The cohomology of $SL(3, \mathbb{Z}[1/2])$

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Abstract

We compute the cohomology of $SL(3, \mathbb{Z}[1/2])$ with coefficients in the prime fields and in the integers. On the way we obtain the cohomology of certain mod - 2 congruence subgroups of $SL(3, \mathbb{Z})$ with coefficients in $\mathbb{F}_p$ for $p > 2$. Finally we compute the cohomology of $GL(3, \mathbb{Z}[1/2])$.

Contents

1 Introduction

So far there exist only very few complete computations of integral or mod - p cohomology rings of arithmetic or more generally S - arithmetic groups. Among the known results we mention the calculations for $SL(2,\mathbb{Z})$ (which is straightforward from the well-known amalgamated product decomposition $SL(2,\mathbb{Z}) \cong \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$), of $SL(2,\mathbb{Z}[1/2])$ [Mi] and that of $SL(3,\mathbb{Z})$ [So]. Soulé's computation is already fairly involved; e.g. he obtains that the integral cohomology ring of $SL(3,\mathbb{Z})$, after localization at the prime 2, is generated by 7 elements which are subject to 22 relations. His result suggests that the answer for $SL(n,\mathbb{Z})$ would not be easily digestable (one should add that it also seems to be completely out of reach at this point).

From a conceptual point of view the complexity of the answer in Soulé's calculation can also be explained by Quillen's work [Q] which says among other things that the minimal prime ideals in the mod - p cohomology ring $H^*(\Gamma;\mathbb{F}_p)$ of an S - arithmetic group Γ are in one to one correspondence with the conjugacy classes of maximal elementary abelian p - subgroups of Γ. (We recall that an elementary abelian p - group is a group isomorphic to $(\mathbb{Z}/p)^k$ for some natural number k.) From this point of view those cases in which there exists a unique conjugacy class of maximal elementary abelian p - subgroups look more favourable than others. In the case of $SL(n,\mathbb{Z})$ or $GL(n,\mathbb{Z})$ it is very difficult to determine the precise number of conjugacy classes of maximal elementary abelian p - subgroups (this is essentially a problem of the integral representation theory of elementary abelian p - groups) and thus the mod - p cohomology of these groups must be complicated. The situation improves if one inverts p and adjoins p - th roots of unity. In particular in the case of $SL(n,\mathbb{Z}[1/2])$ and $GL(n,\mathbb{Z}[1/2])$ every elementary abelian 2 - subgroup is diagonalizable and there is a unique maximal one up to conjugacy.

This observation was presumably the basis of Quillen's conjecture (p. 591 of [Q]), which in the case of $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ claims that the inclusion of rings $\mathbb{Z}[1/2] \subset \mathbb{R}$ (and identifying $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ as usual with the mod 2 - cohomology of the classifying space $BGL(n,\mathbb{Z}[1/2])$) makes $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ into a free, in particular a torsion free module over the polynomial ring $\mathbb{F}_2[w_1,...,w_n] \cong H^*(BGL(n,\mathbb{R});\mathbb{F}_2)$ with w_i denoting as usual the i - th universal Stiefel - Whitney class. In [HLS] it was shown that torsion-freeness implies that the restriction map $\rho_n : H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2) \longrightarrow H^*(D_n;\mathbb{F}_2)$ (with $D_n \cong \prod_{i=1}^n(\mathbb{Z}[1/2])^\times \cong \prod_{i=1}^n(\mathbb{Z}\times\mathbb{Z}/2)$ denoting the subgroup of diagonal matrices of $GL(n,\mathbb{Z}[1/2])$) is injective. Quillen also remarked that with his conjecture a calculation of $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ should be within reach. In fact, the image $\operatorname{Im}\rho_n$ has been computed by Mitchell. In order to state his result we identify the classes w_i with their image under restriction in $H^*(D_n;\mathbb{F}_2) \cong \mathbb{F}[x_1,...,x_n]\otimes E(a_1,...,a_n)$ (with E as usual denoting an exterior algebra, and with all generators of dimension 1), namely with the i - th elementary symmetric polynomial in the variables x_i. We also need classes $e_i \in H^{2i-1}(D_n;\mathbb{F}_2)$: they are the symmetrizations of the elements $x_1^2...x_{i-1}^2a_i$ with respect to the canonical action of the symmetric group $\mathfrak{S}_n$ on n letters.

Now Mitchell's result reads as follows.

Theorem 1.1 *[Mi]* $\operatorname{Im}\rho_n \cong \mathbb{F}_2[w_1,...,w_n]\otimes E(e_1,...,e_{2n-1})$. □

Note that with this result Quillen's conjecture would imply an isomorphism $H^*(GL(\infty,\mathbb{Z}[1/2]);\mathbb{F}_2) \cong \mathbb{F}_2[w_1,w_2,...]\otimes E(e_1,e_3,...)$ and hence the Dwyer - Friedlander version [DF] of the Lichtenbaum - Quillen conjecture at $p=2$. Unfortunately Quillen's conjecture was too optimistic. Dwyer has recently shown.

Theorem 1.2 *[D] The restriction map ρ_n is not injective for all n.* □

The only previous complete computation of $H^*(GL(n,\mathbb{Z}[1/2]);\mathbb{F}_2)$ was that of [Mi] for $n=2$, and in this case ρ_n turned out to be injective. Some qualitative information on the size of the kernel of ρ_n as n grows is provided in [H2]. Dwyer shows, in fact, that ρ_{32} is not injective, so that the case $n=3$ becomes an interesting test case in which one also has a nice candidate, namely $\operatorname{Im}\rho_3$, for the answer.

In fact, one of the main results of this paper shows that this candidate is correct.

Theorem 1.3 *The restriction homomorphism maps $H^*(GL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)$ isomorphically onto the subalgebra $\mathbb{F}_2[w_1,w_2,w_3]\otimes E(e_1,e_3,e_5)$ of $H^*(D_3;\mathbb{F}_2)$.*

This result is really an easy consequence of the following companion result for $SL(3,\mathbb{Z}[1/2])$. We denote its subgroup of diagonal matrices by SD_3. Note that the restriction map from $H^*(D_3;\mathbb{F}_2)$ to $H^*(SD_3;\mathbb{F}_2)$ kills the elements w_1 and e_1. Let v_i be the image of w_i and d_{2i-1} the image of e_{2i-1}, $i=2,3$.

Theorem 1.4 *The restriction homomorphism maps $H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)$ isomorphically onto the subalgebra $\mathbb{F}_2[v_2,v_3]\otimes E(d_3,d_5)$ of $H^*(SD_3;\mathbb{F}_2)$.*

We remark that the corresponding result does not hold in the same way for $n=2$, i.e. the restriction map is not an isomorphism in this case, although there is an abstract isomorphism $H^*(SL(2,\mathbb{Z}[1/2]);\mathbb{F}_2)\cong\mathbb{F}_2[v_2]\otimes E(d_3)$ [Mi].

How can Theorem 1.4 be proved? The standard approach would be to take a suitable finite dimensional contractible space X on which $\Gamma:=SL(3,\mathbb{Z}[1/2])$ acts properly and with finite isotropy groups (there is a canonical such candidate, namely the product of the symmetric space $SL(3,\mathbb{R})/SO(3)$ and the Bruhat-Tits-building for $SL(3,\mathbb{Q}_2)$, see Section 2.1 below). Then one would take the Borel construction $E\Gamma\times_\Gamma X$ as model for the classifying space $B\Gamma$ and study its mod - 2 cohomology $H^*_\Gamma(X;\mathbb{F}_2)$ via the cohomology spectral sequence of the map $E\Gamma\times_\Gamma X\longrightarrow\Gamma\backslash X$. If X has the structure of a Γ - CW - complex then the E_1 - term of this spectral sequence is given as $E_1^{s,t}=\bigoplus_\sigma H^t(\Gamma_\sigma;\mathbb{F}_2)$ where σ runs through a set of representatives of the Γ - orbits of s - dimensional cells of X and Γ_σ denotes the isotropy group of σ. This is how Soulé studied the cohomology of $SL(3,\mathbb{Z})$ [So]. However, in our case the space X looks too complicated to make this spectral sequence manageable: in Section 2.6 we actually analyze the canonical X above and we essentially produce a Γ - equivariant deformation retract with finitely many Γ - orbits of cells; however, finite means 474 (!) orbits (see the table at the beginning of Section 3) and so this standard approach looks unfeasible. Instead we use a more manageable "centralizer spectral sequence"

$$E_2^{s,t}\cong\lim^s_{E\in\mathcal{A}_*(\Gamma)}H^t(C_\Gamma(E);\mathbb{F}_2)\Longrightarrow H^{s+t}_\Gamma(X_s;\mathbb{F}_2)$$

converging to the mod - 2 cohomology of the Borel - construction of the 2 - singular locus X_s, i.e. the subspace of X consisting of all points whose isotropy group contains an element of order 2. Here $\mathcal{A}_*(\Gamma)$ is the category of elementary abelian 2 - subgroups of Γ, $\lim^s$ is the s - th derived functor of the inverse limit functor and $C_\Gamma(E)$ is the centralizer in Γ of the elementary abelian 2 - subgroup $E\subset\Gamma$. This spectral sequence is based on a homotopy colimit decomposition of $E\Gamma\times_\Gamma X_s$ and was introduced in [H1]. In this paper we also evaluated this spectral sequence and obtained the following result in which Σ denotes as usual the suspension functor, e.g. $\Sigma^4\mathbb{F}_p$ denotes the graded $\mathbb{F}_p$ - vectorspace which is trivial in all dimensions except in dimension 4 where it is $\mathbb{F}_p$.

Theorem 1.5 *[H1] Let $\Gamma=SL(3,\mathbb{Z}[1/2])$ and let X be any mod - 2 acyclic finite dimensional Γ - CW - complex for which the stabilizers of all cells are* *finite. Then there is a short exact sequence*

$$0 \longrightarrow \Sigma^4\mathbb{F}_2 \longrightarrow H^*_\Gamma(X_s;\mathbb{F}_2) \xrightarrow{\rho} \mathbb{F}_2[v_2,v_3]\otimes E(d_3,d_5) \longrightarrow 0$$

in which ρ is an algebra homomorphism. Furthermore, if π denotes the projection map from $E\Gamma\times_\Gamma X_s$ to the classifying space $B\Gamma$ then the composition

$$H^*(\Gamma;\mathbb{F}_2) \xrightarrow{\pi^*} H^*_\Gamma(X_s;\mathbb{F}_2) \xrightarrow{\rho} \mathbb{F}_2[v_2,v_3]\otimes E(d_3,d_5) \subset H^*(SD_3;\mathbb{F}_2)$$

agrees with the restriction homomorphism of 1.4. □

Now for $*$ exceeding $\dim X$, the dimension of X, we have isomorphisms $H^*_\Gamma(X_s;\mathbb{F}_2)\cong H^*_\Gamma(X;\mathbb{F}_2)\cong H^*(\Gamma;\mathbb{F}_2)$ and hence Theorem 1.5 is also a computation of $H^*(\Gamma;\mathbb{F}_2)$ in large dimensions. In fact, X can be chosen to be of dimension 5 (see [BS] or Section 2 below) and Theorem 1.5 gives encouraging evidence for Theorem 1.4.

In this paper we complete the proof of Theorem 1.4 by computing for the canonical space X mentioned above, the relative groups $H^*_\Gamma(X,X_s;\mathbb{F}_2)$ and the boundary homomorphism of the appropriate long exact cohomology sequence. Note that, because the isotropy groups outside of X_s are finite of order prime to 2, we have the following isomorphisms for the relative groups: $H^*_\Gamma(X,X_s;\mathbb{F}_2)\cong H^*(\Gamma\backslash(X,X_s);\mathbb{F}_2)$.

As a byproduct of our investigations we obtain the following results which are of independent interest. In these results we abbreviate $SL(3,\mathbb{Z}[1/2])$ by Γ, $SL(3,\mathbb{Z})$ by Γ_0, and we denote the subgroup of $SL(3,\mathbb{Z})$ consisting of all matrices whose first column agrees with the first standard basis vector modulo 2 by Γ_1, and the subgroup of all matrices which are upper triangular modulo 2 by Γ_2.

Theorem 1.6 *Let $\mathcal{X}_\infty$ denote the symmetric space $SL(3,\mathbb{R})/SO(3)$, $\mathcal{X}_2$ the Bruhat-Tits-building of $SL(3,\mathbb{Q}_2)$, $\mathcal{X}=\mathcal{X}_\infty\times\mathcal{X}_2$ and let p be any prime. Then the reduced cohomology of the quotient spaces by the obvious action of the respective groups is given as follows:*

a) $\widetilde{H}^*(\Gamma_0\backslash\mathcal{X}_\infty;\mathbb{F}_p)=0$

b) $\widetilde{H}^*(\Gamma_1\backslash\mathcal{X}_\infty;\mathbb{F}_p)=0$

c) $\widetilde{H}^*(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_p)=\Sigma^3\mathbb{F}_p$

d) $\widetilde{H}^*(\Gamma\backslash\mathcal{X};\mathbb{F}_p)=\Sigma^5\mathbb{F}_p$.

For $p>3$ there are no elements of order p in these groups (because there are obviously no elements of order p in $SL(3,\mathbb{Q})$) and hence we obtain the following Corollary. For $SL(3,\mathbb{Z})$ this was already known by [So] and for $SL(3,\mathbb{Z})[1/2]$ by [Mo]. The results for Γ_1 and Γ_2 are compatible with the Euler chararacteristic computations in [Mo].

Corollary 1.7 *Assume $p > 3$. Then*

a) $\widetilde{H}^*(\Gamma_0;\mathbb{F}_p) = 0$

b) $\widetilde{H}^*(\Gamma_1;\mathbb{F}_p) = 0$

c) $\widetilde{H}^*(\Gamma_2;\mathbb{F}_p) = \Sigma^3\mathbb{F}_p$

d) $\widetilde{H}^*(\Gamma;\mathbb{F}_p) = \Sigma^5\mathbb{F}_p$. □

Theorem 1.8 *Let $\mathcal{X}_\infty$, $\mathcal{X}_2$, $\mathcal{X}$ and p be as in the previous theorem. Then we get the following relative cohomology groups (where ($\mathcal{X}_{\infty,s}(i)$ denotes the 2 - singular locus of $\mathcal{X}_\infty$ with respect to the action of Γ_i, and $\mathcal{X}_s$ the 2 - singular locus of $\mathcal{X}$ with respect to the action of Γ):*

a) $H^*(\Gamma_0\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(0));\mathbb{F}_p) = 0$

b) $H^*(\Gamma_1\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(1));\mathbb{F}_p) = \Sigma^2(\mathbb{F}_p)^2$

c) $H^*(\Gamma_2\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(2));\mathbb{F}_p) = \Sigma^3\mathbb{F}_p\oplus\Sigma^2(\mathbb{F}_p)^6$

d) $H^*(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_2) = \Sigma^5\mathbb{F}_2$.

(Observe that we restrict to the case $p = 2$ for the last part of the Theorem.)

The next result together with Theorem 1.5 and the last part of Theorem 1.8 finishes the proof of Theorem 1.4.

Proposition 1.9 *The boundary homomorphism*

$$H^4_\Gamma(\mathcal{X}_s;\mathbb{F}_2)\longrightarrow H^5_\Gamma(\mathcal{X},\mathcal{X}_s;\mathbb{F}_2)$$

is an epimorphism.

With the help of Theorem 1.6 we are also able to compute the mod - 3 cohomology. Again this was known for $SL(3,\mathbb{Z})$ by [So].

Theorem 1.10 *There are isomorphisms of $\mathbb{F}_3$ - algebras (without unit) which in the case of a), b) and d) are induced by restrictions to appropriate subgroups:*

a) $\widetilde{H}^*(\Gamma_0;\mathbb{F}_3)\cong\prod_{i=1}^2\widetilde{H}^*(\mathfrak{S}_3;\mathbb{F}_3)$

b) $\widetilde{H}^*(\Gamma_1;\mathbb{F}_3)\cong\prod_{i=1}^2\widetilde{H}^*(\mathfrak{S}_3;\mathbb{F}_3)$

c) $\widetilde{H}^*(\Gamma_2;\mathbb{F}_3)\cong\Sigma^3\mathbb{F}_3$

d) $\widetilde{H}^*(\Gamma;\mathbb{F}_3)$ *is isomorphic to the subalgebra of* $\prod_{i=1}^2\widetilde{H}^*(\mathfrak{S}_3\times\mathbb{Z};\mathbb{F}_3)$ *which can be characterized as follows: it is all of* $\prod_{i=1}^2\widetilde{H}^*(\mathfrak{S}_3\times\mathbb{Z};\mathbb{F}_3)$ *except in degrees 1 and 4; in degree 1 it is trivial, and in degree 4 it is of dimension 3 and is generated by the image of the Bockstein of H^3 and one further element which restricts non-trivially to both factors.*

The paper is organized as follows: In Section 2 we recall the symmetric space $\mathcal{X}_\infty$ and the Bruhat - Tits building $\mathcal{X}_2$. We discuss the Soulé - Lannes method of replacing the symmetric space by a smaller space $\mathcal{Z}$ for which the quotients by $\Gamma_0 = SL(3,\mathbb{Z})$ and the congruence subgroups Γ_1 and Γ_2 are compact. The bulk of this long section is then devoted to patiently working out an explicit cell structure of the quotients $\Gamma_i\backslash\mathcal{Z}$, $i = 0, 1, 2$, in fact even a Γ_i - equivariant cell structure on $\mathcal{Z}$. This is straightforward but it is crucial for the remainder of the paper; for $i = 0$ it is a variation of Soulé's investigations [So]. In Section 3 we use these cell structures to prove Theorem 1.6 and Theorem 1.8 as well as the corresponding results for the cohomology of the quotients of the singular locus $\mathcal{X}_{\infty,s}(i)$ resp. $\mathcal{X}_s$. This is quite an elaborate calculation but apart from the last part of Theorem 1.8 it is straightforward given the results in Section 2. The last part of Theorem 1.8 is more tricky and to settle it we use low dimensional information on $H^*_\Gamma(\mathcal{X}_s;\mathbb{F}_2)$ as provided by Theorem 1.5. In Section 4 we apply the results of Section 2 and Section 3 and derive the remaining results listed in this introduction. We also determine the height of torsion in $H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{Z})$ (Proposition 4.15). In Section 5 we compute $H^*(GL(3,\mathbb{Z}[1/2]);\mathbb{F}_p)$ for primes $p > 2$ (Proposition 5.1, 5.2) and for $p = 2$, i.e. we derive Theorem 1.3.

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2 Contractible spaces with actions of $SL(3,\mathbb{Z})$ and $SL(3,\mathbb{Z}[1/2])$

2.1 The symmetric space and the Bruhat-Tits-building

We start by recalling the contractible spaces on which our groups act with finite stabilizer groups.

The symmetric space. The space $\mathcal{Q}(n)$ of positive definite quadratic forms on $\mathbb{R}^n$ is equipped with an action of the multiplicative group $\mathbb{R}_+$ of positive real numbers, given by $(rq)(x) = rq(x)$ for $r \in \mathbb{R}_+$, $q \in \mathcal{Q}(n)$ and $x \in \mathbb{R}^n$. The quotient will be denoted by $\mathcal{X}_\infty(n)$, or simply by $\mathcal{X}_\infty$ if n is clear from the context. The space $\mathcal{X}_\infty(n)$ is contractible because $\mathcal{Q}(n)$ is a convex open cone in $\mathbb{R}^{n^2}$. Furthermore, $\mathcal{X}_\infty(n)$ can be identified with the symmetric space of $SL(n,\mathbb{R})$, i.e. the space of left cosets $SL(n,\mathbb{R})/SO(n)$, via the map which sends a matrix A to the equivalence class of the positive definite quadratic form q, given by $q(x) = ||A^{-1}x||$ where $||\ \ ||$ denotes the euclidean norm in $\mathbb{R}^n$. The group $SL(n,\mathbb{Z})$ acts on this coset space from the left, and this action is proper, i.e. if $C \subset \mathcal{X}_\infty(n)$ is compact then there are only finitely many $g \in SL(n,\mathbb{Z})$ for which $gC \cap C \neq \emptyset$; in particular the isotropy groups of the action are all finite.

The Bruhat-Tits-building. The group $SL(n,\mathbb{Z}[1/2])$ acts on the coset space $\mathcal{X}_\infty(n)$ as well. However, in this case the action is not proper. In order to get a contractible space with proper action, the space $\mathcal{X}_\infty(n)$ has to be enlarged by the appropriate Bruhat-Tits-building $\mathcal{X}_2(n)$ (or simply $\mathcal{X}_2$ if n is clear from the context) for the group $SL(n,\mathbb{Q}_2)$. As reference for more on this bulding we recommend [B2]. We recall here only some basic properties.

The space $\mathcal{X}_2(n)$ is an $(n-1)$-dimensional simplicial complex which can be described as follows: an n - dimensional 2 - adic lattice L is a $\mathbb{Z}_2$ - submodule of $\mathbb{Q}_2^n$ which is free of rank n. The group $\mathbb{Q}_2^\times$ of units in $\mathbb{Q}_2$ acts on the set of all such lattices via scalar multiplication, and the set of equivalence classes is the set of vertices in $\mathcal{X}_2(n)$. A finite subset $\{l_0, l_1, ..., l_n\}$ of vertices spans an n - dimensional simplex in $\mathcal{X}_2(n)$ if and only if there are representative lattices L_i in the class of l_i for $i = 0, ..., n$ such that $L_0 \subsetneq L_1 \subsetneq ... \subsetneq L_n \subsetneq \frac{1}{2}L_0$. The space $\mathcal{X}_2(n)$ is contractible (see Section V.8 and Theorem VI.3 in [B2]). Furthermore the set of all 2 - adic lattices can be identified with the set of left cosets $GL(n,\mathbb{Q}_2)/GL(n,\mathbb{Z}_2)$ via the map which sends a matrix A to the lattice $A(\mathbb{Z}_2^n)$. The natural left action of $SL(n,\mathbb{Q}_2)$ on this coset space induces a simplicial left action of $SL(n,\mathbb{Q}_2)$ on $\mathcal{X}_2(n)$ and the quotient of $\mathcal{X}_2(n)$ by the action of $SL(n,\mathbb{Q}_2)$ is an $(n-1)$ - dimensional simplex Δ^{n-1}.

Furthermore $SL(n,\mathbb{Z}[1/2])$ is dense in $SL(n,\mathbb{Q}_2)$ and therefore the quotient of $\mathcal{X}_2(n)$ by the action of $SL(n,\mathbb{Z}[1/2])$ agrees with the quotient by the group $SL(n,\mathbb{Q}_2)$.

The group $SL(n,\mathbb{Z}[1/2])$ embedds diagonally as a discrete subgroup into $SL(3,\mathbb{R})\times SL(n,\mathbb{Q}_2)$ and acts properly on the contractible space $\mathcal{X} := \mathcal{X}_\infty \times \mathcal{X}_2$.

The projection maps and congruence subgroups. From now on we concentrate on the case $n = 3$. We will be interested in the $SL(3,\mathbb{Z}[1/2])$ - equivariant projection map $p : \mathcal{X} \longrightarrow \mathcal{X}_2$.

With respect to the action of $GL(3,\mathbb{Z}[1/2])$ all vertices in $\mathcal{X}_2$ fall into a single orbit and hence their isotropy groups (in $SL(3,\mathbb{Z}[1/2])$) are conjugate in the larger group $GL(3,\mathbb{Z}[1/2])$, in particular they are abstractly isomorphic; similarly with simplices of dimension one. For the vertex l_0 corresponding to the standard lattice L_0 (which is spanned over $\mathbb{Z}_2^3$ by the standard basis vectors e_1,e_2 and e_3, i.e. $L_0 = \langle e_1, e_2, e_3\rangle$), the isotropy group is $SL(3,\mathbb{Z}_2)\cap SL(3,\mathbb{Z}[1/2]) = SL(3,\mathbb{Z}) =: \Gamma_0$. For the edge consisting of the set $\{l_0, l_1\}$ with l_0 the class of L_0 and l_1 the class of the lattice $L_1 = \langle \frac{1}{2}e_1, e_2, e_3\rangle$, the isotropy group is the subgroup Γ_1 of Γ_0 consisting of matrices whose first column is equal to e_1 modulo 2; for the two-dimensional simplex spanned by the set $\{l_0, l_1, l_2\}$ with l_2 the class of the lattice $L_2 = \langle \frac{1}{2}e_1, \frac{1}{2}e_2, e_3\rangle$, the isotropy group is the subgroup Γ_2 of Γ_0 consisting of all matrices which are upper triangular modulo 2. For simplicity of notation we will write Γ instead of $SL(3,\mathbb{Z}[1/2])$.

The "fibres" of the map (which is induced by p)

$$\widetilde{p} : E\Gamma \times_\Gamma \mathcal{X} \longrightarrow \Gamma\backslash\mathcal{X}_2 \cong \Delta^2$$

over the 0 -, 1 - resp. 2 - dimensional simplices respectively are homotopy - equivalent to the classifying spaces $B\Gamma_0$, $B\Gamma_1$ and $B\Gamma_2$ respectively. We will have to study the mod - 2 (co)homology spectral sequence of $\widetilde{p}$ as well as that of the map (which is also induced by p)

$$\overline{p} : \Gamma\backslash\mathcal{X} \longrightarrow \Gamma\backslash\mathcal{X}_2 \cong \Delta^2 \ .$$

In particular we need to understand the "fibres" of $\overline{p}$, i.e. the quotients $\Gamma_i\backslash\mathcal{X}_\infty$, $i = 0, 1, 2$. These quotients are not compact and in the next section we recall the Soulé-Lannes method of finding a deformation retract of $\Gamma_i\backslash\mathcal{X}_\infty$ which is compact, even a finite 3 - dimensional complex (see [A]).

2.2 Well-rounded lattices and the deformation retractions

Well-rounded lattices. We note that $\Gamma_i\backslash\mathcal{X}_\infty \cong \Gamma_i\backslash(SL(3,\mathbb{R})/SO(3))$ may also be obtained as quotient of $\Gamma_i\backslash SL(3,\mathbb{R})$ by the right action of $SO(3)$. Now the right $SO(3)$ - space $GL(3,\mathbb{Z})\backslash GL(3,\mathbb{R})$ can be identified with the space of all integral lattices in $\mathbb{R}^3$ (via the correspondence which sends a matrix g to the lattice $g^{-1}(\mathbb{Z}^n)$), and the space $\Gamma_0\backslash SL(3,\mathbb{R})$ can be identified with the space of equivalence classes (with respect to scalar multiplication) of integral lattices L in $\mathbb{R}^3$, or equivalently with the space of integral lattices whose minimal vectors are of length 1, i.e. for which $m(L) := \min\{\|x\| | x \in L - \{0\}\} = 1$. We will denote this latter space by $\mathcal{L}_0$. Note that, in terms of lattices, the right action of $SO(3)$ on $\Gamma_0\backslash SL(3,\mathbb{R})$ is given by $L \cdot g := g^{-1}L$ for $L \in \mathcal{L}_0$ and $g \in SO(3)$.

Similarly the space $\Gamma_1\backslash SL(3,\mathbb{R})$ can be identified with the space $\mathcal{L}_1$ of pairs (L_0, L_1) of lattices such that $m(L_0) = 1$ and $L_0 \subsetneq L_1 \subsetneq \frac{1}{2}L_0$, and the space $\Gamma_2\backslash SL(3,\mathbb{R})$ can be identified with the space $\mathcal{L}_2$ of triples (L_0, L_1, L_2) of lattices such that $m(L_0) = 1$ and $L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq \frac{1}{2}L_0$.

We recall that a lattice L in $\mathbb{R}^3$ is called *well-rounded* if its set of minimal vectors, i.e. $\{x \in L - \{0\} | \|x\| = m(L)\}$ spans $\mathbb{R}^3$. For $i = 0, 1, 2$ let $\mathcal{W}_i$ denote the subspace of $\mathcal{L}_i$ consisting of all tuples $(L_0, ..., L_i)$ for which L_0 is well-rounded.

The deformation retractions. There is a beautiful geometric argument which shows that $\mathcal{W}_i$ is an $SO(3)$ - equivariant deformation retract of $\mathcal{L}_i$, hence $\mathcal{W}_i/SO(3)$ is a deformation retract of $\mathcal{L}_i/SO(3) \cong \Gamma_i\backslash\mathcal{X}_\infty$. We recall the construction ([A]).

For $i = 0, 1, 2$ and $1 \leq p \leq 3$ let $\mathcal{W}_i^p$ be the set of tuples $(L_0, ..., L_i)$ of lattices such that the dimension of the subspace of $\mathbb{R}^3$ spanned by the set of minimal vectors in L_0 is at least p. Then $\mathcal{W}_i^1 = \mathcal{L}_i$, $\mathcal{W}_i^3 = \mathcal{W}_i$ and therefore it suffices to show that $\mathcal{W}_i^{p+1}$ is an $SO(3)$ - equivariant deformation retract of $\mathcal{W}_i^p$ for $p = 1, 2$. So assume that the set of minimal vectors in L_0 spans a subspace U of dimension $q \geq p$. If $q > p$ then nothing happens to our tuple in the next step of the deformation. Otherwise, consider a radial contracting homotopy in the subspace $U^\perp$ of $\mathbb{R}^3$ perpendicular to U, and extend linearly to a deformation of $\mathbb{R}^3$ by leaving U fixed. This defines a deformation $L_j(t)$, $0 < t \leq 1$ of lattices (for $0 \leq j \leq i$) with $L_j(1) = L_j$ and there will be a maximal t_0 with $0 < t_0 < 1$ for which $L_0(t_0)$ has a new vector of minimal length 1. The corresponding tuple $(L_0(t_0), ..., L_i(t_0))$ of lattices lies in $\mathcal{W}_i^{p+1}$ and is the image under the next step in the deformation. It is easy to see that these constructions describe continuous $SO(3)$ - equivariant maps which combine to give an $SO(3)$ - equivariant deformation retraction from $\mathcal{L}_i$ to $\mathcal{W}_i$ and induce a deformation retraction from $\Gamma_i\backslash\mathcal{X}_\infty \cong \mathcal{L}_i/SO(3)$ to $\mathcal{W}_i/SO(3)$. We will see in the next section that the spaces $\mathcal{W}_i/SO(3)$ are compact and of dimension 3.

We can do even a bit better: the $SO(3)$ - equivariant deformation retraction of $\mathcal{L}_0$ can be lifted to give a left $SL(3,\mathbb{Z})$ - equivariant and right $SO(3)$ - equivariant deformation retraction of $SL(3,\mathbb{R})$ onto the subspace $\mathcal{Y} := \{g \in SL(3,\mathbb{R}) | g^{-1}(\mathbb{Z}^n)$ is a wellrounded lattice$\}$. Dividing out by the $SO(3)$ - action gives a left $SL(3,\mathbb{Z})$ - space $\mathcal{Z}$ and an $SL(3,\mathbb{Z})$ - equivariant deformation retraction from $\mathcal{X}_\infty$ to $\mathcal{Z}$. The space $\mathcal{Z}$ will also be called the space of (equivalence classes of) well - rounded quadratic forms.

The remainder of Section 2 is devoted to a detailed analysis of the spaces $\Gamma_i\backslash\mathcal{Z} \cong \mathcal{W}_i/SO(3)$, in particular we will exhibit explicit finite cell structures on them.

2.3 The space $\mathcal{W}_0/SO(3)$

Our first task is to understand the space $\Gamma_0\backslash\mathcal{Z} \cong \mathcal{W}_0/SO(3)$. This space agrees with Soulé's deformation retract of the space $\Gamma_0\backslash SL(3,\mathbb{R})/SO(3)$ [So]; however, our point of view is a bit different in so far as we emphasize lattices rather than quadratic forms, i.e. we prefer to think in terms of $\mathcal{W}_0/SO(3)$, the space of wellrounded 3 - dimensional lattices L with $m(L) = 1$, modulo the action of $SO(3)$.

We will see in a moment that in dimension 3 (unlike in higher dimensions) the sublattice spanned by any set of 3 linearly independent vectors of minimal length in a well-rounded lattice L is all of L, and therefore L is (up to the action of $SO(3)$) determined by $m(L)$ and the 3 scalar products between these vectors. We will analyze which of these 3 - tuples occur in this way and which tuples give the same lattice, up to the action of $SO(3)$. This analysis will lead to an explicit description of the spaces $\mathcal{W}_i/SO(3)$. In this section we will first concentrate on the case $i = 0$. Our first step is given by the following Lemma.

Lemma 2.1 *Suppose* $L \subset \mathbb{R}^3$ *is a well-rounded lattice and let* v_1, v_2 *and* v_3 *be linearly independent vectors of minimal length* $m(L)$ *in* L. *Then the sublattice* L' *spanned by these vectors is all of* L.

Proof. By scaling and rotating L we may assume that $m(L) = 1$, $v_1 = (1,0,0)$ and v_2, v_3 have the form: $v_2 = (a,x,0)$ and $v_3 = (b,y,z)$. Assume there exists $w = (w_1,w_2,w_3) \in L - L'$. By adding a suitable vector in L' we may assume that $|w_3| \leq \frac{1}{2}|z| \ \leq \frac{1}{2}$, $|w_2| \leq \frac{1}{2}|x| \leq \frac{1}{2}$ and $|w_1| \leq \frac{1}{2}$. But then $||w|| < 1$ and we obtain a contradiction to the assumption that $m(L) = 1$. □

The next two results will enable us to give an explicit description of the space $\mathcal{W}_0/SO(3)$. They will be proved together.

Proposition 2.2 *Suppose v_1, v_2 and v_3 are linearly independent vectors of length 1 in $\mathbb{R}^3$ with scalar products $a = \langle v_1, v_2\rangle$, $b = \langle v_1, v_3\rangle$ and $c = \langle v_2, v_3\rangle$. Assume that $a \geq 0$ and $b \geq 0$. Then the lattice L spanned by v_1, v_2 and v_3 is well-rounded with $m(L) = 1$ if and only if*

1. *$c \geq 0$ and $a, b, c \leq \frac{1}{2}$, or*
2. *$c \leq 0$, $a, b, |c| \leq \frac{1}{2}$ and $a + b - c \leq 1$.*

Clearly, the assumption on a and b can be assured by replacing, if necessary, one of the vectors v_i by its negative.

Proposition 2.3 *Suppose v_1, v_2 and v_3, a, b, c and L are as in Proposition 2.2. Furthermore assume $a \geq b \geq |c|$. Then the set of minimal vectors in L contains $\pm v_1, \pm v_2, \pm v_3$ and in addition only the following vectors:*

1. *$\pm(v_1 - v_2)$ if $a = \frac{1}{2}$, $b \neq \frac{1}{2}$ and $a + b - c \neq 1$.*
2. *$\pm(v_1 - v_2 - v_3)$ if $a \neq \frac{1}{2}$, $b \neq \frac{1}{2}$ and $a + b - c = 1$.*
3. *$\pm(v_1 - v_2)$ and $\pm(v_1 - v_3)$ if $a = b = \frac{1}{2}$ and $c \neq 0, \frac{1}{2}$.*
4. *$\pm(v_1 - v_2)$ and $\pm(v_1 - v_2 - v_3)$ if $a = \frac{1}{2}$, $b \neq \frac{1}{2}$ and $a + b - c = 1$.*
5. *$\pm(v_1 - v_2)$, $\pm(v_1 - v_3)$ and $\pm(v_2 - v_3)$ if $a = b = c = \frac{1}{2}$.*
6. *$\pm(v_1 - v_2)$, $\pm(v_1 - v_3)$ and $\pm(v_1 - v_2 - v_3)$ if $a = b = \frac{1}{2}$ and $c = 0$.*

Again the assumption on a, b and c can always be assured by permuting the vectors v_i and passing to negatives if necessary.

Proof. 1. Let us first consider the case $c \geq 0$.

Consider a vector w in L and write

$$w = n_1 v_1 + n_2 v_2 + n_3 v_3, \qquad n_i \in \mathbb{Z}, \quad i = 1, 2, 3 \ .$$

Then

$$||w||^2 = n_1^2 + n_2^2 + n_3^2 + 2an_1n_2 + 2bn_1n_3 + 2cn_2n_3 \ , \tag{2.1}$$

or equivalently

$$\begin{aligned} ||w||^2 = \ & a(n_1 + n_2)^2 + b(n_1 + n_3)^2 + c(n_2 + n_3)^2 \\ & +(1 - a - b)n_1^2 + (1 - a - c)n_2^2 + (1 - b - c)n_3^2. \end{aligned} \tag{2.2}$$

If $a > \frac{1}{2}$ then $n_1 = -n_2 = 1$, $n_3 = 0$ gives a vector w with $||w|| = 2 - 2a < 1$. The same argument for b and c shows that, if $m(L) = 1$, then $b, c \leq \frac{1}{2}$. Now assume that $a, b, c \leq \frac{1}{2}$. We distinguish different cases.

1.1. At least one $n_i = 0$, w.l.o.g. $n_3 = 0$. Then we obtain

$$||w||^2 = n_1^2 + n_2^2 + 2an_1n_2 = a(n_1+n_2)^2 + (1-a)n_1^2 + (1-a)n_2^2. \qquad (2.3)$$

Because $1-a \geq \frac{1}{2}$ and $1-b \geq \frac{1}{2}$ it is clear from (2.3) that $||w||^2 \geq 1$ unless $w = 0$.

We also observe that the only vectors of length 1 in L with $n_3 = 0$ are the vectors $\pm v_1$, $\pm v_2$, and if $a = \frac{1}{2}$, the vector $\pm(v_1 - v_2)$. Similarly, the only vectors with $n_2 = 0$ are the vectors $\pm v_1$, $\pm v_3$, and if $b = \frac{1}{2}$, the vector $\pm(v_1 - v_3)$. The only vectors with $n_1 = 0$ are the vectors $\pm v_2$, $\pm v_3$, and if $c = \frac{1}{2}$, the vector $\pm(v_2 - v_3)$.

1.2. We may now assume that all $n_i \neq 0$. Then at least one of the sums n_1+n_2, n_1+n_3, n_2+n_3 must be different from 0. If precisely one of the sums is non-zero, say n_2+n_3, then $n_1 = -n_3$, $n_1 = -n_2$ and $|n_2+n_3| \geq 2$ and (2.2) yields $||w||^2 \geq 3-2a-2b+2c \geq 1$; equality holds iff $c = 0$, $a = b = \frac{1}{2}$, $n_1 = -n_2 = -n_3 = \pm 1$, i.e. $w = \pm(v_1 - v_2 - v_3)$. If at least two of the sums are non-zero, say n_1+n_3 and n_2+n_3, then $|n_1+n_3| \geq 2$ and $|n_2+n_3| \geq 2$ and (2.2) yields $||w||^2 \geq 3-2a+2b+2c > 1$, in particular there are no such vectors of length 1.

2. Now consider the case $c \leq 0$. Then we write

$$\begin{aligned} ||w||^2 &= a(n_1+n_2)^2 + b(n_1+n_3)^2 - c(n_2-n_3)^2 \\ &\quad +(1-a-b)n_1^2 + (1-a+c)n_2^2 + (1-b+c)n_3^2. \end{aligned} \qquad (2.4)$$

As before we see that $a, b, |c| \leq \frac{1}{2}$ is necessary for L to satisfy $m(L) = 1$. Now assume these inequalities hold. Again we distinguish different cases.

2.1. If at least one $n_i = 0$ and $w \neq 0$, then we see as above that $||w||^2 \geq 1$ and we only obtain additional vectors of length 1 iff $a = \frac{1}{2}$ resp. $b = \frac{1}{2}$ resp. $c = -\frac{1}{2}$, namely the vectors $\pm(v_1 - v_2)$ resp. $\pm(v_1 - v_3)$ resp. $\pm(v_2 + v_3)$.

2.2. We may now assume that all $n_i \neq 0$. Consider the sums n_1+n_2, n_1+n_3, $n_2 - n_3$. We subdivide into further cases. In case all sums are zero we have $n_1 = -n_2 = -n_3$ and from (2.4) we obtain again $||w||^2 \geq 3-2a-2b+2c$. By taking $n_1 = -n_2 = -n_3 = \pm 1$ we see that the condition $a+b-c \leq 1$ is necessary for L to satisfy $m(L) = 1$, and there are further vectors of length 1 iff $a+b-c = 1$, namely the vectors $\pm(v_1 - v_2 - v_3)$.

If two sums are zero, then the third one is as well, hence we may next assume that at most one sum is zero, hence at least two of the terms $|n_1+n_2|$ and $|n_1+n_3|$, $|n_2-n_3|$ are ≥ 2. In case $|n_1+n_2|$ and $|n_1+n_3|$ are ≥ 2, (2.4) yields $\|w\|^2 \geq 3+2a+2b+2c > 1$, in particular there are no such vectors of length 1. The other two cases are analogous. □

After these preparations we can now describe the space $\mathcal{W}_0/SO(3)$. Consider the following subspace $\mathcal{D}_0$ of $\mathbb{R}^3$ (see figure 1):

$$\mathcal{D}_0 := \{(a,b,c) \in \mathbb{R}^3 |\ |c| \leq b \leq a \leq \frac{1}{2},\ \text{and}\ \ a+b-c \leq 1 \text{ if } c \leq 0\}\ .$$

We define a map

$$\gamma : \mathcal{D}_0 \longrightarrow \mathcal{Y} = \{g \in SL(3,\mathbb{R}) | g^{-1}(\mathbb{Z}^n) \text{ is a wellrounded lattice}\}$$

by sending the triple (a,b,c) to the unique matrix $\gamma(a,b,c)$ with the following properties: $\gamma(a,b,c)$ is (up to a scalar multiple guaranteeing $\gamma(a,b,c) \in SL(3,\mathbb{R})$) the inverse of the matrix whose i - th column is the basis vector v_i, where $v_1 = (1,0,0)$, $v_2 = (a,x,0)$, $v_3 = (b,y,z)$ and x, y and z are uniquely determined by the requirements $x \geq 0$, $ab + xy = c$, $z \geq 0$ and $||v_i|| = 1$ for $i = 1,2,3$. By construction and Proposition 2.2 the lattice $\gamma(a,b,c)^{-1}(\mathbb{Z}^n)$ is well - rounded, hence $\gamma(a,b,c) \in \mathcal{Y}$. Let $\lambda : \mathcal{D}_0 \longrightarrow \mathcal{W}_0$ denote the composition of γ with the canonical projection $\mathcal{Y} \longrightarrow \mathcal{W}_0$; then $\lambda(a,b,c)$ is the well - rounded lattice spanned by the vectors v_1, v_2 and v_3. Note that by construction $a = \langle v_1, v_2 \rangle$, $b = \langle v_1, v_3 \rangle$ and $c = \langle v_2, v_3 \rangle$. Finally let $\varphi : \mathcal{D}_0 \longrightarrow \mathcal{W}_0/SO(3)$ be the composition of λ with the canonical projection $\mathcal{W}_0 \longrightarrow \mathcal{W}_0/SO(3)$. Clearly all these maps are continuous.

Finally we define an equivalence relation $\sim$ on $\mathcal{D}_0$ by declaring the points $(\frac{1}{2}, b, c)$ with $0 \leq c \leq \frac{1}{2}b$ equivalent to $(\frac{1}{2}, b, b-c)$ and equivalent to $(\frac{1}{2}, b-c, -c)$ (cf. figure 1).

Theorem 2.4 *The map $\varphi : \mathcal{D}_0 \longrightarrow \mathcal{W}_0/SO(3)$ is onto and induces a homeomorphism $\widetilde{\varphi} : \mathcal{D}_0/\sim \longrightarrow \mathcal{W}_0/SO(3)$.*

In the proof we will make repeated use of the following elementary fact.

Lemma 2.5 *Assume v_1, v_2, v_3 and $v_1{}'$, $v_2{}'$, $v_3{}'$ are two sets of linearly independent vectors of length 1 in $\mathbb{R}^3$ such that $\langle v_i, v_j \rangle = \langle v_i{}', v_j{}' \rangle$ for all $1 \leq i < j \leq 3$. Then there exist unique rotations $R, S \in O(3)$ such that $Rv_i = v_i{}'$ and $Sv_i = -v_i{}'$ for $i = 1,2,3$, and either R or S is in $SO(3)$.* □

Proof of Theorem. That φ is onto can be seen as follows. Assume we are given a well-rounded lattice L with minimal vectors of length 1. By Lemma 2.1 we can find spanning vectors w_1, w_2 and w_3 in L of length 1, and after a suitable permutation (and passing to additive inverses, if necessary) we may assume that the scalar products a, b and c are as in 2.2 and 2.3. Then Lemma 2.5 implies $\varphi(a,b,c) = [L]$ where $[L]$ denotes the image of L in $\mathcal{W}_0/SO(3)$.

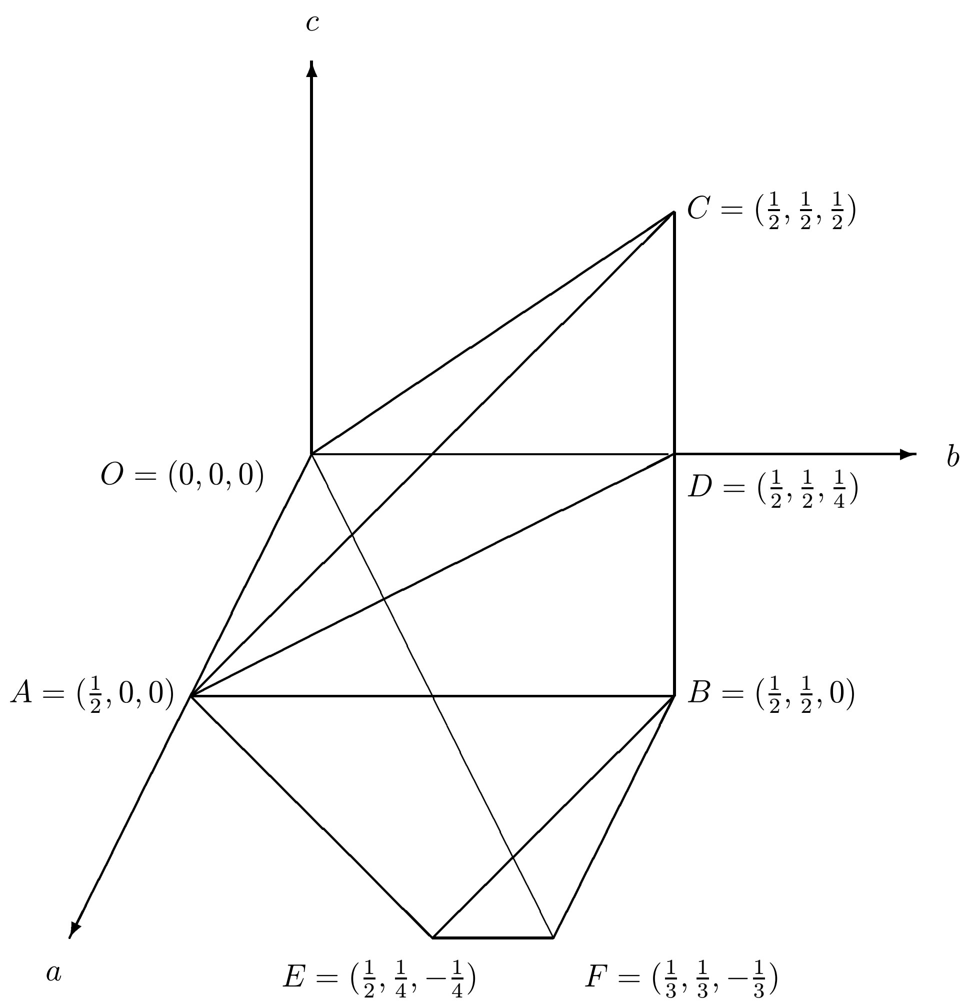


Figure 1: The space $\mathcal{D}_0$ and the equivalence relation $\sim$. The equivalence relation is given by identifying the triangle ΔABD with the triangles ΔACD and ΔABE via reflections at the edges AD and AB.

Next we show that equivalent triples have the same image under φ so that φ induces a continuous map $\widetilde{\varphi}$. So assume $0 \le c \le \frac{1}{2}b$ and consider the lattice $\lambda(\frac{1}{2}, b, c)$. This has (at least) 4 pairs of minimal vectors, namely $\pm v_1 = \pm(1,0,0)$, $\pm v_2 = \pm(a,x,0)$, $\pm v_3 = \pm(b,y,z)$ and $\pm(v_1 - v_2)$. (Here x, y and z are as before.) Then it is straightforward to check that the scalar products between the vectors $v_1' := v_1$, $v_2' := v_1 - v_2$ and $v_3' := v_3$ are $(\frac{1}{2}, b, b-c)$ and those between the vectors $v_1'' := v_2 - v_1$, $v_2'' := v_2$ and $v_3'' := -v_3$ are $(\frac{1}{2}, b-c, -c)$ and Lemma 2.5 implies again that the image under φ of these triples agree.

Now we turn to injectivity of $\widetilde{\varphi}$. As $\mathcal{D}_0/\sim$ is compact and $\mathcal{W}_0/SO(3)$ is Hausdorff, this will show that $\widetilde{\varphi}$ is a homeomorphism and finish the proof. So assume $\varphi(a,b,c) = \varphi(a',b',c')$. By assumption the corresponding lattices $L := \lambda(a,b,c)$ and $L' := \lambda(a',b',c')$ agree up to a rotation $R \in SO(3)$, i.e. $L = RL'$. In particular, L and L' have the same number of minimal vectors, the vectors Rv_1', Rv_2', Rv_3' form a set of linearly independent vectors of length 1 in L and the triple (a',b',c') occurs as a triple of scalar products between three linearly independent vectors of length 1 of L. We have to show that this happens only if (a,b,c) and (a',b',c') are equivalent under the relation $\sim$.

In the "generic" case, i.e. if $a \ne \frac{1}{2}$ and $a+b-c \ne 1$, L has only the minimal vectors $\pm v_1$, $\pm v_2$ and $\pm v_3$ (cf. Proposition 2.3), and in this case it is obvious that the triple of scalar products is uniquely determined by L and by the condition $a \ge b \ge |c|$.

Now assume $(a,b,c) \ne (a',b',c')$ and we have 6 pairs of minimal vectors. By Proposition 2.3 this can only happen if w.l.o.g. $(a,b,c) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(a',b',c') = (\frac{1}{2}, \frac{1}{2}, 0)$. However, these points are clearly equivalent under $\sim$.

Next assume we have precisely 5 pairs of minimal vectors in L. By Proposition 2.3 and because φ is constant on $\sim$ - equivalence classes we may assume that our triples are of the form $(\frac{1}{2}, \frac{1}{2}, c)$ and $(\frac{1}{2}, \frac{1}{2}, c')$ with $0 < c, c' \le \frac{1}{4}$ and we have to show $c = c'$. The lattice $L = \lambda(\frac{1}{2}, \frac{1}{2}, c)$ has the following pairs of minimal vectors of length 1: $\pm v_1$, $\pm v_2$, $\pm v_3$, $\pm(v_1 - v_2)$ and $\pm(v_1 - v_3)$, and the triple $(\frac{1}{2}, \frac{1}{2}, c')$ must occur as a triple of scalar products of 3 linearly independent vectors taken from those 5 pairs. It is now straightforward to check that this can happen only if $c = c'$.

Finally assume we have exactly 4 pairs of linearly independent vectors of length 1 in $L = \lambda(a,b,c)$. Again by Proposition 2.3 and because φ is constant on $\sim$ - equivalence classes we may assume that the triple (a,b,c) satisfies either $a = \frac{1}{2}$ and $0 \le c \le \frac{1}{2}b < \frac{1}{4}$, or $a \ne \frac{1}{2}$, $c \le 0$ and $a+b-c = 1$. We have to show that L determines uniquely a triple of this form. In the first case the set of minimal vectors consists of $\pm v_1$, $\pm v_2$, $\pm v_3$, $\pm(v_1 - v_2)$, in the second case of $\pm v_1$, $\pm v_2$, $\pm v_3$, $\pm(v_1 - v_2 - v_3)$. Again it is straightforward to see that all triples of linearly independent vectors taken from those sets which lead to scalar products of the required form, lead indeed to the same scalar products, and thus the proof is complete. □

2.4 Γ_i - equivariant cell structures on $\mathcal{Z}$

2.4.1 The case $i = 0$

We recall the map $\gamma : \mathcal{D}_0 \longrightarrow \mathcal{Y}$ which sends $d = (a, b, c) \in \mathcal{D}_0$, up to a scalar multiple, to the inverse of the matrix whose i - th colum is the vector v_i specified in the last section (cf. the discussion before Theorem 2.4). The composition of the map $\gamma : \mathcal{D}_0 \longrightarrow \mathcal{Y}$ with the quotient map $\pi_{\mathcal{Z}} : \mathcal{Y} \longrightarrow \mathcal{Y}/SO(3) \cong \mathcal{Z}$ will be denoted by ψ_0. Note that $\psi_0(d)$ is the equivalence class of the positive definite quadratic form for which the scalar products between the standard basis vectors e_1, e_2, e_3 are given by $\langle e_i, e_j\rangle = \langle v_i, v_j\rangle$ for $i \leq i, j \leq 3$, i.e. $\langle e_i, e_i\rangle = 1$ for $i = 1, 2, 3$, $\langle e_1, e_2\rangle = a$, $\langle e_1, e_3\rangle = b$ and $\langle e_2, e_3\rangle = c$. In particular this map is injective and a homeomorphism from $\mathcal{D}_0$ to $\psi_0(\mathcal{D}_0)$. The Γ_0 - equivariant extension $\Gamma_0 \times \mathcal{D}_0 \longrightarrow \mathcal{Z}$ which sends (g, d) to $g\psi_0(d)$ will still be denoted by ψ_0. Let $\sim_0$ be the equivalence relation on $\Gamma_0 \times \mathcal{D}_0$ induced by the map ψ_0, i.e. defined by $(g, d) \sim_0 (g', d')$ iff $\psi_0(g, d) = \psi_0(g', d')$. Then $\sim_0$ is Γ_0 - equivariant, i.e. if $(g, d) \sim_0 (g', d')$ then $(hg, d) \sim_0 (hg', d')$ for every $h \in \Gamma_0$.

Let $g^{AD} \in \Gamma_0$ be given by $g^{AD}(e_1) = -e_1$, $g^{AD}(e_2) = -e_1 + e_2$ and $g^{AD}(e_3) = -e_3$, and let $g^{AB} \in \Gamma_0$ be given by $g^{AB}(e_1) = e_2 - e_1$, $g^{AB}(e_2) = e_2$ and $g^{AB}(e_3) = -e_3$. Then we have the following result which is a refinement of Theorem 2.4.

Theorem 2.6 *The equivalence relation $\sim_0$ on $\Gamma_0 \times \mathcal{D}_0$ induced by the map $\psi_0 : \Gamma_0 \times \mathcal{D}_0 \longrightarrow \mathcal{Z}$ is the smallest Γ_0 - equivariant equivalence relation generated by the following elementary relations: (g, d) and (g', d') are elementary equivalent if either*

1. $g' = 1$, $d = d'$ and g belongs to the isotropy group $H_d \subset \Gamma_0$ of the (class of the) quadratic form $\psi_0(d)$.

2. $g' = 1$, $d = (\frac{1}{2}, b, c)$, $0 \leq c \leq \frac{1}{2}b$, and either

$$d' = (\frac{1}{2}, b, b - c), \quad g = g^{AD} \text{ ,or}$$

$$d' = (\frac{1}{2}, b - c, -c), \quad g = g^{AB} \text{ .}$$

Furthermore the induced map $\Psi_0 : \Gamma_0 \times \mathcal{D}_0/\sim_0 \longrightarrow \mathcal{Z}$ *is a homeomorphism of* Γ_0 *- spaces.*

Proof. First we observe that points of $\Gamma_0 \times \mathcal{D}_0$ which are elementary equivalent are mapped to the same point in $\mathcal{Z}$ under ψ_0. This is trivial for the first elementary relation. For the second one it follows because by definition of g^{AD} and Lemma 2.5 we have $g^{AD}\gamma(\frac{1}{2}, b, c) \in \gamma(\frac{1}{2}, b, b-c)SO(3)$, i.e. $g^{AD}\psi_0(\frac{1}{2}, b, c) = \psi_0(\frac{1}{2}, b, b-c)$. Similarly, g^{AB} is defined such that $g^{AB}\gamma(\frac{1}{2}, b, c) \in \gamma(\frac{1}{2}, b-c, -c)SO(3)$, i.e. $g^{AB}\psi_0(\frac{1}{2}, b, c) = \psi_0(\frac{1}{2}, b-c, -c)$. It follows that ψ_0 induces a map Ψ_0 as claimed.

Furthermore ψ_0 induces (on passing to the quotients with respect to the actions of Γ_0) the surjection φ of Theorem 2.4. In particular, if follows that ψ_0 and hence Ψ_0 is surjective. Next assume that $\psi_0(g, d) = \psi_0(g', d')$. Then Theorem 2.4 shows that $d \sim d'$ and by definition of $\sim_0$ we may therefore assume that $d = d'$. But then we clearly have $g^{-1}g' \in H_d$ and by Γ_0 - equivariance of $\sim_0$ we see that $(g, d) \sim_0 (g', d')$ and injectivity of Ψ_0 follows.

Finally it is easy to see that the map Ψ_0 is an open map and hence a homeomorphism (e.g. by using that the actions of Γ_0 on $\Gamma_0 \times \mathcal{D}_0/\sim_0$ and $\mathcal{Z}$ are proper, and that the induced map on the quotient spaces is a homeomorphism by Theorem 2.4). □

Cell structures on $\mathcal{D}_0$, $\mathcal{D}_0/\sim$ and a Γ_0 - equivariant cell structure on $\mathcal{Z}$. Theorem 2.6 allows us to establish a Γ_0 - equivariant cell structure on $\mathcal{Z}$ in terms of a cell structure on $\mathcal{D}_0$ resp. on $\mathcal{D}_0/\sim$. We start with cell structures on $\mathcal{D}_0$ and $\mathcal{D}_0/\sim$ (see figure 1).

0. The 0 - dimensional cells of $\mathcal{D}_0$ are the vertices $O = (0, 0, 0)$, $A = (\frac{1}{2}, 0, 0)$, $B = (\frac{1}{2}, \frac{1}{2}, 0)$, $C = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $D = (\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$, $E = (\frac{1}{2}, \frac{1}{4}, -\frac{1}{4})$ and $F = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. On $\mathcal{D}_0/\sim$ this gives 5 cells which will still be labelled O, A, $B \sim C$, $D \sim E$ and F.

1. The 1 - dimensional cells of $\mathcal{D}_0$ are the edges OC, OF, OA, EF, BF, AB, AC, AD, AE, BD, CD and BE. On $\mathcal{D}_0/\sim$ this gives 8 cells labelled OC, OF, OA, EF, BF, $AB \sim AC$, $AD \sim AE$ and $BD \sim CD \sim BE$.

2. The 2 - dimensional cells of $\mathcal{D}_0$ are the quadrangles $OAEF$ (characterized by $b = -c$) and $OCBF$ ($a = b$), and the triangles OAC ($b = c$), BEF ($a + b - c = 1$) and ABD, ACD and ABE. On $\mathcal{D}_0/\sim$ this gives 5 cells labelled $OAEF$, $OCBF$, OAC, BEF and $ABD \sim ACD \sim ABE$.

3. $\mathcal{D}_0$ has one cell of dimension 3, namely the interior of $\mathcal{D}_0$, and this gives also one cell for $\mathcal{D}_0/\sim$.

It follows easily from Proposition 2.3 (see also Section 2.5 below) that the isotropy groups H_d of the action of Γ_0 on $\mathcal{Z}$ at $\psi_0(d)$ are constant within the interior of each cell e of $\mathcal{D}_0$ and this is the reason for the choice of our cell structure on $\mathcal{D}_0$. If we denote this isotropy group by H_e then Theorem 2.6 shows that $\mathcal{Z}$ has an equivariant cell structure with one orbit $(\Gamma_0/H_e)\times e$ of cells for each equivalence class of cells in $\mathcal{D}_0/\sim$. The attaching maps can be read off from figure 1.

2.4.2 The cases $i=1$ and $i=2$

We consider the right Γ_0 - spaces $\Gamma_i\backslash\Gamma_0$. In case $i=1$ this coset space can be identified with the set S_1 of non-zero vectors in $(\mathbb{F}_2^3-\{0\})$ and in case $i=2$ with the set S_2 of pairs consisting of a line in $\mathbb{F}_2^3$ and a plane in $\mathbb{F}_2^3$ containing the line, i.e. with the set of complete flags in $\mathbb{F}_2^3$. In fact, there is a canonical left action of Γ_0 on the sets S_i, and if we convert this into a right action in the usual way via $s\cdot g:=g^{-1}s$, then the map $\Gamma_0\longrightarrow S_1, g\mapsto g^{-1}(e_1)$ mod 2 induces an isomorphism of right Γ_0 - spaces $\Gamma_1\backslash\Gamma_0\longrightarrow S_1$; similarly the map $\Gamma_0\longrightarrow S_2$ which sends g to the flag $(\langle g^{-1}(100)\rangle\subset\langle g^{-1}(100),g^{-1}(010)\rangle$ mod 2) induces an isomorphism of right Γ_0 - spaces $\Gamma_2\backslash\Gamma_0\longrightarrow S_2$. (Here $\langle\ \ \rangle$ denotes the subgroup generated by the elements within the brackets and 100, 010 are standard basis vectors in $\mathbb{F}_2^3$.) The sets $\mathcal{D}_0\times S_i$ will be denoted by $\mathcal{D}_i$.

Now we choose representatives for the right cosets of Γ_i in Γ_0. Such a choice of a representative g_s for each $s\in S_i$ gives an explicit Γ_i - equivariant homeomorphism

$$\Gamma_i\times\mathcal{D}_0\times S_i\longrightarrow\Gamma_0\times\mathcal{D}_0,\ (g,d,s)\mapsto(gg_s,d)\ .$$

In order to obtain a Γ_i - equivariant cell structure on $\mathcal{Z}$ we will carry over the Γ_0 - equivariant equivalence relation $\sim_0$ on $\Gamma_0\times\mathcal{D}_0$ to a Γ_i - equivariant equivalence relation $\sim_i$ on $\Gamma_i\times\mathcal{D}_i$. For this we note that the isotropy groups H_d act from the right on the coset spaces S_i. Likewise the matrices g^{AB} and g^{AD} act from the right on S_i. The following result is now a straightforward consequence of Theorem 2.6.

Theorem 2.7 *The equivalence relation $\sim_i$ on $\Gamma_i\times\mathcal{D}_i$ induced by the map*

$$\psi_i:\Gamma_i\times\mathcal{D}_i\longrightarrow\mathcal{Z},\quad(g,d,s)\mapsto gg_s\psi_0(d)$$

is the smallest Γ_i - equivariant equivalence relation generated by the following elementary relations: (g,d,s) and (g',d',s') are elementary equivalent if either

1. $g'=1$, $d=d'$, there exists an element $h\in H_d$ with $s=s'h$ (in particular s and s' belong to the same H_d - orbit with respect to the right action of H_d on the set S_i) and g is determined by $gg_s=g_{s'}h$.

2. $g' = 1$, $d = (\frac{1}{2}, b, c)$, $0 \leq c \leq \frac{1}{2}b$ *and either*

$$d' = (\frac{1}{2}, b, b - c), \quad s = s' g^{AD}, \quad g g_s = g_{s'} g^{AD} \text{ , or}$$

$$d' = (\frac{1}{2}, b - c, -c), \quad s = s' g^{AB}, \quad g g_s = g_{s'} g^{AB} \ .$$

Furthermore the induced map $\Psi_i : \Gamma_i \times \mathcal{D}_i / \sim_i \longrightarrow \mathcal{Z}$ *is a homeomorphism of* Γ_i *- spaces, the* Γ_i *- equivariant equivalence relation* $\sim_i$ *induces an equivalence relation (denoted by* $\sim_{(i)}$*) on the quotient* $\mathcal{D}_i$ *of* $\Gamma_i \times \mathcal{D}_i$ *such that the induced map* $\widetilde{\Psi} : \mathcal{D}_i / \sim_{(i)} \longrightarrow \Gamma_i \backslash \mathcal{Z}$ *is a homeomorphism.* □

Γ_i - equivariant cell structures on $\mathcal{Z}$ and cell structures on $\Gamma_i \backslash \mathcal{Z}$. Theorem 2.7 yields Γ_i - equivariant cell strucures on the space $\mathcal{Z}$ (and then ordinary cell structures on the quotients $\Gamma_i \backslash \mathcal{Z}$). The indexing set for the Γ_i - orbits of cells on $\mathcal{Z}$ (resp. the cells on the quotients $\Gamma_i \backslash \mathcal{Z}$) are equivalence classes of pairs (e, s) with e a cell in $\mathcal{D}_0$ and $s \in S_i$, with the equivalence relation generated by the following elementary relations: $(e, s) \sim_i (e', s')$ iff either

1. $e = e'$ and s and s' are in the same H_e - orbit, or

2a. e is ABD or a face of it, e' is ACD or the corresponding face of it and $s = s' g^{AD}$, or

2b. e is ABD or a face of it, e' is ABE or the corresponding face of it and $s = s' g^{AB}$.

The Γ_i - orbits of cells of $\mathcal{Z}$ are then of the form $\Gamma_i / H_{(e,s)} \times (e, s)$ where (e, s) runs through a set of representatives of equivalence classes of such pairs and the isotropy group $H_{(e,s)}$ of the cell (e, s) is given by $\Gamma_i \cap g_s H_e {g_s}^{-1}$, i.e agrees up to conjugation by g_s with ${g_s}^{-1} \Gamma_i g_s \cap H_e$ which is the isotropy group of s with respect to the right action of H_e on S_i. The attaching maps can again be read off from figure 1.

In the next two sections we will make this concrete, i.e. we will describe in explicit form the isotropy groups H_e, their actions on the sets S_i, and also the effect of the action of g^{AB} and g^{AD} on S_i.

2.5 Symmetries of well-rounded quadratic forms

In order to make the equivariant cell structure of the spaces $\Gamma_i \backslash \mathcal{Z}$ concrete we need to determine the isotropy groups $H_{(a,b,c)}$ of the action of Γ_0 on $\mathcal{Z}$ at $\psi_0(a, b, c)$. Of course, $H_{(a,b,c)}$ preserves the length of vectors and the scalar products between them (both taken, of course, with respect to a representative quadratic form of (the equivalence class of quadratic forms) $\psi_0(a, b, c)$), and hence $H_{(a,b,c)}$ acts on the set of minimal vectors in the standard lattice. These sets have been determined in Theorem 2.4 (we just have to replace the letter v by e everywhere). The standard basis vectors are always minimal vectors and so $H_{(a,b,c)}$ is determined by this action. It is clear that the groups $H_{(a,b,c)}$ are constant in the interior of each cell of $\mathcal{D}_0$ and this gives the justification for the choice of our cell structure on $\mathcal{D}_0$.

The case of the 3 - dimensional cell is particularly simple. If (a, b, c) is in its interior then we have only the 3 standard basis vectors and their negatives in the set of minimal vectors and it is easy to check that $H_{(a,b,c)} = \{1\}$.

Tables 1, 2 and 3 below give the isotropy groups on the open cells of $\mathcal{D}_0$ of dimension 2, 1 and 0. In fact it will be enough for us to take one cell from each $\sim$ - equivalence class of cells. The first column lists the name of the cell, the second one the set of minimal vectors on the standard lattice with respect to (a representative quadratic form of) $\psi_0(a, b, c)$ if (a, b, c) is an interior point of the appropriate cell and the third column gives the isotropy group. The last column describes the action of the isotropy group on the tuple (e_1, e_2, e_3) of minimal vectors explicitly; the 3 - tuples in this column are the images of the tuple (e_1, e_2, e_3) under the action of appropriate generators. The proofs are straightforward and are left to the reader.

We use the following notation in these tables: for the symmetric group on n - letters we write $\mathfrak{S}_n$, $\wr$ denotes the wreath product construction, and the dihedral group with n elements is denoted by D_n.

Table 1: Symmetries on the 2-dimensional cells

Cell	Minimal vectors	Isotropy	Generators
ABD	$\pm e_1, \pm e_2, \pm e_3,$ $\pm(e_1 - e_2)$	trivial	
OAC	$\pm e_1, \pm e_2, \pm e_3$	$\mathbb{Z}/2$	$(-e_2, -e_1, -e_3)$
OAEF	$\pm e_1, \pm e_2, \pm e_3$	$\mathbb{Z}/2$	$(e_2, e_1, -e_3)$
OCBF	$\pm e_1, \pm e_2, \pm e_3$	$\mathbb{Z}/2$	$(-e_1, -e_3, -e_2)$
BEF	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2 - e_3)$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(-e_2, -e_1, e_1 - e_2 - e_3)$ $(-e_3, e_1 - e_2 - e_3, -e_1)$

Table 2: Symmetries on the 1-dimensional cells

Cell	Minimal vectors	Isotropy	Generators
OC	$\pm e_1, \pm e_2, \pm e_3$	$\mathfrak{S}_3$	$(-e_1, -e_3, -e_2)$ $(-e_2, -e_1, -e_3)$
OF	$\pm e_1, \pm e_2, \pm e_3$	$\mathfrak{S}_3$	$(-e_1, -e_3, -e_2)$ $(e_2, e_1, -e_3)$
OA	$\pm e_1, \pm e_2, \pm e_3$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(-e_2, -e_1, -e_3)$ $(e_2, e_1, -e_3)$
AB	$\pm e_1, \pm e_2, \pm e_3,$ $\pm(e_1 - e_2)$	$\mathbb{Z}/2$	$(e_2 - e_1, e_2, -e_3)$
AD	$\pm e_1, \pm e_2, \pm e_3,$ $\pm(e_1 - e_2)$	$\mathbb{Z}/2$	$(-e_1, e_2 - e_1, -e_3)$
BD	$\pm e_1, \pm e_2, \pm e_3,$ $\pm(e_1 - e_2), \pm(e_1 - e_3)$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$(-e_1, -e_3, -e_2)$ $(-e_1, e_3 - e_1, e_2 - e_1)$
BF	$\pm e_1, \pm e_2, \pm e_3,$ $\pm(e_1 - e_2 - e_3)$	D_8	BEF symmetries, $(-e_1, -e_3, -e_2)$
EF	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2 - e_3)$	D_8	BEF symmetries, $(e_2, e_1, -e_3)$

Table 3: Symmetries on the 0-dimensional cells

Cell	Minimal vectors	Isotropy	Description
O	$\pm e_1, \pm e_2, \pm e_3$	$\mathfrak{S}_4 \cong (\mathbb{Z}/2)^2 \wr \mathfrak{S}_3$	symmetry of a cube; index 2 in $(\mathbb{Z}/2)^3 \wr \mathfrak{S}_3$ permuting the set of pairs $\{\pm e_1\}, \{\pm e_2\}, \{\pm e_3\}$
C	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2)$ $\pm(e_1 - e_3)$ $\pm(e_2 - e_3)$	$\mathfrak{S}_4 \cong (\mathbb{Z}/2)^2 \wr \mathfrak{S}_3$	$\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by: $(e_2 - e_3, e_1 - e_3, -e_3)$ $(e_3 - e_2, -e_2, e_1 - e_2)$; $\mathfrak{S}_3$ symmetry as on OC
F	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2 - e_3)$	$\mathfrak{S}_4 \cong (\mathbb{Z}/2)^2 \wr \mathfrak{S}_3$	$\mathbb{Z}/2 \times \mathbb{Z}/2$ action as on BEF; $\mathfrak{S}_3$ symmetry as on OF
A	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2)$	D_{12}	$\{\pm e_3\}$ is invariant. Standard action on the regular planar hexagon formed by $\pm e_1, \pm e_2, \pm(e_1 - e_2)$
D	$\pm e_1, \pm e_2, \pm e_3$ $\pm(e_1 - e_2)$ $\pm(e_1 - e_3)$	D_8	$\mathbb{Z}/2 \times \mathbb{Z}/2$ action as on BD; additional generator: $(-e_1, e_2 - e_1, -e_3)$

2.6 The equivalence relations $\sim_i$ on the spaces $\Gamma_i \times \mathcal{D}_i$

By Theorem 2.7 the equivalence relations $\sim_i$, as well as Γ_i - equivariant cell structures on $\mathcal{Z}$ and ordinary cell structures on $\Gamma_i\backslash\mathcal{Z}$, are determined by the right actions of the isotropy groups $H_d \subset \Gamma_0$, $d \in \mathcal{D}$, on the sets S_i together with the right action of g^{AB} and g^{AD} on these sets. Clearly, the associated left action of H_d on the sets S_i has identical orbits and isotropy groups as the right action; in the case of g^{AB} and g^{AD} the left and right actions are even identical because both elements agree with their own inverses; we prefer to work with the left actions.

As remarked above the isotropy groups and hence their actions are constant on the cells of $\mathcal{D}$ and we consider cell by cell separately. In our analysis the elements in S_2 will be labelled by pairs consisting of a non-zero vector in $\mathbb{F}_2^3$ and a plane in $\mathbb{F}_2^3$ containing this vector, e.g. $010x = z$, $011x = 0, \ldots$. The plane with equation $x + y + z = 0$ will be abbreviated by $\Sigma = 0$. We will also abbreviate $(ABD, 100y = z)$ by $ABD100y = z$ and so on. The proofs in this section are all straightforward and are left to the reader.

2.6.1 3 - cells

By Theorem 2.7 and by Section 2.5 there are no identifications in the interior of the 3 - cells. As there is only one 3 - cell in $\mathcal{D}_0$, the 3 - cells in $\Gamma_1\backslash\mathcal{Z}$ will be labelled just by the non-zero vectors $\mathbb{F}_2^3$. So there are seven 3 - cells in $\Gamma_1\backslash\mathcal{Z}$, labelled:

$$100,\ 010,\ 001,\ 110,\ 101,\ 011,\ 111\ .$$

Similarly there are 21 cells of dimension 3 in $\Gamma_1\backslash\mathcal{Z}$ which will be labelled:

$$\begin{array}{lll}
100y = 0, & 100z = 0, & 100y = z \\
010x = 0, & 010z = 0, & 010x = z, \\
001x = 0, & 001y = 0, & 001x = y, \\
110z = 0, & 110x = y, & 110\Sigma = 0, \\
101y = 0, & 101x = z, & 101\Sigma = 0, \\
011x = 0, & 011y = z, & 011\Sigma = 0, \\
111x = y, & 111x = z, & 111y = z.
\end{array}$$

The isotropy group of the 3 - cell is trivial, so there is nothing else to do in this case.

2.6.2 2 - cells

1. ABD By Section 2.5 and Theorem 2.7 all the relations involving these 2 - cells are of type 2, i.e. the following cells become equivalent.

$$ABDs\sim_i ACDg^{AD}s\sim_i ABEg^{AB}s\ . \tag{2.5}$$

Clearly the isotropy groups are trivial for all these cells.

We will now make the maps g^{AD} and g^{AB} explicit. We recall that by definition these matrices induce the linear maps on $\mathbb{F}_2^3$ given by

$$g^{AD}(100) = 100,\quad g^{AD}(010) = 110,\quad g^{AD}(001) = 001\ ,$$

$$g^{AB}(100) = 110,\quad g^{AB}(010) = 010,\quad g^{AB}(001) = 001\ .$$

Explicit knowledge of the action of these maps on the sets S_i will be used repeatedly later on and therefore these maps are explicitly described in tables 4 and 5 below.

Table 4: Action of g^{AD} and g^{AB} on S_1

s	100	010	001	110	101	011	111
$g^{AD}s$	100	110	001	010	101	111	011
$g^{AB}s$	110	010	001	100	111	011	101

Table 5: Action of g^{AD} and g^{AB} on S_2

s	$100y=0$	$100z=0$	$100y=z$	$010x=0$	$010z=0$	$010x=z$
$g^{AD}s$	$100y=0$	$100z=0$	$100y=z$	$110x=y$	$110z=0$	$110\Sigma=0$
$g^{AB}s$	$110x=y$	$110z=0$	$110\Sigma=0$	$010x=0$	$010z=0$	$010x=z$
s	$001x=0$	$001y=0$	$001x=y$	$110z=0$	$110x=y$	$110\Sigma=0$
$g^{AD}s$	$001x=y$	$001y=0$	$001x=0$	$010z=0$	$010x=0$	$010x=z$
$g^{AB}s$	$001x=0$	$001x=y$	$001y=0$	$100z=0$	$100y=0$	$100y=z$
s	$101y=0$	$101x=z$	$101\Sigma=0$	$011x=0$	$011y=z$	$011\Sigma=0$
$g^{AD}s$	$101y=0$	$101\Sigma=0$	$101x=z$	$111x=y$	$111y=z$	$111x=z$
$g^{AB}s$	$111x=y$	$111x=z$	$111y=z$	$011x=0$	$011\Sigma=0$	$011y=z$
s	$111x=y$	$111x=z$	$111y=z$			
$g^{AD}s$	$011x=0$	$011\Sigma=0$	$011y=z$			
$g^{AB}s$	$101y=0$	$101x=z$	$101\Sigma=0$			

2. OAC In the interior of these cells all relations are of type 1. In other words we have to determine the action of the group $H_{OAC} \cong \mathbb{Z}/2$ on the vector space $\mathbb{F}_2^3$. By table 1 the action of the non-trivial element $h \in H_{OAC}$ on $\mathbb{F}_2^3$ is given by

$$h100 = 010, \quad h010 = 100, \quad h001 = 001 \ .$$

Hence we get the following orbits and isotropy groups for the action on S_1:

Isotropy groups	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	100, 010	101, 011	001	110	111

For the action on S_2 we obtain:

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
Orbits	$100y = 0$ $010x = 0$	$100z = 0$ $010z = 0$	$100y = z$ $010x = z$	$101y = 0$ $011x = 0$	$101x = z$ $011y = z$
Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$101\Sigma = 0$ $011\Sigma = 0$	$001x = 0$ $001y = 0$	$111x = z$ $111y = z$	$001x = y$	$111x = y$
Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
Orbits	$110x = y$	$110z = 0$	$110\Sigma = 0$		

3. OAEF Again all relations are of type 1. Furthermore the action of H_{OAEF} on $\mathbb{F}_2^3$ is the same as in the case of OAC. Therefore we obtain the same list of orbits and isotropy groups.

4. OCBF Once again all relations are of type 1. By table 1 the action of the non-trivial element $h \in H_{OCBF}$ on $\mathbb{F}_2^3$ is given by

$$h100 = 100, \quad h010 = 001, \quad h001 = 010$$

and we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	010, 001	110, 101	100	011	111

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
Orbits	$010x = 0$ $001x = 0$	$010z = 0$ $001y = 0$	$010x = z$ $001x = y$	$110z = 0$ $101y = 0$	$110x = y$ $101x = z$
Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$110\Sigma = 0$ $101\Sigma = 0$	$100y = 0$ $100z = 0$	$111x = y$ $111x = z$	$100y = z$	$111y = z$
Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
Orbits	$011x = 0$	$011y = z$	$011\Sigma = 0$		

5. BEF All relations are of type 1. By table 1 the action of two generators h_1 and h_2 of $H_{BEF} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ on $\mathbb{F}_2^3$ is given by

$$h_1 100 = 010, \quad h_1 010 = 100, \quad h_1 001 = 111 \ ,$$

$$h_2 100 = 001, \quad h_2 010 = 111, \quad h_2 001 = 100 \ .$$

Hence we get the following orbits and (types of) isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	100, 111, 010, 001	110	101	011

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$100y = 0$ $111x = z$ $010x = z$ $001y = 0$	$100z = 0$ $111x = y$ $010z = 0$ $001x = y$	$100y = z$ $111y = z$ $010x = 0$ $001x = 0$	$110x = y$ $110z = 0$	$101x = z$ $101y = 0$
Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
Orbits	$011y = z$ $011x = 0$	$110\Sigma = 0$	$101\Sigma = 0$	$011\Sigma = 0$	

2.6.3 1 - cells

1. OC There are only relations of type 1. By table 1 the action of two generators h_1 and h_2 of $H_{OC} \cong \mathfrak{S}_3$ on $\mathbb{F}_2^3$ is given by

$$h_1 100 = 100, \quad h_1 010 = 001, \quad h_1 001 = 010,$$

$$h_2 100 = 010, \quad h_2 010 = 100, \quad h_2 001 = 001,$$

and we get the following orbits and isotropy groups for the actions on S_1 resp. S_2:

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathfrak{S}_3$
Orbits	100, 010, 001	110, 101, 011	111

Isotropy groups	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$100y=0,\ 100z=0$	$100y=z$	$110x=y$
	$010x=0,\ 010z=0$	$010x=z$	$101x=z$
	$001x=0,\ 001y=0$	$001x=y$	$011y=z$
Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$110z=0$	$110\Sigma=0$	$111x=y$
	$101y=0$	$101\Sigma=0$	$111x=z$
	$011x=0$	$011\Sigma=0$	$111y=z$

2. OF Again there are only relations of type 1, and furthermore the action of H_{OF} on $\mathbb{F}_2^3$ is the same as in the case of OC. Therefore we obtain the same list of orbits and isotropy groups.

3. OA There are only relations of type 1. By table 1 the action of two generators h_1 and h_2 of $H_{OA} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ on $\mathbb{F}_2^3$ is given by

$$h_1 100 = 100, \quad h_1 010 = 010, \quad h_1 001 = 001\ ,$$

$$h_2 100 = 010, \quad h_2 010 = 100, \quad h_2 001 = 001\ .$$

Hence we get the same orbits as in the case of the 2 - cells OAC resp. $OAEF$. However, the isotropy groups are now larger: the trivial ones get replaced by $\mathbb{Z}/2$ generated by h_1h_2, the $\mathbb{Z}/2$ gets replaced by $\mathbb{Z}/2 \times \mathbb{Z}/2$.

4. AB and AC There are relations of both types. Those of type 2 lead to $ABs \sim_i ACg^{AD}s$ and are described by tables 4 and 5. As far as relations of type 1 are concerned we can concentrate on the edge AB. By table 2 the action of the non-trivial element $h \in H_{AB}$ on $\mathbb{F}_2^3$ is here given by

$$h100 = 110, \quad h010 = 010, \quad h001 = 001\ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2 (the orbits for AB and those for AC in the same column correspond to each other via g^{AD}; the same conventions will hold in later tables of this section):

Isotropy groups	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for AB	100, 110	101, 111	010	001	011
Orbits for AC	100, 010	101, 011	110	001	111

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
Orbits for AB	$100y = 0$ $110x = y$	$100z = 0$ $110z = 0$	$100y = z$ $110\Sigma = 0$	$101y = 0$ $111x = y$	$101\Sigma = 0$ $111y = z$
Orbits for AC	$100y = 0$ $010x = 0$	$100z = 0$ $010z = 0$	$100y = z$ $010x = z$	$101y = 0$ $011x = 0$	$101x = z$ $011y = z$

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for AB	$101x = z$ $111x = z$	$001x = y$ $001y = 0$	$011\Sigma = 0$ $011y = z$	$010x = 0$	$010z = 0$
Orbits for AC	$101\Sigma = 0$ $011\Sigma = 0$	$001x = 0$ $001y = 0$	$111x = z$ $111y = z$	$110x = y$	$110z = 0$

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
Orbits for AB	$010x = z$	$001x = 0$	$011x = 0$		
Orbits for AC	$110\Sigma = 0$	$001x = y$	$111x = y$		

5. AD and AE There are again relations of both types. Those of type 2 lead to $ADs \sim_i AEg^{AB}s$ and are described by tables 4 and 5. As far as relations of type 1 are concerned we can concentrate on the edge AD. By table 2 the action of the non-trivial element $h \in H_{AD}$ on $\mathbb{F}_2^3$ is here given by

$$h100 = 100, \quad h010 = 110, \quad h001 = 001 \ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for AD	$110,\ 010$	$111,\ 011$	001	100	101
Orbits for AE	$100,\ 010$	$101,\ 011$	001	110	111

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
Orbits for AD	$110x = y$ $010x = 0$	$110z = 0$ $010z = 0$	$110\Sigma = 0$ $010x = z$	$111x = y$ $011x = 0$	$111x = z$ $011\Sigma = 0$
Orbits for AE	$100y = 0$ $010x = 0$	$100z = 0$ $010z = 0$	$100y = z$ $010x = z$	$101y = 0$ $011x = 0$	$101x = z$ $011y = z$

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for AD	$111y = z$ $011y = z$	$001x = 0$ $001x = y$	$101x = z$ $101\Sigma = 0$	$100y = 0$	$100z = 0$
Orbits for AE	$101\Sigma = 0$ $011\Sigma = 0$	$001x = 0$ $001y = 0$	$111x = z$ $111y = z$	$110x = y$	$110z = 0$

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$		
Orbits for AD	$100y = z$	$001y = 0$	$101y = 0$		
Orbits for AE	$110\Sigma = 0$	$001x = y$	$111x = y$		

6. BD, CD and BE There are again relations of both types. Those of type 2 lead to $BDs \sim_i BEg^{AB}s$ resp. $BDs \sim_i CDg^{AD}s$ and are described by tables 4 and 5. As far as relations of type 1 are concerned we can concentrate on the edge BD. By table 2 the action of two generators h_1 and h_2 of $H_{BD} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ on $\mathbb{F}_2^3$ is here given by

$$h_1 100 = 100, \quad h_1 010 = 001, \quad h_1 001 = 010 \ ,$$

$$h_2 100 = 100, \quad h_2 010 = 101, \quad h_2 001 = 110 \ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\{1\}$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
Orbits for BD	010, 001 101, 110	100	011	111
Orbits for CD	110, 001 101, 010	100	111	011
Orbits for BE	010, 001 111, 100	110	011	101

Isotropy groups	$\{1\}$	$\{1\}$	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for BD	$010x = 0$ $001x = 0$ $101\Sigma = 0$ $110\Sigma = 0$	$010z = 0$ $001y = 0$ $101y = 0$ $110z = 0$	$010x = z$ $001x = y$ $101x = z$ $110x = y$	$100y = 0$ $100z = 0$	$011x = 0$ $011\Sigma = 0$
Orbits for CD	$110x = y$ $001x = y$ $101x = z$ $010x = z$	$110z = 0$ $001y = 0$ $101y = 0$ $010z = 0$	$110\Sigma = 0$ $001x = 0$ $101\Sigma = 0$ $010x = 0$	$100y = 0$ $100z = 0$	$111x = y$ $111x = z$
Orbits for BE	$010x = 0$ $001x = 0$ $111y = z$ $100y = z$	$010z = 0$ $001x = y$ $111x = y$ $100z = 0$	$010x = z$ $001y = 0$ $111x = z$ $100y = 0$	$110x = y$ $110z = 0$	$011x = 0$ $011y = z$
Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	
Orbits for BD	$111x = y$ $111x = z$	$100y = z$	$011y = z$	$111y = z$	
Orbits for CD	$011x = 0$ $011\Sigma = 0$	$100y = z$	$111y = z$	$011y = z$	
Orbits for BE	$101y = 0$ $101x = z$	$110\Sigma = 0$	$011\Sigma = 0$	$101\Sigma = 0$	

7. BF There are only relations of type 1. By table 1 and table 2 we know the action of three generating involutions of $H_{BF} \cong D_8$ on $\mathbb{F}_2^3$

$$h_1 100 = 010, \quad h_1 010 = 100, \quad h_1 001 = 111\ ,$$
$$h_2 100 = 001, \quad h_2 010 = 111, \quad h_2 001 = 100\ ,$$
$$h_3 100 = 100, \quad h_2 010 = 001, \quad h_2 001 = 010\ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp S_2:

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits	100, 111, 010, 001	110, 101	011

Isotropy groups	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$100y = 0,\ 111x = z$ $010x = z,\ 001y = 0$ $100z = 0,\ 111x = y$ $001x = y,\ 010z = 0$	$100y = z$ $111y = z$ $010x = 0$ $001x = 0$	$110x = y$ $110z = 0$ $101x = z$ $101y = 0$
Isotropy groups	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits	$110\Sigma = 0$ $101\Sigma = 0$	$011y = z$ $011x = 0$	$011\Sigma = 0$

8. EF Again there are only relations of type 1 and by table 1 and table 2 we know the action of three generating involutions of $H_{EF} \cong D_8$ on $\mathbb{F}_2^3$

$$h_1 100 = 010, \quad h_1 010 = 100, \quad h_1 001 = 111\ ,$$
$$h_2 100 = 001, \quad h_2 010 = 111, \quad h_2 001 = 100\ ,$$
$$h_3 100 = 010, \quad h_3 010 = 100, \quad h_3 001 = 001\ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits	100, 111, 010, 001	101, 011	110

Isotropy groups	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits	$100y = 0,\ 111x = z$ $010x = z,\ 001y = 0$ $010x = 0,\ 111y = z$ $100y = z,\ 001x = 0$	$100z = 0$ $111x = y$ $010z = 0$ $001x = y$	$101x = z$ $101y = 0$ $011y = z$ $011x = 0$
Isotropy groups	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits	$101\Sigma = 0$ $011\Sigma = 0$	$110x = y$ $110z = 0$	$110\Sigma = 0$

2.6.4 0 - cells

1. O There are only relations of type 1. By table 3 the action of $H_O \cong \mathfrak{S}_4$ factors through an action of $\mathfrak{S}_3$, and this action agrees with that in the case of the edge OC. Therefore we get the same orbits; the isotropy groups "grow" by $\mathbb{Z}/2\times\mathbb{Z}/2$, more precisely the trivial isotropy group gets replaced by $\mathbb{Z}/2\times\mathbb{Z}/2$, $\mathbb{Z}/2$ gets replaced by D_8 and $\mathfrak{S}_3$ by $\mathfrak{S}_4$.

2. B and C There are relations of both types. Those of type 2 lead to $Bs \sim_i Cg^{AD}s$ and are described by table 4 and 5. As far as relations of type 1 are concerned we can concentrate on C. By table 3 the action of $H_C \cong \mathfrak{S}_4 \cong \mathbb{Z}/2\times\mathbb{Z}/2 \wr \mathfrak{S}_3$ is described as follows: for the generators h_1 and h_2 of $\mathbb{Z}/2\times\mathbb{Z}/2$ we have

$$h_1 100 = 011, \quad h_1 010 = 101, \quad h_1 001 = 001 \ ,$$

$$h_2 100 = 011, \quad h_2 010 = 010, \quad h_2 001 = 110 \ .$$

The action of $\mathfrak{S}_3$ is again as in the case of the edge OC. Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\mathbb{Z}/2\times\mathbb{Z}/2$	$\mathfrak{S}_4$
Orbits for C	100, 010, 001, 110, 101, 011	111
Orbits for B	100, 110, 001, 010, 101, 111	011

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2\times\mathbb{Z}/2$	D_8
Orbits for C	$100y=0,\ 100z=0$	$100y=z$	$111x=y$
	$010x=0,\ 010z=0$	$010x=z$	$111x=z$
	$001x=0,\ 001y=0$	$001x=y$	$111y=z$
	$110z=0,\ 110\Sigma=0$	$110x=y$	
	$101y=0,\ 101\Sigma=0$	$101x=z$	
	$011x=0,\ 011\Sigma=0$	$011y=z$	
Orbits for B	$100y=0,\ 100z=0$	$100y=z$	$011x=0$
	$010x=z,\ 010z=0$	$010x=0$	$011y=z$
	$001y=0,\ 001x=y$	$001x=0$	$011\Sigma=0$
	$110z=0,\ 110x=y$	$110\Sigma=0$	
	$101y=0,\ 101x=z$	$101\Sigma=0$	
	$111x=y,\ 111x=z$	$111y=z$	

3. F Here there are only relations of type 1. By table 3 the action of $H_F \cong \mathfrak{S}_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \wr \mathfrak{S}_3$ is described as follows: for the generators h_1 and h_2 of $\mathbb{Z}/2 \times \mathbb{Z}/2$ we have

$$h_1 100 = 010, \quad h_1 010 = 100, \quad h_1 001 = 111 \ ,$$

$$h_2 100 = 001, \quad h_2 010 = 111, \quad h_2 001 = 100 \ .$$

The action of $\mathfrak{S}_3$ is again as in the case of the edge OC resp. OF. Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\mathfrak{S}_3$	D_8
Orbits	$100,\ 010,\ 001,\ 111$	$110,\ 101,\ 011$

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits	$100y = 0,\ 010x = 0$	$110x = y$	$110\Sigma = 0$
	$100z = 0,\ 010z = 0$	$110z = 0$	$101\Sigma = 0$
	$100y = z,\ 010x = z$	$101x = z$	$011\Sigma = 0$
	$001x = 0,\ 111x = y$	$101y = 0$	
	$001y = 0,\ 111x = z$	$011x = 0$	
	$001x = y,\ 111y = z$	$011y = z$	

4. A Again there are only relations of type 1. By table 3 the action of $H_A \cong D_{12}$ factors through an action of $\mathfrak{S}_3$ and permutes the elements 100, 010 and 110 while 001 is fixed under the action. Therefore we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_{12}
Orbits	$100,\ 110,\ 010$	$101,\ 111,\ 011$	001

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
Orbits	$101x = z,\ 101\Sigma = 0$	$100y = z$	$100y = 0$
	$011y = z,\ 011\Sigma = 0$	$110\Sigma = 0$	$110x = y$
	$111x = z,\ 111y = z$	$010x = z$	$010x = 0$
Isotropy groups	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$
Orbits	$100z = 0$	$101y = 0$	$001x = 0$
	$110z = 0$	$011x = 0$	$001x = y$
	$010z = 0$	$111x = y$	$001y = 0$

5. D and E There are relations of both types. Those of type 2 lead to $Ds \sim_i Eg^{AB}s$ and are described by table 4 and 5. As far as relations of type 1 are concerned we can concentrate on D. By table 3 we know the action of three generating involutions of $H_D \cong D_8$

$$h_1 100 = 100, \quad h_1 010 = 001, \quad h_1 001 = 010 \ ,$$

$$h_2 100 = 100, \quad h_2 010 = 101, \quad h_2 001 = 110 \ ,$$

$$h_3 100 = 100, \quad h_3 010 = 110, \quad h_3 001 = 001 \ .$$

Hence we get the following orbits and isotropy groups for the action on S_1 resp. S_2:

Isotropy groups	$\mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits for D	110, 101, 010, 001	111, 011	100
Orbits for E	100, 111, 010, 001	101, 011	110

Isotropy groups	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Orbits for D	$010x = 0,\ 010x = z$	$010z = 0$	$011x = 0$
	$001x = 0,\ 001x = y$	$001y = 0$	$011\Sigma = 0$
	$101x = z,\ 101\Sigma = 0$	$110z = 0$	$111x = y$
	$110x = y,\ 110\Sigma = 0$	$101y = 0$	$111x = z$
Orbits for E	$100y = 0,\ 100y = z$	$100z = 0$	$101y = 0$
	$010x = 0,\ 010x = z$	$010z = 0$	$101x = z$
	$001x = 0,\ 001y = 0$	$001x = y$	$011x = 0$
	$111x = z,\ 111y = z$	$111x = y$	$011y = z$

Isotropy groups	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	D_8
Orbits for D	$100y = 0$	$011y = z$	$100y = z$
	$100z = 0$	$111y = z$	
Orbits for E	$110z = 0$	$101\Sigma = 0$	$110\Sigma = 0$
	$110x = y$	$011\Sigma = 0$	

3 The homology of the quotient spaces

Let p be any prime. In this section we will compute the mod - p cohomology resp. homology of the quotients of $\mathcal{X}_\infty$, $\mathcal{X}_{\infty,s}(i)$ and the pair $(\mathcal{X}_\infty, \mathcal{X}_{\infty,s}(i))$ by the groups Γ_i, and also the cohomology of the quotients of $\mathcal{X}$, $\mathcal{X}_s$ and $(\mathcal{X}, \mathcal{X}_s)$ by $\Gamma := SL(3, \mathbb{Z}[1/2])$; in particular we prove Theorem 1.6, Corollary 1.7 and Theorem 1.8.

In Sections 2.4 and 2.6 we described cell structures on the spaces $\Gamma_i\backslash\mathcal{Z} \simeq \Gamma_i\backslash\mathcal{X}_\infty$. Let $\mathcal{Z}_s(i)$ be the 2 - singular locus of $\mathcal{Z}$ with respect to the action of Γ_i, $i = 0, 1, 2$, so that $\Gamma_i\backslash\mathcal{Z}_s(i) \simeq \Gamma_i\backslash\mathcal{X}_{\infty,s}(i)$. We will use the results of Section 2 to give the boundary homomorphisms of the chain complexes $C_*(\Gamma_i\backslash(\mathcal{Z}, \mathcal{Z}_s(i)))$ and $C_*(\Gamma_i\backslash\mathcal{Z}_s(i))$ (with integral coefficients) in an explicit form. Then we compute the homology groups of interest from these complexes. As these complexes are quite big our computations will be simplified by Euler characteristic considerations. We summarize the discussion of Section 2 relevant for the Euler characteristic χ in the following table.

	0-cells	1-cells	2-cells	3-cells	number of all cells	χ
$\Gamma_0\backslash\mathcal{Z}$	5	8	5	1	19	1
$\Gamma_0\backslash\mathcal{Z}_s(0)$	5	8	4	0	17	1
$\Gamma_0\backslash(\mathcal{Z}, \mathcal{Z}_s(0))$	0	0	1	1	2	0
$\Gamma_1\backslash\mathcal{Z}$	13	31	26	7	77	1
$\Gamma_1\backslash\mathcal{Z}_s(1)$	13	26	12	0	51	-1
$\Gamma_1\backslash(\mathcal{Z}, \mathcal{Z}_s(1))$	0	5	14	7	26	2
$\Gamma_2\backslash\mathcal{Z}$	24	72	69	21	186	0
$\Gamma_2\backslash\mathcal{Z}_s(2)$	23	49	21	0	93	-5
$\Gamma_2\backslash(\mathcal{Z}, \mathcal{Z}_s(2))$	1	23	48	21	93	5

In order to determine the incidence matrices, i.e. the boundary homomorphisms in the relevant cellular chain complexes, we will have to choose orientations for our cells. We will choose the orientation of the edges and triangles in $\mathcal{D}_0$ in accordance with the ordering of the vertices in their names so that for example $[ACD] = -[ADC]$ and for the boundary of $[ABD]$ we obtain $[AB] + [BD] - [AD]$. (Here $[ACD], [ABD]$ etc. denote the basis elements in the chain complex given by the cells ACD, ABD etc.; similar notation will be used below.) The 2 - dimensional cell $OAEF$ is oriented such that its boundary is $[OA] + [AE] + [EF] - [OF]$; likewise with $OCBF$. The 3 - dimensional cell in $\mathcal{D}$ can then be oriented such that its boundary is given by $[OAEF] - [ABD] - [ADC] - [AEB] - [BEF] - [OCBF] - [OAC]$.

Then we get an orientation of the cells (e, s) in $\mathcal{D}_0 \times S_i$ (by choosing the orientation of e) and finally we get induced orientations for the cells in $\Gamma_i \backslash \mathcal{Z}$. For example in $C_*(\Gamma_0 \backslash \mathcal{Z})$ we obtain $[ABD] = [ACD] = -[ADC]$.

3.1 Quotients of $(\mathcal{X}_\infty, \mathcal{X}_{\infty,s}(i))$ by Γ_i

We will compute the homology of the homotopy-equivalent quotients of the pairs $(\mathcal{Z}, \mathcal{Z}_s(i))$ by Γ_i.

1. Γ_0 **:** There is only one 2 - and one 3 - dimensional cell in $\Gamma_0 \backslash (\mathcal{Z}, \mathcal{Z}_s(0))$ and it is clear that the boundary map $\partial_3 : C_3 \longrightarrow C_2$ in the cellular chain complex $C_*(\Gamma_0 \backslash (\mathcal{Z}, \mathcal{Z}_s(0))$ is an isomorphism. This implies part a) of Theorem 1.8.

2. Γ_1 **:** Using the description of $\Gamma_1 \backslash \mathcal{Z}$ that we gave in Section 2.4 and 2.6 it is straightforward to check that the boundary maps ∂_2 and ∂_3 in the cellular complex $C_*(\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1))$ are given by the matrices in tables 6 and 7 below. In these matrices the columns and rows are labelled by cells in $\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1))$, i.e. by equivalence classes of "nonsingular" cells in $\mathcal{D}_1$ (cf. Section 2.4.2), and we have chosen representatives from equivalence classes where necessary. Furthermore all zero entries in these matrices have been omitted.

One sees at once that ∂_3 has trivial kernel, i.e. $H_3(\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1)); \mathbb{F}_p) = 0$ and that ∂_2 is onto, i.e. $H_1(\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1)); \mathbb{F}_p) = 0$. Then the Euler characteristic argument implies that $H_2(\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1)); \mathbb{F}_p) \cong (\mathbb{F}_p)^2$ and we obtain part b) of Theorem 1.8. For later use we specify two 2 - dimensional cycles which form a basis of H_2. We can take the cycles

$$[ABD100] - [OAC100] \quad \text{and} \quad [ABD011] + [OAEF101] \ . \tag{3.1}$$

3. Γ_2 **:** Now consider the complex $C_*(\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2)))$. First of all it is clear that $\partial_1 : C_1 \longrightarrow C_0$ is onto and hence we obtain $H_0(\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2)); \mathbb{F}_p) = 0$. Furthermore, using our description in Section 2.6 again, it is straightforward to check that ∂_2 and ∂_3 are given by the matrices in tables 8 - 11 below.

These matrices show that the kernel of ∂_3 is of dimension 1 and is generated by the cycle:

$$[100y = 0] - [100z = 0] - [010x = 0] + [010z = 0] + [001x = 0] - [001y = 0] \ . \tag{3.2}$$

In particular we get $H_3(\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2)); \mathbb{F}_p) \cong \mathbb{F}_p$. (Here $[100y = 0]$ etc. denote the 3 - dimensional cells in $\Gamma_2 \backslash \mathcal{Z}$ corresponding to the elements $100y = 0$ etc. in S_2.) Furthermore, the image of ∂_2 is of dimension 22, i.e. $H_1(\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2)); \mathbb{F}_p) = 0$, and then the Euler characteristic argument implies $H_3(\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2)); \mathbb{F}_p) \cong (\mathbb{F}_p)^6$ and hence part c) of Theorem 1.8.

Again for later use we specify six 2 - dimensional cycles whose homology classes form a basis of H_2. We can take the cycles

$$c_1 : \;=\; [OAC100y = z] - [ABD100y = z] \tag{3.3}$$

$$c_2 : \;=\; [OAC100z=0] + [ABD100y=0] - [ABD100z=0] - -[OAC100y=0] \tag{3.4}$$

$$c_3 : \;=\; -[ABD001y=0] + [BEF100z=0] + [OAC001x=0] + +[ABD100y=0] - [OAC100y=0] + [OCBF111x=y] \tag{3.5}$$

$$c_4 : \;=\; [ABD011x=0] + [OAEF101y=0] \tag{3.6}$$

$$c_5 : \;=\; [ABD011y=z] - [OAC111x=z] + [OAEF101\Sigma=0] \tag{3.7}$$

$$c_6 : \;=\; [ABD011\Sigma=0] - [OAC111x=z] + [OAEF101x=z] \;. \tag{3.8}$$

Table 6: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1))$

		ABD							BEF	OAC		OAEF		OCBF	
		100	010	001	110	101	011	111	100	100	101	100	101	010	110
BD	010		1	1	1	1			1						
AB	100	1			1					1					
	101					1		1			1				
AD	110		−1		−1							1			
	111						−1	−1					1		

Table 7: The boundary homomorphism $C_3 \to C_2$ for $\Gamma_1 \backslash (\mathcal{Z}, \mathcal{Z}_s(1))$

		100	010	001	110	101	011	111
ABD	100				1			
	010				1			
	001			1				
	110	1	1		−1			
	101							1
	011							1
	111					1	1	−1
BEF	100	−1	−1	−1				−1
OAC	100	−1	−1					
	101					−1	−1	
OAEF	100	1	1					
	101					1	1	
OCBF	010		−1	−1				
	110				−1	−1		

Table 8: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_2\backslash(\mathcal{Z}, \mathcal{Z}_s(2))$; Part I

			ABD																						BEF		
			100			010			001			110			101			011			111			100		
			$y=0$	$z=0$	$y=z$	$x=0$	$z=0$	$x=z$	$x=0$	$y=0$	$x=y$	$z=0$	$x=y$	$\Sigma=0$	$y=0$	$x=z$	$\Sigma=0$	$x=0$	$y=z$	$\Sigma=0$	$x=y$	$x=z$	$y=z$	$y=0$	$z=0$	$y=z$
BD	010	$x=0$				1			1					1			1									1
	010	$z=0$					1			1		1			1										1	
	010	$x=z$						1			1		1			1								1		
AB	001	$y=0$								1	1															
	011	$y=z$																	1	1						
	100	$y=0$	1										1													
	100	$z=0$		1								1														
	100	$y=z$			1									1												
	101	$y=0$													1						1					
	101	$\Sigma=0$															1						1			
	101	$x=z$														1						1				
AD	001	$x=0$							−1		−1															
	101	$x=z$														−1	−1									
	110	$x=y$				−1							−1													
	110	$z=0$					−1					−1														
	110	$\Sigma=0$						−1						−1												
	111	$x=y$																−1			−1					
	111	$x=z$																		−1		−1				
	111	$y=z$																	−1				−1			
OC	100	$y=0$																								
OF	100	$y=0$																								
EF	100	$y=0$																						1		1
BF	100	$y=0$																						−1	−1	

Table 9: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2))$; Part II

		OAC								OAEF								OCBF							
		100			101			001	111	100			101			001	111	010			110			100	111
		$y=0$	$z=0$	$y=z$	$y=0$	$x=z$	$\Sigma=0$	$x=0$	$x=z$	$y=0$	$z=0$	$y=z$	$y=0$	$x=z$	$\Sigma=0$	$x=0$	$x=z$	$x=0$	$z=0$	$x=z$	$z=0$	$x=y$	$\Sigma=0$	$y=0$	$x=y$
BD	010 $x=0$																	-1		1		1	-1		
	010 $z=0$																								
	010 $x=z$																	1		-1		-1	1		
AB	001 $y=0$							1																	
	011 $y=z$								1																
	100 $y=0$	1																							
	100 $z=0$		1																						
	100 $y=z$			1																					
	101 $y=0$				1																				
	101 $\Sigma=0$					1																			
	101 $x=z$						1																		
AD	001 $x=0$															1									
	101 $x=z$																1								
	110 $x=y$									1															
	110 $z=0$										1														
	110 $\Sigma=0$											1													
	111 $x=y$												1												
	111 $x=z$													1											
	111 $y=z$														1										
OC	100 $y=0$	-1	-1					-1										1	1					1	
OF	100 $y=0$									-1	-1					-1		-1	-1					-1	
EF	100 $y=0$									1		1				1	1								
BF	100 $y=0$																		1	1				1	1

Table 10: The boundary homomorphism $C_3 \to C_2$ for $\Gamma_2\backslash(\mathcal{Z},\mathcal{Z}_s(2))$; Part I

		100			010			001			110			101			011			111		
		$y=0$	$z=0$	$y=z$	$x=0$	$z=0$	$x=z$	$x=0$	$y=0$	$x=y$	$z=0$	$x=y$	$\Sigma=0$	$y=0$	$x=z$	$\Sigma=0$	$x=0$	$y=z$	$\Sigma=0$	$x=y$	$x=z$	$y=z$
ABD	100 $y=0$											1										
	100 $z=0$										1											
	100 $y=z$												1									
	010 $x=0$											1										
	010 $z=0$										1											
	010 $x=z$												1									
	001 $x=0$									1												
	001 $y=0$									1												
	001 $x=y$							1	1	-1												
	110 $z=0$		1			1					-1											
	110 $x=y$	1			1							-1										
	110 $\Sigma=0$			1			1						-1									
	101 $y=0$																			1		
	101 $x=z$														-1	1					1	
	101 $\Sigma=0$														1	-1						1
	011 $x=0$																			1		
	011 $y=z$																	-1	1			1
	011 $\Sigma=0$																	1	-1		1	
	111 $x=y$													1			1			-1		
	111 $x=z$														1				1		-1	
	111 $y=z$															1		1				-1
BEF	100 $y=0$	-1					-1		-1												-1	
	100 $z=0$		-1			-1				-1										-1		
	100 $y=z$			-1	-1			-1														-1

Table 11: The boundary homomorphism $C_3 \to C_2$ for $\Gamma_2 \backslash (\mathcal{Z}, \mathcal{Z}_s(2))$; Part II

		100			010			001			110			101			011			111		
		$y=0$	$z=0$	$y=z$	$x=0$	$z=0$	$x=z$	$x=0$	$y=0$	$x=y$	$z=0$	$x=y$	$\Sigma=0$	$y=0$	$x=z$	$\Sigma=0$	$x=0$	$y=z$	$\Sigma=0$	$x=y$	$x=z$	$y=z$
OAC	100 $y=0$	-1			-1																	
	100 $z=0$		-1			-1																
	100 $y=z$			-1			-1															
	101 $y=0$													-1			-1					
	101 $x=z$														-1			-1				
	101 $\Sigma=0$															-1			-1			
	001 $x=0$							-1	-1													
	111 $x=z$																				-1	-1
OAEF	100 $y=0$	1			1																	
	100 $z=0$		1			1																
	100 $y=z$			1			1															
	101 $y=0$													1			1					
	101 $x=z$														1			1				
	101 $\Sigma=0$															1			1			
	001 $x=0$							1	1													
	111 $x=z$																				1	1
OCBF	010 $x=0$				-1			-1														
	010 $z=0$					-1			-1													
	010 $x=z$						-1			-1												
	110 $z=0$										-1			-1								
	110 $x=y$											-1			-1							
	110 $\Sigma=0$												-1			-1						
	100 $y=0$	-1	-1																			
	111 $x=y$																			-1	-1	

3.2 Quotients of $\mathcal{X}_{\infty,s}(i)$ by Γ_i

We will derive Theorem 1.6 from Theorem 1.8 and from the following result.

Theorem 3.1 *Let p be any prime. Then the reduced cohomology of the quotients of $\mathcal{X}_{\infty,s}(i)$ by the action of the respective groups is given as follows.*

a) $\widetilde{H}^*(\Gamma_0\backslash\mathcal{X}_{\infty,s}(0);\mathbb{F}_p) = 0$

b) $\widetilde{H}^*(\Gamma_1\backslash\mathcal{X}_{\infty,s}(1);\mathbb{F}_p) = \Sigma(\mathbb{F}_p)^2$

c) $\widetilde{H}^*(\Gamma_2\backslash\mathcal{X}_{\infty,s}(2);\mathbb{F}_p) = \Sigma(\mathbb{F}_p)^6$

Proof. We will compute the mod - p (co-)homology from the cell complexes of the homotopy equivalent spaces $\Gamma_i\backslash\mathcal{Z}_s(i)$. First of all we note that in all cases we have $\widetilde{H}_0(\Gamma_i\backslash\mathcal{Z}_s(i);\mathbb{F}_p) = 0$ (e.g. because $\mathcal{Z}$ is connected and because of Theorem 1.8).

a) Because of Euler characteristic considerations it suffices in the case of Γ_0 to show that the boundary map $\partial_2 : C_2 \longrightarrow C_1$ in the cellular complex $C_*(\Gamma_0\backslash\mathcal{Z}_s(0))$ is a monomorphism. This can be easily seen from figure 1.

b) In the case of Γ_1 the Euler characteristic argument shows that is enough to verify $\widetilde{H}_2(\Gamma_1\backslash\mathcal{Z}_s(1);\mathbb{F}_p) = 0$. For this we need to show that the boundary homomorphism ∂_2 is injective. This boundary homomorphism can be easily determined by the information provided in Section 2.6 and is described in table 12 below. Injectivity is now easily checked.

c) Finally we consider the case of Γ_2. Again by the Euler characteristic argument it suffices to show $\widetilde{H}_2(\Gamma_2\backslash\mathcal{Z}_s(2);\mathbb{F}_p) = 0$. The boundary map ∂_2 is now described in tables 13 and 14 and again injectivity is easily checked. □

3.3 Quotients of $\mathcal{X}_\infty$ by Γ_i

In order to determine $\widetilde{H}_*(\Gamma_i\backslash\mathcal{X}_\infty;\mathbb{F}_p) \cong \widetilde{H}_*(\Gamma_i\backslash\mathcal{Z}_\infty;\mathbb{F}_p)$ it remains to compute the relevant connecting homomorphismsm in the long exact sequence for the homology of the pair $\Gamma_i\backslash(\mathcal{Z},\mathcal{Z}_s(i))$.

In case $i = 0$ we obtain clearly $\widetilde{H}_*(\Gamma_0\backslash\mathcal{Z};\mathbb{F}_p) = 0$ and in the other two cases one checks easily that under $\partial_2 : C_2(\Gamma_i\backslash\mathcal{Z}) \longrightarrow C_1(\Gamma_i\backslash\mathcal{Z})$ the images of the relative cycles in (3.1) resp. (3.3) ff. are linearly independent in the quotient of $C_1(\Gamma_i\backslash\mathcal{Z}_s)$ by the image of $\partial_2 : C_2(\Gamma_i\backslash\mathcal{Z}_s(i)) \longrightarrow C_1(\Gamma_i\backslash\mathcal{Z}_s(i))$; in fact, to see this it is enough to determine the "OA" - part of the total boundary of these relative cycles and compare with tables 12 resp. 13 and 14. In other words, the corresponding connecting homomorphism in the long exact sequence is injective and then even an isomorphism because of dimension reasons. Part a), b) and c) of Theorem 1.6 follow.

Table 12: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_1 \backslash \mathcal{Z}_s(1)$

		OAC			OAEF			OCBF			BEF		
		001	110	111	001	110	111	100	011	111	110	101	011
BD	100										1		
	011								−1	1			1
	111								1	−1		1	
AC	001	1											
	110		1										
	111			1									
AE	001				1								
	110					1							
	111						1						
OC	100	−1						1					
	110		−1						1				
	111			−1						1			
OF	100				−1			−1					
	110					−1			−1				
	111						−1			−1			
EF	110					1					1		
	101											1	1
	100				1		1						
BF	011								1				−1
	110										−1	−1	
	100							1		1			
OA	001	1			1								
	110		1			1							
	111			1			1						
	100												
	101												

Table 13: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_2 \backslash \mathcal{Z}_s(2)$; Part I

		OAC					OAEF					OCBF					BEF					
		001	111	110			001	111	110			100	111	011			101		110		011	
		$x=y$	$x=y$	$x=y$	$z=0$	$\Sigma=0$	$x=y$	$x=y$	$x=y$	$z=0$	$\Sigma=0$	$y=z$	$y=z$	$y=z$	$x=0$	$\Sigma=0$	$x=z$	$\Sigma=0$	$x=y$	$\Sigma=0$	$y=z$	$\Sigma=0$
BD	100 $y=0$																		1			
	100 $y=z$																			1		
	011 $x=0$														-1	-1					1	
	011 $y=z$												1	-1								1
	111 $x=y$														1	1	1					
	111 $y=z$												-1	1				1				
AC	001 $x=y$	1																				
	111 $x=y$		1																			
	110 $x=y$			1																		
	110 $z=0$				1																	
	110 $\Sigma=0$					1																
AE	001 $x=y$						1															
	111 $x=y$							1														
	110 $x=y$								1													
	110 $z=0$									1												
	110 $\Sigma=0$										1											
OC	100 $y=z$	-1										1										
	111 $x=y$		-1										1									
	110 $x=y$			-1										1								
	110 $z=0$				-1										1							
	110 $\Sigma=0$					-1										1						
OF	100 $y=z$						-1					-1										
	111 $x=y$							-1					-1									
	110 $x=y$								-1					-1								
	110 $z=0$									-1					-1							
	110 $\Sigma=0$										-1					-1						

Table 14: The boundary homomorphism $C_2 \to C_1$ for $\Gamma_2 \backslash \mathcal{Z}_s(2)$; Part II

		OAC					OAEF					OCBF					BEF					
		001	111	110			001	111	110			100	111	011			101		110		011	
		$x=y$	$x=y$	$x=y$	$z=0$	$\Sigma=0$	$x=y$	$x=y$	$x=y$	$z=0$	$\Sigma=0$	$y=z$	$y=z$	$y=z$	$x=0$	$\Sigma=0$	$x=z$	$\Sigma=0$	$x=y$	$\Sigma=0$	$y=z$	$\Sigma=0$
EF	100 $z=0$						1	1														
	110 $x=y$								1	1									1			
	110 $\Sigma=0$										1									1		
	101 $x=z$																1				1	
	101 $\Sigma=0$																	1				1
BF	100 $y=z$											1	1									
	110 $x=y$																-1		-1			
	110 $\Sigma=0$																	-1		-1		
	011 $y=z$													1	1						-1	
	011 $\Sigma=0$															1						-1
OA	100 $y=0$																					
	100 $z=0$																					
	100 $y=z$																					
	001 $x=0$																					
	001 $x=y$	1					1															
	101 $y=0$																					
	101 $x=z$																					
	101 $\Sigma=0$																					
	110 $x=y$			1					1													
	110 $z=0$				1					1												
	110 $\Sigma=0$					1					1											
	111 $x=y$		1					1														
	111 $x=z$																					

3.4 Quotients by $SL(3,\mathbb{Z}[1/2])$

As before we abbreviate $SL(3,\mathbb{Z}[1/2])$ by Γ. In this section we are concerned with the proof of part d) of Theorem 1.6 and Theorem 1.8, i.e. with the computation of the mod - p cohomology of $\Gamma\backslash\mathcal{X}$ and the mod - 2 cohomology of the pair $(\Gamma\backslash\mathcal{X},\Gamma\backslash\mathcal{X}_s)$. For this we consider the Γ - equivariant projection map $p:\mathcal{X}\longrightarrow\mathcal{X}_2$ and the spectral sequences (which arise from the skeletal filtrations of the bases) of the following associated maps

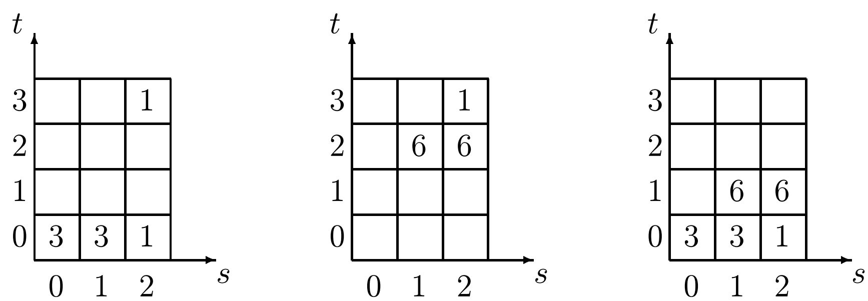
$$\overline{p}_{\mathcal{X}}:\Gamma\backslash\mathcal{X}\longrightarrow\Gamma\backslash\mathcal{X}_2\cong\Delta^2\ ,$$

$$\overline{p}_{(\mathcal{X},\mathcal{X}_s)}:\Gamma\backslash(\mathcal{X},\mathcal{X}_s)\longrightarrow\Gamma\backslash\mathcal{X}_2\cong\Delta^2\ ,$$

and also that of

$$\overline{p}_{\mathcal{X}_s}:\Gamma\backslash\mathcal{X}_s\longrightarrow\Gamma\backslash\mathcal{X}_2\cong\Delta^2\ .$$

The fibres of these maps over the i - simplices in Δ^2 are homeomorphic to the spaces $\Gamma_i\backslash\mathcal{X}_\infty$ resp. to $\Gamma_i\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(i))$ resp. to $\Gamma_i\backslash\mathcal{X}_{\infty,s}(i)$ (cf. Section 2.1). Therefore Theorem 3.1 and the already proven parts a), b) and c) of Theorem 1.6 and Theorem 1.8 immediately give the following E_1 - terms for the cohomology spectral sequences converging to $H^*(\Gamma\backslash\mathcal{X};\mathbb{F}_p)$, $H^*(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_p)$ resp. $H^*(\Gamma\backslash\mathcal{X}_s;\mathbb{F}_p)$.



$E_1^{s,t}(\Gamma\backslash\mathcal{X})$ $E_1^{s,t}(\Gamma\backslash(\mathcal{X},\mathcal{X}_s))$ $E_1^{s,t}(\Gamma\backslash\mathcal{X}_s)$

The numbers in these diagramms give the dimension of $E_1^{s,t}$ as an $\mathbb{F}_p$ - vector space. Missing numbers are to be interpreted as 0. The differential d_1 on the line $t=0$ (in the first and the third case) is as in the case of the simplicial chains on Δ^2, in particular we get in these cases $E_{0,0}^2\cong\mathbb{F}_p$ and $E_{s,0}^2=0$ if $s>0$. In particular, we immediately obtain $\widetilde{H}^*(\Gamma\backslash\mathcal{X};\mathbb{F}_p)\cong\Sigma^5\mathbb{F}_p$, i.e.

part d) of Theorem 1.6. We also see that $H^i(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_p)=0$ for $i\leq 2$, independent of the precise behaviour of the spectral sequences.

What remains to be calculated is the differential $d_1^{1,2}:E_1^{1,2}\longrightarrow E_1^{2,2}$ in the second case and the differential $d_1^{1,1}:E_1^{1,1}\longrightarrow E_1^{2,1}$ in the third case. The connecting homomorphisms of the long exact sequences associated to the pairs $\Gamma_i\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(i))$ induce (by Section 3.3) isomorphisms between $E_1^{i,1}(\Gamma\backslash\mathcal{X}_s)$ and $E_1^{i,2}(\Gamma\backslash(\mathcal{X},\mathcal{X}_s))$ for $i=1,2$, hence it suffices to do the calculation in one case. We will show that in the third case $d^1_{1,1}$ is an isomorphism if $p=2$ and this will finish the proof of Theorem 1.8.

For this consider the mod - 2 cohomology spectral sequences (arising from a skeletal filtration of the base) of the maps

$$E\Gamma\times_\Gamma\mathcal{X}_s\longrightarrow\Gamma\backslash\mathcal{X}_s$$

and

$$E\Gamma\times_\Gamma\mathcal{X}\longrightarrow\Gamma\backslash\mathcal{X}\ .$$

(For a discussion of the existence of a cellular structure on these bases we refer to the remark at the end of this section.) The E_1 - terms of both spectral sequences agree except on the line $t=0$ because the mod - 2 cohomology of a fibre outside of $\Gamma\backslash\mathcal{X}_s$ vanishes in positive dimensions. Consequently the E_2 - terms of both spectral sequences also agree except on the line $t=0$ and there we get $E_2^{s,0}\cong H^s(\Gamma\backslash\mathcal{X}_s;\mathbb{F}_2)$ resp. $E_2^{s,0}\cong H^s(\Gamma\backslash\mathcal{X};\mathbb{F}_2)$.

Claim 1: $E_2^{0,1}=0$ in both spectral sequences.

Proof. We consider the spectral sequence converging to $H^*(E\Gamma\times_\Gamma\mathcal{X};\mathbb{F}_2)$. As we have seen above the groups $E_2^{i,0}\cong H^i(\Gamma\backslash\mathcal{X},\mathbb{F}_2)$ are trivial for $i=1,2$. Therefore we have $E_2^{0,1}=0$ if and only if $H^1(E\Gamma\times_\Gamma\mathcal{X};\mathbb{F}_2)=0$. From Theorem 1.5 we know that $H^1(E\Gamma\times_\Gamma\mathcal{X}_s;\mathbb{F}_2)=0$ and from the discussion above we know that $H^1(E\Gamma\times_\Gamma(\mathcal{X},\mathcal{X}_s);\mathbb{F}_2)\cong H^1(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_2)=0$. Then the long exact sequence of the pair $E\Gamma\times_\Gamma(\mathcal{X},\mathcal{X}_s)$ shows $H^1(E\Gamma\times_\Gamma\mathcal{X};\mathbb{F}_2)=0$ and we are done. □

Now consider the class $v_2\in H^2(E\Gamma\times_\Gamma\mathcal{X}_s;\mathbb{F}_2)$ which is pulled back from the second universal Stiefel Whitney class in $H^*(BSL(3,\mathbb{R});\mathbb{F}_2)$ under the induced map of the composition $E\Gamma\times_\Gamma\mathcal{X}_s\xrightarrow{\pi}B\Gamma\xrightarrow{i}BSL(3,\mathbb{R})$ (where π is given by sending $\mathcal{X}_s$ to a point and i by the canonical inclusion $\Gamma\hookrightarrow SL(3,\mathbb{R})$).

Claim 2: In the spectral sequence converging to $H^*(E\Gamma\times_\Gamma\mathcal{X}_s;\mathbb{F}_2)$ the class v_2 is detected on $E_1^{0,2}$.

Proof. We have $E_1^{0,2} \cong \bigoplus_e H^2(\Gamma_e;\mathbb{F}_2)$ where e runs through a set of Γ - orbits of 0 - dimensional cells in $\mathcal{X}_s$ and Γ_e denotes the isotropy group of the cell e. We may write $e = (e_2, e_\infty)$; if the $\mathcal{X}_2$ - component e_2 is given by the vertex l_0 defined by the standard $\mathbb{Z}_2$ - lattice in $\mathbb{Q}_2$ then we have $\Gamma_e = \Gamma_{e_\infty}$, the isotropy group of the cell $e_\infty \subset \mathcal{X}_{\infty,s}$ with respect to the action of $SL(3,\mathbb{Z})$. For any such cell the class v_2 restricts to the Stiefel-Whitney class of the representation of Γ_e arising from the embedding $\Gamma_e \hookrightarrow SL(3,\mathbb{Z}[1/2]) \hookrightarrow SL(3,\mathbb{R})$. Now Γ_e contains at least one element of order 2 and all such elements are conjugate in $SL(3,\mathbb{R})$. It follows that v_2 restricts non-trivially to any subgroup of order 2 in Γ_e and hence the claim is proved. □

Now assume that the differential $d_1^{1,1}(\Gamma\backslash\mathcal{X}_s)$ is not an isomorphism. Then $H^2(\Gamma\backslash\mathcal{X}_s;\mathbb{F}_2) \neq 0$ and in the spectral sequence converging to $H^*(E\Gamma \times_\Gamma \mathcal{X}_s;\mathbb{F}_2)$ we have $E_2^{2,0} \cong H^2(\Gamma\backslash\mathcal{X}_s;\mathbb{F}_2) \neq 0$. Because of Claim 1 we conclude that all of $E_2^{2,0}$ survives to E_∞, and because of Claim 2 we see that the assumption implies that $H^2(E\Gamma \times_\Gamma \mathcal{X}_s;\mathbb{F}_2)$ is a vector space of dimension bigger than 1 in contradiction to Theorem 1.5. This finishes the proof of part d) of Theorem 1.8. □

Remark. a) Our approach to the computation of $H^*(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_2)$ is rather indirect and one might wonder why we did not analyze the differential

$$E_1^{1,2} \cong \bigoplus_{i=1}^{3} H^2(\Gamma_1\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(1));\mathbb{F}_2) \longrightarrow H^2(\Gamma_2\backslash(\mathcal{X}_\infty,\mathcal{X}_{\infty,s}(2));\mathbb{F}_2) \cong E_1^{2,2}$$

directly? The reason is that the three summands in the source (corresponding to the three Γ - orbits of 1 - dimensional cells in $\mathcal{X}_2$ resp. the three edges in Δ^2) are mapped differently under this differential; only on one summand is the map induced by the inclusion $\Gamma_2 \subset \Gamma_1$, on the other two summands it is induced by the inclusion of Γ_2 into the isotropy groups of the edges $\{l_0, l_2\}$ resp. $\{l_1, l_2\}$ where as in Section 2.1 l_0, l_1, l_2 are the classes of the $\mathbb{Z}_2$ - lattices $L_0 = \langle e_1, e_2, e_3\rangle$, $L_1 = \langle \frac{1}{2}e_1, e_2, e_3\rangle$ and $L_2 = \langle \frac{1}{2}e_1, \frac{1}{2}e_2, e_3\rangle$ respectively. The component of the differential corresponding to the inclusion $\Gamma_2 \subset \Gamma_1$ (corresponding to the edge $\{l_0, l_1\}$) is straightforward to determine: with respect to our cell structures on the spaces $\Gamma_i\backslash\mathcal{Z}$ it is induced by a cellular map which is determined by the forgetful map $S_2 \longrightarrow S_1$. The component corresponding to the edge $\{l_0, l_2\}$ can also be worked out on the level of the spaces $\Gamma_i\backslash\mathcal{Z}$. In contrast the isotropy group $H_{\{l_1,l_2\}}$ of the edge $\{l_1, l_2\}$ is not contained in Γ_0, hence the deformation retraction $\mathcal{X}_\infty \longrightarrow \mathcal{Z}$ is not $H_{\{l_1,l_2\}}$ - equivariant and this makes this component of the differential much harder to evaluate. In terms of integral lattices in $\mathbb{R}^3$ and the spaces $\mathcal{W}_i/SO(3)$ this last component is induced by the map which associates to the triple (L_0, L_1, L_2) of lattices (with L_0 being well-rounded and $m(L_0) = 1$) the pair (L_1, L_2). Because L_1 need not be well-rounded one has to work out the effect of the deformation retraction $\mathcal{L}_1/SO(3) \longrightarrow \mathcal{W}_1/SO(3)$ of Section 2.2 explicitly. Although one would not expect that this could cause unsurmountable problems it would be at the very least very laborious and the author found his initial attempts to carry this out very frustrating.

b) We have tacitly used above that $\mathcal{X}$ is a Γ - equivariant CW - complex and we will use it again, namely in the final step of the proof of 1.4 which combines Proposition 1.9 and Theorem 1.5. It is quite likely that there is such a stucture but we do not know of a suitable reference. However, it is easy to show that $\mathcal{X}$ has the equivariant homotopy type of a Γ - CW complex, and this will be enough: for example, we can do induction on the skeleta $\mathcal{X}_2^k$ of the evident Γ - equivariant cell structure of the simplicial complex $\mathcal{X}_2$ using that the preimages of the space $\mathcal{X}_2^k$ and $\mathcal{X}_2^k - \mathcal{X}_2^{k-1}$ with respect to the projection map $\mathcal{X} \longrightarrow \mathcal{X}_2$ are understood by Section 2.4.

4 The cohomology of $SL(3,\mathbb{Z}[1/2])$

4.1 Mod - 2 cohomology

In this section we will complete the proof of Theorem 1.4. Because of Theorem 1.5 and Theorem 1.8 it is enough to prove Proposition 1.9, i.e. that the connecting homomorphism

$$H^4_\Gamma(\mathcal{X}_s;\mathbb{F}_2) \longrightarrow H^5_\Gamma(\mathcal{X},\mathcal{X}_s;\mathbb{F}_2) \cong \mathbb{F}_2$$

in the long exact sequence of the pair $E\Gamma \times_\Gamma (\mathcal{X},\mathcal{X}_s)$ is an epimorphism, or equivalently that the natural map

$$H^5_\Gamma(\mathcal{X},\mathcal{X}_s;\mathbb{F}_2) \longrightarrow H^5_\Gamma(\mathcal{X};\mathbb{F}_2)$$

is trivial.

For this consider the following commutative diagramm in which the horizontal maps are induced by inclusions and the vertical maps by projections:

$$\begin{array}{ccc} H^5_\Gamma(\mathcal{X},\mathcal{X}_s;\mathbb{F}_2) & \longrightarrow & H^5_\Gamma(\mathcal{X};\mathbb{F}_2) \\ {\scriptstyle\cong}\Big\uparrow & & {\scriptstyle q^*}\Big\uparrow \\ H^5(\Gamma\backslash(\mathcal{X},\mathcal{X}_s);\mathbb{F}_2) & \overset{\cong}{\longrightarrow} & H^5(\Gamma\backslash\mathcal{X};\mathbb{F}_2) \end{array}$$

The indicated arrows are isomorphisms because of Section 3.4 resp. because the isotropy groups in $\mathcal{X}-\mathcal{X}_s$ are of odd order. Therefore we have to show that the map $q^*: H^5(\Gamma\backslash\mathcal{X};\mathbb{F}_2) \longrightarrow H^5_\Gamma(\mathcal{X};\mathbb{F}_2)$ is trivial.

This will be a consequence of the following two results.

Lemma 4.1 *If the map* ${q_2}^* : H^3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_2) \longrightarrow H^3_{\Gamma_2}(\mathcal{X}_\infty;\mathbb{F}_2)$ *(induced by projection) is trivial then so is* $q^*: H^5(\Gamma\backslash\mathcal{X};\mathbb{F}_2) \longrightarrow H^5_\Gamma(\mathcal{X};\mathbb{F}_2)$.

Lemma 4.2 *The map* ${q_2}^* : H^3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_2) \longrightarrow H^3_{\Gamma_2}(\mathcal{X}_\infty;\mathbb{F}_2)$ *is trivial.*

Proof of Lemma 4.1 This follows immediately from naturality applied to the following situation. If $\mathcal{X}^1$ denotes the preimage of the 1-skeleton $\partial\Delta^2$ of Δ^2 under the projection map $\mathcal{X} \longrightarrow \mathcal{X}_2 \longrightarrow \Gamma\backslash\mathcal{X}_2 \cong \Delta^2$ then consider the following commutative diagram in which the vertical maps are induced by projections and the horizontal maps by inclusions:

$$\begin{array}{ccc} H^5_\Gamma(\mathcal{X},\mathcal{X}^1;\mathbb{F}_2) & \longrightarrow & H^5_\Gamma(\mathcal{X};\mathbb{F}_2) \\ \Big\uparrow & & \Big\uparrow \\ H^5(\Gamma\backslash(\mathcal{X},\mathcal{X}^1);\mathbb{F}_2) & \longrightarrow & H^5(\Gamma\backslash\mathcal{X};\mathbb{F}_2)\ . \end{array}$$

It is clear from Section 3.4 that the horizontal arrow on the bottom line of the diagram is an isomorphism. By excision we see

$$H^5(\Gamma\backslash(\mathcal{X},\mathcal{X}^1);\mathbb{F}_2)\cong H^5((\Delta^2,\partial\Delta^2)\times(\Gamma_2\backslash\mathcal{X}_\infty);\mathbb{F}_2)\cong\Sigma^2H^3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_2)$$

and

$$H^5_\Gamma(\mathcal{X},\mathcal{X}^1;\mathbb{F}_2)\cong H^5_{\Gamma_2}((\Delta^2,\partial\Delta^2)\times\mathcal{X}_\infty;\mathbb{F}_2)\cong\Sigma^2H^3_{\Gamma_2}(\mathcal{X}_\infty;\mathbb{F}_2)$$

and the claim follows. □

Proof of Lemma 4.2 This is more involved. We prefer to work in homology and there we have to show that the map $q_{2*}:H_3^{\Gamma_2}(\mathcal{X}_\infty;\mathbb{F}_2)\longrightarrow H_3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_2)$ is trivial, or equivalently that the non-trivial element (cf. Theorem 1.6) of $H_3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_2)$ is not a permanent cycle in the homology spectral sequence of the projection map. For this consider the short exact sequence of $\mathbb{F}_2\Gamma_2$ - modules

$$0\longrightarrow C_3\xrightarrow{\partial_3}C_2\xrightarrow{\partial_2}I\longrightarrow 0 \tag{4.1}$$

in which C_i denotes the i-th cellular chain group (with coefficients in $\mathbb{F}_2$) of the contractible Γ_2 - space $\mathcal{Z}$ and I is the image of the boundary map $\partial_2:C_2\longrightarrow C_1$. Note that C_i can be identified with $\bigoplus_\tau\mathbb{F}_2[\Gamma_2/\Gamma_\tau]$ where τ runs through a set of representatives of Γ_2 - orbits of i - cells in $\mathcal{Z}$ and Γ_τ is the isotropy subgroup of the chosen representative τ.

The E^1 - term of the spectral sequence of the projection map is given as $E^1_{s,t}\cong H_t(\Gamma_2;C_s)$ and the differential $d^2_{3,0}:E^2_{3,0}\longrightarrow E^2_{1,1}$ can be described as follows: The group $E^2_{3,0}$ is given as

$$E^2_{3,0}\cong H_3(\Gamma_2\backslash\mathcal{Z})\cong\text{Ker }(H_0(\Gamma_2;C_3)\longrightarrow H_0(\Gamma_2;C_2))$$

while $E^2_{1,1}$ is given as quotient

$$E^2_{1,1}\cong\frac{\text{Ker }(H_1(\Gamma_2;C_1)\longrightarrow H_1(\Gamma_2;C_0))}{\text{Im }(H_1(\Gamma_2;C_2)\longrightarrow H_1(\Gamma_2;C_1))}\ .$$

All maps are, of course, induced by the differentials in the chain complex C_*. If z is an element in $E^2_{3,0}\subset H_0(\Gamma_2,C_3)$ then $z=\partial y$ for some $y\in H_1(\Gamma_2;I)$ (with ∂ denoting the connecting homomorphism associated to the exact sequence (4.1)) and $d^2_{3,0}z$ is represented by $i_*y\in H_1(\Gamma_2;C_1)$ (with i denoting the inclusion of I into C_1. In particular we see that the following two conditions are equivalent:

1. $d^2_{3,0} : E^2_{3,0} \cong \mathbb{F}_2 \longrightarrow E^2_{1,1}$ is non-trivial.

2. Im $(H_1(\Gamma_2; I) \xrightarrow{i_*} H_1(\Gamma_2; C_1))$ is strictly larger than Im $(H_1(\Gamma_2; C_2) \xrightarrow{d^1_{2,1}} H_1(\Gamma_2; C_1))$.

We will verify the second condition and this will complete the proof of Lemma 4.2. In fact, it suffices to verify the second condition after projecting off to a suitable summand in $H_1(\Gamma_2; C_1) \cong \bigoplus_\tau H_1(\Gamma_2; \mathbb{F}_2[\Gamma_2/\Gamma_\tau])$ where as above τ runs through a set of representatives of Γ_2 - orbits of 1 - cells in $\mathcal{Z}$. We choose the 1 - dimensional cells τ_1 resp. τ_2 given by $1 \cdot (AC, 001x = y) \subset \Gamma_2 \times \mathcal{D}_2/ \sim_2$ and $1 \cdot (OA, 001x = y) \subset \Gamma_2 \times \mathcal{D}_2/ \sim_2$ (where $1 \in \Gamma$ and our conventions for labelling the cells in $\mathcal{Z} \cong \Gamma_2 \times \mathcal{D}_2/ \sim_2$ are those of Section 2.4.2). We will denote the corresponding projections by π_1 and π_2 respectively. Lemma 4.2 will then follow from the following two results. □

Lemma 4.3 *If $y \in H_1(\Gamma_2; C_2)$ is mapped non-trivially by the map $H_1(\Gamma_2; C_2) \xrightarrow{d^1_{2,1}} H_1(\Gamma_2; C_1) \xrightarrow{\pi_1} H_1(\Gamma_2; \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_1}])$ then y is also mapped non-trivially by $H_1(\Gamma_2; C_2) \xrightarrow{d^1_{2,1}} H_1(\Gamma_2; C_1) \xrightarrow{\pi_2} H_1(\Gamma_2; \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_2}])$.*

Lemma 4.4 *There is an element $u \in H_1(\Gamma_2; I)$ which maps non-trivially by the map $H_1(\Gamma_2; I) \xrightarrow{i_*} H_1(\Gamma_2; C_1) \xrightarrow{\pi_1} H_1(\Gamma_2; \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_1}])$ and trivially by $H_1(\Gamma_2; I) \xrightarrow{i_*} H_1(\Gamma_2; C_1) \xrightarrow{\pi_2} H_1(\Gamma_2; \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_2}])$.*

Proof of Lemma 4.3. The differential $d^1_{s,*}$ can be described as follows (cf. chapter VII.8 of [B1]): via the identifications of C_s with $\bigoplus_\sigma \mathbb{F}_2[\Gamma_2/\Gamma_\sigma]$ and of C_{s-1} with $\bigoplus_\tau \mathbb{F}_2[\Gamma_2/\Gamma_\tau]$ the $\sigma\tau$ component of $d^1_{s,*}$ is induced by the corresponding component $\mathbb{F}_2[\Gamma_2/\Gamma_\sigma] \xrightarrow{\partial_{\sigma,\tau}} \mathbb{F}_2[\Gamma_2/\Gamma_\tau]$ of the boundary map $C_s \longrightarrow C_{s-1}$. By equivariance this component is determined by the image of the coset $1 \cdot \Gamma_\sigma \in \mathbb{F}_2[\Gamma_2/\Gamma_\sigma]$, i.e. by understanding the incidence numbers $[\sigma : g\tau]$ between the cell σ and all cells in the Γ_2 - orbit of τ. We obtain $\partial_{\sigma,\tau}(1 \cdot \Gamma_\sigma) = \sum_{g\Gamma_\tau} [\sigma : g\tau] g\Gamma_\tau$ where the sum is over the Γ_2 - orbit of τ.

Because we project off via π_1 and π_2 and we are interested in homology in degree 1 only it suffices to consider "singular" 2 - dimensional cells $\sigma \subset \mathcal{Z}_s(2)$ for which $[\sigma : g\tau_i]$ is non-trivial for some cell in the orbit of τ_i; in particular all cells of the form $g \cdot (ACD, s)$, $g \cdot (ABD, s)$ and $g \cdot (ABE, s)$ in $\Gamma_2 \backslash \mathcal{Z} \cong \Gamma_2 \times \mathcal{D}_2/ \sim_2$ are "non-singular" and can be ignored. By going through the discussion of the relevant 2 - cells in Section 2.6 and using Theorem 2.7 we see that we only get contributions to $\partial_{\sigma,\tau_i}(1 \cdot \Gamma_\sigma)$ for $\sigma = \sigma_1 := 1 \cdot (OAC001, x = y)$ in the case of τ_1, and $\sigma = \sigma_1$ or $\sigma = \sigma_2 := 1 \cdot (OAEF, 001x = y)$ in the case of τ_2. Furthermore σ_1 and σ_2 are the only cells in their Γ_2 - orbit for which the incidence numbers are non-trivial, namely equal to 1.

Therefore it suffices to consider the following situation in which we identify $H_1(\Gamma_2;\mathbb{F}_2[\Gamma_2/\Gamma_\sigma])$ with $H_1(\Gamma_\sigma;\mathbb{F}_2)$ for $\sigma\in\{\sigma_1,\sigma_2,\tau_1,\tau_2\}$ and drop the coefficients from the notation; the maps i_1 resp. i_2 denote the inclusions of Γ_{σ_1} resp. Γ_{σ_2} into Γ_{τ_2} (cf. table 1 and table 2 for the isotropy groups and their inclusions).

$$\begin{array}{ccc} H_1(\Gamma_{\sigma_1})\oplus H_1(\Gamma_{\sigma_2})\cong H_1(\mathbb{Z}/2)\oplus H_1(\mathbb{Z}/2) & & (a,b) \\ \downarrow & & \downarrow \\ H_1(\Gamma_{\tau_1})\oplus H_1(\Gamma_{\tau_2})\cong H_1(\mathbb{Z}/2)\oplus H_1(\mathbb{Z}/2\oplus\mathbb{Z}/2) & & (a,i_{1*}a+i_{2*}b)\ . \end{array}$$

The proof of the Lemma is now reduced to showing that $a\neq 0$ implies $i_{1*}a+i_{2*}b\neq 0$, and this is clear. □

Proof of Lemma 4.4. Of course, the connecting homomorphism associated to the exact sequence (4.1) has to send the element u in question to the element in $H_3(\Gamma_2\backslash\mathcal{Z};\mathbb{F}_2)\subset H_0(\Gamma_2;C_3)$ given by the cycle of (3.2):

$$[100y=0]+[100z=0]+[010x=0]+[010z=0]+[001x=0]+[001y=0]\ .$$

Consider now the following chain in C_3 whose class in $H_0(\Gamma_2;C_3)$ agrees with this cycle:

$$\begin{aligned} z: &= [1\cdot(100y=0)]+[1\cdot(100z=0)]+[1\cdot(010x=0)]+ \\ &\quad +[1\cdot(010z=0)]+[1\cdot(001x=0)]+[1\cdot(001y=0)]\ . \end{aligned}$$

Let σ denote the 2 - dimensional cell $1\cdot(ABD,001x=0)$ in $\Gamma_2\times\mathcal{D}_2/\sim_2\cong\mathcal{Z}$. This cell generates a free $\mathbb{F}_2[\Gamma_2]$ - submodule that we denote by $\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle$. Let $\pi_3: C_2\longrightarrow\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle$ denote the projection map. Then Section 2.4.2 and inspection of table 5 in Section 2.6 yield $\pi_3\partial_3 z=\lambda\sigma$ where $\lambda=(1+g_sg^{AB}{g_s}^{-1})\in\mathbb{F}_2[\Gamma_2]$, $s\in S_2\cong\Gamma_2\backslash\Gamma_0$ is the element $001x=0$, $g_s\in\Gamma_0$ is a fixed chosen coset representative of s and $g^{AB}\in\Gamma_0$ is as in Section 2.4. Note that because of $sg^{AB}=s$ we have $g_sg^{AB}{g_s}^{-1}\in\Gamma_2$, and in fact, by Section 2.4.2, the element $g_sg^{AB}{g_s}^{-1}$ is the unique non-trivial element in the isotropy group of the cell $1\cdot(AB,001x=0)$. In $\mathcal{Z}$ the cell $g\cdot(AB,001x=0)$ gets identified with the cell $1\cdot(AC,001x=y)=\tau_1$ if g is determined by the equation $gg_s=g_{s'}g^{AD}$, $s'\in S_2$ is the element $001x=y$, $g_{s'}$ is a fixed chosen coset representative of s' and g^{AD} is again as in Section 2.4.2. It follows that the assignment $\sigma\mapsto g^{-1}\tau_1$ induces an isomorphism $\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle/\lambda\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle\cong\mathbb{F}_2[\Gamma_2/\Gamma_{g^{-1}\tau_1}]$ of $\mathbb{F}_2[\Gamma_2]$ - modules.

Now let $\mathbb{F}_2[\Gamma_2]\langle z\rangle$ be the $\mathbb{F}_2[\Gamma_2]$ - submodule of C_3 generated by the cycle z; obviously this is a free $\mathbb{F}_2[\Gamma_2]$ - module. Let $C_2^{\partial z}$ be the $\mathbb{F}_2[\Gamma_2]$ - submodule of C_2 which is generated by all cells appearing in $\partial_3 z$ if this is written as linear combination of cells; observe that $C_2^{\partial z}$ is a direct summand of C_2 as a $\mathbb{F}_2[\Gamma_2]$ - module. Next let $I^{\partial z}$ be the quotient of $C_2^{\partial z}$ by the $\mathbb{F}_2[\Gamma_2]$ - submodule generated by $\partial_3 z$. Then we get the following diagramm of exact sequences of $\mathbb{F}_2[\Gamma_2]$ - modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_3 & \xrightarrow{\partial_3} & C_2 & \longrightarrow & I & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow i & & \uparrow j & & \\
0 & \longrightarrow & \mathbb{F}_2[\Gamma_2]\langle z\rangle & \longrightarrow & C_2^{\partial z} & \longrightarrow & I^{\partial z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \pi_3 & & \downarrow \widetilde{\pi}_3 & & \\
0 & \longrightarrow & \lambda\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle & \longrightarrow & \mathbb{F}_2[\Gamma_2]\langle\sigma\rangle & \longrightarrow & \mathbb{F}_2[\Gamma_2/\Gamma_{g^{-1}\tau_1}] & \longrightarrow & 0
\end{array}$$

where the left hand vertical arrow in the upper half of the diagram is an inclusion, i is the inclusion of a direct summand, the upper left hand rectangle commutes and induces j. The lower left hand vertical arrow is given by $z \mapsto \lambda\sigma$, hence the lower left hand rectangle commutes by the formula for $\pi_3\partial_3(z)$ and induces the map $\widetilde{\pi}_3$. By going through table 5 in Section 2.6 one checks that $\widetilde{\pi}_3$ agrees with the composition $I^{\partial z} \xrightarrow{j} I \xrightarrow{i} C_1 \xrightarrow{\widetilde{\pi}_1} \mathbb{F}_2[\Gamma_2/\Gamma_{g^{-1}\tau_1}]$ where $\widetilde{\pi}_1$ denotes the composition of π_1 with left multiplication by g^{-1}.

Now the upper half of the diagramm shows that there is an element $u' \in H_1(\Gamma_2; I^{\partial z})$ such that $\partial j_* u' = z$. (Use that $z \in H_0(\Gamma_2; C_3)$ comes from an element, still denoted by z, in $H_0(\Gamma_2; \mathbb{F}_2[\Gamma_2]\langle z\rangle)$ whose image in $H_0(\Gamma_2; C_2^{\partial z})$ vanishes because i is split injective.) Pick any such u' and let $u := j_* u'$. Then the lower half of the diagramm shows that the connecting homomorphism maps $\widetilde{\pi}_{3*}(u')$ to the non-trivial element in $H_0(\Gamma_2; \lambda\mathbb{F}_2[\Gamma_2]\langle\sigma\rangle) \cong \mathbb{F}_2$, in particular $\widetilde{\pi}_{3*}(u') \neq 0$ and hence $\pi_{1*} i_* u \neq 0$.

Finally, using Section 2.6 once more, it is straightforward to check that $\pi_2\partial_2 : C_2 \longrightarrow \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_2}]$ vanishes on $C_2^{\partial z}$ and hence $\pi_2 ij : I^{\partial z} \longrightarrow \mathbb{F}_2[\Gamma_2/\Gamma_{\tau_2}]$ is the zero map and the second statement of the Lemma follows. □

4.2 Mod - 3 cohomology

In this section we prove Theorem 1.10. We take advantage of our investigations in Sections 2.4, 2.5 and 2.6. In particular we will make use of the description of the Γ_j - space $\mathcal{Z}$ given by Theorem 2.6 resp. Theorem 2.7, i.e.

we will identify $\mathcal{Z}$ with $\Gamma_j \times \mathcal{D}_j / \sim_j$ and write A, B, ... for the points of this space given by the class of $(1, A)$, $(1, B)$, ...

We break the proof into two parts.

Proof of Theorem 1.10 (a)-(c). Let $\mathcal{Z}_{s,3}(j)$ denote the 3 - singular locus of $\mathcal{Z}$ with respect to the action of Γ_j.

Part c) of the Theorem follows immediately from Section 2.6 because in this case $\mathcal{Z}_{s,3}(2) = \emptyset$, and therefore we get $\widetilde{H}^*(\Gamma_2; \mathbb{F}_3) \cong \widetilde{H}^*_{\Gamma_2}(\mathcal{Z}; \mathbb{F}_3) \cong \widetilde{H}^*(\Gamma_2 \backslash \mathcal{Z}; \mathbb{F}_3) \cong \Sigma^3 \mathbb{F}_3$ by Theorem 1.6.

In the cases of Γ_0 and Γ_1 we derive from Section 2.6 that the space $\Gamma_j \backslash \mathcal{Z}_{s,3}(j)$ consists of two components. In the case of Γ_0 one of the components consists of the image of the Γ_0 - orbit of the 0 - cell A in $\mathcal{Z}_{s,3}(0)$ (with isotropy group isomorphic to D_{12}). The other one consists of the image of the Γ_0 - orbit of the subcomplex with the two 1 - simplices OC and OF in $\mathcal{Z}_{s,3}(0)$; the Γ_0 - orbits of the 1 - dimensional cells have isotropy isomorphic to $\mathfrak{S}_3$ and the Γ_0 - orbits of the three 0 - dimensional cells have isotropy group isomorphic to $\mathfrak{S}_4$. In the case of Γ_1 one component in $\Gamma_1 \backslash \mathcal{Z}_{s,3}(1)$ comes from the 0 - cell $A001$ (where we use the convention of Section 2.6 for labelling the cells), again with isotropy group isomorphic to D_{12}; the other one comes from the subcomplex with 1 - cells $OC111$ and $OF111$, with isotropy groups isomorphic to $\mathfrak{S}_3$ for the 1 - dimensional cells and the 0 - dimensional cell $F111$, and isomorphic to $\mathfrak{S}_4$ for the 0 - dimensional cells $O111$ and $C111$.

Now we consider the spectral sequences associated to the maps

$$E\Gamma_j \times_{\Gamma_j} \mathcal{Z}_s(j) \longrightarrow \Gamma_j \backslash \mathcal{Z}_{s,3}(j), \quad j = 0, 1 \ .$$

Because the inclusions of $\mathfrak{S}_3$ into $\mathfrak{S}_4$ and of $\mathfrak{S}_3$ into D_{12} induce isomorphisms in mod - 3 cohomology we find in both cases an isomorphism

$$H^*_{\Gamma_j}(\mathcal{Z}_{s,3}(j); \mathbb{F}_3) \cong \prod_{i=1}^{2} H^*(\mathfrak{S}_3; \mathbb{F}_3) \ .$$

Furthermore it is clear that $H^*(\Gamma_j \backslash \mathcal{Z}_s(j); \mathbb{F}_3) = \mathbb{F}_3 \oplus \mathbb{F}_3$; from Theorem 1.6 we know $\widetilde{H}^*(\Gamma_j \backslash \mathcal{Z}; \mathbb{F}_3) = 0$ and therefore we conclude $H^*(\Gamma_j \backslash (\mathcal{Z}, \mathcal{Z}_{s,3}(j)); \mathbb{F}_3) = \Sigma \mathbb{F}_3$. Finally the long exact sequence for the Borel cohomology of the pair $(\mathcal{Z}, \mathcal{Z}_{s,3}(j))$ gives part a) and b). □

To prove part (d) of Theorem 1.10 one could now consider the spectral sequence of the map $E\Gamma \times_\Gamma \mathcal{X} \longrightarrow \Delta^2$ and we will actually make some use of this spectral sequence. However, both for the final description of the result as well as for the proofs centralizers of elementary abelian 3 - subgroups turn out to be helpful again. In fact, we will combine information derived from the knowledge of these centralizers with information coming from the analysis of the spectral sequence of the map $E\Gamma\times_{\Gamma}\mathcal{X} \longrightarrow \Delta^2$. We start by analyzing the elementary abelian 3 - subgroups of $GL(3,\mathbb{Z}[1/2])$. First it is clear that there are no elementary abelian 3 - subgroups of rank 2 (isomorphic to $\mathbb{Z}/3\times Z/3$) because there are none in $GL(3,\mathbb{R})$.

Proposition 4.5 *In $GL(3,\mathbb{Z}[1/2])$ there are precisely two conjugacy classes of subgroups isomorphic to $\mathbb{Z}/3$.*

Proof. These conjugacy classes are in one to one correspondence with the isomorphism classes of modules M over the group algebra $\mathbb{Z}[1/2][\mathbb{Z}/3]$ which are free of rank 3 as $\mathbb{Z}[1/2]$ - modules and on which $\mathbb{Z}/3$ acts faithfully. Such modules are classified by the (obvious modification for the ring $\mathbb{Z}[1/2]$ of the) Diederichsen-Reiner Theorem (cf. Theorem (74.3) of [CR]); one of the two classes corresponds to the free $\mathbb{Z}[1/2][\mathbb{Z}/3]$ - module F on one generator, the other one to $T\oplus R$, the direct sum of the trivial module T and the module $R:=\mathbb{Z}[1/2][\zeta_3]$ where a fixed chosen generator of $\mathbb{Z}/3$ acts by multiplication with ζ_3, a fixed chosen third root of unity. □

We pick a subgroup E_1 corresponding to the module F and a subgroup E_2 corresponding to $T\oplus R$. If E is a subgroup of a group G we write $C_G(E)$ for the centralizer of E in G and $N_G(E)$ for the normalizer of E in G. The units in a ring R will be denoted by $R^\times$. We will now analyze the centralizers and normalizers of E_i. We start with the case of E_2.

Proposition 4.6 *The centralizer $C_{GL(3,\mathbb{Z}[1/2])}(E_2)$ is isomorphic to $\mathbb{Z}/3\times\mathbb{Z}[1/2]^\times\times\mathbb{Z}[1/2]^\times$.*

Proof. The centralizer is isomorphic to the group of automorphisms of the corresponding $\mathbb{Z}[1/2][\mathbb{Z}/3]$ - module, i.e. to the group of units in its endomorphism ring. After tensoring with $\mathbb{Q}$ both F and $R\oplus T$ become isomorphic; both $R\otimes\mathbb{Q}$ and $T\otimes\mathbb{Q}$ are irreducible, in particular there are no $\mathbb{Z}[1/2][\mathbb{Z}/3]$ - module maps between R and T.

Therefore we obtain $C_{GL(3,\mathbb{Z}[1/2])}(E_2)\cong\mathbb{Z}[1/2]^\times\times(\mathbb{Z}[1/2][\zeta_3])^\times$ and it is easy to check (say by using the norm map from the cyclotomic extension $\mathbb{Q}[\zeta_3]$ down to $\mathbb{Q}$) that the map

$$\mathbb{Z}/3\times\mathbb{Z}[1/2]^\times\to(\mathbb{Z}[1/2][\zeta_3])^\times$$

$$(a,b)\mapsto\zeta_3^a b$$

is an isomorphism. □

Remark. The norm can also be used to show that $\mathbb{Z}[\zeta_3]$ is a Euclidean ring and therefore a principal ideal domain. This simplifies the proof and statement of the Diederichsen-Reiner Theorem for modules over $\mathbb{Z}[\mathbb{Z}/3]$ and $\mathbb{Z}[1/2][\mathbb{Z}/3]$.

Corollary 4.7 *E_2 is contained in $SL(3,\mathbb{Z}[1/2])$ and there is a unique conjugacy class of elementary abelian 3 - subgroups in $SL(3,\mathbb{Z}[1/2])$ which maps to the $GL(3,\mathbb{Z}[1/2])$ - conjugacy class of E_2. Furthermore*

$$C_{SL(3,\mathbb{Z}[1/2])}(E_2) \cong \mathbb{Z}/3 \times \mathbb{Z}[1/2]^\times ,$$

$$N_{SL(3,\mathbb{Z}[1/2])}(E_2) \cong C_{SL(3,\mathbb{Z}[1/2])}(E_2) \rtimes \mathbb{Z}/2$$

and the isomorphism can be chosen such that the conjugation action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ is non-trivial while it is trivial on $\mathbb{Z}[1/2]^\times$.

Proof. It is clear that E_2 is contained in $SL(3,\mathbb{Z}[1/2])$ and also that $C_{SL(3,\mathbb{Z}[1/2])}(E_2)$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}[1/2]^\times$. Furthermore, if σ denotes the Galois automorphism of $\mathbb{Z}[1/2][\zeta_3]$ then $\sigma \oplus (-id)$ normalizes E_2 and this shows $N_{SL(3,\mathbb{Z}[1/2])}(E_2) \cong C_{SL(3,\mathbb{Z}[1/2]}(E_2) \rtimes \mathbb{Z}/2$ with the conjugation action as claimed.

Now assume E' is another subgroup of $SL(3,\mathbb{Z}[1/2])$ which becomes conjugate in $GL(3,\mathbb{Z}[1/2])$ to E_2, say be an element g. Now the determinant from $GL(3,\mathbb{Z}[1/2])$ to $\mathbb{Z}[1/2]^\times$ remains onto when restricted to $C_{GL(3,\mathbb{Z}[1/2])}(E_2)$, hence we can write $g = g_1 g_2$ with $g_1 \in C_{GL(3,\mathbb{Z}[1/2])}(E_2)$ and $g_2 \in SL(3,\mathbb{Z}[1/2])$ and this implies that E and E' are already conjugate in $SL(3,\mathbb{Z}[1/2])$. $\square$

Proposition 4.8 *The centralizer $C_{GL(3,\mathbb{Z}[1/2])}(E_1)$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}[1/2]^\times \times \mathbb{Z}$.*

Proof. The module F contains the direct sum of the submodules $\mathrm{Ker}(g-1)$ (generated as abelian group by $1+g+g^2$) and $\mathrm{Ker}(1+g+g^2)$ (generated by $1-g$ and $1-g^2$) with quotient isomorphic to $\mathbb{Z}/3$ (observe that $3 = (1+g+g^2)+(1-g)+(1-g^2)$). These submodules are isomorphic to T resp. R and are preserved by any automorphism of F. In other words we get a homomorphism

$$\mathrm{Aut}(F) \longrightarrow \mathrm{Aut}(R \oplus T) \cong \mathbb{Z}/3 \times \mathbb{Z}[1/2]^\times \times \mathbb{Z}[1/2]^\times .$$

This is obviously injective and we claim that its image is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}[1/2]^\times \times \mathbb{Z}$. To see this note that the subgroup $\mathbb{Z}/3$ is clearly in the image; scalar automorphisms show that the diagonal of $\mathbb{Z}[1/2]^\times \times \mathbb{Z}[1/2]^\times$ is also in the image. Therefore it suffices to determine which of the automorphisms $\alpha_{\epsilon,n}: R\oplus T\to R\oplus T$, $(r,t)\mapsto(r,\epsilon 2^n t)$ (with $\epsilon\in\{0,1\}$ and $n\in\mathbb{Z}$) extends to one of F. Because of $3=(1+g+g^2)+(1-g)+(1-g^2)$ an extension exists iff $\epsilon 2^n(1+g+g^2)+(1-g)+(1-g^2)=(g+g^2)(\epsilon 2^n-1)+(\epsilon 2^n+2)$ is divisible by 3. This happens iff $\epsilon 2^n-1$ is divisible by 3. In other words, n may be chosen arbitrarily but ϵ is then determined by n. □

Corollary 4.9 *E_1 is contained in $SL(3,\mathbb{Z}[1/2])$ and there is a unique conjugacy class of elementary abelian 3 - subgroups in $SL(3,\mathbb{Z}[1/2])$ which maps to the $GL(3,\mathbb{Z}[1/2])$ - conjugacy class of E_1. Furthermore*

$$C_{SL(3,\mathbb{Z}[1/2])}(E_1)\cong\mathbb{Z}/3\times\mathbb{Z}\ ,$$

$$N_{SL(3,\mathbb{Z}[1/2])}(E_1)\cong C_{SL(3,\mathbb{Z}[1/2])}(E_1)\rtimes\mathbb{Z}/2$$

and the isomorphism can be chosen such that the conjugation action of $\mathbb{Z}/2$ on $\mathbb{Z}/3$ is non-trivial while it is trivial on $\mathbb{Z}$.

Proof. The proof is analogous to that of Proposition 4.7. One only has to check that the restriction of the determinant to $C_{GL(3,\mathbb{Z}[1/2])}(E_1)$ remains onto and that the automorphism $-\sigma\oplus$ (-id) of $R\oplus T$ (with σ the Galois automorphism of $\mathbb{Z}[1/2][\zeta_3]$) extends to an automorphism of F. □

We can now use the "centralizer spectral sequence"

$$E_2^{s,t}\cong\lim\nolimits_E^s H^t(C_\Gamma(E);\mathbb{F}_3)\Longrightarrow H_\Gamma^{s+t}(\mathcal{X}_{s,3};\mathbb{F}_3)$$

of [H1] to compute $H^*_\Gamma(\mathcal{X}_{s,3};\mathbb{F}_3)$ where as before $\Gamma=SL(3,\mathbb{Z}[1/2])$, $\mathcal{X}$ is the space $\mathcal{X}_\infty\times\mathcal{X}_2$, but now $\mathcal{X}_{s,3}$ denotes the 3 - singular locus of $\mathcal{X}$ with respect to the action of Γ; the limit is here taken over the category of elementary abelian 3 - subgroups of Γ. Because the 3 - rank of Γ is equal to 1, the spectral sequence degenerates into an isomorphism

$$H^*_\Gamma(\mathcal{X}_{s,3};\mathbb{F}_3)\cong\prod_{(E)}(H^*(C_\Gamma(E);\mathbb{F}_3))^{N_\Gamma(E)/C_\Gamma(E)}\cong\prod_{(E)}H^*(N_\Gamma(E);\mathbb{F}_3)$$

where the product is indexed by conjugacy classes of elementary abelian 3 - subgroups of Γ (see 3.3.1 of [H1]). In our case there are two conjugacy classes whose normalizers are isomorphic to $\mathfrak{S}_3\times\mathbb{Z}\times\mathbb{Z}/2$ resp. $\mathfrak{S}_3\times\mathbb{Z}$ resp., so we obtain the following result.

Proposition 4.10

$$H^*_\Gamma(\mathcal{X}_{s,3};\mathbb{F}_3)\cong\prod_{i=1}^{2}\widetilde{H}^*(\mathfrak{S}_3\times\mathbb{Z};\mathbb{F}_3)$$

We now turn towards the proof of part (d) of Theorem 1.10. By Proposition 4.10 it suffices to prove the following result.

Proposition 4.11 *a)* $H^*_\Gamma(\mathcal{X},\mathcal{X}_{s,3};\mathbb{F}_3)\cong\Sigma\mathbb{F}_3\oplus\Sigma^2(\mathbb{F}_3)^2\oplus\Sigma^5(\mathbb{F}_3)$.
b) The boundary map

$$H^*_\Gamma(\mathcal{X}_{s,3};\mathbb{F}_3)\longrightarrow H^{*+1}_\Gamma(\mathcal{X},\mathcal{X}_{s,3};\mathbb{F}_3)$$

is surjective. Its kernel in degree 4 *is of dimension* 3 *and is generated by the image of the Bockstein of* H^3 *and one further element which has non-trivial restriction to the two factors in Proposition 4.10.*

The proof of Proposition 4.11 depends on another result whose proof we give at the end of this section.

Proposition 4.12 *The restriction map* $H^*(\Gamma_1;\mathbb{F}_3)\longrightarrow H^*(\Gamma_2;\mathbb{F}_3)$ *is onto, and with respect to the isomorphism* $H^3(\Gamma_1;\mathbb{F}_3)\cong H^3(\mathfrak{S}_3;\mathbb{F}_3)\oplus H^3(\mathfrak{S}_3;\mathbb{F}_3)$ *of part (b) of Theorem 1.10 the kernel in degree* 3 *restricts non-trivially to both factors.*

Proof of Proposition 4.11. We consider the E_1 - term of the spectral sequence of the map

$$E\Gamma\times_\Gamma(\mathcal{X},\mathcal{X}_{s,3})\to\Delta^2\cong\Gamma\backslash\mathcal{X}_2\ .$$

By the proof of part (a) - (c) of Theorem 1.10 we get $E_1^{0,1}\cong E_1^{1,1}\cong(\mathbb{F}_3)^3$, $E_1^{2,0}\cong E_1^{2,3}\cong\mathbb{F}_3$ and $E_1^{s,t}=0$ in all other cases. In particular, we see that $H^3_\Gamma(\mathcal{X},\mathcal{X}_{s,3};\mathbb{F}_3)=H^4_\Gamma(\mathcal{X},\mathcal{X}_{s,3};\mathbb{F}_3)=0$.

Now one could try to directly compute the differentials in order to prove (a). This can presumably be done directly, but we proceed in a different way which at the same time turns out to be quite useful for the proof of part (b).

We use the spectral sequence of the map

$$E\Gamma\times_\Gamma\mathcal{X}\to\Delta^2\cong\Gamma\backslash\mathcal{X}_2\ .$$

By Theorem 1.10 its E_1 - term is given by

$$E_1^{s,*}\cong\prod_{i=1}^3H^*(\Gamma_s;\mathbb{F}_3)\text{ if }s=0,1\text{ and }E_1^{2,*}\cong H^*(\Gamma_2;\mathbb{F}_3)\cong\mathbb{F}_3\oplus\Sigma^3\mathbb{F}_3\ .$$

By Proposition 4.10 we already know $H^*(\Gamma;\mathbb{F}_3)$ in large dimensions. This together with the multiplicative structure of the spectral sequence forces the behaviour of $d_1 : E_1^{0,*} \longrightarrow E_1^{1,*}$ and gives for all $* > 0$ with $* \equiv 3, 4 \bmod 4$ that the kernel and cokernel of the map

$$d_1 : (\mathbb{F}_3)^6 \cong E_1^{0,*} \longrightarrow E_1^{1,*} \cong (\mathbb{F}_3)^6$$

is of dimension 2. By Proposition 4.12 the restriction map $H^*(\Gamma_1; \mathbb{F}_3) \longrightarrow H^*(\Gamma_2; \mathbb{F}_3)$ is onto and therefore $d_1 : E_1^{1,*} \longrightarrow E_1^{2,*}$ is onto as well. In particular we find that the spectral sequence collapses at E_2 and with some extra effort one could probably also determine the multiplicative structure. Here we need only the additive result in dimensions up to 5 where we find $H^0(\Gamma; \mathbb{F}_3) \cong \mathbb{F}_3$, $H^1(\Gamma; \mathbb{F}_3) = H^2(\Gamma; \mathbb{F}_3) = 0$, $H^3(\Gamma; \mathbb{F}_3) \cong H^5(\Gamma; \mathbb{F}_3) = (\mathbb{F}_3)^2$, $H^4(\Gamma; \mathbb{F}_3) \cong (\mathbb{F}_3)^3$.

Part (a) and the surjectivity in part (b) of the proposition follow now immediately from the long exact sequence of the pair $(\mathcal{X}, \mathcal{X}_{s,3})$ together with the knowledge that $H^3_\Gamma(\mathcal{X}, \mathcal{X}_{s,3}; \mathbb{F}_3) = H^4_\Gamma(\mathcal{X}, \mathcal{X}_{s,3}; \mathbb{F}_3) = 0$. It is also clear that the kernel in degree 4 is of dimension 3 and contains the image of the Bockstein of H^3. The following proposition finishes the proof. □

Proposition 4.13 *The restriction maps*

$$H^*(\Gamma; \mathbb{F}_3) \longrightarrow H^*(N_\Gamma(E_i); \mathbb{F}_3) \cong H^*(\mathfrak{S}_3 \times \mathbb{Z}); \mathbb{F}_3)$$

are surjective for $i = 1, 2$ *except in degree* 1.

Proof of Proposition 4.13. We abbreviate $N_\Gamma(E_i)$ by N_i. By Smith theory the space $\mathcal{X}^{E_i}$ is mod 3 - acyclic, so we try to understand the N_i - space $\mathcal{X}^{E_i}$ and for this we consider the canonical N_i - equivariant map $\mathcal{X}^{E_i} \longrightarrow \mathcal{X}_2^{E_i}$ induced by the Γ - equivariant projection $\mathcal{X} \longrightarrow \mathcal{X}_2$. The quotient of $\mathcal{X}_2$ by the action of Γ is Δ^2. It is an elementary exercise to verify that the quotient of $\mathcal{X}_2^{E_i}$ by N_i is the 1 - skeleton $\partial\Delta^2$ of Δ^2, and furthermore that the isotropy group of the j - simplices in $\partial\Delta^2$ are isomorphic to $N_i \cap \Gamma_j$ for $j = 0, 1$.

Now we compare the mod - p cohomology spectral sequences of the maps

$$E\Gamma \times_\Gamma \mathcal{X} \to \Delta^2 \cong \Gamma \backslash \mathcal{X}_2$$

and

$$E\Gamma \times_{N_i} \mathcal{X}^{E_i} \to \partial\Delta^2 \cong N_i \backslash \mathcal{X}_2^{E_i} \ .$$

As observed before the first spectral spectral sequence has as E_1 - terms

$$E_1^{s,*} \cong (H^*(\Gamma_s; \mathbb{F}_3))^{\oplus 3} \text{ if } s = 0, 1 \text{ and } E_1^{2,*} \cong H^*(\Gamma_2; \mathbb{F}_3) \cong \mathbb{F}_3 \oplus \Sigma^3\mathbb{F}_3 \ ,$$

while the second has

$$\widetilde{E_1}^{s,*} \cong (H^*(\Gamma_s \cap N_i; \mathbb{F}_3))^{\oplus 3} \text{ if } s = 0, 1 \text{ and } \widetilde{E_1}^{2,*} = 0 \ .$$

The map on E_1 - terms is induced by the restriction maps $H^*(\Gamma_s;\mathbb{F}_3)\longrightarrow H^*(\Gamma_s\cap N_i;\mathbb{F}_3)$ for $s=0,1$. The groups $\Gamma_s\cap N_i$ are easily identified with $\mathfrak{S}_3$ (for E_1) resp. $\mathfrak{S}_3\times\mathbb{Z}/2$ (for E_2). The map on $E_1^{s,*}$ corresponds for $s=0,1$ to the projection onto the i-th factor, $i=1,2$ (with respect to the product decomposition of the source, cf. Theorem 1.10(a+b)). Now we use Proposition 4.12 to finish the proof. □

Finally we turn towards the proof of Proposition 4.12.

Proof of Proposition 4.12. We dualize and work in homology. Furthermore we use Theorem 1.10(a+b) to identify $H_*(\Gamma_0;\mathbb{F}_3)$ with $H_*(\Gamma_1;\mathbb{F}_3)$ via the map induced by inclusion. Therefore we may consider the homomorphism $H_*(\Gamma_2;\mathbb{F}_3)\longrightarrow H_*(\Gamma_0;\mathbb{F}_3)$ again induced by inclusion. By Shapiro's lemma this homomorphism can be identified with the map

$$H_*(\Gamma_0;\mathbb{F}_3[\Gamma_0/\Gamma_2])\longrightarrow H_*(\Gamma_0;\mathbb{F}_3)$$

induced by the canonical map of Γ_0 - modules $\mathbb{F}_3[\Gamma_0/\Gamma_2]\longrightarrow\mathbb{F}_3$. We denote the kernel of this map by K. The following lemma is the main step in the proof.

Lemma 4.14 *$H_2(\Gamma_0;K)\cong\mathbb{F}_3$.*

We continue with the proof of Proposition 4.12. The lemma immediately implies the first part of 4.12. For the second part we consider the two non-conjugate elementary abelian 3 - subgroups E_1 and E_2 of Γ_0; they are the 3 - Sylow subgroups of the two $\mathfrak{S}_3$'s which detect $H^*(\Gamma_0;\mathbb{F}_3)$. The proof of the proposition will be complete once we have seen that the inclusions of both $\mathfrak{S}_3$'s into Γ_0 induce isomorphisms $H_2(\mathfrak{S}_3;K)\longrightarrow H_2(\Gamma_0;K)$. (We note that $\mathbb{F}_3[\Gamma_0/\Gamma_2]$ is projective as $\mathbb{F}_3[\mathfrak{S}_3]$ - module, and hence $H_2(\mathfrak{S}_3;K)\cong H_3(\mathfrak{S}_3;\mathbb{F}_3)\cong\mathbb{F}_3$.) In fact, this follows at once from the following observation: via mod - 2 reduction both $\mathfrak{S}_3$'s map monomorphically to $SL(3,\mathbb{F}_2)$ and there they agree with the normalizer of a 3 - Sylow subgroup; therefore the composition

$$H_2(\mathfrak{S}_3;K)\longrightarrow H_2(\Gamma_0;K)\longrightarrow H_2(SL(3,\mathbb{F}_2);K)$$

(the second arrow being induced by mod - 2 reduction) is an isomorphism. □

We turn towards the proof of Lemma 4.14. We could explicitly work out a projective resolution of the trivial $\mathbb{F}_3[\Gamma_0]$ - module $\mathbb{F}_3$ from the resolution provided by the cellular chains of $\mathcal{Z}$, and then compute $H_2(\Gamma_0;K)$ from this projective resolution. As this would be quite involved we construct just as much of this resolution as necessary.

Proof of Lemma 4.14. We consider the mod - 3 cellular chain complex of $\mathcal{Z}$ and break it apart into the following exact sequences of Γ_0 - modules where $i_1\delta_2 = \partial_2$ and $i_0\delta_1 = \partial_1$:

$$0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\delta_2} I_2 \longrightarrow 0 \tag{4.2}$$

$$0 \longrightarrow I_2 \xrightarrow{i_1} C_1 \xrightarrow{\delta_1} I_1 \longrightarrow 0 \tag{4.3}$$

$$0 \longrightarrow I_1 \xrightarrow{i_0} C_0 \xrightarrow{\varepsilon} \mathbb{F}_3 \longrightarrow 0 \tag{4.4}$$

The lemma will follow from the long exact sequence in $\mathrm{Tor}_*^{\Gamma_0}(-;K)$ associated to the exact sequence (4.4) and the following claims. (Here and in the sequel we abbreviate $\mathrm{Tor}_*^{\mathbb{F}_3[\Gamma_0]}(-;K)$ by $\mathrm{Tor}_*^{\Gamma_0}(-;K)$.)

Claim 1: $\mathrm{Tor}_2^{\Gamma_0}(C_0;K) \cong (\mathbb{F}_3)^4$.

Claim 2: $\mathrm{Tor}_1^{\Gamma_0}(I_1;K) = 0$ and $\mathrm{Tor}_2^{\Gamma_0}(I_1;K) \cong (\mathbb{F}_3)^3$.

Claim 3: The map $\mathrm{Tor}_2^{\Gamma_0}(I_1;K) \longrightarrow \mathrm{Tor}_2^{\Gamma_0}(C_0;K)$ induced by i_0 is injective.

Proof of Claim 1. The Γ_0 - modules C_i are direct sums $\bigoplus_\sigma \mathbb{F}_3[\Gamma_0/\Gamma_\sigma]$ where σ runs through the set of Γ_0 - orbits of i - dimensional cells in $\mathcal{Z}$ and Γ_σ is the isotropy group of a chosen representative of the orbit σ. By Sections 2.4 and 2.5 we have 5 orbits of 0 - cells in $\mathcal{Z}$ corresponding to the vertices A, O, C, F and D in $\Gamma_0\backslash\mathcal{Z}$, with respective isotropy groups D_{12} (for A), $\mathfrak{S}_4$ (for O, C and F) and D_8 (for D). By Shapiro's lemma we have therefore isomorphisms

$$\mathrm{Tor}_2^{\Gamma_0}(C_0;K) \cong H_2(D_{12},K) \oplus (H_2(\mathfrak{S}_4;K))^{\oplus 3} \oplus H_2(D_8,K)\ .$$

The contribution coming from D_8 is trivial because the order of D_8 is prime to 3. Furthermore Γ_2 has no 3 - torsion, hence the 3 - Sylow subgroup of all the other finite subgroups acts freely on $\mathbb{F}_3[\Gamma_0/\Gamma_2]$ and hence this module is projective when restricted to the other finite subgroups. As a consequence we obtain $H_2(D_{12};K) \cong H_3(D_{12};\mathbb{F}_3) \cong \mathbb{F}_3$ and $H_2(\mathfrak{S}_4;K) \cong H_3(\mathfrak{S}_4;\mathbb{F}_3) \cong \mathbb{F}_3$ and the claim follows.

□

Proof of Claim 2. Here we use the exact sequences (4.2) and (4.3). Just as above we find

$$\mathrm{Tor}_i^{\Gamma_0}(C_s;K) \cong \bigoplus_\sigma H_{i+1}(\Gamma_\sigma;\mathbb{F}_3) \tag{4.5}$$

if $s = 0,1,2,3$ and all $i > 0$. If $i = 0$, we observe that for all σ we have $H_1(\Gamma_\sigma;\mathbb{F}_3) = 0$, and hence the functors $\mathrm{Tor}_0^{\Gamma_0}(C_s;-)$ carry the short exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{F}_3[\Gamma_0/\Gamma_2] \longrightarrow \mathbb{F}_3 \longrightarrow 0$$

into short exact sequences. Hence we have a short exact sequence of complexes

$$0 \longrightarrow \mathrm{Tor}_0^{\Gamma_0}(C_*;K) \longrightarrow \mathrm{Tor}_0^{\Gamma_0}(C_*;\mathbb{F}_3[\Gamma_2/\Gamma_0]) \longrightarrow \mathrm{Tor}_0^{\Gamma_0}(C_*;\mathbb{F}_3) \longrightarrow 0 \quad (4.6)$$

for which the homology is known in the middle and on the right by Theorem 1.6.

From the exact sequence (4.2), formula (4.5) and our analysis of the cell structures and their symmetries in Sections 2.4, 2.5 and 2.6 we deduce that $\mathrm{Tor}_i^{\Gamma_0}(I_2;K) = 0$ if $i > 1$. For $i = 1$ it is isomorphic to the homology in dimension 3 of the complex $\mathrm{Tor}_0^{\Gamma_0}(C_*;K)$; this in turn is isomorphic to the homology in dimension 3 of the complex $\mathrm{Tor}_0^{\Gamma_0}(C_*;\mathbb{F}_3[\Gamma_2/\Gamma_0])$, i.e. to $H_3(\Gamma_2\backslash\mathcal{X}_\infty;\mathbb{F}_3) \cong \mathbb{F}_3$ by Theorem 1.6. For $i = 0$ we obtain

$$\mathrm{Tor}_0^{\Gamma_0}(I_2;K) \cong \mathrm{Coker}(\mathrm{Tor}_0^{\Gamma_0}(C_3;K) \xrightarrow{\partial_3} \mathrm{Tor}_0^{\Gamma_0}(C_2;K)) \ .$$

Now we can compute $\mathrm{Tor}_i^{\Gamma_0}(I_1;K)$ for $i = 1,2$ from the long exact sequence in Tor which is associated to the exact sequence (4.3). Using once more our analysis in section 2.4, 2.5 and 2.6 we see that $\mathrm{Tor}_1^{\Gamma_0}(C_1;K) = 0$ and we obtain a short exact sequence

$$0 \longrightarrow (\mathbb{F}_3)^2 \cong \mathrm{Tor}_2^{\Gamma_0}(C_1;K) \longrightarrow \mathrm{Tor}_2^{\Gamma_0}(I_1;K) \longrightarrow \mathrm{Tor}_1^{\Gamma_0}(I_2;K) \cong \mathbb{F}_3 \longrightarrow 0 \quad (4.7)$$

where the contribution to $\mathrm{Tor}_2^{\Gamma_0}(C_1;K)$ comes from the two Γ_0 - orbits of 1 - dimensional cells corresponding to OC and OF with symmetry group isomorphic to $\mathfrak{S}_3$ in both cases. For $\mathrm{Tor}_1^{\Gamma_0}(I_1;K)$ we use again that $\mathrm{Tor}_1^{\Gamma_0}(C_1;K) = 0$ and that the map $\mathrm{Tor}_0^{\Gamma_0}(I_2;K) \longrightarrow \mathrm{Tor}_0^{\Gamma_0}(C_1;K)$ is injective (because the complex $\mathrm{Tor}_0^{\Gamma_0}(C_*;K)$ has no homology in degree 2 by Theorem 1.6). □

Proof of Claim 3. We proceed in several steps. In a first step we reduce the evaluation of

$$i_{0*} : \mathrm{Tor}_2^{\Gamma_0}(I_1;K) \longrightarrow \mathrm{Tor}_2^{\Gamma_0}(C_0;K) \cong H_2(D_{12},K) \oplus (H_2(\mathfrak{S}_4;K))^{\oplus 3}$$

to the study of a particular chain map. In a second step we descroibe this chain map and in a final step we finish the computation of i_{0*}.

First step. We consider the restriction of the map i_{0*} to the subgroup $\mathrm{Tor}_2^{\Gamma_0}(C_1;K) \cong (H_2(\mathfrak{S}_3;K))^{\oplus 2} \cong (\mathbb{F}_3)^2$ (cf. (4.7)). This restriction is induced by injections of the isotropy groups (isomorphic to $\mathfrak{S}_3$) of the edges OC and OF into the isotropy groups (isomorphic to $\mathfrak{S}_4$) of the vertices O, C and F. Each of these injections induces an isomorphism of cohomology in $H_2(-;K) \cong H_3(-;\mathbb{F}_3)$, hence these injections map $(H_2(\mathfrak{S}_3;K))^{\oplus 2}$ monomorphically to the summand $(H_2(\mathfrak{S}_4;K))^{\oplus 3}$ of $\mathrm{Tor}_2^{\Gamma_0}(C_0;K)$. It suffices therefore to show that the composition of $i_0 : I_1 \longrightarrow C_0$ with the projection $\pi : C_0 \longrightarrow \mathbb{F}_3[\Gamma_0/D_{12}]$ induces a non-trivial map in $\mathrm{Tor}_2^{\Gamma_0}(-;K)$. To this end we should construct $\mathbb{F}_3[\Gamma_0]$ - projective resolutions P_* of I_1 and Q_* of $\mathbb{F}_3[\Gamma_0/D_{12}]$ and lift πi_0 to a chain map $\alpha : P_* \longrightarrow Q_*$.

In the case of Q_* we start with the minimal projective resolution Q'_* for $\mathbb{F}_3$ as a $\mathbb{F}_3[D_{12}]$ - module and take $Q_* = \mathbb{F}_3[\Gamma_0] \otimes_{\mathbb{F}_3[D_{12}]} Q'_*$. In the case of I_1 we obtain a projective resolution P_* by taking first projective resolutions R_{2*} of I_2 and R_{1*} of C_1; then we lift $i_1 : I_2 \longrightarrow C_1$ to a chain map $\widetilde{i_1} : R_{2*} \longrightarrow R_{1*}$ and obtain a double complex R_{**} whose total complex gives the desired projective resolution P_*. More concretely, we can take the exact sequence $0 \longrightarrow C_3 \longrightarrow C_2 \longrightarrow I_2 \longrightarrow 0$ as a projective resolution R_{2*} of I_2. For C_1 we use the direct sum decomposition $C_1 \cong \bigoplus_\sigma \mathbb{F}_3[\Gamma_0/\Gamma_\sigma]$ and, if R^σ_{1*} is a minimal projective resolution of the trivial $\mathbb{F}_3[\Gamma_\sigma]$ - module $\mathbb{F}_3$, then we take $R_{1*} = \bigoplus_\sigma \mathbb{F}_3[\Gamma_0] \otimes_{\mathbb{F}_3[\Gamma_\sigma]} R^\sigma_{1*}$. In these terms, the projective resolution P_* of I_1 looks as follows:

$$\cdots \longrightarrow R_{14} \xrightarrow{\partial_3^P} R_{13} \xrightarrow{\partial_2^P} R_{12} \oplus C_3 \xrightarrow{\partial_1^P} R_{11} \oplus C_2 \xrightarrow{\partial_0^P} R_{10}$$

with $\partial_3^P = \partial_3^{R_1}$, $\partial_2^P = \begin{pmatrix} \partial_2^{R_1} \\ 0 \end{pmatrix}$, $\partial_1^P = \begin{pmatrix} \partial_1^{R_1} & \widetilde{i_{11}} \\ 0 & -\partial_0^{R_2} \end{pmatrix}$ and $\partial_0^P = (\partial_0^{R_1} \;\; \widetilde{i_{10}})$.

We denote the direct summands of C_2 resp. R_1 corresponding to the 2 - dimensional cells BEF and $OCBF$ resp. the 1 - dimensional faces of these 2 - dimensional cells by $\overline{C_2}$ resp. $\overline{R_1}$. It is clear that the lift $\widetilde{i_1}$ of i_1 can be chosen in such a way that $\widetilde{i_{10}}(\overline{C_2}) \subset \overline{R_{10}}$ so that

$$\cdots \longrightarrow \overline{R_{14}} \xrightarrow{\partial_3^P} \overline{R_{13}} \xrightarrow{\partial_2^P} \overline{R_{12}} \xrightarrow{\partial_1^P} \overline{R_{11}} \oplus \overline{C_2} \xrightarrow{\partial_0^P} \overline{R_{10}}$$

is a subcomplex of P_*. Furthermore the lift α of πi_0 can be chosen such that this subcomplex maps trivially to Q_*. Therefore, if we denote the direct summands of C_2 resp. C_1 correponding to the other 2 - dimensional resp. 1 - dimensional cells by $\widetilde{C_2}$ resp. $\widetilde{C_1}$ and if we use that $\widetilde{C_1}$ is projective (because all isotropy groups are of even order), i.e. $\widetilde{R_{10}} = \widetilde{C_1}$ and $\widetilde{R_{1*}} = 0$ for $* > 0$, we obtain a factorization of α to a map $\widetilde{\alpha}$ of complexes $\widetilde{P_*} \longrightarrow Q_*$ as follows (with i_{10} being induced by ∂_1^C) :

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_3 & \xrightarrow{-\partial_0^{R_2}} & \widetilde{C_2} & \xrightarrow{\widetilde{i_{10}}} & \widetilde{C_1} \\ & & \alpha_2 \downarrow & & \alpha_1 \downarrow & & \alpha_0 \downarrow \\ Q_3 & \xrightarrow{\partial_3^Q} & Q_2 & \xrightarrow{\partial_2^Q} & Q_1 & \xrightarrow{\partial_1^Q} & Q_0 \end{array}$$

Second step. Now we construct $\widetilde{\alpha}$ in detail. First we observe that the complexes $\widetilde{P_*}$ resp. Q are induced from $\mathbb{F}_3[D_{12}]$ - complexes P'_* resp. Q'_* and that $\widetilde{\alpha}$ can also be constructed as a map induced from a chain map α' of D_{12} - chain complexes. So it is enough to construct α'.

We denote the natural $\mathfrak{S}_3$ - modules $\mathbb{F}_3[\mathfrak{S}_3/\mathfrak{S}_2]$ by S and the tensor product of S with the non-trivial one-dimensional $\mathfrak{S}_3$ - module by $S(-1)$. With respect to the action of the 3 - Sylow subgroup of $\mathfrak{S}_3$ the two module structures agree, so if we denote a fixed generator of this 3 - Sylow subgroup by t, we have well defined linear maps $S(-1) \longrightarrow S$ and $S(-1) \longrightarrow S$ which deserve to be labelled t^2-t. We consider all these modules as D_{12} - modules via the natural homomorphism $D_{12} \longrightarrow \mathfrak{S}_3$ and leave it to the reader to verify that the minimal resolution Q'_* of the trivial $\mathbb{F}_3[D_{12}]$ - module $\mathbb{F}_3$ is periodic of order 4 and can be described as follows:

$$\cdots S(-1) \xrightarrow{t^2-t} S \xrightarrow{1+t+t^2} S \xrightarrow{t^2-t} S(-1) \xrightarrow{1+t+t^2} S(-1) \xrightarrow{t^2-t} S \ .$$

Now we define maps α'_i for $i = 1, 2, 3$ and leave it to the reader to verify that they fit together to give a D_{12} - equivariant chain map $\alpha' : P'_* \longrightarrow Q'_*$ as desired.

As before we denote the isotropy groups of the cells OA, AB, ... by Γ_{OA}, Γ_{AB}, ... Then the map

$$\alpha'_0 : \mathbb{F}_3[D_{12}/\Gamma_{OA}] \oplus \mathbb{F}_3[D_{12}/\Gamma_{AB}] \oplus \mathbb{F}_3[D_{12}/\Gamma_{AD}] \longrightarrow S$$

is given as follows (we choose the letter e as a generic letter for the generators of the various modules while g^{AB} and g^{AD} are the elements in $\Gamma_A = D_{12}$ which have been introduced in section 2.4):

$$\alpha'_0(e_{OA}) = e_S, \quad \alpha'_0(e_{AB}) = -g^{AD}e_S, \quad \alpha'_0(e_{AD}) = -g^{AB}e_S \ .$$

This is D_{12} - equivariant if the subgroup $\mathfrak{S}_2$ which occurs in the definition of the module S is chosen as the subgroup generated by the image of $g^{AB}g^{AD}g^{AB}$ with respect to the projection $D_{12} \longrightarrow \mathfrak{S}_3$; for t we take the image of $g^{AD}g^{AB}$. The map

$$\alpha'_1 : \mathbb{F}_3[D_{12}/\Gamma_{OAC}] \oplus \mathbb{F}_3[D_{12}/\Gamma_{OAF}] \oplus \mathbb{F}_3[D_{12}/\Gamma_{ABD}] \longrightarrow S(-1)$$

may then be given by

$$\alpha_1'(e_{OAC}) = 0, \quad \alpha_1'(e_{OAF}) = 0, \quad \alpha_1'(e_{ABD}) = -e_{S(-1)} \ .$$

Finally we have

$$\alpha_2' : \mathbb{F}_3[D_{12}] \longrightarrow S(-1), \quad \alpha_2'(e) = e_{S(-1)} \ .$$

Third step. We are now ready to finish the calculation of i_{0*}. The following element of $K \subset \mathbb{F}_3[\Gamma_0/\Gamma_2]$ (cf. formula (3.2) in section 3.1)

$$[100y=0]-[100z=0]-[010x=0]+[010z=0]+[001x=0]-[001y=0]$$

represents a class in $H_2(P_*' \otimes_{\mathbb{F}_3[D_{12}]} K)$ and it suffices to show that its image via

$$\alpha_2' \otimes id_K : K \cong \mathbb{F}_3[D_{12}] \otimes_{\mathbb{F}_3[D_{12}]} K \longrightarrow S(-1) \otimes_{\mathbb{F}_3[D_{12}]} K$$

is non-trivial in $H_2(Q_*' \otimes_{\mathbb{F}_3[D_{12}]} K)$, i.e. is not in the image of

$$S \otimes_{\mathbb{F}_3[D_{12}]} K \xrightarrow{t^2-t} S(-1) \otimes_{\mathbb{F}_3[D_{12}]} K \ .$$

We leave this verification to the patient reader with the hint that the calulation can be significantly simplified by making use of the decomposition of $\mathbb{F}_3[\Gamma_0/\Gamma_2]$ as a $\mathbb{F}_3[D_{12}]$ - module (cf. section 2.6). □

4.3 Higher torsion in the integral cohomology

It is clear from Corollary 1.7 that the p - torsion in $H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{Z})$ is trivial for primes $p > 3$. Furthermore the mod - 3 Bockstein spectral sequence and Theorem 1.10 shows that the 3 - torsion is all of order 3 and is easily understood from the results in the last section. Therefore we restrict attention to higher 2 - torsion.

For this consider the mod - 2 Bockstein spectral sequence for $SL(3,\mathbb{Z}[1/2])$: We know from Theorem 1.4 that $H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)$ maps injectively onto the subalgebra of $H^*(SD_3;\mathbb{F}_2) \cong \mathbb{F}_2[x,y] \otimes E(f,g)$ generated by $v_2 = x^2 + xy + y^2$, $v_3 = x^2y + xy^2$, $d_3 = x^2g + y^2f$ and $d_5 = x^4g + y^4f$ (cf. [H1]). Therefore we have $Sq^1 v_2 = v_3$ while Sq^1 is zero on the other algebra generators and thus we see that the E_2 - term of this spectral sequence is isomorphic to $\mathbb{F}_2[v_2^2] \otimes E(d_3, d_5)$. The crucial point is now which order Bockstein of d_3 kills v_2^2.

To settle this we consider the mod - 2 Bockstein spectral sequence for $GL(2,\mathbb{Z}[1/2])$: In this case $H^*(GL(2,\mathbb{Z}[1/2]);\mathbb{F}_2)$ maps injectively onto the subalgebra of $H^*(D_2;\mathbb{F}_2) \cong \mathbb{F}_2[x,y] \otimes E(f,g)$ generated by $w_1 = x + y$,

$w_2 = xy$, $e_1 = e + f$ and $e_3 = x^2g + y^2f$ [H1] and the E_2 - term identifies with $\mathbb{F}_2[w_2^2] \otimes E(e_1, e_3)$. The restriction map from $H^*(SL(3, \mathbb{Z}[1/2]); \mathbb{F}_2)$ to $H^*(GL(2, \mathbb{Z}[1/2]); \mathbb{F}_2)$ maps d_3 to e_3 and v_2 to $w_2 + w_1^2$, hence v_3^2 to $w_2^2 + w_1^4$ which in the E_2 - term is identified with w_2^2. Therefore it suffices to determine which higher order Bockstein of e_3 kills w_2^2.

Now we compare the mod - 2 Bockstein spectral sequence for $GL(2, \mathbb{Z}[1/2])$ with that of $SL(2, \mathbb{Z}[1/2])$; we recall that $H^*(SL(3, \mathbb{Z}[1/2]); \mathbb{F}_2) \cong \mathbb{F}_2[w_2] \otimes E(e_3)$ (cf. [Mi]). The notation suggests the behaviour of the restriction map, i.e. the elements w_2 and e_3 of $H^*(GL(2, \mathbb{Z}[1/2]); \mathbb{F}_2)$ map to the elements in $H^*(SL(2, \mathbb{Z}[1/2]); \mathbb{F}_2)$ with the same name. Furthermore, the element w_2 comes from an integral class in $H^*(SL(2, \mathbb{Z}[1/2]); \mathbb{Z})$ (namely the first Chern class c_1), hence Sq^1 acts trivially on it. Therefore, the E_2 - term in the case of $SL(2, \mathbb{Z}[1/2])$ is isomorphic to $\mathbb{F}_2[w_2] \otimes E(e_3)$ and hence it is enough to determine which higher order Bockstein of e_3 kills w_2^2 in the Bockstein spectral sequence for $SL(2, \mathbb{Z}[1/2])$, or equivalently which is the additive order of the second power of the integral lift c_1 of w_2. This can be checked to be of order 8, e.g. by playing off the mod - 2 cohomology computation [Mi] against an integral cohomology computation based on the amalgam description $SL(2, \mathbb{Z}[1/2]) \cong SL(2, \mathbb{Z}) *_\Delta SL(2, \mathbb{Z})$ [Se]. Here Δ is the subgroup of $SL(2, \mathbb{Z})$ consisting of all matrices which are upper triangular modulo 2.

We summarize our discussion in the following result.

Proposition 4.15 *The higher* 2 *- torsion in* $H^*(SL(3, \mathbb{Z}[1/2]); \mathbb{Z})$ *is all of order* 8 *and is represented in the mod - 2 Bockstein spectral sequence by the classes* ${v_2}^{2n}$ *and* $d_5{v_2}^{2n}$ *(*$n > 0$*); the classes* 1 *and* d_5 *represent classes of infinite order.* □

The integral cohomology of $SL(3, \mathbb{Z}[1/2])$ can now be easily written down explicitly. We leave the details to the interested reader.

5 The cohomology of $GL(3,\mathbb{Z}[1/2])$

Let $GL_{\pm}(3,\mathbb{Z}[1/2])$ be the preimage of the subgroup $\{\pm 1\}$ of $(\mathbb{Z}[1/2])^{\times}$ under the determinant $GL(3,\mathbb{Z}[1/2]) \longrightarrow (\mathbb{Z}[1/2])^{\times}$. The group $GL_{\pm}(3,\mathbb{Z}[1/2])$ splits as $SL(3,\mathbb{Z}[1/2]) \times \mathbb{Z}/2$, so we understand its mod p - cohomology by Theorem 1.4, Corollary 1.7 and Theorem 1.10. We will work out the mod - p cohomology spectral sequences of the group extension

$$1 \longrightarrow GL_{\pm}(3,\mathbb{Z}[1/2]) \longrightarrow GL(3,\mathbb{Z}[1/2]) \longrightarrow \mathbb{Z} \longrightarrow 1 \qquad (5.1)$$

where the homomorphism from $GL(3,\mathbb{Z}[1/2])$ to $\mathbb{Z}$ is the determinant followed by the quotient map $(\mathbb{Z}[1/2])^{\times} \longrightarrow (\mathbb{Z}[1/2])^{\times}/\{\pm 1\} \cong \mathbb{Z}$.

Note that the matrix $2 \cdot \mathrm{id}$ is central in $GL(3,\mathbb{Z}[1/2])$, hence it acts trivially on $GL_{\pm}(3,\mathbb{Z}[1/2])$ by conjugation. Its determinant is $8 = 2^3$ which corresponds to the element 3 in $\mathbb{Z}$ under the determinant map. It follows that the conjugation action of $\mathbb{Z}$ on $H^*(GL_{\pm}(3,\mathbb{Z}[1/2]);\mathbb{F}_p)$ factors through an action of $\mathbb{Z}/3$.

The case $p > 3$. The conjugation action of $\mathbb{Z}/3$ on $H^5(GL_{\pm}(3,\mathbb{Z}[1/2]);\mathbb{F}_p) \cong \mathbb{F}_p$ comes from one on integral cohomology, hence it is necessarily trivial. Furthermore the spectral sequence necessarily collapses at E_2 and we obtain the following result.

Proposition 5.1 *Assume $p > 3$. Then there is an isomorphism of algebras*

$$H^*(GL(3,\mathbb{Z}[1/2]);\mathbb{F}_p) \cong H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_p) \otimes H^*((\mathbb{Z}[1/2])^{\times};\mathbb{F}_p)\ .$$

We have chosen $\mathbb{Z}[1/2]^{\times}$ as second factor in order to get "symmetric statements" for the different primes.

The case $p = 2$. Again we look at the spectral sequence of the group extension (5.1). As in the case of primes $p > 3$ we claim that the conjugation action of $\mathbb{Z}/3$ on $H^*(GL_{\pm}(3,\mathbb{Z}[1/2]);\mathbb{F}_2) \cong H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2) \otimes H^*(\mathbb{Z}/2;\mathbb{F}_2)$ is trivial.

First we note that this action leaves the two factors $H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)$ and $H^*(\mathbb{Z}/2;\mathbb{F}_2)$ invariant and is clearly trivial on the second factor. By dimensional reasons it is clear that the action is trivial on v_2, and because of $Sq^1 v_2 = v_3$ it is also trivial on v_3. Now we know that the action of $\mathbb{Z}/3$ on $H^3(SL(3,\mathbb{Z}[1/2];\mathbb{F}_2) \cong (\mathbb{F}_2)^2$ has an invariant subspace (namely the subspace generated by v_3) and this forces it to be also trivial on d_3. Next the formula $Sq^2 d_3 = d_5$ and multiplicativity of the action shows that $\mathbb{Z}/3$ acts trivially as claimed. We obtain $E_2 \cong H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_2) \otimes H^*((\mathbb{Z}[1/2])^{\times};\mathbb{F}_2)$ as algebras. By Theorem 1.1 E_2 consists of permanent cycles, i.e. the spectral sequence collapses and we have finally proved Theorem 1.3.

The case $p = 3$. Once more we look at the spectral sequence of the group extension (5.1). Using the restriction map to the cohomology of the centralizers $C_{SL(3,\mathbb{Z}[1/2])}(E_i)$ together with the description of these groups as provided by Section 4.2, it is easy to see that the action of $\mathbb{Z}/3$ on $\widetilde{H}^*(GL_{\pm}(3,\mathbb{Z}[1/2]);\mathbb{F}_3)$ is trivial. So as before we obtain an isomorphism $E_2 \cong H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_3) \otimes H^*((\mathbb{Z}[1/2])^\times;\mathbb{F}_3)$ as algebras, i.e. there is no room for differentials and the spectral sequence collapses. By using the restriction maps to the centralizers $C_{GL(3,\mathbb{Z}[1/2])}(E_i)$ we see that the E_2 - term gives also the algebra structure. We state the result of our discussion in the following result.

Proposition 5.2 *There is an isomorphism of algebras*

$$H^*(GL(3,\mathbb{Z}[1/2]);\mathbb{F}_3) \cong H^*(SL(3,\mathbb{Z}[1/2]);\mathbb{F}_3) \otimes H^*((\mathbb{Z}[1/2])^\times;\mathbb{F}_3) \ .$$

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