mathcal{O}(\mathfrak{a})} H^*_{C_G(E)}(X^E) \longrightarrow R^1F_{\mathfrak{a}}H^*_GX \longrightarrow 0$$

in which the components of ρ are induced by the obvious inclusions on the group and space level. In particular, the kernel and cokernel of ρ are unstable finitely generated $\mathfrak{a}$ - torsion modules. Furthermore, ρ is localization away from the subcategory $\mathcal{T}ors(\mathfrak{a})$.

b) There are natural isomorphisms

$$\lim^i_{\mathcal{O}(\mathfrak{a})} H^*_{C_G(E)}(X^E) \cong R^{i+1}F_{\mathfrak{a}}H^*_GX$$

for all $i > 0$. In particular, $\lim^i_{\mathcal{O}(\mathfrak{a})} H^*_{C_G(E)}(X^E)$ is an unstable finitely generated $\mathfrak{a}$ - torsion module for all $i > 0$.

For the generalization of this result to the case where K is noetherian and M is in $K_{fg} - \mathcal{U}$ the reader is referred to 3.9.

If one ignores the assertion about finite generation (which is a consequence of Theorem 0.1) this result is formally analogous to the classical situation [Gr]. There one considers an ideal $\mathfrak{a}$ in a noetherian commutative ring R and the derived functors of the functor $\Gamma_{\mathfrak{a}}$, which associates to an R - module M its $\mathfrak{a}$ - torsion submodule. These derived funcors are identified with the cohomology groups $H^*_{V(\mathfrak{a})}(\operatorname{spec} R; \widetilde{M})$ of $\operatorname{spec}(R)$ with support in the closed set $V(\mathfrak{a})$ and coefficients in the sheaf $\widetilde{M}$. The inverse limit and its derived functors correspond in this picture to the cohomology of the open complement of the closed set $V(\mathfrak{a})$ with coefficients in $\widetilde{M}$ (The analogy can be made more precise by noting that $\mathcal{O}(\mathfrak{a})$ is, in fact, an open subobject in the topos of sheafs on the site $\mathcal{A}(G)^{op}$ [AGV,IV.8.4], and by refering to the topos theoretic versions of cohomology with support (cf. [AGV,V.6.5]).

In the important special case where $\mathfrak{a}$ is the invariant maximal ideal $\mathfrak{m}$ of all positive dimensional elements in H^*BG, the submodule $F_{\mathfrak{m}}$ is the largest unstable finite H^*BG - submodule of M and we will write F instead of $F_{\mathfrak{m}}$. The category $\mathcal{O}(\mathfrak{m})$ turns out to be the full subcategory of $\mathcal{A}(G)$ consisting of all non-trivial E. We will write $\mathcal{A}_*(G)$ for this category. Then Theorem 0.4 takes the following form.

COROLLARY 0.5. *Let G and X be as in Theorem 0.2.*

a) Then there is a natural exact sequence

$$0 \longrightarrow FH^*_G X \longrightarrow H^*_G X \stackrel{\rho}{\longrightarrow} \lim\nolimits_{\mathcal{A}_*(G)} H^*_{C_G(E)}(X^E) \longrightarrow R^1 F H^*_G X \longrightarrow 0$$

in which the components of ρ are induced by the obvious inclusions on the group and space level. In particular, the kernel and cokernel of ρ are finite. Furthermore, ρ is localization away from the subcategory of unstable finite H^*BG - modules.

b) There are natural isomorphisms

$$\lim\nolimits^i_{\mathcal{A}_*(G)} H^*_{C_G(E)}(X^E) \cong R^{i+1} F H^*_G X$$

for all $i > 0$. In particular, $\lim^i_{\mathcal{A}_*(G)} H^*_{C_G(E)}(X^E)$ is is finite for all $i > 0$.

If G is compact Lie and X is a point it is a theorem of Jackowski and McClure that ρ is an isomorphism and all higher limits vanish in 0.5 [JM]. An algebraic version of the theorem of Jackowski and McClure was proved by Dwyer and Wilkerson [DW2] and their result motivated the investigations in this paper. Applications in the case where ρ is an isomorphism (in particular in the cases considered in [JM] and [DW2]) were discussed by Mislin [M].

For applications of Theorem 0.4 and Corollary 0.5 the reader is referred to [H2,H3,H4] where we study the mod - p cohomology of groups of units in maximal orders of certain p - adic division algebras, of the general linear groups $GL(p-1, \mathbb{Z}_p)$ (with $\mathbb{Z}_p$ the ring of p - adic integers) and the mod - 2 cohomology of $SL(3, \mathbb{Z}[1/2])$ and $GL(3, \mathbb{Z}[1/2])$. Applications to mapping classs groups will be considered in joint work with F. Cohen and Y. Xia.

The higher limits resp. the derived funcors of F in the case $H^*_G X$, G elementary abelian will be investigated in [HLO]. They yield new invariants for G - complexes which give obstructions for equivariant embeddings of finite G - complexes into smooth G - manifolds.

As another concrete application of our theory in the case of group cohomology we offer the following result. Here D_n is the subgroup of diagonal matrices in the general linear group $GL(n, \mathbb{Z}[1/2])$, cohomology is with coefficients in $\mathbb{F}_2$ and the size of a graded finitely generated module M over a connected finitely generated $\mathbb{F}_p$ - algebra is given by the order of the pole at $t = 1$ of the power series series $\sum_i dim_{\mathbb{F}_p} M^i t^i$. Note that the size measures the growth of the sequence of numbers $\dim_{\mathbb{F}_p} M^i$.

THEOREM 0.6. *The kernel of the restriction map* ρ_n *from* $H^*BGL(n, \mathbb{Z}[1/2])$ *to* H^*BD_n *has size precisely equal to* $n - n_0 + 1$ *where* n_0 *denotes the smallest natural number such that* ρ_{n_0} *is not a monomorphism and* $n \geq n_0$. *In particular, the size of the kernel of* ρ_{n_0} *itself is* 1, *i.e.* $\operatorname{Ker} \rho_{n_0}$ *is periodic in large degrees.*

By unpublished work of Dwyer it is known that n_0 is finite but no element in these kernels seems to be known explicitly.

Here is a brief outline of the paper. In section 1 we review what we need to know about Lannes' functor, discuss injectives and prove Theorem 0.1. Section 2 is concerned with concepts of commutative algebra in $K_{fg} - \mathcal{U}$: we discuss invariant ideals and the category $\mathcal{R}(K)$, torsion modules, the T - support of an unstable finitely generated module over a noetherian algebra K and its relation to the classical support. In section 3 we study localization away from subcategories of torsion modules, we prove the generalization of Theorem 0.4 in the context of $K_{fg} - \mathcal{U}$ and discuss its consequences. We also show how to derive the theorem of Jackowski and McClure resp. Dwyer and Wilkerson with our methods. Section 4 is devoted to the proof of Theorem 0.6 and in an appendix we prove part (b) of Theorem 0.2.

Acknowledgements: The research in this paper was inspired by the work of Jackowski and McClure [JM] and in particular by the algebraic approach to it by Dwyer and Wilkerson [DW2]. The paper should also be considered as a sequal to [HLS2]. In fact, my first proof (in 1990) of the fact that the kernel and cokernel of the map ρ in Corollary 0.5 (resp. 3.10) were finite used the main result of [HLS2] in an essential way. I had helpful discussions with many different people on the subject matter of this paper and I am especially happy to acknowledge numerous inspiring discussions with Jean Lannes. In particular, he first showed me a result like Corollary 0.5 (resp. 3.10) in the case $K = H^*BV$. I would also like to thank John Greenlees for a timely conversation on local cohomology and Bob Oliver for comments on a preliminary version of this paper. During the research presented in this paper I was supported by a Heisenberg fellowship of the DFG.

1. Review of Lannes' T - functor; injectives in $K_{fg} - \mathcal{U}$

We begin by recalling some terminology and facts about Lannes' T - functor. As general reference for background information we refer to [L2,L3] and [S].

1.1 Let p be a fixed prime. Let $\mathcal{U}$ resp. $\mathcal{K}$ denote the category of unstable modules resp. unstable algebras over the mod - p Steenrod algebra A. The Steenrod algebra is actually a Hopf algebra and its diagonal gives rise to a tensor product on the categories $\mathcal{U}$ resp. $\mathcal{K}$.

For a fixed unstable algebra K we consider the following category $K-\mathcal{U}$: its objects, which we call unstable K - A -modules (or unstable K - modules for short), are unstable A - modules M together with A - linear structure maps $K \otimes M \longrightarrow M$ which make M into a K - module; its morphisms are all A - linear maps which are also K - linear. The full subcategory of $K - \mathcal{U}$ consisting of those objects which are finitely generated as K - modules is denoted by $K_{fg} - \mathcal{U}$. Its objects will be called unstable finitely generated K - modules.

1.2. Now let V be an elementary abelian p - group (i.e. $V \cong (\mathbb{Z}/p)^n$ for some natural number n). Let $T_V : \mathcal{U} \longrightarrow \mathcal{U}$ be the functor introduced by Lannes [L2,L3]. It is left adjoint to tensoring with H^*BV, so there are natural isomorphisms $\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*BV \otimes N)$ for all unstable modules M and N. T_V has a number of remarkable properties. In particular, T_V commutes with tensor products and lifts to a functor from $\mathcal{K}$ to itself and the adjunction relation continues to hold in $\mathcal{K}$: $\mathrm{Hom}_{\mathcal{K}}(T_V K, L) \cong \mathrm{Hom}_{\mathcal{K}}(K, H^*BV \otimes L)$ for all unstable algebras K and L. Similarly, T_V lifts to a functor from $K - \mathcal{U}$ to $T_V K - \mathcal{U}$.

1.3. To an unstable algebra K we associate a category $\mathcal{S}(K)$ as follows. Its objects are the morphisms of unstable algebras $\varphi : K \to H^*BV$ with V an elementary abelian p - group; it will be convenient to denote such an object as (V, φ). Then the set of morphisms from (V_1, φ_1) to (V_2, φ_2) are all homomorphisms $V_1 \xrightarrow{\alpha} V_2$ of abelian groups such that $\varphi_1 = \alpha^* \varphi_2$. The full subcategory of $\mathcal{S}(K)$ of objects (V, φ) for which H^*BV becomes a finitely generated K - module via φ will be denoted by $\mathcal{R}(K)$. Note that in this case the homomorphisms α has to be injective. If K is a noetherian algebra then this category is equivalent to a finite category and it (resp. its opposite) was first investigated by Rector [R]. The full subcategory of $\mathcal{R}(K)$ having as objects all (V, φ) with V non-trivial will play an important role for us. We will denote it by $\mathcal{R}_*(K)$.

1.4. Now consider the unstable algebra $T_V K$. A morphism of unstable algebras $\varphi : K \longrightarrow H^*BV$ determines a connected component $T_V(K;\varphi)$ of $T_V K$: it is defined as $T_V(K;\varphi) := \mathbb{F}_p(\varphi) \otimes_{T_V^0 K} T_V K$ where $\mathbb{F}_p(\varphi)$ denotes $\mathbb{F}_p$ considered as a module over $T_V^0 K$ (the subalgebra of homogeneous elements of degree 0) via the adjoint of φ. Similarly, if M is in $K-\mathcal{U}$, we have a "component" $T_V(M;\varphi) := \mathbb{F}_p(\varphi) \otimes_{T_V^0 K} T_V M$ which has an obvious structure of an unstable $T_V(K;\varphi)$ - module. Furthermore there is a canonical map of unstable algebras $\rho_{K,(V,\varphi)}$: $K \longrightarrow T_V(K;\varphi)$ which makes $T_V(M;\varphi)$ into an unstable K - module; the map $\rho_{K,(V,\varphi)}$ is the composition of the map $\gamma_{K,(V,\varphi)} : K \longrightarrow H^*BV \otimes T_V(K;\varphi)$, which is adjoint to the projection map $T_VK \longrightarrow T_V(K;\varphi)$, followed by the projection map $H^*BV \otimes T_V(K;\varphi) \longrightarrow T_V(K;\varphi)$. Similarly, there are maps $\rho_{M,(V,\varphi)} : M \longrightarrow T_V(M;\varphi)$.

Now it is straightforward to check that the assignment $(V,\varphi) \mapsto T_V(M;\varphi)$ extends to a functor $\mathcal{S}(K) \to K - \mathcal{U}$. If K is noetherian and $M \in K_{fg} - \mathcal{U}$, then we obtain a functor $\mathcal{R}(K) \longrightarrow K_{fg} - \mathcal{U}$ (cf. 1.8 and 1.12 below).

1.5. Next we discuss injectives in the category $K - \mathcal{U}$. First we have the analogues of the Brown - Gitler modules in the category $\mathcal{U}$, i.e for each natural number n there is an unstable K - module $J_K(n)$ representing the functor $M \mapsto (M^n)^*$. In fact, if $F(n)$ denotes the free unstable A - module on a generator in degree n, then we define $J_K(n)^l$, the subspace of elements of degree l, as $((K \otimes F(l))^n)^*$ with $(\ )^*$ denoting the vector space dual. The A - and K - module structure on $J_K(n)$ can then be defined by appropriate maps between the modules $K \otimes F(\cdot)$, just as in the case of the modules $J(n)$ (cf. [LZ1]).

This description makes it clear that the module $J_K(n)$ is trivial in degrees bigger than n and is of finite type if K is of finite type. Furthermore, in $\mathcal{U}$ the module $J_K(n)$ is isomorphic to $\bigoplus_i J(i) \otimes (K^{n-i})^*$ (with $(K^{n-i})^*$ denoting the dual of the subspace of homogeneous elements of degree $n-i$ and being considered as unstable module concentrated in degree 0).

To get more injectives we use the following refinement of the adjunction property of the T - functor.

PROPOSITION 1.6 [LZ2]. *Let K and L be two unstable algebras, V an elementary abelian p - group and $g : K \longrightarrow H^*BV \otimes L$ a homomorphism of unstable algebras. For any unstable K - module M and unstable L - module N the adjunction* $\mathrm{Hom}_{\mathcal{U}}(T_VM, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*BV \otimes N)$ *induces an isomorphism*

$$\mathrm{Hom}_{T_VK-\mathcal{U}}(T_VM, N) \cong \mathrm{Hom}_{K-\mathcal{U}}(M, H^*BV \otimes N)\ .$$

(Here $H^*BV \otimes N$ is a K - module via g and N is a T_VK - module via $\widetilde{g} : T_VK \longrightarrow L$, the adjoint of g.) □

Now consider a map $\varphi : K \longrightarrow H^*BV$ of unstable algebras and apply this proposition to the map $\gamma_{K,(V,\varphi)} : K \longrightarrow H^*BV \otimes T_V(K;\varphi)$. If N is an unstable $T_V(K;\varphi)$ - module, we will also write $H^*BV(\varphi) \otimes N$ for the unstable K - module $H^*BV \otimes N$ if its K - module structure is defined via the map $\gamma_{K,(V,\varphi)}$. Because injectives in $T_V(K;\varphi)-\mathcal{U}$ are also injective in $T_VK - \mathcal{U}$, exactness of T_V [L2,L3] implies the following result.

PROPOSITION 1.7. *Let $\varphi : K \longrightarrow H^*BV$ be a map of unstable algebras and I be any injective in $T_V(K;\varphi) - \mathcal{U}$. Then $H^*BV(\varphi) \otimes I$ is injective in $K - \mathcal{U}$.* □

In particular, all the objects $H^*BV(\varphi) \otimes J_{T_V(K;\varphi)}(n)$ are injective. If K is understood from the context, and if (V,φ) is in $\mathcal{S}(K)$, we will also write $I_{(V,\varphi)}(n)$ for this injective.

1.8. We will be mainly concerned with the case of unstable noetherian algebras and finitely generated unstable modules over them. We recall that in this case $T_V K$ is again noetherian and $T_V M$ is finitely generated over $T_V K$ ([DW1], [H1]). Furthermore, the canonical map $K \longrightarrow T_V K$ makes $T_V K$ into a finitely generated K - module and hence $T_V M$ becomes a finitely generated K - module (cf. 1.12 below).

If K is also connected, then the map $\gamma_{M,(0.\epsilon)} : M \longrightarrow T_0(M;\epsilon)$ is an isomorphism (cf. [S, Prop. 3.9.7]). Here 0 is the trivial elementary abelian p - group and ϵ is the augmentation of the connected algebra K. In particular, we see that in this case the modules $J_K(n)$ and $I_{(0,\epsilon)}(n)$ agree.

If K is noetherian, then it is easy to check that the modules $I_{(V,\varphi)}(n)$ are finitely generated K - modules for any $(V,\varphi) \in \mathcal{R}(K)$. The following result shows that these modules give enough injectives in the category $K_{fg} - \mathcal{U}$. We will give two proofs of this result which both rely crucially on the main result of [H1].

THEOREM 1.9. (Existence of enough injectives). *Let K be an unstable noetherian algebra and M an unstable finitely generated K - module. Then there is an embedding $M \longrightarrow I$ in the category $K_{fg} - \mathcal{U}$ in which I is isomorphic to a finite direct product of injective modules $I_{(V,\varphi)}(n)$ for suitable (V,φ) in $\mathcal{R}(K)$ and natural numbers n.*

First Proof of 1.9: From Theorem I of [H1] we know that there is a finite filtration $0 = M_0 \subset M_1 \subset ... \subset M_n = M$ such that the successive filtration quotients M_i/M_{i-1} are k_i - fold suspensions of modules L_i which can be embedded (in the category $K_{fg} - \mathcal{U}$) into a finite direct product of modules $H^*BV(\varphi) = I_{(V,\varphi)}(0)$. It is enough to show that the theorem holds for the quotients M_i/M_{i-1}. Now M_i/M_{i-1} can be embedded into a finite direct sum of modules $H^*BV(\varphi) \otimes \Sigma^{k_i}\mathbb{F}_p$. The K - module structure on $H^*BV(\varphi) \otimes \Sigma^{k_i}\mathbb{F}_p$ is clearly pulled back from the obvious $H^*BV \otimes T_V(K;\varphi)$ - module structure via the map $\gamma_{K,(V,\varphi)} : K \longrightarrow H^*BV \otimes T_V(K;\varphi)$. Now $H^*BV \otimes \Sigma^{k_i}\mathbb{F}_p$ can be embedded (as $H^*BV \otimes T_V(K;\varphi)$ - module) into $H^*BV \otimes J_{T_V(K;\varphi)}(k_i)$, hence M_i/M_{i-1} can be embedded in $K_{fg} - \mathcal{U}$ into a finite direct product of modules $I_{(V,\varphi)}(k_i)$ and we are done. $\square$

The second proof relies on the following result of [HLS2] which again depends on [H1,Thm.I].

THEOREM 1.10. *Let K be an unstable noetherian algebra and M an unstable finitely generated K - module. Then there is a natural number n such that the maps $\gamma_{M,(V,\varphi)}$ (which are adjoint to the projection maps $T_V M \longrightarrow T_V(M;\varphi)$) induce an embedding*

$$\gamma : M \longrightarrow \prod_{(V,\varphi)} H^*BV \otimes T_V(M;\varphi)^{<n}$$

in the category $K_{fg} - \mathcal{U}$. (Here and elsewhere in the paper $(\ )^{<n}$ stands for the quotient of $(\ )$ by the submodule of elements which have degree n or bigger.) $\square$

Second Proof of 1.9: Each of the modules $T_V(M;\varphi)^{<n}$ can be embedded into a finite direct product of modules $J_{T_V(K;\varphi)}(k)$ (with $k < n$) and hence 1.9 follows. $\square$

The two proofs are actually very close to each other. The first one is more direct, the second one is shorter; both depend on [H1]. We remark that 1.10 can be easily deduced from 1.9 which is another reason for giving the first proof.

COROLLARY 1.11. *Let K be an unstable noetherian algebra. Then injective resolutions exist in the category $K_{fg} - \mathcal{U}$.* □

This corollary improves on Theorem II of [H1]. There we constructed resolutions in the category $\mathcal{U}$ (with certain finiteness properties) for unstable finitely generated K - modules.

If K is an unstable noetherian algebra and M is an unstable finitely generated K - module of the form $H^*BV(\varphi) \otimes F$ for $(V, \varphi) \in \mathcal{R}(K)$ and an unstable finite $T_V(K;\varphi)$ - module F, then it is easy to see that $T_V M$ is again finitely generated as K - module (cf. 3.6 below where $T_V M$ is explicitly described for such M) and therefore 1.9 or 1.10 give the following result.

COROLLARY 1.12. *Let K be an unstable noetherian algebra and let M be an unstable finitely generated K - module. Then $T_V M$ is finitely generated as a K - module.* □

2. Commutative algebra for unstable K - modules.

For reasons of completeness we start this section with a short discussion of invariant ideals and of the category $\mathcal{R}(K)$. Propositions 2.3 and 2.4 below are undoubtedly known to the experts but we do not know any convenient reference for them (although variatons can be found in [R]).

DEFINITION 2.1. *Let K be an unstable algebra. A homogeneous ideal $\mathfrak{a}$ in K is called invariant iff it is closed under the action of all operations Sq^i if $p = 2$, and of all P^i if p is odd.*

It is immediate from the Cartan formula that the radical of an invariant ideal is again invariant. For odd primes, however, the example of the zero ideal in $H^*B\mathbb{Z}/p$ shows that this would not be true if we included the Bockstein β in the definition of invariant ideals.

<u>2.2</u> Given an object (V, φ) of $\mathcal{S}(K)$, consider its kernel $\operatorname{Ker}\varphi$. For $p = 2$ this is a prime ideal and for p odd the radical of $\operatorname{Ker}\varphi$ is a prime ideal. It is clear that this prime ideal is invariant with respect to the action of the reduced power operations P^i (with respect to all of the Steenrod algebra if $p = 2$, but not necessarily with respect to the Bockstein if p is odd). In fact, the following proposition shows that the converse is true as well.

PROPOSITION 2.3. *Let K be an unstable algebra.*

a) Let $\mathfrak{p}$ be an invariant ideal in K. Then there exists an object (V,φ) of $\mathcal{S}(K)$ such that $\mathfrak{p} = \mathrm{Rad}(\mathrm{Ker}\,\varphi)$. Furthermore, if K is noetherian then we can assume that (V,φ) is in $\mathcal{R}(K)$.

b) Let $\mathfrak{a}$ be an invariant ideal in K which is radical. Then $\mathfrak{a}$ is the intersection of all invariant ideals containing $\mathfrak{a}$.

Proof of a): First consider the case $p = 2$. Then the map $\varphi : K \longrightarrow H^*BV$ is the composition of the quotient map from K to $K/\mathfrak{p}$ followed by an Adams - Wilkerson embedding [AW] for the unstable integral domain $K/\mathfrak{p}$.

If p is odd then the version of the Adams Wilkerson theorem proved in [HLS1, II.Thm.3] gives again an embedding ψ of the unstable algebra $K/\mathfrak{p}$ into some H^*BV, but the quotient map $\pi : K \longrightarrow K/\mathfrak{p}$ need not commute with the Bockstein. However, the restriction of this quotient map to $\mathcal{O}\widetilde{\mathcal{O}}K$, the largest unstable subalgebra of K which is concentrated in even degrees, does commute with β, and the resulting map $\mathcal{O}\widetilde{\mathcal{O}}K \longrightarrow H^*BV$ extends to a map $\varphi : K \longrightarrow H^*BV$ of unstable algebras because H^*BV is injective in $\mathcal{K}$ by [L2,L3]. Finally, the map $\varphi - \psi$ commutes with the operations P^i, in particular with the p - th power map, and factors through the quotient $K/\mathcal{O}\widetilde{\mathcal{O}}K$. This quotient is a suspension [Z], and from this $\mathrm{Rad}(\mathrm{Ker}\,\varphi) = \mathfrak{p}$ follows easily.

If K is noetherian then H^*BV can be taken finitely generated over K by [AW, Thm.1.8] (see also [HLS1,II.Cor.7.7]), and therefore we can asssume that (V,φ) is in $\mathcal{R}(K)$.

Proof of b): By passing to the quotient $K/\mathfrak{a}$ we may assume that K has no nilpotent elements (if p is odd then one has to justify this reduction as in case a) above). If K has no nilpotent elemenents then there exists an embedding of unstable algebras $K \longrightarrow \prod_\alpha H^*BV_\alpha$ ([HLS1]) and hence the zero ideal is the intersection of the prime ideals $\mathfrak{p}_\alpha$ which are given as the kernel of the α - th component of the embedding. □

COROLLARY 2.4. *Let K be an unstable noetherian algebra. Then the assignment $(V,\varphi) \mapsto \mathrm{Rad}(\mathrm{Ker}\,\varphi)$ induces a bijection between the isomorphism classes of objects of $\mathcal{R}(K)$ and the invariant prime ideals in K.*

Proof: It is clear that this map is well defined and by 2.3 it is onto. Injectivity follows from injectivity of H^*BV in the category $\mathcal{K}$ [L2,L3]. □

Now we introduce a topology on the set of objects of the category $\mathcal{R}(K)$ if K is noetherian. We remark that our open (resp. closed) subsets correspond to the open (resp. closed) subobjects in the Grothendieck topos of sheafs on the site (equipped with the trivial Grothendieck topology) $\mathcal{R}(K)^{op}$ (cf. [AGV,IV.8.4]). Further justification for the definition of this topology comes from 2.7 below.

DEFINITION 2.5. *Assume K is an unstable noetherian algebra.*

a) A subset $\mathcal{C}$ of $\mathcal{R}(K)$ is called closed if it has the following property: if $(V,\varphi) \in \mathcal{C}$ then $(W,\psi) \in \mathcal{C}$ for all (W,ψ) for which there exists a morphism $(W,\psi) \longrightarrow (V,\varphi)$ in $\mathcal{R}(K)$.

b) A subset $\mathcal{O}$ of $\mathcal{R}(K)$ is called open if its complement is closed.

By injectivity of H^*BW in the category $\mathcal{K}$ there is a morphism $(W,\psi) \longrightarrow (V,\varphi)$ if and only if $\text{Rad}(\text{Ker}\,\varphi) \subset \text{Rad}(\text{Ker}\,\psi)$. In particular, for every invariant ideal $\mathfrak{a}$ the set $\mathcal{C}(\mathfrak{a})$ of those (V,φ) with $\mathfrak{a} \subset \text{Rad}(\text{Ker}\,\varphi)$ is closed. We will denote its open complement by $\mathcal{O}(\mathfrak{a})$. In 2.7 below we will see that all closed subsets arise in this way, and therefore this topology on $\mathcal{R}(K)$ corresponds to the Zariski topology on the prime ideal spectrum restricted to the set of invariant prime ideals. Note however, that $\mathcal{R}(K)$ has automorphisms built in (essentially the "Galois automorphisms" of the extensions $K/\,\text{Ker}\,\varphi \subset H^*BV$), while this is not part of the data of the Zariski topology.

PROPOSITION 2.6 (cf. [CH, Prop. 3]). *Assume K is an unstable noetherian algebra and let $\mathcal{O}$ be any open subset of $\mathcal{R}(K)$. Then there exists an element $x \in K$ such that $\mathcal{O}$ is equal to the set $D(x)$ of all those (V,φ) in $\mathcal{R}(K)$ such that φx is not nilpotent.*

Proof: The crucial point is Corollary 2.4 and that $\mathcal{R}(K)$ is equivalent to a finite category. With these two facts the proof proceeds just as in [CH, Prop. 3] where the case of the category $\mathcal{A}(G)$ and $K = H^*BG$ was treated. □

PROPOSITION 2.7. *Assume K is an unstable noetherian algebra. Then there is a bijection between the radical invariant ideals of K and closed subsets of $\mathcal{R}(K)$. Given a radical invariant ideal, it determines the closed subset $\mathcal{C}(\mathfrak{a})$, and given a closed subset $\mathcal{C}$ it determines the radical invariant ideal $\mathfrak{a}(\mathcal{C}) := \bigcap_{(V,\varphi)} \text{Rad}(\text{Ker}\,\varphi)$ with (V,φ) running through $\mathcal{C}$.*

Proof: This is immediate from 2.3.b) and 2.6. □

Our next topic is the notion of support. If M is an unstable K - module then the classical support of the underlying graded K - module will be denoted by $\text{supp}(M)$.

DEFINITION 2.8. *Assume K is an unstable noetherian algebra and M is an unstable K - module. The set of $(V,\varphi) \in \mathcal{R}(K)$ with $T_V(M,\varphi) \neq 0$ is called the T - support of M and is denoted by $T - \text{supp}(M)$.*

The following two results provide justification of our terminology.

LEMMA 2.9. *Assume K is an unstable noetherian algebra and M is any unstable K - module. Then $T - \text{supp}(M)$ is closed.*

In the case of Borel cohomology this proposition follows immediately from Lannes T computation (Theorem 0.2). In fact, this is a guiding example that the reader should keep in mind.

Proof: Assume (V,φ) is in $T - \text{supp}(M)$ and consider a morphism $\alpha : (W,\psi) \to (V,\varphi)$. We have to show $T_W(M;\psi) \neq 0$.

If $T_V(M;\varphi) \neq 0$ then the map $\gamma_{M,(V,\varphi)} : M \longrightarrow H^*BV \otimes T_V(M;\varphi)$ is nontrivial. Because α is a monomorphism we may identify W with $\alpha W \subset V$ and choose a complement V' of

W. Then we can write $\gamma_{M,(V,\varphi)} : M \longrightarrow H^*BW \otimes H^*BV' \otimes T_V(M;\varphi)$ and by adjointness (cf. Proposition 1.6) we get a $T_W K$- linear (hence K - linear) non-trivial map $T_W M \longrightarrow H^*BV' \otimes T_V(M;\varphi)$ which factors through $T_W(M;\psi)$. $\Box$

PROPOSITION 2.10. *Assume K is an unstable noetherian algebra and M is an unstable finitely generated K - module. Then (V,φ) is in $T-\mathrm{supp}(M)$ iff $\mathrm{Rad}(\mathrm{Ker}\,\varphi)$ is in $\mathrm{supp}(M)$.*

Proof: Suppose (V,φ) is in the T - support. Then there is a non-trivial morphism $M \longrightarrow H^*BV(\varphi)\otimes F$ in $K-\mathcal{U}$ where F is some finite $T_V(K;\varphi)$ - module (e.g. $F = (T_V(M;\varphi))^{<n}$ for n large enough). It is straightforward to show that $H^*BV(\varphi)\otimes F$ injects into its localization at the prime ideal $\mathrm{Rad}(\mathrm{Ker}\,\varphi)$ and hence M does not localize to 0.

Conversely, assume localization at $\mathrm{Rad}(\mathrm{Ker}\,\varphi)$ does not kill M. Consider an embedding i of M as in Theorem 1.9 or 1.10. Because localization is exact there exists a factor $H^*BW(\psi)\otimes F$ of I which is not killed by it. It is clear that for such a factor we must have that $\mathrm{Rad}(\mathrm{Ker}\,\psi) \subset \mathrm{Rad}(\mathrm{Ker}\,\varphi)$, i.e. there is morphism $\alpha : (V,\varphi) \longrightarrow (W,\psi)$. Now assume that i is minimal, i.e. i followed by the projection to any proper subproduct is no longer an embedding. Then $T_W(M;\psi) \neq 0$ for each factor and Lemma 2.9 gives $T_V(M;\varphi) \neq 0$. $\Box$

Next we consider the concept of torsion with respect to a given invariant ideal. We will restrict our attention to the noetherian case.

DEFINITION 2.11. *Let K be an unstable noetherian algebra and let $\mathfrak{a}$ be an invariant ideal of K. An unstable finitely generated K - module M is called an $\mathfrak{a}$ - torsion module (or briefly $\mathfrak{a}$ - torsion) if $\mathfrak{a}^n M = 0$ for some natural number n. We will say in this case that M is an $\mathfrak{a}$ - torsion module of height n.*

Because we are in a noetherian situation, M is $\mathfrak{a}$ - torsion iff there exists a number n such that $a^n M = 0$ for each $a \in \mathfrak{a}$, i.e. if $\mathfrak{a}$ is contained in $\mathrm{Rad}(\mathrm{Ann}(M))$, the radical of the annihilator ideal. Furthermore, it follows immediately from [LS, Prop.3] (and can also be deduced from the embeddings in 1.9 and 1.10) that the radical of the annihilator ideal is an invariant ideal, so it is no restriction to assume that $\mathfrak{a}$ is invariant because we can always pass to the invariant ideal generated by $\mathfrak{a}$. From 2.3, 2.10 and classical results about the support we derive that $\mathrm{Rad}(\mathrm{Ann}(M)$ is the intersection of all invariant primes in the T - support of M, and hence we obtain the following result.

PROPOSITION 2.12. *Let K be an unstable noetherian algebra and M an unstable finitely generated K - module. Let $\mathfrak{a}$ be an invariant ideal of K. Then the following are equivalent.*

a) M is $\mathfrak{a}$ - torsion.

b) The ideal $\mathfrak{a}$ is contained in $\mathrm{Rad}(\mathrm{Ann}(M))$.

c) $T_V(M,\varphi) = 0$ for all $(V,\varphi) \in \mathcal{R}(K)$ with $\mathfrak{a} \not\subset \mathrm{Rad}(\mathrm{Ker}\,\varphi)$, i.e. $(V,\varphi) \in \mathcal{O}(\mathfrak{a})$. $\Box$

An unstable finitely generated K - module N is clearly finite iff it is torsion with respect to the ideal K^+ of all elements of positive degree. It is clear that in this case the set $\mathcal{O}(K^+)$ agrees with the set of objects in $\mathcal{R}_*(K)$ and therefore we obtain the following result.

COROLLARY 2.13. *Assume K is an unstable noetherian algebra and N an unstable finitely generated K - module. Then the following two conditions are equivalent.*

a) N is a finite module.

b) $T_V(N,\varphi) = 0$ for all $(V,\varphi) \in \mathcal{R}_*(K)$. □

Now we introduce the $\mathfrak{a}$ - torsion submodule functors $F_{\mathfrak{a}}$ which will be further studied in the next section.

DEFINITION/PROPOSITION 2.14. *Let K be an unstable noetherian algebra, $\mathfrak{a}$ be an invariant ideal of K and M be in $K_{fg} - \mathcal{U}$. Then $\{x \in M | \mathfrak{a}^n x = 0 \ for\ some\ n > 0\}$ is an unstable K - submodule which we denote by $F_{\mathfrak{a}}M$. In fact, this assignment extends to a functor $F_{\mathfrak{a}}$ from $K_{fg} - \mathcal{U}$ to itself which we call the $\mathfrak{a}$ - torsion submodule functor. It is left exact and its right derived functors are denoted by $R^iF_{\mathfrak{a}}$. If $\mathfrak{a} = K^+$ we will write F instead of F_{K^+} and call F the "finite submodule functor".*

<u>Proof:</u> The only part which is not obvious is that $F_{\mathfrak{a}}M$ is closed under Steenrod operations. For simplicity assume that the prime is 2. Let $x \in F_{\mathfrak{a}}M$, $a \in \mathfrak{a}$, so $a^n x = 0$ for some large n. We may assume that $n = 2^k$ and k is large. Then we apply the total Steenrod operation to this equation and obtain $Sq(a)^{2^k} Sq(x) = 0$. Now assume that $i < 2^k$ and consider the homogeneous part of degree $|a^{2^k}x| + i$ ($|y|$ denoting the degree of y). We obtain $a^{2^k} Sq^i x = 0$, i.e. $Sq^i x$ is again $\mathfrak{a}$ - torsion. The argument for odd primes is analogous. □

The final result in this section together with the embedding results of 1.9 and 1.10 shows that primary decompositions (cf. [L]) exist in $K_{fg} - \mathcal{U}$ if K is noetherian. We leave it to the interested reader to state and prove the appropriate existence and uniqueness results for such decompositions.

PROPOSITION 2.15. *Assume K is an unstable algebra and M is an unstable K - module. Assume we have a map $M \longrightarrow H^*BV(\varphi) \otimes F$ of unstable K - modules with F a finite $T_V(K;\varphi)$ - module for some $(V,\varphi) \in \mathcal{R}(K)$. Then the kernel of this map is primary with respect to the prime ideal $\mathrm{Rad}(\mathrm{Ker}\,\varphi)$. Furthermore, if F vanishes in degrees n and bigger, then $(\mathrm{Ker}\,\varphi)^n$ kills $H^*BV(\varphi) \otimes F$.*

<u>Proof:</u> To see this take an element $x \in K$ and consider $\gamma_{K,(V,\varphi)}(x) \in H^*BV \otimes T_V(K;\varphi)$. This can be written as $\gamma_{K,(V,\varphi)}(x) = \varphi x \otimes 1 + y$ with $y \in H^*BV \otimes T_V(K;\varphi)^+$ and $T_V(K;\varphi)^+$ denoting as before the ideal of elements of positive degree. From this formula it is clear that x acts nilpotently on $H^*BV(\varphi) \otimes F$ iff $x \in \mathrm{Rad}(\mathrm{Ker}\,\varphi)$, and if $x_1, \ldots, x_n$ are in $\mathrm{Ker}\,\varphi$, then the product $x_1 \ldots x_n$ kills $H^*BV(\varphi) \otimes F$. □

3. Localizations in $K_{fg} - \mathcal{U}$ away from torsion modules

<u>3.1.</u> Throughout this section K will be an unstable noetherian algebra. We will study localizations on the category $K_{fg} - \mathcal{U}$ of unstable finitely generated K - modules away from the

full subcategory $\mathcal{T}ors(\mathfrak{a})$ of $\mathfrak{a}$ - torsion modules for some fixed invariant ideal $\mathfrak{a}$. We begin with some formal definitions.

DEFINITION 3.2. *Let K be an unstable noetherian algebra and let $\mathfrak{a}$ be an invariant ideal of K. Let M be an unstable finitely generated K - module.*

a) M is called $\mathcal{T}ors(\mathfrak{a})$ - reduced (or $\mathfrak{a}$ - reduced) iff $\mathrm{Hom}_{K-\mathcal{U}}(N, M) = 0$ *for each $\mathfrak{a}$ - torsion module $N \in K_{fg} - \mathcal{U}$.*

b) M is called $\mathcal{T}ors(\mathfrak{a})$ - closed (or $\mathfrak{a}$ - closed) iff $\mathrm{Ext}^i_{K-\mathcal{U}}(N, M) = 0$, *$i = 0, 1$ for all $\mathfrak{a}$ - torsion modules $N \in K_{fg} - \mathcal{U}$.*

It is clear that M is $\mathfrak{a}$ - reduced iff it does not contain any non-trivial $\mathfrak{a}$ - torsion submodules. Furthermore, M is $\mathfrak{a}$ - closed iff for any morphism $\alpha : A \longrightarrow B$ of unstable finitely generated K - modules whose kernel and cokernel is $\mathfrak{a}$ - torsion, the induced map $\mathrm{Hom}_{K-\mathcal{U}}(B, M) \longrightarrow \mathrm{Hom}_{K-\mathcal{U}}(A, M)$ is an isomorphism.

The following proposition follows immediately from the definitions (cf. [G], or [HLS1,2] where the same concept has been investigated in other settings).

PROPOSITION 3.3. *Let K be an unstable noetherian algebra and let $\mathfrak{a}$ be an invariant ideal.*

a) If $0 \longrightarrow M \longrightarrow M_1 \longrightarrow M_2$ is exact and M_1 is $\mathfrak{a}$ - closed and M_2 is $\mathfrak{a}$ - reduced then M is $\mathfrak{a}$ - closed.

b) Any finite inverse limit of $\mathfrak{a}$ - closed modules is $\mathfrak{a}$ - closed.

c) Any summand of an $\mathfrak{a}$ - closed module is $\mathfrak{a}$ - closed. $\square$

3.4. The categories $\mathcal{T}ors(\mathfrak{a})$ are "Serre subcategories", i.e. their set of objects form a Serre class. They are localizing in the sense of Gabriel [G, III.3. Cor. 1] so there are functors $L_{\mathfrak{a}} : K_{fg} - \mathcal{U} \to K_{fg} - \mathcal{U}$ and natural transformations $\lambda_{\mathfrak{a}} : 1_{K_{fg} - \mathcal{U}} \to L_{\mathfrak{a}}$ such that for each $M \in K_{fg} - \mathcal{U}$:

- $L_{\mathfrak{a}} M$ is $\mathfrak{a}$ - closed and
- kernel and cokernel of $\lambda_{\mathfrak{a}, M}$ are $\mathfrak{a}$ - torsion.

We will study the localization away from these subcategories. The following result provides general examples for $\mathfrak{a}$ - closed modules.

PROPOSITION 3.5. *Let K be an unstable noetherian algebra, let $\mathfrak{a}$ be an invariant ideal and $(V, \varphi) \in \mathcal{R}(K)$ be such that* $\mathrm{Rad}(\mathrm{Ker}\, \varphi)$ *does not contain $\mathfrak{a}$, in other words $(V, \varphi) \in \mathcal{O}(\mathfrak{a})$. Then the following assertions hold for each unstable finitely generated K - module M.*

a) $Ext^i_{K-\mathcal{U}}(N, T_V(M; \varphi)) = 0$ for each $\mathfrak{a}$ - torsion module N in $K_{fg} - \mathcal{U}$ and each i. In particular $T_V(M; \varphi)$ is $\mathfrak{a}$ - closed.

b) $R^i F_{\mathfrak{a}}(T_V(M; \varphi)) = 0$ for all i.

The proof of the proposition relies on the following key lemma.

LEMMA 3.6. *Let F be an unstable finite $T_W(K;\psi)$ - module for some $(W,\psi)\in\mathcal{R}(K)$ and let (V,φ) be in $\mathcal{R}(K)$. Then there is an isomorphism of unstable K - modules*

$$T_V(H^*BW(\psi)\otimes F;\varphi)\cong\prod_{\mathrm{Hom}_{\mathcal{R}(K)}((V,\varphi),(W,\psi))}H^*BW(\psi)\otimes F\ .$$

Proof of Lemma 3.6: By definition $T_V(H^*BW(\psi)\otimes F;\varphi)\cong\mathbb{F}_p(\varphi)\otimes_{T_V^0K}T_V(H^*BW(\psi)\otimes F)$. Futhermore, because F is finite and T_V commutes with tensor products we have an isomorphism of unstable modules (see [L2,L3])

$$T_V(H^*BW\otimes F;\varphi)\cong\prod_{\mathrm{Hom}(V,W)}H^*BW\otimes F\ ,$$

and we have to identify the T_VK - module structure, and in particular the T_V^0K - module structure on this. Now F is bounded above, so the action of $T_W(K;\psi)$ on F factors through an action of $(T_W(K;\psi))^{<n}$ for n sufficiently large. In addition, $T_V((T_W(K;\psi))^{<n})\cong(T_W(K;\psi))^{<n}$, hence

$$T_V(H^*BW\otimes(T_W(K;\psi))^{<n})\cong\prod_{\mathrm{Hom}(V,W)}H^*BW\otimes(T_W(K;\psi))^{<n}\ .$$

Now the action of T_V^0K on $T_V(H^*BW(\psi)\otimes F;\varphi)$ is pulled back from the obvious action of $T_V^0(H^*BW\otimes(T_W(K;\psi))^{<n})\cong T_V^0(H^*BW)$ via the map $T_V^0K\longrightarrow T_V^0(H^*BW)$ induced by ψ. If we identify T_V^0K with $\mathbb{F}_p^{\mathrm{Hom}_{\mathcal{K}}(K,H^*BV)}$, the algebra of $\mathbb{F}_p$ - valued functions on $\mathrm{Hom}_{\mathcal{K}}(K,H^*BV)$, and similarly $T_V^0(H^*BW)$ with $\mathbb{F}_p^{\mathrm{Hom}(V,W)}$ then this map is induced by the map $\mathrm{Hom}(V,W)\to\mathrm{Hom}_{\mathcal{K}}(K,H^*BV)$, $\alpha\mapsto\alpha^*\psi$. The proposition follows. □

Proof of Proposition 3.5: Take any injective resolution of M in $K_{fg}-\mathcal{U}$ as provided by Theorem 1.9. Because $T_V(-;\varphi)$ is exact we may apply $T_V(-;\varphi)$ to it to get an injective resolution of $T_V(M;\varphi)$. In fact, by 3.6 this resolution consists of injectives of the form $I_{(W,\psi)}(n)$ for which $\mathrm{Hom}_{\mathcal{R}(K)}((V,\varphi),(W,\psi))\neq\emptyset$. However, because of $\mathfrak{a}\not\subset\mathrm{Rad}(\mathrm{Ker}\,\varphi)$ and $\mathrm{Rad}(\mathrm{Ker}\,\psi)\subset\mathrm{Rad}(\mathrm{Ker}\,\varphi)$ these modules are clearly $\mathfrak{a}$ - reduced (cf. 2.15) and the result follows. □

For the reader familiar with [DW2] we remark that the assumption that a module is of the form $T_V(M;\varphi)$ plays in this section the role of the centrality assumptions in [DW2].

Of course, one tries to take $\mathfrak{a}$ as small as possible in this proposition. By 2.7 the minimal $\mathfrak{a}$ is the ideal for which the set $\mathcal{O}(\mathfrak{a})$ consists precisely of those (W,ψ) for which there exists a morphism from (V,φ) to (W,ψ).

3.7 Example. Let G and X be as in Theorem 0.2, assume C is a central subgroup of G which acts trivially on X and let $\mathfrak{a}$ be any invariant ideal in H^*BG such that $\mathrm{Rad}(\mathrm{Ker}\,\mathrm{res}_{G,C})$ does not contain $\mathfrak{a}$. Then Proposition 3.5 together with Theorem 0.2 gives that $H^*_G X$ is $\mathfrak{a}$ - closed considered as an object of $H^*BG_{fg} - \mathcal{U}$.

COROLLARY 3.8. *Let G be a compact Lie group and let $\mathfrak{m}$ be the invariant maximal ideal consisting of all elements of positive degree. Then the following assertions hold:*

a) $Ext^i_{H^*BG-\mathcal{U}}(N, H^*BG) = 0$ for all unstable finite H^*BG - modules N. In particular, H^*BG is $\mathfrak{m}$ - closed.

b) $R^iF_{\mathfrak{m}}H^*BG = 0$ for all i.

Proof: Choose a non-trivial elementary abelian p - subgroup E which is central in some p - Sylow subgroup G_p (i.e. G_p is a pullback of a p - Sylow subgroup of the Weyl group $N_G(T)/T$, T denoting a maximal torus of G and $N_G(T)$ its normalizer in G). Then the result holds (by 3.5 and by Theorem 0.2) if we replace H^*BG by $H^*BC_G(E)$. Because $C_G(E)$ contains G_p, the usual transfer argument shows that H^*BG is a summand in $H^*BC_G(E)$ as an unstable H^*BG - module and the result follows. □

We now come to our main result (the algebraic version of Theorem 0.4 of the introduction) which describes the localization away from $\mathcal{T}ors(\mathfrak{a})$ in terms of the T - functor and relates it and its derived functors to the derived functors $R^iF_{\mathfrak{a}}$.

THEOREM 3.9. *Let K be an unstable noetherian algebra and $\mathfrak{a}$ be an invariant ideal with associated open subset $\mathcal{O}(\mathfrak{a})$ of $\mathcal{R}(K)$. Denote the full subcategory with $\mathcal{O}(\mathfrak{a})$ as object set again by $\mathcal{O}(\mathfrak{a})$. Let M be an unstable finitely generated K - module.*

a) Then there is a natural exact sequence

$$0 \longrightarrow F_{\mathfrak{a}}M \longrightarrow M \xrightarrow{\rho} \lim_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi) \longrightarrow R^1F_{\mathfrak{a}}M \longrightarrow 0$$

in which the components of ρ are induced by the maps $\rho_{M,(V,\varphi)}$ (cf. 1.4). In particular, the kernel and cokernel of ρ are unstable finitely generated $\mathfrak{a}$ - torsion modules. Furthermore, ρ is localization away from the subcategory $\mathcal{T}ors(\mathfrak{a})$.

b) There are natural isomorphisms

$$\lim^i_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi) \cong R^{i+1}F_{\mathfrak{a}}M$$

for all $i > 0$. In particular, $\lim^i_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi)$ is an unstable finitely generated $\mathfrak{a}$ - torsion module for all $i > 0$.

The following special case of this theorem needs to be emphasized.

COROLLARY 3.10. *Let K be an unstable noetherian algebra and let M be an unstable finitely generated K - module.*

a) Then there is a natural exact sequence

$$0 \longrightarrow FM \longrightarrow M \xrightarrow{\rho} \lim_{\mathcal{R}_*(K)} T_V(M;\varphi) \longrightarrow R^1FM \longrightarrow 0$$

in which the components of ρ are induced by the maps $\rho_{M,(V,\varphi)}$. In particular, the kernel and cokernel of ρ are finite. Furthermore, ρ is localization away from the subcategory of finite unstable K - modules.

b) There are natural isomorphisms

$$\lim^i_{\mathcal{R}_*(K)} T_V(M;\varphi) \cong R^{i+1}FM$$

for all $i > 0$. In particular, $\lim^i_{\mathcal{R}_*(K)} T_V(M;\varphi)$ is finite for all $i > 0$. ∎

<u>3.11. Remarks.</u> a) Theorem 0.4 and Corollary 0.5 of the introduction are clearly just special cases of 3.9 and 3.10: the two subsets resp. subcategories both labelled $\mathcal{O}(\mathfrak{a})$ (of $\mathcal{A}(G)$ and $\mathcal{R}(H^*BG)$ respectively) clearly correspond under the equivalence of categories $\mathcal{A}(G) \cong \mathcal{R}(H^*BG)$ of 0.2.I; furthermore for $M = H^*_G X$ the isomorphisms of 0.2.II are compatible with the maps $\rho_{M,(V,\varphi)}$ and the maps $H^*_G X \longrightarrow H^*_{C_G(E)}(X^E)$ induced by the inclusions. This is obvious from the construction of the isomorphisms (as desribed in the appendix).

b) Of course, as in the case of Theorem 0.4 of the introduction we have here the same formal analogy with the classical case considered in [Gr].

c) If $\mathfrak{a}$ is the ideal of positive dimensional elements of K then there is a vanishing result for the higher limits due to Oliver [O]: $\lim^i_{\mathcal{R}_*(K)} T_V(M;\varphi) = 0$ if $i > d(K)$ where $d(K)$ is the maximal rank of an elementary abelian p - group V such that $(V,\varphi) \in \mathcal{R}(K)$ for some $\varphi : K \longrightarrow H^*V$. This is analogous to the vanishing theorem of Grothendieck [Ha, Theorem III.2.7]. However, for more general ideals this analogy breaks down: e.g. if $\mathcal{O}$ is the open (!) set consisting of all (V,φ) with Rad(Ker φ) equal to a fixed minimal prime ideal then $\mathcal{O} = \mathcal{O}(\mathfrak{a})$ where $\mathfrak{a}$ is the intersection of all other invariant primes. In this case $\mathcal{O}(\mathfrak{a})$ is equivalent to the one-object-category associated to the automorphism group $\mathrm{Aut}_{\mathcal{R}(K)}((V,\varphi))$ and the higher limits can be identified with group cohomology and can be non-zero in arbitrary high degrees.

The following result represents the key step in the proof of Theorem 3.9.

LEMMA 3.12. *Let K, $\mathfrak{a}$ and $\mathcal{O}(\mathfrak{a})$ be as in 3.9, let (W,ψ) be in $\mathcal{R}(K)$, F be an unstable finite $T_W(K;\psi)$ - module and consider the unstable K - module $M = H^*BW(\psi) \otimes F$.*

a) If (W,ψ) is in $\mathcal{O}(\mathfrak{a})$ then

$$\rho : M \longrightarrow \lim_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi)$$

is an isomorphism and for each $i > 0$

$$\lim^i_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi) = 0 \ .$$

b) If (W,ψ) is not in $\mathcal{O}(\mathfrak{a})$ then for each $i \geq 0$

$$\lim^i_{\mathcal{O}(\mathfrak{a})} T_V(M;\varphi) = 0 \ .$$

Proof: By Lemma 3.6 the functor on $\mathcal{O}(\mathfrak{a})$ which sends (V,φ) to $T_V(M;\varphi)$ is given by $T_V(H^*BW(\psi)\otimes F;\varphi) \cong \prod_{\mathrm{Hom}_{\mathcal{R}(K)}((V,\varphi),(W,\psi))}(H^*BW(\psi)\otimes F)$. In other words it is induced from the graded vector space $H^*BW(\psi)\otimes F$ if $(W,\psi)\in\mathcal{O}$, and trivial otherwise. (Here we identify the category of graded vector spaces with the category of functors from the "trivial category", with one object (W,ψ) and the identity morphism only, to the category of graded vector spaces.) The Proposition follows. $\square$

Proof of Theorem 3.9: Consider an injective resolution $I_\bullet$ of M in $K_{fg}-\mathcal{U}$ as provided by Theorem 1.9 and Corollary 1.11. Because T is exact the complex of functors $T_-(I_\bullet;-)$ is a resolution of the functor $T_-(M;-)$. By the previous proposition the higher derived functors of $\lim_{\mathcal{O}(\mathfrak{a})}$ vanish on the functors $T_-(I_k;-)$. This together with the fact that $\lim_{\mathcal{O}(\mathfrak{a})}$ is left exact implies that $\lim^i_{\mathcal{O}(\mathfrak{a})} T_-(M;-)$ can be computed as the cohomolgy of the complex $\lim_{\mathcal{O}} T_-(I_\bullet;-)$

Again by the previous proposition the complex $\lim_{\mathcal{O}} T_-(I_\bullet;-)$ is obtained from $I_\bullet$ by throwing away those $I_{(V,\varphi)}(n)$ for which (V,φ) is not in $\mathcal{O}(\mathfrak{a})$ and keeping all the others. In other words, we get an exact sequence of complexes

$$0 \longrightarrow F_{\mathfrak{a}} I_\bullet \longrightarrow I_\bullet \longrightarrow \lim_{\mathcal{O}(\mathfrak{a})} T_V(I_\bullet;\varphi) \longrightarrow 0 \ .$$

The long exact sequence belonging to this short exact sequence yields the exact sequence of a) and the isomorphisms in b).

Finally the inverse limit is $\mathfrak{a}$ - closed by 3.3 and 3.5. $\square$

3.13. The proof of 3.9 together with 2.15 gives also information about the height of the $\mathfrak{a}$ - torsion modules $R^iF_{\mathfrak{a}}M$ in terms of a given injective resolution. For example, if $p=2$ and if n_i is the maximum of all n such that for some $(V,\varphi)\in\mathcal{C}(\mathfrak{a})$ the injective $I_{(V,\varphi)}(n)$ occurs as a summand in I_k , then the height of the $\mathfrak{a}$ - torsion module $R^iF_{\mathfrak{a}}M$ is at most n_k.

In the case of equivariant cohomology information about injective resolutions is often available (see [HLS2, II.1 and II.2], [HLO]). In joint work with F.Cohen and Y. Xia we will apply this in the case of mapping class groups to get new classes in their cohomology.

Theorem 3.9 together with Theorem 0.2 and the obvious generalization of Corollary 3.8 imply the following result. It contains the theorem of Jackowski and McClure [JM] (see also [DW2]) as a special case. (Take $\mathcal{O}=\mathcal{A}_*(G)$ and C be isomorphic to $\mathbb{Z}/p$!) Note also that under the equivalence of categories of Theorem 0.2.I a subset $\mathcal{O}$ of $\mathcal{A}(G)$ will be called open if $E\in\mathcal{O}$ implies $E'\in\mathcal{O}$ whenever E is subconjugate to E'.

COROLLARY 3.14. *Let p be a prime. Assume G is a compact Lie group and let C be an elementary abelian p - subgroup of G which is central in some p - Sylow subgroup. Let $\mathcal{O}$ be any open subset of $\mathcal{A}(G)$ containing C.*

a) Then the restriction maps $H^*BG \longrightarrow H^*BC_G(E)$ induce an isomorphism

$$\rho : H^*BG \longrightarrow \lim_{\mathcal{O}} H^*BC_G(E) \ .$$

b) Furthermore $\lim^i_{\mathcal{O}} H^*BC_G(E) = 0$ for all $i > 0$. $\square$

The following change of rings type result gives us some flexibility for computing the higher limits. In particular it allows a change of groups in the situation of equivariant cohomology. The result is implicit in [DW2].

PROPOSITION 3.15. *Let K, L be unstable noetherian algebras and let $f : K \longrightarrow L$ be a homomorphism of unstable algebras which makes L into an unstable finitely generated K - module. Let M be an object in $L_{fg} - \mathcal{U}$ which we also consider as an object of $K_{fg} - \mathcal{U}$ via the map f. Then for any open set $\mathcal{O} \in \mathcal{R}(K)$ there are natural isomorphisms for all $i > 0$*

$$\lim^i_{\mathcal{O}} T_V(M;\varphi) \cong \lim^i_{f^{*-1}\mathcal{O}} T_V(M;\psi) \ .$$

(Of course, f^* denotes the induced map $\mathcal{R}(L) \longrightarrow \mathcal{R}(K)$ and $f^{*-1}\mathcal{O}$ is the full subcategory of $\mathcal{R}(L)$ whose objects are mapped to $\mathcal{O}$ under f^*.)

<u>Proof:</u> The map f induces a functor $f^* : f^{*-1}\mathcal{O} \longrightarrow \mathcal{O}$ and a functor $f_! : \mathcal{O} - \text{mod} \longrightarrow f^{*-1}\mathcal{O} - \text{mod}$. Here we write $\mathcal{D} - \text{mod}$ for the category of functors from a category $\mathcal{D}$ to the category of abelian groups. The objects of $\mathcal{D} - \text{mod}$ are called $\mathcal{D}$ - modules. Let $\widetilde{f_!} : f^{*-1}\mathcal{O} - \text{mod} \longrightarrow \mathcal{O} - \text{mod}$ be the right Kan - extension along $f_!$ so that we have natural isomorphisms

$$\text{Hom}_{\mathcal{O}-\text{mod}}(F, \widetilde{f_!}G) \cong \text{Hom}_{f^{*-1}\mathcal{O}-\text{mod}}(f_!F, G)$$

for any $\mathcal{O}$ - module F and any $f^{*-1}\mathcal{O}$ - module G.

We recall that the right Kan - extension of an $f^{*-1}\mathcal{O}$ - module G is given on the object $(V,\varphi) \in \mathcal{O}$ as follows: Let $(V,\varphi) \downarrow f_!$ be the under category with respect to $f_!$. Then $(\widetilde{f_!}G)(V,\varphi) = \lim_{(V,\varphi)\downarrow f_!} G$.

We will need the following lemma. We omit its proof which is not difficult and completely analogous to that of Lemma 2.9 of [H3].

LEMMA 3.16. *Let (V,φ) be an object in $\mathcal{O}$. Then the under category $(V,\varphi) \downarrow f_!$ is a disjoint union of categories each of which has an initial object. The components are indexed by those $(V,\psi) \in f^{*-1}\mathcal{O}$ which extend (V,φ), i.e. for which $\varphi = \psi f$ holds. These objects are at the same time the initial objects of the components.* $\square$

The lemma implies that the Kan extension is given by the following formula:

$$(\widetilde{f_!}G)(V,\varphi) = \prod_{(V,\psi)} G(V,\psi) \ .$$

The product in this formula is indexed by those $(V,\psi) \in f^{*-1}\mathcal{O}$ which extend (V,φ). In particular, the Kan - extension $\widetilde{f_!}$ is exact. The functor $f_!$ is clearly exact and we conclude that it carries projective resolutions to projective resolutions. Taking a projective resolution of the constant functor with value $\mathbb{Z}$ we obtain for any $\mathcal{R}(L)$ - module G:

$$\lim^i_{\mathcal{O}} \widetilde{f_!}G \cong \lim^i_{f^{*-1}\mathcal{O}} G \ .$$

Now let M be an unstable L - module and consider the $f^{*-1}\mathcal{O}$ - module $T_-(M;-)$. Now let $(V,\varphi) \in \mathcal{O}$. It is easy to check that (with an obvious abuse of notation) we have $(f_! T_-(M;-))(V,\varphi) \cong T_V(M;\varphi)$. In other words, we obtain the right Kan - extension and the result follows. ∎

4. An application to $H^*BGL(n,\mathbb{Z}[1/2])$

4.1. We give an application of 3.9 to a qualitative study of the mod - 2 cohomology ring $H^*BGL(n,\mathbb{Z}[1/2])$. Here $GL(n,\mathbb{Z}[1/2])$ is the general linear group of rank n over the ring $\mathbb{Z}[1/2]$. Let D_n be the subgroup of diagonal matrices with diagonal entries ± 1. The image of the restriction map $\rho_n : H^*BGL(n,\mathbb{Z}[1/2]) \longrightarrow H^*BD_n$ has recently been determined by Mitchell [Mt]; it is isomorphic to a tensor product $\mathbb{F}_2[w_1,\ldots,w_n]\otimes E(e_1,\ldots,e_{2n-1})$ of a polynomial algebra on generators $w_1,\ldots,w_n$ and an exterior algebra on generators $e_1,\ldots,e_{2n-1}$, with indices giving the degrees of the elements. Let n_0 be minimal such that ρ_{n_0} is not injective. According to Dwyer (private communication) n_0 is finite. We will show in [H4] that $n_0 > 3$. The theory developed here can be used to get some qualitative information about the size of $\operatorname{Ker}\rho_n$.

To do this we recall (cf. [HLS2,II.5.2]) that the objects of $\mathcal{R}_n := \mathcal{R}(H^*BGL(n,\mathbb{Z}[1/2]))$ can be identified with the faithful representations of elementary abelian 2 - groups V, i.e. with formal sums $\Sigma_\chi n_\chi \chi$ where χ runs through the characters of V and the n_χ are non-negative integers such that $\Sigma_\chi n_\chi = n$ and such that the set of χ with $n_\chi > 0$ spans V^*, the group of characters of V. (The representation φ is identified with the object (V,φ^*) in $\mathcal{R}(H^*BGL(n,\mathbb{Z}[1/2]))$, if φ^* denotes the induced map in cohomology, and corresponds to the object $\operatorname{Im}\rho$ in $\mathcal{A}(GL(n,\mathbb{Z}[1/2]))$ under the equivalence of categories in 0.2.I.) Furthermore, if $\varphi = \Sigma_\chi n_\chi \chi$ (resp. $\varphi' = \Sigma_{\chi'} n_{\chi'}\chi'$) are faithful representations of V (resp. V') then $\operatorname{Hom}_{\mathcal{R}_n}(\varphi,\varphi') = \{\alpha \in \operatorname{Hom}(V,V') | \varphi = \Sigma_{\chi'} n_{\chi'}\chi'\alpha\}$. We note that the centralizer of $\operatorname{Im}\varphi$ is isomorphic to $\prod_\chi GL(n_\chi,\mathbb{Z}[1/2])$.

Let $\mathcal{O}_n \subset \mathcal{R}_n$ be the subset consisting of those representations φ for which $n_\chi < n_0$ for all χ. This is clearly an open subset and we have $\mathcal{O}_n = \mathcal{O}(\mathfrak{a}_n)$ where $\mathfrak{a}_n$ is the invariant ideal given as the intersection of all $\operatorname{Rad}(\operatorname{Ker}\varphi^*)$ with $(V,\varphi^*) \notin \mathcal{O}_n$. Then $T_V(H^*BGL(n,\mathbb{Z}[1/2]);\varphi) \cong \prod_\chi H^*BGL(n_\chi,\mathbb{Z}[1/2])$ by Theorem 0.2 and hence embedds into $\prod_\chi H^*BD_{n_\chi} \cong H^*BD_n$ for all $\varphi \in \mathcal{O}_n$. Consequently the kernel of the localization map away from $\mathcal{T}ors(\mathfrak{a}_n)$ modules agrees with the kernel of ρ_n. In particular, $\operatorname{Ker}\rho_n$ is $\mathfrak{a}_n$ - torsion. If φ is a representation of V which is not in $\mathcal{O}_n$ then V has rank at most equal to $n - n_0 + 1$, therefore the size (as defined in the introduction) of $H^*BGL(n, \mathbb{Z}[1/2])/\operatorname{Rad}(\mathfrak{a}_n)$ is at most equal to $n - n_0 + 1$. Clearly this number is an upper bound for the size of the $\mathfrak{a}_n$ - torsion module $\operatorname{Ker}\rho_{n_0}$. In fact, we have the following theorem.

THEOREM 4.2. *The kernel of the restriction map* $H^*BGL(n, \mathbb{Z}[1/2]) \xrightarrow{\rho_n} H^*BD_n$ *is the largest* $\mathfrak{a}_n$ *- torsion submodule and for* $n \geq n_0$ *it has size* $n - n_0 + 1$ *where* n_0 *denotes the smallest natural number such that* ρ_{n_0} *is not a monomorphism. In particular, the size of* $\operatorname{Ker}\rho_{n_0}$ *is one, i.e.* $\operatorname{Ker}\rho_{n_0}$ *is periodic in large degrees.*

Proof: We have already seen that the size is at most $n - n_0 + 1$.

Let d_n be equal to the size of $\operatorname{Ker}\rho_n$. Consider an embedding of $\operatorname{Ker}\rho_{n_0}$ as in 1.9, and assume it is minimal in the sense that no factor $I_{(V,\varphi)}(n)$ can be dropped without loosing the embedding property. Then it is clear that the size of $\operatorname{Ker}\rho_n$ is equal to the maximum of the sizes of the injectives $I_{(V,\varphi)}(n)$ involved, and hence exactness of T and Lemma 3.6. imply that each component $T_V(\operatorname{Ker}\rho_n; \varphi^*)$ has size at most d_n. It suffices therefore to find a faithful representation $\varphi : V \longrightarrow GL(n, \mathbb{Z}[1/2])$ such that $T_V(\operatorname{Ker}\rho_n; \varphi^*)$ has size $n - n_0 + 1$.

Such a representation can be obtained as follows. Let V be elementary abelian of rank $n - n_0 + 1$ and let χ_i, $1 \leq i \leq n - n_0 + 1$ be a dual basis of V. Then consider $\varphi = n_0\chi_1 + \Sigma_{i \neq 1}\chi_i$. Applying $T_V(-, \varphi^*)$ to the exact sequence

$$0 \longrightarrow \operatorname{Ker}\rho_n \longrightarrow H^*BGL(n, \mathbb{Z}[1/2]) \longrightarrow H^*BD_n$$

yields an exact sequence

$$0 \longrightarrow T_V(\operatorname{Ker}\rho_n; \varphi^*) \longrightarrow H^*BGL(n_0, \mathbb{Z}[1/2]) \otimes H^*BD_{n-n_0} \longrightarrow \prod H^*BD_n$$

where the product is taken over all homomorphisms $\psi : V \longrightarrow D_n$ such that its composition with the inclusion of D_n into $GL(n, \mathbb{Z}[1/2])$ is conjugate to φ (cp. the discussion of the Kan extension in the proof of 3.15.) Now the different maps $H^*BGL(n_0, \mathbb{Z}[1/2]) \otimes H^*BD_{n-n_0} \longrightarrow H^*BD_n$ differ only by the action of an appropriate element of the symmetric group $\mathfrak{S}_n$ on the target, in particular all these maps have the same kernel, which is equal to $\operatorname{Ker}\rho_{n_0} \otimes H^*BD_{n-n_0}$. This has size $n - n_0 + d_{n_0}$ and hence we only have to show that the size of $\operatorname{Ker}\rho_{n_0}$ is positive. However, Theorem 0.2 and Proposition 3.5 show that $H^*BGL(n, \mathbb{Z}[1/2])$ does not contain any unstable finite ideals (take for $\mathfrak{a}$ the ideal of elements in positive degrees and for φ the restriction map to the central $\mathbb{Z}/2$), so the size must be positive. $\square$

The method used in the discussion above should lead to similar results for general linear groups over rings of S - integers in other number fields.

Appendix. Lannes' T - functor and Borel constructions of discrete groups

Let p be a fixed prime. As before we surpress the coefficients from our notation.

In this appendix we prove part (b) of Theorem 0.4. We will use freely results and terminology of [HLS1].

Let V be an elementary abelian p - group and ρ a homomorphism from V to the (discrete) group Γ. Let X be a Γ - space. Denote the centralizer of the image of ρ in Γ by Γ_ρ and the fixed point set with respect to the image of ρ by X^ρ. Then the homomorphism $V \times \Gamma_\rho \to \Gamma$, $(v, g) \mapsto \rho(v)g$ induces a map $BV \times (E\Gamma_\rho \times_{\Gamma_\rho} X^\rho) \longrightarrow E\Gamma \times_\Gamma X$ which we denote by c_ρ . Passing to cohomology and using adjointness we obtain a map $ad(c_\rho^*) : T_V H_\Gamma^* X \longrightarrow H_{\Gamma_\rho}^* X^\rho$.

THEOREM A.1. *Let Γ be a discrete group and X a finite dimensional Γ - CW - complex of finite orbit type whose isotropy groups are all finite. Then the natural map*

$$T_V(H_\Gamma^* X) \longrightarrow \prod_{\rho \in \mathrm{Rep}(V,\Gamma)} H_{\Gamma_\rho}^*(X^\rho)$$

with components $ad(c_\rho^*)$ is an isomorphism for each elementary abelian p - group V. (Here Rep(V, Γ) *denotes the set of Γ - conjugacy classes of homomorphisms from V to Γ and we have chosen a representative ρ from each conjugacy class.)*

<u>Proof:</u> Because T_V is exact and commutes with direct sums, it is enough to do the case of an orbit, i.e. $X = G/H$ with H finite. In this case we have natural isomorphisms

$$T_V H_\Gamma^* X \cong T_V H^* BH \cong \prod_{\rho \in \mathrm{Rep}(V,H)} H^* BH_\rho$$

where the second isomorphism comes from Lannes' Theorem [L1,L3] and has components $ad(c_\rho^*)$ as in the statement of the theorem. Now it follows from [H3, Lemma 2.8] that there is a natural isomorphism

$$(A.1.1.) \qquad \prod_{\rho \in \mathrm{Rep}(V,H)} H^* BC_H(\rho) \cong \prod_{\rho \in \mathrm{Rep}(V,\Gamma)} H_{\Gamma_\rho}^*(X^\rho) \ .$$

It is straightforward to check that this string of isomorphisms is given by the natural map in question. □

If X can also be chosen mod - p acyclic then by Smith theory the fixed point sets X^ρ are mod - p acyclic as well. Hence we have $H_\Gamma^* X \cong H^* B\Gamma$ and $H_{\Gamma_\rho}^* X^\rho \cong H^* B\Gamma_\rho$. Consequently we obtain the following result.

COROLLARY A.2. *Let Γ be a discrete group which admits an action on a finite dimensional mod - p acyclic Γ - CW - complex with finite orbit type and with finite isotropy groups. Then the natural map*

$$T_V(H^*B\Gamma) \longrightarrow \prod_{\rho\in \mathrm{Rep}(V,\Gamma)} H^*(B\Gamma_\rho)$$

is an isomorphism for each elementary abelian p - group V. □

THEOREM A.3. *Let Γ be a discrete group and X be a Γ - CW - complex with finitely many equivariant cells and finite isotropy groups. Then $H^*_\Gamma(X)$ is a finitely generated algebra.*

Proof: We will first prove that $H^*_\Gamma(X)$ is noetherian up to F - isomorphism, or more precisely, that its Nil - closure is noetherian.

Consider the contravariant functor $g(H^*_\Gamma X)$ from elementary abelian p - groups to sets which associates to an elementary abelian p - group V the set $\mathrm{Hom}_{\mathcal{K}}(H^*_\Gamma X, H^*BV)$. This set is given by the spectrum of the p - Boolean algebra $T^0_V H^*_\Gamma X$ [L2,L3] which by Theorem A.1 above can be identified with the disjoint union

$$\coprod_{\rho\in \mathrm{Rep}(V,\Gamma)} \pi_0(E\Gamma_\rho \times_{\Gamma_\rho} X^\rho) \cong \coprod_{\rho\in \mathrm{Rep}(V,\Gamma)} \pi_0(X^\rho)/\Gamma_\rho \ .$$

It follows from (A.1.1) together with our finiteness assumptions that this set is finite (although the set $\mathrm{Rep}(V,\Gamma)$ need not be finite). The functor $g(H^*_\Gamma X)$ has finite transcendance degree d in the sense of [HLS1,II.5] and, in fact, d is equal to the maximal rank of an elementary abelian subgroup which occurs as isotropy subgroup in X. Furthermore the $\mathrm{End}(V)$ - set $g(H^*_\Gamma X)(V)$ is noetherian in the sense of [HLS1], which means that the Nil - closure of $H^*_\Gamma X$ (which is given, up to isomorphism, by Quillen's inverse limit) is noetherian [HLS1,II.7].

We can now pick an unstable noetherian subalgebra K of $H^*_\Gamma X$ which is F - isomorphic to the Nil - closure of $H^*_\Gamma X$. The spectral sequence of the map $E\Gamma \times_\Gamma X \longrightarrow \Gamma\backslash X$ is a spectral sequence of $H^*_\Gamma X$ - modules and hence one of K - modules. Because K is noetherian and this spectral sequence has only finitely many columns it is enough to show that its E_1 - term is a finitely generated K - module. The E_1 - term is given as $E_1^{s,*} = \bigoplus_\sigma H^*(B\Gamma_\sigma)$ where σ runs over the set of s - cells in $\Gamma\backslash X$ and Γ_σ denotes the isotropy group of a chosen representative of the set of cells in X which project to the cell σ in $\Gamma\backslash X$. It is clearly enough to consider a single cell, i.e. to show that $H^*(B\Gamma_\sigma)$ is a finitely generated K - module. However, this follows immediately from [HLS1, II. Prop. 7.8]. □

In particular, if X can also be chosen mod - p acyclic we obtain.

COROLLARY A.4. *Let Γ be a discrete group which acts on a mod - p acyclic Γ - CW - complex with finitely many Γ - cells and with finite isotropy groups. Then $H^*B\Gamma$ is a finitely generated algebra.* □

We repeat that interesting examples of such groups are (S)-arithmetic groups, mapping class groups, outer automorphism groups of free groups and word-hyperbolic groups in the sense of Gromov.

Proof of Theorem 0.2.b: The functor $g(H^*B\Gamma)$ which by A.2 sends V to $\mathrm{Rep}(V,\Gamma)$ determines the category $\mathcal{R}(H^*B\Gamma)$ (see [HLS1,II.7]); as in the other cases of Theorem 0.2 its objects can be identified with the conjugacy classes of monomorphisms from elementary abelian p - groups to Γ. This category is (for any group Γ) equivalent to the category $\mathcal{A}(\Gamma)$ and part I follows. Part II follows immediately from Theorem A.1. $\square$

A.5 Remark. In [H3] we use that Corollary 0.5 of this paper holds under the assumptions of Theorem A.3 which are a bit more general than those of 0.5 because it is not assumed that $H^*B\Gamma$ is noetherian. However, by an argument as in 3.15 one shows that Theorem A.1 implies $\lim^i_{\mathcal{A}_*(\Gamma)} H^*_{C_\Gamma(E)}(X^E) \cong \lim^i_{\mathcal{R}_*(H^*_\Gamma X)} T_V(H^*_\Gamma X;\varphi)$ and then 3.10 gives the more general case as well.

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