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K-theory of mapping class groups III:
Odd torsion

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1. Introduction.

Let Γ^n be the mapping class group of a 2-sphere with n punctures ($n \geq 3$), an interesting infinite group of finite virtual cohomological dimension. The aim of this series of papers is to compute the (topological) K-theory of the classifying spaces $B\Gamma^n$. It is known (see [6], [10]) that the reduced K-theory of $B\Gamma^n$ is profinite. In [10] the torsion-free part of $K^*(B\Gamma^n)$ was found for all n, but it was also shown that there is sometimes torsion. The aim of this paper is to find when this torsion occurs (for odd primes), and to describe it.

Where the work of [10] was based on a formula of A. Adem using elements of finite order in the group, and hence on the relation between the group's structure and its geometry, for this one we use homotopy theoretic methods. These were applied to find the cohomology of $B\Gamma^n$ by Bödigheimer, Cohen and Peim in [7]; we use them to find the rank of mod p K-theory and by comparing with [10] we locate the torsion. We are also able to apply similar methods in the p-local case to compute its order (which is an improvement over the cohomology result). The basic tool is the spectral sequence in K-theory of a fibration relating the whole family of $B\Gamma^n$'s, $BSO(3)$, and the space $\Lambda^2 S^{2q+2}$ of maps from a 2-sphere to a $(2q+2)$-sphere. The computation of this spectral sequence is straightforward on the whole, if a bit lengthy. Our main result is:

Theorem 1.1. *Let p be an odd prime. Then the p-torsion in $K^*(B\Gamma^n; \mathbf{Z})$ is all in K^1, and is as follows:*
Suppose that either $n = q \equiv 3, \ldots, p-1 \pmod p$, or $n = qp^r + \alpha$ where $\alpha \in \{0, 1, 2\}$ and q is prime to p. Let $v_p(t)$ denote the p-adic value of t. Then for each integer t such that $2pt < q$, Tors$(K^*(B\Gamma^n; \mathbf{Z}))$ has one finite cyclic p-power summand $C(t)$; and the order of $C(t)$ is $p^{1+v_p(t)}$. The p-torsion is therefore a sum of $\left[\frac{q}{2p}\right]$ components.
In particular, the K-theory is p-torsion free if and only if $q < 2p$.

On the way, we shall find the additive structure of $K^*(B\Gamma^n; \mathbf{Z}/p)$, indeed, it will be necessary for the proof of theorem 1.1. We shall also find a considerably more detailed description of the mod p and p-local K-theories of $B\Gamma_n$, with generators and some extra structure. The detailed results will be given in §§5, 6.

In order to describe the layout of this paper more fully, I shall first recall the way in which results for odd primes are arrived at in [7]. We begin with the 'fibred configuration space' for a fibration $\pi : E \to B$ with fibre Y. This is defined as

$$E(\pi, k) = \{(e_1, ..., e_k) \in E^k | e_i \neq e_j \text{ and } \pi(e_i) = \pi(e_j) \text{ if } i \neq j\}$$

The symmetric group Σ_k acts freely on $E(\pi, k)$, and the important case for us is that of the fibration $\eta : BSO(2) \to BSO(3)$ whose fibre is S^2. Here we have:

Fact 1. *For $k \geq 3$, $E(\eta, k)/\Sigma_k$ is a model for $B\Gamma^k$.*

This is proposition 1.1 of [7], and it gives a relation between mapping class groups and configuration spaces. To find the K-theory of the configuration spaces they in their turn need first to be combined into a construction which contains all values of k at once, and then to be related to loopspaces. We accordingly define for any based CW-complex X:

$$E(\eta, X) = \Big[\coprod_{k\geq 0} E(\eta, k) \times_{\Sigma_k} X^k\Big]/(\sim)$$

where $\sim$ is the natural contraction (compare reduced product spaces) which forgets the basepoint when it occurs. This space has a filtration coming from k, and if we define $D_k(\eta, X)$ to be the filtration quotient $E_k(\eta, X)/E_{k-1}(\eta, X)$, then there is a stable splitting generalizing the Snaith splitting [17]:

$$E(\eta, X) \underset{s}{\simeq} \bigvee_{k\geq 1} D_k(\eta, X) \tag{1}$$

(This is proposition 1.3. of [7].)
We are particularly interested in $D_k(\eta, S^m)$ for appropriate values of m, since it is closely related to $E(\eta, k)/\Sigma_k$ and so to the mapping class group. For this we follow the paper [8] — [7] uses a homology argument at the chain level which is inappropriate for K-theory. Consider the vector bundle

$$\xi : E(\eta, k) \underset{\Sigma_k}{\times} \mathbf{R}^k \to E(\eta, k)/\Sigma_k$$

By the argument of Milgram [15] quoted in [8] 2.6, the Thom space $T(m.\xi)$ can be identified with $D_k(\eta, S^m)$ We cannot, as in [8], choose m so that $m.\xi$ is trivial, since the group K^0 of the base contains, as we know, numerous elements of infinite order in general. (The base is equivalent to $B\Gamma^k$, and not a finite complex.) However, it is easy to choose m so that $n.\xi$ is at least orientable for K-theory with any coefficients; in fact we only need that the bundle admits a complex structure (see [4]), so that m even is sufficient. For such m, the reduced K-theory of $T(m.\xi)$ is the same as the K-theory of the base. We deduce:

Proposition 1.1. *There is an isomorphism, for m even,*

$$K^*(B\Gamma^k; R) \to \tilde{K}^*(D_k(\eta, S^m); R) \tag{2}$$

for any ring of coefficients R.

Corollary. *$\tilde{K}^*(E(\eta, S^m); R)$ splits as a direct sum corresponding to the filtration quotients D_k; and for $k \geq 3$ the kth summand is isomorphic to the K-theory of $B\Gamma^k$.*

Note 1. In the cohomology version of this result (cf. [7] proposition 3.1.), there is a shift in dimension coming from the Thom isomorphism. Since the bundle is even dimensional, the shift is not perceived by K-theory, and we needn't bother with it.

Note 2. We can make the isomorphism (2) canonical, since the natural complex structure on $m.\xi$ gives a preferred choice of Thom class.

The proposition and its corollary shift the focus of interest to the space $E(\eta, S^m)$, and to find the K-theory here we have our last important result, Proposition 1.2 of [7] in our particular case:

Fact 2. *There is a homotopy equivalence*

$$E(\eta, S^m) \to ESO(3) \times_{SO(3)} \Lambda^2 S^{m+2} \tag{3}$$

where $\Lambda^2 X$ denotes the free space of all maps from S^2 to X, and $SO(3)$ acts on the mapping space via the usual action on S^2. In consequence, there is a fibration

$$\Lambda^2 S^{m+2} \to E(\eta, S^m) \to BSO(3) \tag{4}$$

Note. We shall also want a related result 'over a point'. Namely, consider the one-point S^2-bundle $\pi : S^2 \to *$. The space $E(\pi, S^m)$ can be naturally identified with $\Lambda^2 S^{m+2}$ itself. Using the decomposition of $E(\pi, S^m)$ into filtration quotients, and the identification of those quotients as Thom spaces, we find that for m even:

$$\tilde{K}^*(\Lambda^2 S^{m+2}; R) \cong \tilde{K}^*(E(\pi, S^m); R) \cong \bigoplus_{k=1}^{\infty} K^*(E(\pi, k)/\Sigma_k; R) \tag{5}$$

The spaces $E(\pi, k)/\Sigma_k$ are the configuration spaces of unordered sets of points on S^2; we shall abbreviate them to $F_k(S^2)$.

From the above summary of the theory of [7], the following plan of attack emerges. First in section 2 we recall the (known) mod p K-theory of the loopspace $\Omega^2 S^{m+2}$ for $m = 2n$ even, and deduce it for the free mapping space $\Lambda^2 S^{2n+2}$. We discuss the nature of the torsion in the p-local theory in each case. In section 3 we begin the calculation of the spectral sequence of the fibration (4) in K-theory mod p, with the aim of finding $K^*(E(\eta, S^{2n}); \mathbf{Z}/p)$ and so (by the corollary to proposition 1.1) $K^*(B\Gamma^k; \mathbf{Z}/p)$ for all k. We begin with the cases where the weight k is p^r or $2p^r$, which are particularly easy, find the main differentials in section 4, and deduce the K-theory in these cases. We then in section 5 compute the mod p spectral sequence in general — this has to be divided into cases according to the residue class of k mod p. Along the way it is necessary to compare our results with those of [10] so as to know which elements are to be identified as torsion. Finally in section 6, we calculate the p-local spectral sequence and find the order and location of the torsion summands identified in section 5.

In what follows, p will denote a fixed odd prime and the letter q, wherever it occurs, will be used for a number which is prime to p (not necessarily odd).

2. K-theory of $\Lambda^2 S^{2n+2}$.

We begin — again following closely the lines of [7] — by recalling that, localized at an odd prime, an odd sphere is an H-space, and there is a decomposition

$$\Omega S^{2n+1} \times \Omega^2 S^{4n+3} \xrightarrow{\simeq} \Omega^2 S^{2n+2} \tag{6}$$

This is in fact derived from the 'Hopf fibration' of S^{4n+3} and so deloops. We consider the simpler K-homology for the present; with coefficients in a field it is of course a Hopf algebra. However, any homology theory applied to ΩS^{2n+1} is very simple using the standard reduced product space decomposition [12]. We find here that $K_*(\Omega S^{2n+1}; R)$ is the polynomial algebra $R[x]$, where x is the fundamental class of the $2n$-sphere mapped in by the inclusion.We can deduce

Lemma 2.1. *Over any ring of coefficients R, the K-homology $K_*(\Omega^2 S^{2n+2}; R)$ is naturally isomorphic to the tensor product $R[x] \otimes K_*(\Omega^2 S^{4n+3}; R)$.*

On the other hand, the K-theory of second loopspaces presents more complications. To begin with we concentrate on the mod p theory.

Proposition 2.1. *There are generators $y_0, y_1 \in K_1(\Omega^2 S^{2q+1}; \mathbf{Z}/p)$ and $z_1, z_2, z_3, \ldots \in K_0(\Omega^2 S^{2q+1}; \mathbf{Z}/p)$ so that*

$$K_*(\Omega^2 S^{2q+1}; \mathbf{Z}/p) = E(y_0, y_1) \otimes \mathbf{Z}/p\,[z_1, z_2, z_3, \ldots]/((z_1)^p, (z_2)^p, \ldots) \tag{7}$$

Proof. The computation has been done for the Morava K-theories $K(n)$, for all n and all odd primes, by Yamaguchi [18]; see also Langsetmo [13] for K-theory in particular. The diagonal — and so the full product structure in K-cohomology — are not given in [18]; however, they have recently been determined, in our case by Ravenel [16]. (For the general space $\Omega^k S^n$, see Langsetmo [14].) The results are as follows (again general for the first loopspace and mod p for the second):

Proposition 2.2. *(i) For any ring of coefficients R, $K^*(\Omega S^{2q+1}; R)$ is a 'completed divided polynomial algebra on one generator'; more precisely, there are generators $\xi_1, \xi_2, \ldots \in K^0(\Omega S^{2q+1}; R)$ such that*

$$K^*(\Omega S^{2q+1}; R) = R[[\xi_1, \xi_2, \ldots]]/(\xi_i \xi_j = (i,j)\xi_{i+j}; i, j = 1, 2, \ldots) \tag{8}$$

as an algebra. (Here (i,j) is the binomial coefficient as usual.)
(ii) There are generators $\eta_0, \eta_1 \in K^1(\Omega^2 S^{2q+1}; \mathbf{Z}/p)$ and $\zeta_1 \in K^0(\Omega^2 S^{2q+1}; \mathbf{Z}/p)$ so that

$$K^*(\Omega^2 S^{2q+1}; \mathbf{Z}/p) = E(\eta_0, \eta_1) \otimes \mathbf{Z}/p\,[[\zeta_1]] \tag{9}$$

— the tensor product of an exterior and a power series algebra.

Note. Here and henceforth it is important that Snaith's stable splitting [17] of the spaces $\Omega^2 S^{2q+1}$ etc. gives a grading by 'weight' on the K-theory algebras. Specifically, ξ_i has weight i and η_i, ζ_i have weight $2p^i$ when defined. (These are well known verifications.)

Proposition 2.2 (i) is well known; while part (ii) is proved in [16].

Corollary. *$K^*(\Omega^2 S^{2n+2}; \mathbf{Z}/p)$ is a completed tensor product of:*
(i) a divided power series algebra on ξ_1
(ii) an exterior algebra on η_0,η_1
(iii) a power series algebra on ζ_1.

It is also convenient, as an aid in calculating the p-local theory, to know the Bockstein spectral sequence (see [2]) of the second loopspace. Here the essential initial result is to be found in [14], although there it is stated for K-homology. In our terms it is:

Proposition 2.3. *For $r = 2, 3, \ldots$, let $\eta_r = \eta_1 \cdot \zeta_1^{p^{r-1}-1} \in K^1(\Omega^2 S^{2n+1}; \mathbf{Z}/p)$. Then in the Bockstein spectral sequence of $\Omega^2 S^{2n+1}$ we have*

$$E_r^\beta = E(\eta_0, \eta_r) \otimes \mathbf{Z}/p\,[[\zeta_1^{p^{r-1}}]]; \qquad d_r(\zeta_1^{p^{r-1}}) = \eta_r$$

In other words, η_0 is an integral class, while η_r is the reduction of an element of order p^r in $K^1(\Omega^2 S^{2n+1}; \mathbf{Z}_p)$. Clearly, essentially the same relations hold in $\Omega^2 S^{2n+2}$.

It is obviously more convenient to use the K-cohomology once we start investigating the free mapping space $\Lambda^2 S^{2n+2}$, since it is not an H-space, and the Hopf algebra structure is not available. We have as usual the Serre type spectral sequence of the fibration

$$\Omega^2 S^{2n+2} \to \Lambda^2 S^{2n+2} \to S^{2n+2} \tag{10}$$

with $E_2 = H^*(S^{2n+2}; K^*(\Omega^2 S^{2n+2}; \mathbf{Z}/p))$ and $E_\infty \sim K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}/p)$. The only differential is d_{2n+2}. Let ι be the generator of $H^{2n+2}(S^{2n+2}; \mathbf{Z}/p)$.

Lemma 2.2. *In the Serre spectral sequence of the fibration (11), we have:*

$$d_{2n+2}(\eta_0) = \iota \otimes \xi_1$$
$$d_{2n+2}(\xi_i) = d_{2n+2}(\zeta_1) = d_{2n+2}(\eta_1) = 0$$

Proof. The essential point is that we can compare the spectral sequence with the similar one for cohomology which is computed in [7]. By the same method, first of all, we can establish that the ξ_i's are cycles (the argument for this involves no peculiarities of cohomology). Now we use the splitting (5) to identify $K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}/p)$ with the direct sum for all k of $K^*(F_k(S^2); \mathbf{Z}/p)$ — and similarly for the cohomology. We have that η_0 has weight 2 in the corresponding decomposition of $\Omega^2 S^{2n+2}$ (into braid spaces, [CLM]); and so $d_{2n+2}(\eta_0)$ can be found by looking at the mod p K-theory of $F_2(S^2)$. This, like the cohomology, is trivial (the space is homotopy equivalent to RP_2). It follows that η_0, $\iota \otimes \xi_1$ (the only non-trivial generators of weight 2) must cancel under d_{2n+2} as stated.
ζ_1 and η_1 on the other hand have weight $2p$ and to investigate them we need to look at $H^*(F_{2p}(S^2); \mathbf{Z}/p)$. An inspection of the results and methods of [7] section 7 (in particular lemma 7.2) shows that this has Poincaré polynomial $1 + t^3 + t^{2p-2} + t^{2p-1}$. There is therefore no possibility of non-zero differentials in the Atiyah-Hirzebruch spectral

sequence (the first one would be d_{2p-1}). The rank of $K^*(F_{2p}(S^2);\mathbf{Z}/p)$ over $\mathbf{Z}/p$ is therefore 4 and it follows immediately that $d_{2n+2}(\zeta_1) = d_{2n+2}(\eta_1) = 0$.

The structure of $K^*(\Lambda^2 S^{2n+2};\mathbf{Z}/p)$ is a direct consequence of this, although it is not as simple to state as one could wish. (In particular, the topology of the K-groups has to be included.) Imitating precisely the statement in [7], we let A_d be the $(\mathbf{Z}\oplus\mathbf{Z}/2)$-graded $\mathbf{Z}/p$-module with generators:

ξ_j, $j \geq 0$
$\eta_0 \cdot \xi_j$, $j \equiv -1 \,(\mathrm{mod}\, p)$
$\iota \cdot \xi_j$, $j \equiv 0 \,(\mathrm{mod}\, p)$
$\eta_0 \cdot \iota \cdot \xi_j$, $j \geq 0$

where the notional bidegrees assigned are $(j,0)$ to ξ_j, $(2,1)$ to η_0, and $(1,0)$ to ι. The first grading is the weight, the second that of K-theory. There is a natural weight filtration on A_d, and A is the completion of A_d with respect to the corresponding topology.

Proposition 2.4. *$K^*(\Lambda^2 S^{2n+2};\mathbf{Z}/p)$ is the completed tensor product of A as defined above and $B = E(\eta_1)\otimes\mathbf{Z}/p[[\zeta_1]]$; where η_1 has bidegree $(2p,1)$ and ζ_1 has bidegree $(2p,0)$.*

The proof is immediate from lemma 2.2.

Note. Unlike the cohomology case, not all the generators as given above are obviously unique. The ξ_i's are, of course (as units in the components of the decomposition). The arguments of lemma 2.2 show that the classes we have called $\eta_0\cdot\iota$ (which is not in fact a product) and η_1, ζ_1 are well defined; hence so are all powers of ζ_1 and all products of any of them with a ξ. We can also fix $\eta_0\cdot\xi_{p-1}$ in the K-theory of the $(p+1)$-component, since it's low-dimensional; and this enables us to define $\eta_0\xi_{qp-1}$ for all q by multiplication. On the other hand, we have a certain freedom in choosing the classes $\iota\cdot\xi_{qp^r}$ in the (qp^r+1)-component, which by the arguments of §1 can be identified with $K^0(B\Gamma_{qp^r+1};\mathbf{Z}/p)$, which will be useful.

We now proceed to study the p-local theory. First we look at $\Omega^2 S^{2n+1}$. Since clearly ξ_i and η_0 are integral classes, we consider them as elements of the p-local K-theory using (without too much confusion) the same names. If q is prime to p, then it follows from proposition 2.3 that $\eta_1\cdot\zeta_1^{qp^{r-1}-1}$ is the reduction mod p of a class in $K^1(\Omega^2 S^{2n+1};\mathbf{Z}_p)$, whose weight is $2qp^r$ and whose order is p^r. Choose one such class and call it λ_{qp^r}. Let M be the topological $\mathbf{Z}_p$ module generated by 1 (in degree zero) and the λ_{qp^r}'s (in degree 1), completed with respect to the weight filtration topology. Then,

$$K^*(\Omega^2 S^{2n+1};\mathbf{Z}_p) = E(\eta_0)\otimes M \tag{11}$$

And the p-local K-theory of $\Omega^2 S^{2n+2}$ follows from this using Proposition 2.2:

Proposition 2.5. *With the above notation,*

$$K^*(\Omega^2 S^{2n+2};\mathbf{Z}_p) = R[[\xi_1,\xi_2,\ldots]]/(\xi_i\xi_j = (i,j)\xi_{i+j}; i,j=1,2,\ldots)\hat{\otimes}(E(\eta_0)\otimes M)$$

(completed tensor product).

We next proceed to the free loopspace $\Lambda^2 S^{2n+2}$. Here we have:

Lemma 2.3. *In the p-local spectral sequence of the fibration (10), the differentials are given by:*

$$d_{2n+2}(\eta_0) = \iota \otimes \xi_1$$
$$d_{2n+2}(\xi_i) = d_{2n+2}(\lambda_{qp^r}) = 0$$

for all $i, r\ (>0)$ and q prime to p.

The only difficult point concerns the λ's, which is essentially the question of whether λ_{qp^r} is in the image of the restriction $i^* : K^1(\Lambda^2 S^{2n+2}; \mathbf{Z}_p) \to K^1(\Omega^2 S^{2n+2}; \mathbf{Z}_p)$. In any case, we have *integral* classes ξ_i; and an integral class of weight 3 which corresponds to $\eta_0 \cdot \iota$. Since there is no way of expressing this as a product, we shall call it γ_3 (which clarifies the weight). To check that λ_{qp^r} survives in the spectral sequence of (10), it is enough to know that the Bockstein spectral sequence relations of proposition 2.3 still hold in $\Lambda^2 S^{2n+2}$, since then the class $\eta_1 \cdot \zeta_1^{qp^{r-1}-1}$ in $\Lambda^2 S^{2n+2}$ will be the reduction mod p of a local class of order p^r which restricts to λ_{qp^r} on the based loopspace.
To derive the Bockstein relations, we can calculate the Bockstein spectral sequence as before for the free loopspace, weight by weight, forgetting (at least for the moment) the difficult weights which are $\equiv 1 \pmod p$. In E_r^β we find that the only terms of weight $2p^r$ are those we have called η_r, $\zeta_1^{p^{r-1}}$, and integral classes. Since i^* sends the Bockstein spectral sequence of $\Lambda^2 S^{2n+2}$ to that of $\Omega^2 S^{2n+2}$, the required relation for d_r follows. There is therefore an element in $K^1(\Lambda^2 S^{2n+2}; \mathbf{Z}_p)$, which we still call λ_{p^r}, which has order p^r and weight $2p^r$, and which reduces to η_r mod p and must restrict to λ_{p^r} in $K^1(\Omega^2 S^{2n+2}; \mathbf{Z}_p)$. This implies the result for all λ_{qp^r}, q prime to p.

Lemma 2.3 gives straightforward computations of the spectral sequence as before in weights $\not\equiv 1 \pmod p$, the other case having already appeared as 'exceptional' in proposition 2.4. However, we need still more work to clarify what happens in that case. We have in $K^1(\Omega^2 S^{2n+2}; \mathbf{Z}_p)$ classes of the form $\xi_{q'p^s-1}\eta_0\lambda_{qp^r}$, with q', q prime to p; and these are torsion classes of order p^r. We also have that:

$$d_{2n+2}(\xi_{q'p^s-1}\eta_0\lambda_{qp^r}) = \iota \otimes \xi_1\xi_{q'p^s-1}\lambda_{qp^r} = (qp^s)\iota \otimes \xi_{q'p^s}\lambda_{qp^r}$$

(using the formula $(m, 1) = m+1$). A simple computation now shows that if $s \geq r$, the differential is zero and we have contributions $\mathbf{Z}/p^r$ to K^0 and K^1; while if $s < r$, the differential is non-zero, and we have two $\mathbf{Z}/p^s$'s. In the latter case, the odd component is generated by the class of $p^{r-s}\xi_{q'p^s-1}\eta_0\lambda_{qp^r}$, and the even component by $\iota \otimes \xi_{q'p^s}\lambda_{qp^r}$ as before.

Since the multiplicative structure is not so useful for the p-local K-theory as for the mod p, we state our structural result differently from proposition 2.4.

Proposition 2.6. *$K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}_p)$ is generated (as a topological module, complete with respect to the weight filtration as usual) by the following elements:*

ξ_j, $j \geq 0$
$\gamma_3 \cdot \xi_j$, $j \geq 0$
$\iota \cdot \xi_{qp^s}$, $q, s > 0$, *q prime to p (of order* p^s*)*
$\xi_j \cdot \lambda_{qp^r}$, $j \geq 0$, $q, r > 0$, *q prime to p (of order* p^r*)*
$\gamma_3 \cdot \xi_j \cdot \lambda_{qp^r}$, $j \geq 0$, $q, r > 0$, *q prime to p (of order* p^r*)*

and also by elements

$\iota \cdot \lambda_{qp^r}$
$\eta_0 \cdot \xi_{q'p^s-1} \cdot \lambda_{qp^r}$
$\iota \cdot \xi_{q'p^s} \cdot \lambda_{qp^r}$

of order p^r*, with the usual conventions, where* $s \geq r$*, and*

$p^{r-s}\eta_0 \cdot \xi_{q'\cdot p^s-1} \cdot \lambda_{q\cdot p^r}$
$\iota \cdot \xi_{q'\cdot p^s} \cdot \lambda_{q\cdot p^r}$

of order p^s*, where* $s < r$*.*
The bidegrees of generators are as in proposition 2.5; and in addition the bidegree of λ_{qp^r} *is* $(2qp^r, 1)$*.*

By analogy with the previous notation, we write A_0 for the topological module generated by the first three families in the above statement, i.e. by the elements $\xi_j, \gamma_3 \cdot \xi_j$ and $\iota \cdot \xi_{qp^s}$.

3. The auxiliary spectral sequence.

We are now in a position to begin studying the spectral sequence in K-theory (mod p and local) of the fibration

$$\Lambda^2 S^{2n+2} \to E(\eta, S^{2n}) \to BSO(3)$$

which, according to the theory explained in §1, will give the direct sum of the groups $K^*(B\Gamma^k; R)$ for $R = \mathbf{Z}/p, \mathbf{Z}_p$. In this section we shall call the sequence $\{E_r(R)\}$. We have, of course:

$$(12) \quad E_2^{r,s}(R) = H^r(BSO(3); K^s(\Lambda^2 S^{2n+2}; R)) = H^r(BSO(3); R) \otimes K^s(\Lambda^2 S^{2n+2}; R)$$

(recall that $BSO(3)$ has no torsion in its p-local cohomology) and the main task is to find what happens to the generators for $K^*(\Lambda^2 S^{2n+2})$ which we have just found. Let x be the standard generator of $H^4(BSO(3); R)$ (the pontryagin class); we shall aim for relations of form

$$d_{4r}(1 \otimes u) = x^r \otimes v$$

when u is one of the generators found in §2. Our first theorem introduces an auxiliary spectral sequence which gives useful information about the generators in the components we have called A, A_0.

Theorem 3.1. *If* $A \subset K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}/p)$, $A_0 \subset K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}_p)$ *are the submodules described in Propositions 2.4, 2.6, then there exist*
(i) spectral sequences $\{E'_r\}$, $\{E'_r(0)\}$ *with* $E'_2 = H^*(BSO(3); \mathbf{Z}/p) \otimes A$, $E'_2(0) = H^*(BSO(3); \mathbf{Z}_p) \otimes A_0$

(ii) homomorphisms of spectral sequences $j^* : \{E_r(\mathbf{Z}/p)\} \to \{E'_r\}$ *and* $\{E_r(\mathbf{Z}_p)\} \to \{E'_r(0)\}$ *which are the natural epimorphisms on* E_2*.*
Furthermore, the mod p sequence $\{E'_r\}$ *has the following structure:*
(iii) $d_4(1 \otimes \eta_0 \cdot \iota \cdot \xi_k) = x \otimes \xi_{k+3}$ *if* $k \not\equiv -3, -2, -1 \,(\mathrm{mod}\, p)$
(iv) If $k = qp^r - \alpha$ *where* $r \geq 1$, $\alpha \in \{1, 2, 3\}$ *and* q *is prime to* p*, then the first non-vanishing differential on* $\eta_0 \cdot \iota \cdot \xi_k$ *is*

$$d_{2(p^r+1)}(1 \otimes \eta_0 \cdot \iota \cdot \xi_k) = x^{\frac{p^r+1}{2}} \otimes \xi_{k+3}$$

(v) If $k = qp^r - 1$*, with* r, s *as in (iv), then the first non-vanishing differential on* $\eta_0 \cdot \xi_k$ *is* $d_{2(p^r-1)}(1 \otimes \eta_0 \cdot \xi_k) = x^{\frac{p^r-1}{2}} \otimes \iota \cdot \xi_{k+1}$

We shall leave the differentials in the p-local spectral sequence (which are on the whole simpler to calculate) until §6. Let us first note (it is important for the structure) the following fact about the modules A, A_0:

Lemma 3.1. *The component of* A *in weight* k *is generated by:*
(i) ξ_k, $\eta_0 \cdot \iota \cdot \xi_{k-3}$ *if* $k \not\equiv 1 \,(\mathrm{mod}\, p)$
(ii) $\xi_k, \iota \cdot \xi_{k-1}, \eta_0 \cdot \xi_{k-2}, \eta_0 \cdot \iota \cdot \xi_{k-3}$ *if* $k \equiv 1 \,(\mathrm{mod}\, p)$ *and* $k > 1$
(iii) $\xi_1, \iota \cdot \xi_0$ *if* $k = 1$*.*
The component of A_0 *in weight* k *is generated by:*
(i) ξ_k, $\gamma_3 \cdot \xi_{k-3}$ *if* $k \not\equiv 1 \,(\mathrm{mod}\, p)$
(ii) $\xi_k, \iota \cdot \xi_{k-1}$ *(of order as specified above),* $\gamma_3 \cdot \xi_{k-3}$ *if* $k \equiv 1 \,(\mathrm{mod}\, p)$ *and* $k > 1$
(iii) $\xi_1, \iota \cdot \xi_0$ *if* $k = 1$*.*

The proof is immediate from the definitions. What happens in weights 1, 2 is irrelevant to our main purpose, since the groups Γ are not then defined; but the spectral sequence $\{E'_r\}$ is defined nonetheless, and it is neater to work with it consistently.
The key to theorem 3.1 is the embedding in $F_k(S^2)$ — a component of $\Lambda^2 S^{2n+2}$ by (5) — of special point sets which are symmetrical under cyclic and dihedral groups; it will be in terms of such geometrical subsets that $\{E'_r\}$ will be described. Let $k \equiv \alpha \,(\mathrm{mod}\, p)$ where α is 0, 1 or 2, $k \geq p$. Consider the configuration $q(k)$ of $k - \alpha$ points arranged symmetrically around the equator of S^2, with a further α at the poles. The point is that $k - \alpha$ is positive and divisible by p — this situation is discussed in some detail in [10]. Then $q(k) \in F_k(S^2)$, and the stabilizer $C(q(k))$ of $q(k)$ in $SO(3)$ is the dihedral group $D_{k-\alpha}$ for $\alpha = 0, 2$ and the cyclic group C_{k-1} for $\alpha = 1$. Consequently we have a map from $SO(3)/C(q(k))$ to the orbit of $q(k)$, say ϕ or $\phi_k : SO(3)/C(q(k)) \to F_k(S^2)$. If $k = 0, 1$ or 2, the same construction gives more trivial sets $q(k)$ — the empty set, the north pole, and the pair of poles respectively. We find that $SO(3)/C(q(k)$ is a point ($k = 0$), an S^2 ($k = 1$) and a projective plane ($k = 2$). More generally, if k is not as above but $k \geq 3$, we can define $\phi_k : SO(3) \to F_k(S^2)$ to be the map sending g to $g.q(k)$ for some fixed 'maximally unsymmetrical' basepoint $q(k) \in F_k(S^2)$; in this case $C(q(k)) = 1$. The classes which we are interested in, as appears from the next result, are restrictions under ϕ_k^*.

Proposition 3.1. *For $k \geq 0$, ϕ_k^* maps $K^*(F_k(S^2);R)$ onto $K^*(SO(3)/C(q(k));R)$.*

Proof. We can first easily dispose of the cases where $k < 3$. From the remarks above, ϕ_k is a homeomorphism for $k = 0, 1$; while it is easy to see that the image of ϕ_2 in $F_2(S^2)$ is the set of all antipodal point-pairs, which is a deformation retract.
We now turn to the main case where $k \geq 3$. Note that the spaces $SO(3)/C(q(k))$ as defined are orientable 3-manifolds and their K-theory and cohomology are the same. Hence, comparing the cohomology and K-theory of $\Lambda^2 S^{2n+2}$, it will be enough to prove the corresponding result for cohomology in R, or the stronger:

Proposition 3.2. *For $k \geq 3$, the induced homomorphism*

$$\phi_k^* : H^t(F_k(S^2);R) \to H^t(SO(3)/C(q(k));R)$$

is an isomorphism ($t = 1, 2, 3$).

Proof. We begin with the case of H^3; H^1 and H^2 vanish for $SO(3)/C(q(k))$ unless $C(q(k))$ is cyclic, i.e. $\alpha = 1$. Both the H^3 groups in question are isomorphic to R, the first by [7], the second because the space is an orientable 3-manifold. Moreover, in each case the generator is an integral class. Our method is to study the composition

$$H^3(F_k(S^2);\mathbf{Z}) \overset{\phi_k^*}{\to} H^3(SO(3)/C(q(k));\mathbf{Z}) \overset{\pi^*}{\to} H^3(SO(3);\mathbf{Z}) \tag{13}$$

where π is the quotient map. We prove, for each $k \geq 3$,

Lemma 3.2. *The composite homomorphism $\pi^* \circ \phi_k^*$ defined above is multiplication by $k(k-1)(k-2) \in \mathbf{Z}$.*

From the lemma, Proposition 3.2 follows when $t = 3$. In fact, if $k = qp^r + \alpha$ where $r \geq 1$ and $\alpha \in \{0, 1, 2\}$, then $k(k-1)(k-2)$ is a p-local unit times p^r; if not then it is a p-local unit. The same is true of $|C(q(k))|$. Since π^* is multiplication by $|C(q(k))|$, we find that, p-locally, $\pi^* =$ (a unit times) $\pi^* \circ \phi_k^*$; so p-locally, ϕ_k^* is an isomorphism.
To prove the lemma, we start with the case $k = 3$. From the fibration $SO(3) \to F_3(S^2) \to B\Gamma^3$ and the identification of Γ^3 with $\Sigma_3 = D_3 \subset SO(3)$, we obtain that $\phi : SO(3)/D_3 \to F_3(S^2)$ is a homotopy equivalence. Hence ϕ_3^* is an isomorphism and $\pi^* \circ \phi_3^*$ is multiplication by 6.
Now we have an action of $SO(3)$ on $\Lambda^2 S^{2n+2}$, say $\mu : SO(3) \times \Lambda^2 S^{2n+2} \to \Lambda^2 S^{2n+2}$. This with the identifications of §1 gives the usual action (say μ_k) on each $F_k(S^2)$. We therefore turn our attention to $H^{2kn+3}(\Lambda^2 S^{2n+2};\mathbf{Z})$ which is generated (for n large) by the class which corresponds to our $\eta_0 \cdot \iota \cdot \xi_{k-3}$. (Remember that this is integral.) In the notation of [7] this is $\epsilon \cdot i \cdot \gamma_{k-3}$.
It is easy to see that $\phi_k \circ \pi = \mu_k \circ j_k$ where j_k is the inclusion of $SO(3) = SO(3) \times *$ in $SO(3) \times F_k(S^2)$; the basepoint being one of the appropriate symmetry type. Considering $H^*(F_k(S^2;\mathbf{Z})$ as a summand in $H^*(\Lambda^2 S^{2n+2};\mathbf{Z})$ via the identifications we have the obvious formulae in (F_k)-dimensions 0, 3:

$$\mu^*(\gamma_k) = 1 \otimes \gamma_k$$

(14) $$\mu^*(\epsilon \cdot i \cdot \gamma_{k-3}) = 1 \otimes \epsilon \cdot i \cdot \gamma_{k-3} + N(k) \cdot x_3 \otimes \gamma_k$$

where $N(k)$ is some integer. It follows (since j_k^* is $1 \otimes \varepsilon$, where ε is the augmentation in $H^*(F_k(S^2))$, which kills all elements in positive F_k-dimension) that $\pi^* \circ \phi_k^*(\epsilon \cdot i \cdot \gamma_{k-3}) = j_k^* \circ \mu_k^*(\epsilon \cdot i \cdot \gamma_{k-3}) = N(k) \cdot x_3$, and $\pi^* \circ \phi^*$ is multiplication by $\pm N(k)$. We know that $N(3) = 6$, i.e. $\mu^*(\epsilon \cdot i) = 1 \otimes \epsilon \cdot i + 6x_3 \otimes \gamma_3$. Now use the $\Lambda^2 S^{2n+2}$-product to compute $\mu^*(\epsilon \cdot i \cdot \gamma_{k-3})$ in two ways and compare; we find

$$6x_3 \otimes \gamma_3 \gamma_{k-3} = N(k) \cdot x_3 \otimes \gamma_k$$

Now from the formula $\gamma_i \gamma_j = (i,j)\, \gamma_{i+j}$ it is easy to deduce that $N(k) = k(k-1)(k-2)$, and hence the lemma.

It remains to consider the cases of $t = 1, 2$, when (as we have seen) we must have $k = qp^r + 1$ with q prime to p. In this case, $H_1(F_k(S^2); \mathbf{Z}_p)$ is $\mathbf{Z}/p^r$ (see [6]). It follows easily that $H^1(F_k(S^2); \mathbf{Z}/p) = \mathbf{Z}/p$ generated by the element we have called $\gamma_{k-1} \cdot i$, while $H^2(F_k(S^2); \mathbf{Z}/p) = \mathbf{Z}/p$ generated by $\gamma_{k-2} \cdot \epsilon$. Moreover, the two classes are connected by the Bockstein operator of order r, and their product in the mod p cohomology of $F_k(S^2)$ is the generator of H^3. This, however, is also clearly the structure of the cohomology of $SO(3)/C(q(k))$. If ϕ_k^* were zero on H^1, it would follow that it was zero on H^2, H^3. Lemma 3.2 implies therefore that it is non-zero, hence an isomorphism, on all three. It must therefore be an isomorphism with any of the rings of coefficients R.

It follows that the class which we have called $\iota \cdot \xi_{qp^r}$ restricts to the generator of $\tilde{K}^0(SO(3)/C(q(k)); R)$, for $k = qp^r + 1$. While this does not specify it completely (see note following Proposition 2.4), it goes some way towards doing so. For the p-local theory we can do better, since we have a map $H^2(F_{qp^r}(S^2); \mathbf{Z}_p) \to K^0(F_{qp^r}(S^2); \mathbf{Z}_p)$ induced by the inclusion of $BU(1)$ in BU. We can define $\iota \cdot \xi_{qp^r}$ to be the image of a generator under this map; it will then restrict to a generator in $SO(3)/C(q(k))$ as before.

Note. A more 'geometric' proof of proposition 3.2 might proceed by considering the Leray cohomology spectral sequence of the quotient map from $F_k(S^2)$ to $F_k(S^2)/SO(3)$, and showing that (a) under our hypotheses $E_2^{0,3} = H^3(SO(3)/C(q(k)))$ (this is easy), (b) this is the whole of E_∞ and so of $H^3(F_k(S^2))$. Since I have failed to do (b), I pass this idea on to anyone who would like to try it.

Now for each $k \geq 1$ define $\{E_r''\}(k)$ to be the spectral sequence in K-theory with coefficients in $\mathbf{Z}/p$ or $\mathbf{Z}_p$ of the fibration

(15) $$SO(3)/C(q(k)) \to BC(q(k)) \to BSO(3)$$

and let

$$\{E_r''\} = \bigoplus_{k=1}^{\infty} \{E_r''\}(k)$$

Since ϕ_k is a map of $SO(3)$-spaces for each k, we have a homomorphism ϕ^* from $\{E_r(\mathbf{Z}/p)\}$ resp. $\{E_r(\mathbf{Z}_p)\}$ to $\{E_r''\}$ (the sum of the components ϕ_k^*); and from proposition 3.1 we can deduce

Corollary. *ϕ^* induces an isomorphism from the quotients $H^*(BSO(3); \mathbf{Z}/p) \otimes A$, $H^*(BSO(3); \mathbf{Z}_p) \otimes A_0$ of $E_2(\mathbf{Z}/p)$, $E_2(\mathbf{Z}_p)$ to the relevant E_2''. Hence, we can define quotient spectral sequences of the $\{E_r\}$'s called $\{E_r'\}$, $\{E_r'\}(0)$ which are isomorphic to $\{E_r''\}$ and have $E_2' = H^*(BSO(3); \mathbf{Z}/p) \otimes A$ resp $E_2'(0) = H^*(BSO(3); \mathbf{Z}_p) \otimes A_0$*

Writing j^* for the natural quotient map, we have dealt with parts (i) and (ii) of theorem 3.1. It remains to compute the differentials in $\{E_r'\}$, or equivalently in $\{E_r''\}$, the spectral sequences of the fibrations (15). Because we are dealing with finite groups, these are comparatively simple. (I omit the rather trivial cases $k \leq 2$.)

Lemma 3.3. *Let $n = qp^r$, where $r > 0$ and q is prime to p, and let u be a generator of $K^1(SO(3)/D(n); \mathbf{Z}/p) = \mathbf{Z}/p$. Then, in the mod p K-theory spectral sequence of the fibration (15) with $C(q(k)) = D_n$, the only non-vanishing differential is $d_{2(p^r+1)}(1 \otimes u) = x^{\frac{p^r+1}{2}} \otimes 1$.*

Note 1. Because we have allowed $r = 0$, this includes the case $n = 1$, and so in the schema above (setting $D(1) = 1$), accounts for all weights k which are not congruent to 1 mod p. For $n = 1$ of course we have the spectral sequence of the universal $SO(3)$-bundle, with $d_4(1 \otimes u) = x \otimes 1$.

Note 2. Here and elsewhere, by 'the only non-vanishing differential is ♡' I mean, following the usual convention, that the only non-vanishing differentials are the obvious consequences of the relation ♡.

Proof. We have

$$E_2'' = H^*(BSO(3); K^*(SO(3)/D_n; \mathbf{Z}/p)) = \mathbf{Z}/p\,[x] \otimes \Lambda(u)$$

where u and x are as defined. For dimensional reasons, all differentials vanish on $x \otimes 1$. On the other hand, if some differential, necessarily $d_{4t}(1 \otimes u)$, is non-zero it must be equal to $x^t \otimes 1$ up to units; and in this case, we find immediately that $E_{4t+1} = E_\infty$ is all even and has rank t over $\mathbf{Z}/p$. Now we know that the rank of E_∞ is the rank of $K^0(BD_n) \otimes \mathbf{Z}/p$, which (cf [11]) is the number of p-power order conjugacy classes in D_n, i.e. $\frac{p^r+1}{2}$. This implies lemma 3.3.

We now turn to the more complicated case of the cyclic groups C_k.

Lemma 3.4. *Let $n = qp^r$, where $r > 0$ and q is prime to p; and let 1, v, w, $v \cdot w$ generate $K^*(SO(3)/C_n; \mathbf{Z}/p)$ (v odd, w even). Then the only non-vanishing differentials in the spectral sequence of the fibration (15) with $C(q(k)) = C_n$ (i.e. $k = n+1$) are:*

(i) $d_{2(p^s-1)}(1 \otimes v) = x^{\frac{p^r-1}{2}} \otimes w$

(ii) $d_{2(p^s+1)}(1 \otimes v \cdot w) = x^{\frac{p^r+1}{2}} \otimes 1$

Proof. The inclusion of C_n in D_n induces a map of spectral sequences, which takes $1 \otimes u$ in $E_2^{0,1}(D_n)$ to $1 \otimes v \cdot w$ in $E_2^{0,1}(C_n)$. It therefore follows from lemma 3.2 that $d_k(1 \otimes v \cdot w) = 0$ for $k < 2(p^s+1)$. (We can't deduce formula (ii) immediately, though, since the right hand side might have become zero.) Since $K^*(BC_n; \mathbf{Z}/p)$ must be finite

and all in K^0, $x^i \otimes w$ must be a d_{4i}-boundary for some i, while $1 \otimes v$ must have some non-zero differential on it.
If $d_{4t}(1 \otimes v) = x^t \otimes 1$, then necessarily $d_{4q}(1 \otimes v \cdot w) = x^t \otimes w$. This would imply that $\mathrm{rk}(E_{4q+1}) = \mathrm{rk}(E_\infty)$ is $2t$, while in fact by the preceding argument $\mathrm{rk}(E_\infty)$ is the number of p-power order conjugacy classes in C_n, i.e. p^s, which is odd. We must therefore necessarily have $d_{4t}(1 \otimes v) = x^t \otimes w$ for some t, implying $d_{4t}(1 \otimes v \cdot w) = 0$.
From this information, we can deduce that $d_{2(p^r+1)}(1 \otimes v \cdot w) = x^{\frac{p^r+1}{2}} \otimes 1$ as claimed, and $\{x^i \otimes 1; i = 0, 1, \ldots, \frac{p^r-1}{2}\}$ are linearly independent in E_∞. Counting, we see that to make up a basis of the required rank we need $\{x^i \otimes w : i = 0, 1, \ldots, \frac{p^r-3}{2}\}$, which implies formula (i).
Finally, putting together lemmas 3.3 and 3.4 and identifying the isomorphic spectral sequences $\{E'_r\}$ and $\{E''_r\}$, we deduce parts (iii)–(v) of theorem 3.1.

4. The differentials.

We can now state the main theorem about the differentials in the spectral sequence mod p; in this and the next section we shall write this simply as $\{E_r\}$. Here, and in what follows, all relations are up to units of $\mathbf{Z}/p$.

Theorem 4.1. *The generators of $K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}/p)$ behave in the following way in $\{E_r\}$:*
(i) If $k \not\equiv -3, -2, -1 \pmod p$, then $d_4(1 \otimes \eta_0 \cdot \iota \cdot \xi_k) = x \otimes \xi_{k+3}$.
(ii) If $k = qp^r - \alpha$ where $r \geq 1$, $\alpha \in \{1, 2, 3\}$ and q is prime to p, then the first non-vanishing differential on $\eta_0 \cdot \iota \cdot \xi_k$ is

$$d_{2(p^r+1)}(1 \otimes \eta_0 \cdot \iota \cdot \xi_k) = x^{\frac{p^r+1}{2}} \otimes \xi_{k+3}$$

(iii) If $k = qp^r - 1$, with q, r as in (ii), then the first non-vanishing differential on $\eta_0 \cdot \xi_k$ is $d_{2(p^r-1)}(1 \otimes \eta_0 \cdot \xi_k) = x^{\frac{p^r-1}{2}} \otimes \iota \cdot \xi_{k+1}$
(iv) The first non-vanishing differential on $\zeta_1^{p^r}$ (where $r \geq 0$) is

$$d_{2(p^{r+1}-1)}(1 \otimes \zeta_1^{p^r}) = x^{\frac{p^{r+1}-1}{2}} \otimes \zeta_1^{p^r-1} \cdot \eta_1$$

(v) All the generators ξ_k and all $\zeta_1^k \cdot \eta_1$'s are permanent cycles, for $k \geq 0$.

Clearly the relations (i)–(iii) are identical to the relations (iii)–(v) of theorem 3.1. They do not follow immediately from that result, however, since the map j^* of spectral sequences is not split. Our strategy will be to use what we know from [10] of $K^*(B\Gamma^k; \mathbf{Z}/p)$ — that is, the p-adic part. This must be part of the E_∞ term in weight k, and the rest, if any, must come from torsion. We can use induction on k to deal with each new generator of weight k as it arises and find what the first non-vanishing differential must be (if there is one).
To start with we have:

Lemma 4.1. *Every element $1 \otimes \xi_k$ is a permanent cycle.*

Proof. Interpret the component of weight k as coming from the fibration defined in §1 with fibre $F_k(S^2)$ and total space $B\Gamma^k$. Then ξ_k corresponds to the unit element and so is a permanent cycle.
We next dispose of the easy cases of parts (i)–(iii).

Lemma 4.2. *(a) The relation (i) in theorem 4.1 holds for all relevant values of k.*
(b) The relations (ii), (iii) hold when $r = 1$.
(c) If (ii) holds for $k = p^r - 3$ (if (iii) holds for $k = p^r - 1$), then it holds for all $k = qp^r - \alpha$, (for all $k = qp^r - 1$), where q is prime to p.

Proof. Again consider the spectral sequence, and the homomorphism j^* of theorem 3.1 as graded by weight. In weight $\leq 2p$ (and of course ≥ 3), j^* is an isomorphism (there are no generators of form ζ, η), and so we can deduce that the relations (i)–(iii) are true. In particular, (i) and (ii) hold on $\eta_0 \cdot \iota \cdot \xi_k \otimes 1$ for $k = 0$ and for $k = p - \alpha$ respectively; and (iii) holds on $\eta_0 \cdot \xi_k \otimes 1$ for $k = p - 1$.
To prove (a), we now use the relation which follows from lemma 4.1,

$$\begin{aligned} d_4(1 \otimes \eta_0 \cdot \iota \cdot \xi_k) &= (d_4(1 \otimes \eta_0 \cdot \iota)) \cdot (1 \otimes \xi_k) \\ &= 1 \otimes \xi_3 \cdot \xi_k = (3, k)\, x \otimes \xi_{k+3} \end{aligned}$$

(The last equality is the relation in the divided algebra.) As we have seen in §3, $(3, k)$ is non-zero mod p precisely when k is as specified in (i). Hence (a) is proved.
It remains to prove (c); then (b) will follow using what we have just shown about the cases $p - \alpha$. Now, $(qp^r, t) \neq 0$ in $\mathbf{Z}/p$ provided that the sequence of numbers $qp^r + 1, \ldots, qp^r + t$ contains no multiple of p^{r+1}. Hence $\xi_{qp^r-\alpha} = \xi_{p^r-3} \cdot \xi_{(q-1).p^r+3-\alpha}$ up to non-zero multipliers. Applying the same argument as before, suppose $d_{4t}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^r-3}) = x^t \otimes \xi_{p^r}$. Then,

$$\begin{aligned} d_{4t}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{qp^r-\alpha}) &= d_{4t}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^s-3}).\xi_{(q-1)p^r-\alpha+3} \\ &= (p^r, (q-1)p^r - \alpha + 3)(x^t \otimes \xi_{qp^r-\alpha+3}) \end{aligned}$$

And the binomial coefficient is non-zero if q is prime to p. This proves the argument for formula (ii); that for (iii) is similar.

We now come to the first of two propositions which are the main inductive steps in the proof of theorem 4.1.

Proposition 4.1. *If relations (ii), (iv), (v) of theorem 4.1 hold in weights less than p^r, then relation (ii) holds for $k = p^r - 3$ (weight p^r). That is, $d_{2(p^r+1)}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^r-3}) = x^{\frac{p^r+1}{2}} \otimes \xi_{p^r}$*

Proof. We consider the component of the spectral sequence $\{E_r\}$ in weight p^r. The fibre generators (generators of $K^*(F_{p^r}(S^2); \mathbf{Z}/p)$) are:
(a) ξ_{p^r}, $\eta_0 \cdot \iota \cdot \xi_{p^r-3}$
(b) The products of ξ_{p^r-2ps}, $\eta_0 \cdot \iota \cdot \xi_{p^r-2ps-3}$ with $\zeta_1^s, \zeta_1^{s-1}\eta_1$ for $s = 1, 2, \ldots, \frac{p^{r-1}-1}{2}$.

The generators in (a) are those involved in the proposition's statement, while the differentials on those in (b) are given by our induction hypotheses. In fact, for any given $s = qp^{t-1}$ (q prime to p, $1 \leq t < r$) we have that the first non-vanishing differentials on $\eta_0 \cdot \iota \cdot \xi_{p^r-2ps-3}$ (respectively on ζ_1^s) are $d_{2(p^t+1)}$ (respectively $d_{2(p^t-1)}$) by parts (ii) and (iv) of the theorem. The important point is that $p^r - 2ps$ and ps are divisible by the same power p^t of p under our conditions on r. The other two elements involved in the products, i.e. ξ_{p^r-2ps} and $\zeta_1^{s-1}\eta_1$ are cycles by (v).
We shall now and henceforth use notation of form (m, n) for the bigraded rank (in the lattice $\mathbf{Z} \times \mathbf{Z}$) of modules such as $K^*(X)$ or the E_r's with coefficients in a domain; such 'biranks' can and will be added and multiplied by integers where it's helpful.
We now have immediately, using the Leibniz formula:

Lemma 4.3. *Let $s = qp^{t-1}$; then the first non-vanishing differentials involving the four products in (b) above are*

$$d_{2(p^t-1)}(1 \otimes \zeta_1^r \gamma) = x^{\frac{p^t-1}{2}} \otimes \zeta_1^{r-1}\eta_1\gamma$$

where γ is either ξ_{p^r-2ps} or $\eta_0 \cdot \iota \cdot \xi_{p^r-2ps-3}$. Hence, the contribution of such elements to E_∞ is of rank at most $(\frac{p^t-1}{2}, \frac{p^t-1}{2})$.

We also have:

Lemma 4.4. *The differential $d_{2(p^{r-1}+1)}$ vanishes on $\eta_0 \cdot \iota \cdot \xi_{p^r-3}$.*

Proof. As in the proof of lemma 4.2, we have that $\xi_{p^r-3} = \xi_{p^{r-1}-3} \cdot \xi_{(p-1).p^{r-1}}$ up to non-zero multipliers; hence,

$$\begin{aligned} d_{2(p^{r-1}+1)}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^r-3}) &= (1 \otimes \xi_{p^{r-1}})(1 \otimes \xi_{(p-1).p^{r-1}}) \\ &= (p^{r-1}, (p-1)p^{r-1})\, 1 \otimes \xi_{p^r} = 0 \end{aligned}$$

since the binomial coefficient is zero.
We now consider the term $E_{2p^{r-1}+2}$ in the weight p^r part of the spectral sequence. The contribution from part (a) above is just $\mathbf{Z}/p\,[x] \otimes \{\xi_{p^r}, \eta_0 \cdot \iota \cdot \xi_{p^r-3}\}$ by lemma 4.4. On the other hand the part (b) contribution vanishes in base degree $\geq 2(p^{r-1} - 2)$. This enables us to complete the proof of proposition 4.1. In fact, let us consider d_{4m} for $m > \frac{p^{r-1}-1}{2}$. For such r, $E_{4m}^{4m,0}$ is generated by $x^m \otimes \xi_{p^r}$ (the (a) part) only, so that $d_{4m}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^r-3})$ is necessarily some multiple of $x^m \otimes \xi_{p^r}$. Eventually the differential must hit a multiple which is non-zero (in particular, it cannot have been hit by an earlier differential); since otherwise $\eta_0 \cdot \iota \cdot \xi_{p^r-3}$ would be an infinite cycle, and give rise to an infinite summand in E_∞. But the limit of the spectral sequence is equal to $K^*(B\Gamma_k; \mathbf{Z}/p) = K^*(\tilde{F}_k(S^2) \underset{\Sigma_k}{\times} E\Sigma_k; \mathbf{Z}/p)$, which is a finitely generated module over $K^*(B\Sigma_k; \mathbf{Z}/p)$ by the finiteness of orbit types, and so finite over $\mathbf{Z}/p$. Now we can use the auxiliary spectral sequence $\{E'_r\}$ of §3; $d_{4m}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{p^r-3})$ is determined by its image under the map of spectral sequences j^*, and so is zero for $r < \frac{p^r+1}{2}$, while $d_{2(p^r+1)}$ is $x^{\frac{p^r+1}{2}} \otimes \xi_{p^r}$ as claimed.
To complete the induction for theorem 4.1, we need the result which corresponds to proposition 4.1 for the products of ζ_1, η_1.

Proposition 4.2. *If relations (ii), (iv), (v) of theorem 4.1 hold in weights less than $2p^r$, then relations (iv), (v) hold for weight $2p^r$. That is,* $d_{2(p^r-1)}(1 \otimes \zeta_1^{p^{r-1}}) = x^{\frac{p^r-1}{2}} \otimes \zeta_1^{p^{r-1}-1}\eta_1$; $1 \otimes \zeta_1^{p^{r-1}-1}\eta_1$ *is a permanent cycle.*

Proof. As before, we list the generators of $K^*(F_{2p^r}(S^2); \mathbf{Z}/p))$. These are:
(a) ξ_{2p^r}, $\eta_0 \cdot \iota \cdot \xi_{2p^r-3}$
(b) $\zeta_1^{p^{r-1}}$, $\zeta_1^{p^{r-1}-1}\eta_1$
(c) All products of $\xi_{2p^r-2ps}, \eta_0 \cdot \iota \cdot \xi_{2p^r-2ps-3}$ with ζ_1^s, $\zeta_1^{s-1}\eta_1$ for $1 \leq s < p^{r-1}$
We know the first differential on (c) by the induction hypothesis; and the same applies (since we have proved Proposition 4.1) to (a). The analogous result to lemma 4.3, describing the differentials, is:

Lemma 4.5. *(i) The first non-vanishing differential on* $\eta_0 \cdot \iota \cdot \xi_{2p^r-3}$ *is*

$$d_{2(p^r+1)}(1 \otimes \eta_0 \cdot \iota \cdot \xi_{2p^r-3}) = x^{\frac{p^r+1}{2}} \otimes \xi_{2p^r}$$

(ii) Suppose $r = q.p^{t-1}$ with q prime to p, $1 \leq t < s$. Then the first non-vanishing differentials on the four products in (c) above are

$$d_{2(p^t-1)}(1 \otimes \zeta_1^s\gamma) = x^{\frac{p^t-1}{2}} \otimes \zeta_1^{s-1}\eta_1\gamma$$

where γ is either ξ_{p^r-2ps} or $\eta_0 \cdot \iota \cdot \xi_{p^r-2ps-3}$. Hence, the contribution of such elements to E_∞ is of rank at most $(\frac{p^t-1}{2}, \frac{p^t-1}{2})$.

To proceed further, we can't use the auxiliary spectral sequence as before. Instead, we need to recall the results of [10] on the p-adic K-theory of $B\Gamma^n$. Because the K-theory is profinite, it is enough to give (following [1]) the tensor product $K^*(B\Gamma^n) \otimes \mathbf{C}_p$; and to know this additively, we only need the $\mathbf{C}_p$-ranks of K^0, K^1. Writing these again as an ordered pair of integers it is convenient to use the notation

$$\mathrm{Rank}(n) = (\mathrm{rank}(K^0(B\Gamma^n) \otimes \mathbf{C}_p), \mathrm{rank}(K^1(B\Gamma^n) \otimes \mathbf{C}_p))$$

The main results of [10] can be conveniently compressed into the following statement:

Theorem 0. *Let $N(m)$ denote the ordered pair which is (k, k) when $m = 2k$ and $(k+1, k)$ when $m = 2k+1$, and let ϕ be Euler's function. Then*
(i) For q prime to p and $r \geq 1$,

$$\mathrm{Rank}(qp^r) = (1, 0) + \frac{1}{2}\sum_{i=1}^{r} \phi(p^i)N(qp^{r-i}) \tag{16}$$

(ii) $\mathrm{Rank}(qp^r + 2)$ *is the same, while* $\mathrm{Rank}(qp^r + 1)$ *is the same with the factor* $\frac{1}{2}$ *deleted.*
(iii) In all other cases, $\mathrm{Rank}(n) = (1, 0)$ *is trivial.*

For our present purpose we need only the case $n = 2p^r$ of the formula. Using the description of $N(2p^{r-i})$, this gives:

$$\text{Rank}(2p^r) = (1,0) + \frac{1}{2}\sum_{i=1}^{r} \phi(p^i)(p^{r-i}, p^{r-i}) \tag{17}$$

Returning to proposition 4.2, we now consider $E_{2p^{r-1}+2}$ as before. This has an infinite part generated by the elements (a) and (b) tensored with $\mathbf{Z}/p\,[x]$ and a finite part coming from the generators (c). Lemma 4.5 gives a bound on the rank of this finite part, as follows. There are $\phi(p^{r-t})$ numbers $s < p^{r-1}$ of form qp^{t-1}, and each, as we have seen, contributes at most $(\frac{p^t-1}{2}, \frac{p^t-1}{2})$ to the total dimension. Hence the finite part has $\mathbf{Z}/p$-rank $\leq (n_0, n_0)$ where

$$n_0 = \sum_{i=1}^{r-1} \phi(p^i)(\frac{p^{r-i}-1}{2}) \tag{18}$$

(We have written $i = r - t$ to bring the formula in line with (17).) Using the formula $\phi(p^i) = (p-1)p^{i-1}$,

$$\text{Rank}(2p^r) - (n_0, n_0) = (1,0) + \sum_{i=1}^{r} \frac{1}{2}(\phi(p^i), \phi(p^i)) = (\frac{p^r+1}{2}, \frac{p^r-1}{2})$$

Now the rank of E_∞ is equal to the rank of $K^*(B\Gamma^{2p^s}; \mathbf{Z}/p)$ which is $\geq \text{Rank}(2p^r)$. Clearly, therefore, there must be an extra summand of rank $m \geq \frac{p^r-1}{2}$ in the odd part of E_∞, and this can only arise from $\zeta_1^{p^{r-1}-1}\eta_1$ and its products by $1, x, \ldots, x^{m-1}$. Hence, $x^m \otimes \zeta_1^{p^{r-1}-1}\eta_1$ is a d_{4m}-boundary, and since we know what happens to all other generators, it must be $d_{4m}(1 \otimes \zeta_1^{p^{r-1}})$.
We know therefore that $1 \otimes \zeta_1^{p^{r-1}}$ contributes nothing to the even part of E_∞; so the even rank over $\mathbf{Z}/p$, using lemma 4.5 (i), is $n_0 + \frac{p^r+1}{2}$, i.e. is exactly equal to that obtained from the formula (17). However, this means that there cannot be any p-torsion in $K^*(B\Gamma^{2p^r})$, since torsion gives rise (by the universal coefficient theorem) to equal summands in K^0 and K^1 mod p. The rank of E_∞ is therefore exactly $\text{Rank}(2p^r) = (n_0, n_0) + (\frac{p^r+1}{2}, \frac{p^r-1}{2})$. And this implies that $m = \frac{p^r-1}{2}$, i.e. $d_{2p^r-2}(1 \otimes \zeta_1^{p^{r-1}}) = x^{\frac{p^r-1}{2}} \otimes \zeta_1^{p^{r-1}-1}\eta_1$. This completes the proof of proposition 4.2.

There is, however, a problem which we have not encountered before in deducing part (v) of theorem 4.1 from proposition 4.2. This is, that the general statement that the generators $\zeta_1^{k-1}\eta_1$ are permanent cycles does not follow immediately from the cases $k = p^r$. (For the ξ-generators the analogous statement was true for other reasons.) To analyse the general situation we consider a k of form qp^r, where q is prime to p. Inductively we can easily deduce, using proposition 4.2 and the Leibniz formula,

Lemma 4.6. *In the above situation,* $d_{4m}(\zeta_1^{qp^s-1}\eta_1) = 0$ *for all* $m \leq \frac{p^{s+1}-1}{2}$.

The important remaining step is the following:

Lemma 4.7. *In the spectral sequence of* $B\Gamma^{2qp^{r+1}}$,
(a) if $q < p$, *then* $E_{4m}^{4m,*} = 0$ *for all* $m > \frac{p^{r+1}-1}{2}$
(b) in general, $E_{4m}^{4m,*} = 0$ *for all* $m > \frac{p^{r+1}+1}{2}$.

We defer the proof of this to the next section (since it comes most naturally with the detailed calculation of the spectral sequence).

Now for any q write $q = lp + s$ where $1 \leq s < p$. If $l = 0$, then lemmas 4.6 and 4.7(a) imply that $\zeta_1^{qp^r-1}\eta_1$ is an infinite cycle. If not, then

$$d_{4m}(\zeta_1^{qp^r-1}\eta_1) = d_{4m}(\zeta_1^{lp^{r+1}}\zeta_1^{sp^r-1}\eta_1)$$

Using lemma 4.6 we can deduce that this element is a d_{4m}-cycle for all $m \leq \frac{p^{r+2}-1}{2}$; and by lemma 4.7 (b) this means that it is a d_{4m}-cycle for all m.

5. Computing the sequence.

We now have found the necessary differentials; the computation of the spectral sequence $\{E_r\}$ for $K^*(B\Gamma^k; \mathbf{Z}/p)$ using these is more or less routine. We shall divide it into three cases:
(1) $k \equiv 0$ or $2 \pmod p$
(2) $k \equiv 1 \pmod p$
(3) Not as in (1), (2);
and we consider these three in turn. First, a general result:

Lemma 5.1. *Let* $k = q.p^r + \alpha$ *where* $\alpha = 0$ *or* $2 \leq \alpha < p$. *Then the submodule of weight* k *in* $K^*(\Lambda^2 S^{2n+2}; \mathbf{Z}/p)$,*that is,* $K^*(F_k(S^2); \mathbf{Z}/p)$, *is generated by all products of form*

$$u \cdot \zeta_1^m,\ u \cdot \zeta_1^{m-1}\eta_1; \quad (m = 1, 2, \ldots, [\frac{1}{2}qp^{r-1}]) \tag{19}$$

where $u = \xi_{qp^r-2mp+\alpha}$ or $u = \xi_{qp^r-2mp+\alpha-3} \cdot \eta_0 \cdot \iota$;
and by $\xi_{qp^r+\alpha}$, $\xi_{qp^r+\alpha-3} \cdot \eta_0 \cdot \iota$

The proof is straightforward from our previous description of the basis. It should be noted that if $q = 2t$ is even and $\alpha = 0, 2$, the terms of highest weight in ζ_1 are $\xi_\alpha \cdot \zeta_1^{t.p^{r-1}}$ and $\xi_\alpha \cdot \zeta_1^{t.p^{r-1}-1}\eta_1$; there is no term here involving $\eta_0 \cdot \iota$.

Now we restrict attention to case (1). For each element of the basis, we know which is the first non-vanishing differential. The key point will be, of course, to find whether there are any others.

To begin with, consider a product of type (19). Then if $m = sp^t$ and $t < r$, by theorem 4.1 the first differential on the ξ part is $d_{2(p^{t+1}+1)}$; and the first one on the ζ part is $d_{2(p^{t+1}-1)}$. The latter therefore comes first, and accounts for all four terms, giving:

$$d_{2(p^{t+1}-1)}(1 \otimes u \cdot \zeta_1^{sp^t}) = x^{\frac{1}{2}(p^{t+1}-1)} \otimes u \cdot \zeta_1^{sp^t-1}\eta_1 \tag{20}$$

for both possible values of u. Hence the four products contribute a submodule of rank $(\frac{1}{2}(p^{t+1}-1), \frac{1}{2}(p^{t+1}-1))$ to the next stage in the spectral sequence.
However, a different situation arises if m is divisible by p^r. In this case, $qp^r - 2mp$ must have p-value exactly r, while the p-value of $2mp$ is necessarily greater; and the first non-vanishing differential on the four products is :

$$d_{2(p^r+1)}(1 \otimes \xi_{qp^r-2mp-3+\alpha} \cdot \eta_0 \cdot \iota \cdot v) = x^{\frac{1}{2}(p^r+1)} \otimes (\xi_{qp^r-2mp+\alpha} \cdot v) \tag{21}$$

where $v = \zeta_1^m$ or $\zeta_1^{m-1}\eta_1$. In this case the products contribute a submodule of rank $(\frac{1}{2}(p^r+1), \frac{1}{2}(p^r+1))$ to the next stage in the spectral sequence.

From these considerations, we can prove lemma 4.7. In fact we see that if $\alpha = 0, 2$, the products of type (19) are all accounted for by the differentials up to $d_{2(p^r+1)}$; while if $q < p$, $d_{2(p^r-1)}$ will do. This is enough to prove the lemma.

What possibilities remain for non-vanishing differentials in the spectral sequence? These can only arise when one of the products (19) which is annihilated under the rules (20), (21) has a non-vanishing later differential. In the case of the products from (21), i.e. $\xi_{qp^r-2mp+\alpha} \cdot v$ this is impossible; we have already seen that $d_{2(p^r+1)}$ which vanishes on these is the last possible non-vanishing differential. It remains to deal with the products in (20), that is to show:

Lemma 5.2. *Suppose u is as in lemma 5.1, $m = sp^t$, s prime to p and $t < r$. Then $d_i(1 \otimes u \cdot \zeta_1^{sp^t-1}\eta_1) = 0$ for all i.*

Proof. We begin by noting that by theorem 4.1(v), half the given generators are permanent cycles anyway (those where $u = \xi_{q.p^r-2mp+\alpha}$). The remainder are necessarily of form:

$$1 \otimes \xi_{(jp-s)p^{t+1}+\alpha-3} \cdot \eta_0 \cdot \iota \cdot \zeta_1^{sp^t-1}\eta_1 \tag{22}$$

where $j > 0$. If $j = 1$, the generators are of weight exactly p^{t+2}; the result is true for these by proposition 4.1. In general, though, the generator (22) (call it z_j) is the product $\xi_{(j-1)p^{t+2}} \cdot z_1$ by the usual rules for multiplication of the ξ_i's. Since, again from theorem 4.1, all differentials vanish on $\xi_{(j-1)p^{t+2}}$, the result follows.
Our investigation of case (1) concludes with the following result:

Theorem 5.1. *Let q be prime to p, $r \geq 1$, and $\alpha = 0, 2$. Then*

$$\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+\alpha}; \mathbf{Z}/p)) = \mathrm{rk}_{\mathbf{Q}_p}(K^*(B\Gamma^{qp^r+\alpha}; \mathbf{Q}_p)) + (m, m) \tag{23}$$

where m is the integral part of $\frac{q}{2p}$ (note that $\frac{q}{2p}$ cannot be an integer).
Accordingly, there are m p-torsion summands in the integral K-theory of $B\Gamma^{qp^r+\alpha}$ (cyclic groups of p-power order).

Note that this result does not say how the summands are distributed as between K^0 and K^1, nor what their order is.

Proof. Our strategy is a simple one, already employed in the proof of proposition 4.2: namely, to find the rank of the E_∞ term in the spectral sequence, and to compare it with the rank obtained from Theorem 0 above. By the observation following lemma 5.1, the cases of odd and even q need different treatment; let us first suppose q odd. We have the following types of contribution to E_∞, by lemmas 5.1 and 5.2:

$$(A) \qquad x^i \otimes \xi_{qp^r+\alpha} \quad (i = 0, \ldots, \frac{1}{2}(p^r - 1))$$

$$(B_t) \qquad x^i \otimes \xi_{qp^r-2sp^t+\alpha} \cdot \zeta_1^{sp^{t-1}-1} \cdot \eta_1 \quad (i = 0, \ldots, \frac{1}{2}(p^t - 3))$$

$$(C_t) \qquad x^i \otimes \xi_{qp^r-2sp^t+\alpha-3} \cdot \eta_0 \cdot \iota \cdot \zeta_1^{sp^{t-1}-1} \cdot \eta_1 \quad (i = 0, \ldots, \frac{1}{2}(p^t - 3))$$

where $0 < t \le r$ and s is prime to p, $2s < qp^{r-t}$; and

$$(D) \qquad x^i \otimes \xi_{qp^r-2sp^{r+1}+\alpha} \cdot \zeta_1^{sp^r} \quad (i = 0, \ldots, \frac{1}{2}(p^r - 1))$$

$$(E) \qquad x^i \otimes \xi_{qp^r-2sp^{r+1}+\alpha} \cdot \zeta_1^{sp^r-1} \cdot \eta_1 \quad (i = 0, \ldots, \frac{1}{2}(p^r - 1))$$

where now the only restriction on s is that $2ps < q$. (The strict inequalities here and above are because q is odd.)

The cases (D) and (E) correspond to the situation of formula (21); and it is these ones which contribute the torsion. The elements (A) are entirely in even degree, while the others come in pairs ((B) with (C) and (D) with (E)) of equal parts in even and odd. Reckoning up the contributions to E_∞, and so to $K^*(B\Gamma_{qp^r+\alpha}; \mathbf{Z}/p)$, we find:

(A): contributes a submodule of rank $(\frac{1}{2}(p^r + 1), 0)$

(B_t), (C_t), where $t < r$: there are $\frac{1}{2}q\phi(p^{r-t}) = \frac{p-1}{2}qp^{r-t-1}$ choices of s, and each contributes a submodule of rank $(\frac{1}{2}(p^t - 1), \frac{1}{2}(p^t - 1)$ to E_∞.

(B_r), (C_r): the calculation is the same, except that there are only $\left[\frac{q}{2}\right] - \left[\frac{q}{2p}\right]$ choices of s. Note that in fact $\left[\frac{q}{2}\right] = \frac{q-1}{2}$.

(D), (E): there are $\left[\frac{q}{2p}\right]$ choices of s, and each contributes a submodule of rank $(\frac{1}{2}(p^r + 1), \frac{1}{2}(p^r + 1))$.

None of these terms corresponds exactly to those in the sum of theorem 0; however, they are sufficiently near to make a term by term comparison worthwhile. We have from equation (16) of theorem 0,

$$(24) \qquad \mathrm{rk}_{\mathbf{Q}_p}(K^*(B\Gamma^{qp^r+\alpha}; \mathbf{Q}_p)) = (1,0) + \frac{1}{2}\sum_{i=1}^{r} \phi(p^i) N(qp^{r-i})$$

$$= (1,0) + \frac{1}{2}\sum_{i=1}^{r} \phi(p^i)(\frac{qp^{r-i}+1}{2}, \frac{qp^{r-i}-1}{2})$$

while the calculations above (setting $i = r - t$) give

$$(25)\quad \mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+\alpha};\mathbf{Z}/p)) = (\frac{p^r+1}{2},0) + \frac{1}{2}\sum_{i=1}^{r} q.\phi(p^i)(\frac{p^{r-i}-1}{2},\frac{p^{r-i}-1}{2}) + (\frac{q-1}{2})(\frac{p^r-1}{2},\frac{p^r-1}{2}) + ([\frac{q}{2p}],[\frac{q}{2p}])$$

Here we have rearranged the formula by noting that the terms (D), (E) are just one more than the terms $(B)_r$, $(C)_r$; we have grouped the main part of the terms together (last term but one in (25)) and separated out the excess as the last term.
Now, the difference between the even and odd ranks in (25) is clearly $\frac{p^r+1}{2}$, while that in (24) is

$$1 + \frac{1}{2}\sum_{i=1}^{r}\phi(p^i) = 1 + \frac{p-1}{2}(1+p+p^2+\ldots+p^{r-1})$$

which is also equal to $\frac{p^r+1}{2}$. Next, the last term (the excess) in formula (25) is equal to the difference (m,m) between the $\mathbf{Z}/p$-rank and the $\mathbf{Q}_p$-rank as claimed in theorem 5.1. Hence, to establish the theorem, it is sufficient to show that the odd degree component of formula (24) is equal to the odd degree component of formula (25) after the last term has been left out; that is,

Lemma 5.3. *For q odd, we have*

$$\frac{1}{2}\sum_{i=1}^{r}\phi(p^i)(\frac{qp^{r-i}-1}{2}) = \frac{1}{2}\sum_{i=1}^{r} q.\phi(p^i)(\frac{p^{r-i}-1}{2}) + (\frac{q-1}{2})(\frac{p^r-1}{2})$$

Proof. The difference between the two sides in the formula is (subtracting the terms which cancel immediately):

$$\frac{1}{4}[-\sum_{i=1}^{r}\phi(p^i) + q.\sum_{i=1}^{r}\phi(p^i) - (q-1)(p^r-1)]$$

Using the familiar formula $\phi(p)+\ldots+\phi(p^r) = p^r - 1$, the lemma follows immediately; and so therefore does theorem 5.1 in the odd case.
We next consider the case where q is even, say $q = 2l$. In that case, the list $(A)-(E)$ of contributions to E_∞ needs to be supplemented by one other. In fact, in (B_r), but *not* in (C_r), we have the possibility that $2s = q$, giving a term which is different from the others:

$$(F)\qquad x^i \otimes \xi_\alpha \cdot \zeta_1^{p^{r-1}-1} \cdot \eta_1 \quad (i = 0,\ldots,\frac{1}{2}(p^r-3))$$

This contributes an odd module, of rank $(0, \frac{1}{2}(p^r-1)$. And, in enumerating the contribution from type B_r, we must replace $[\frac{q}{2}](= l)$ by $l-1$, since the case $s = l$ is different.

Now we have a similar comparison to make between these results and those of theorem 0. They give:

$$\text{(26)} \qquad \mathrm{rk}_{\mathbf{Q}_p}(K^*(B\Gamma^{qp^r+\alpha};\mathbf{Q}_p)) = (1,0) + \frac{1}{2}\sum_{i=1}^{r}\phi(p^i)N(qp^{r-i})$$
$$= (1,0) + \frac{1}{2}\sum_{i=1}^{r}\phi(p^i)(\frac{qp^{r-i}}{2},\frac{qp^{r-i}}{2})$$

in the even case, while

$$\text{(27)} \quad \mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+\alpha};\mathbf{Z}/p)) = (\frac{p^r+1}{2},0) + \frac{1}{2}\sum_{i=1}^{r} q.\phi(p^i)(\frac{p^{r-i}-1}{2},\frac{p^{r-i}-1}{2})$$
$$+ (\frac{q}{2}-1)(\frac{p^r-1}{2},\frac{p^r-1}{2}) + (0,\frac{p^r-1}{2})$$
$$+ ([\frac{q}{2p}],[\frac{q}{2p}])$$

Once again, we can compare the two; we find (using, basically, the calculation of lemma 5.3.) that the rhs in (27) exceeds that in (26) by

$$-\frac{q}{4}(\phi(p)+\ldots+\phi(p^r))\cdot(1,1) + \frac{q}{4}(p^r-1,p^r-1) + (m,m) = (m,m)$$

where as before $m = \left[\frac{q}{2p}\right]$. This completes the proof of theorem 5.1.

Having found the spectral sequence in case (1) ($\alpha = 0,2$), we proceed to deal more briefly with the other two cases. Case (2), or $\alpha = 1$ is more complicated (compare case (iii) of theorem 4.1). Let us first give the result which corresponds to lemma 5.1.

Lemma 5.4. *Suppose $k = qp^r + 1$. Then the submodule of elements of weight k in $K^*(\Lambda^2 S^{2n+2};\mathbf{Z}/p)$,that is, $K^*(F_k(S^2);\mathbf{Z}/p)$, is generated by all products of form*

$$\text{(28)} \qquad u\cdot\zeta_1^m,\ u\cdot\zeta_1^{m-1}\eta_1; \quad (m = 1,2,\ldots,[\frac{1}{2}qp^{r-1}])$$

where $u = \xi_{qp^r-2mp+1}$ or $\xi_{qp^r-2mp}\cdot\eta_0$ or $\xi_{qp^r-2mp-1}\cdot\iota$ or $\xi_{qp^r-2}\cdot\eta_0\cdot\iota$ and by ξ_{qp^r+1}, $\xi_{qp^r}\cdot\eta_0$, $\xi_{qp^r-1}\cdot\iota$, $\xi_{qp^r-2}\cdot\eta_0\cdot\iota$.

If we now use theorem 4.1 as before to find the first non-vanishing differential, we find that exactly the same results as before hold, for the same reasons, for the elements where $u = \xi_{qp^r-2mp+1}$ or $\xi_{qp^r-2mp-2}\cdot\eta_0\cdot\iota$. We have to be more careful with the other two types of u, since in the 'generic' case the non-vanishing differentials on the ξ part and the ζ part occur at the same time. If we compare $v_p(qp^r-2mp)$ and $v_p(mp)$, we find that
(i) when $v_p(qp^r-2mp) > v_p(mp) = t+1$, the first non-vanishing differential on all products is $d_{2(p^{t+1}-1)}$ and the homology is generated by products $\xi_{qp^r-2mp}\cdot\eta_0\cdot\zeta_1^{m-1}\eta_1$, $\xi_{qp^r-2mp-1}\cdot\iota\cdot\zeta_1^{m-1}\eta_1$;

(ii) when the two p-values are both equal to $t+1$, the first non-vanishing differential on all products is again $d_{2(p^{t+1}-1)}$, and this time the homology is generated by the two cycles $\xi_{qp^r-2mp}\cdot\eta_0\cdot\zeta_1^{m-1}\cdot\eta_1-\xi_{qp^r-2mp-1}\cdot\iota\cdot\zeta_1^m$ and $\xi_{qp^r-2mp-1}\cdot\iota\cdot\zeta_1^{m-1}\cdot\eta_1$ (even and odd respectively);
(iii) when $r=v_p(qp^r-2mp)<v_p(mp)$ then the first non-vanishing differential on all products is $d_{2(p^r-1)}$, and the homology generators are $\xi_{qp^r-2mp}\cdot\iota\cdot\zeta_1^m$ and $\xi_{qp^r-2mp-1}\cdot\iota\cdot\zeta_1^{m-1}\cdot\eta_1$.
We next prove, following the methods of lemma 5.2, that there are no other differentials; this follows essentially the same lines and we shall omit the proof. And finally, it remains to relate the E_∞ which results to the torsion-free K-theory as given by theorem 0. Here the result is perhaps a surprise: the torsion part, unlike the rest of the K-theory, is not doubled.

Theorem 5.2. *Let q be prime to p, $r\geq 1$. Then*

$$\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+1};\mathbf{Z}/p))=\mathrm{rk}_{\mathbf{Q}_p}(K^*(B\Gamma^{qp^r+1};\mathbf{Q}_p))+(m,m) \tag{29}$$

where m is the integral part of $\frac{q}{2p}$.

Proof. We shall just give the case where q is odd, that where q is even being similar. If we look at the list of terms $(A),\ldots,(E)$ in E_∞, we see that each needs to be supplemented by a term involving ι or η_0 (or some more complicated expression). From our computations above, each of the new terms matches the corresponding old term exactly, except that:
(a) in (A), the number of terms of form $x^i\otimes\xi_{qp^r-1}\cdot\iota$ is $\frac{p^r-1}{2}$, i.e. one less than the number of terms of form $x^i\otimes\xi_{qp^r+1}$;
(b) in each exceptional case (D), (E), (m is a multiple of p^r) the rank of the new module is smaller by $(1,1)$.
From these two facts, we see that the rank of $K^*(B\Gamma^{qp^r+1};\mathbf{Z}/p)$ is obtained by doubling $\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r};\mathbf{Z}/p)$ and subtracting $(1,0)$ (from the (A)-terms), together with $(1,1)$ for each s which occurs in the definition of the (D), (E) terms. That is:

$$\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+1};\mathbf{Z}/p))=2.\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp^r+1};\mathbf{Z}/p))-(1,0)-([\frac{q}{2p}][\frac{q}{2p}])$$

From this, and from the statement of theorem 0 in the case $\alpha=1$, theorem 5.2 follows immediately.

The last case of the spectral sequence to be dealt with — case (3) where $k=qp+\alpha$ and $2<\alpha<p$ — is the easiest. (It of course only occurs when $p>3$.) Here there is, as we shall see, only one differential, d_4, to consider. The basis of $K^*(F_k(S^2);\mathbf{Z}/p)$ is given by lemma 5.1 (formula (19)) and we have the following simple relations:

$$\begin{aligned} d_4(1\otimes\xi_{qp-2mp+\alpha-3}\cdot\eta_0\cdot\iota\cdot v)&=x\otimes\xi_{qp-2mp+\alpha}\cdot v\\ d_4(1\otimes\xi_{qp+\alpha-3}\cdot\eta_0\cdot\iota)&=x\otimes\xi_{q.p+\alpha}\end{aligned} \tag{30}$$

(Recall that v is either ζ_1^m or $\zeta_1^{m-1}\cdot\eta_1$, one even and the other odd.) It is easy from this to deduce that E_4 is generated by all elements of form $1\otimes\xi_{q.p-2mp+\alpha}\cdot v$ and by $1\otimes\xi_{q.p+\alpha}$; and that there are no further differentials. Hence, (since, by theorem 0, the p-adic K-theory is trivial) we have the very simple parallel to theorems 5.1 and 5.2:

Theorem 5.3. *Let $2 < \alpha < p$. Then*

(31) $$\mathrm{rk}_{\mathbf{Z}/p}(K^*(B\Gamma^{qp+\alpha};\mathbf{Z}/p)) = \mathrm{rk}_{\mathbf{Q}_p}(K^*(B\Gamma^{qp+\alpha};\mathbf{Q}_p)) + (m,m)$$

where m is the integral part of $\frac{q}{2}$. In consequence, the p-torsion part of the group $K^*(B\Gamma^{qp+\alpha};\mathbf{Z}_p)$ is a sum of $\left[\frac{q}{2}\right]$ cyclic components.

6. The p-local calculations.

We now embark on the p-local case, which will give us the size and location of the torsion components as well as some information about generators. For this, besides the additive structure, it will be important to look at the structure of $K^*(B\Gamma^n;\mathbf{Z}_p)$ as an algebra over $K^*(BSO(3);\mathbf{Z}_p)$ (via the identification of $K^*(B\Gamma^n;\mathbf{Z}_p)$ with $K^*(F_n(S^2)\times_{SO(3)} ESO(3);\mathbf{Z}_p)$). We shall denote the 'ground ring' $K^*(BSO(3);\mathbf{Z}_p)$ by S in this section. We consider the analogue over $\mathbf{Z}_p$ of the spectral sequence used in the preceding sections. Recall that the structure of $K^*(\Lambda^2 S^{2n+2};\mathbf{Z}_p)$ was found in section 2 (proposition 2.6). If we concentrate on a particular weight, then we have:

Lemma 6.1. *(i) Let $k = q.p^r + \alpha$ where $0 \le \alpha < p$ and $\alpha \ne 1$. Then the submodule of weight k in $K^*(\Lambda^2 S^{2n+2};\mathbf{Z}_p)$, that is $K^*(F_k(S^2);\mathbf{Z}_p)$, is generated by all products of form*

(32) $$u\cdot\lambda_m; \qquad (m = 1,2,\ldots,[\frac{1}{2}q.p^{r-1}].p)$$

where $u = \xi_{q.p^r-2m+\alpha}$ or $u = \xi_{q.p^r-2m+\alpha-3}\cdot\gamma_3$ and the products are torsion classes of the appropriate order (depending on the p-value of m);
and by $\xi_{q.p^r+\alpha}, \xi_{q.p^r+\alpha-3}\cdot\gamma_3$ which are torsion-free classes.
(ii) Let $k = q.p^r + 1$, and for $m = q'p^s$, let $a(m,r) = p^{s-r}$ if $s > r$, $a(m,r) = 1$ otherwise. Then the submodule of weight k in $K^*(\Lambda^2 S^{2n+2};\mathbf{Z}_p)$ is generated by the terms in (32) and by

(33) $$\xi_{q.p^r-2m}\cdot\iota\cdot\lambda_m \quad (m = 1,2,\ldots,[\frac{1}{2}q.p^{r-1}].p)$$
$$a(m,r)\xi_{q.p^r-2m-1}\cdot\eta_0\cdot\lambda_m$$

of order $p^{\min(r,s)}$;
and by $\xi_{q.p^r+1}, \xi_{q.p^r-2}\cdot\gamma_3$, (torsion-free) and $\xi_{q.p^r-1}\cdot\iota$ (of order p^r).
If $q = 2t$ is even, then the 'top λ', $\lambda_{t.p^{r-1}}$ appears only multiplied by $u = \xi_\alpha$ (in case (i), $\alpha = 0,2$) and only by ξ_1 or ι (in case (ii)). (Compare lemma 5.1.)

It should be noted that the classes $u\cdot\lambda_m$ are in K^1 (resp. K^0) when u is of form ξ_i (resp. $\xi_i\cdot\gamma_3$).

Now we begin consideration of the spectral sequence

$$\{E_n(\mathbf{Z}_p)\} : H^s(BSO(3);K^\alpha(\Lambda^2 S^{2n+2};\mathbf{Z}_p)) \Rightarrow K^*(\Lambda^2 S^{2n+2}\times_{SO(3)} ESO(3);\mathbf{Z}_p)$$

which will give the p-local K-theory of the $B\Gamma_n$'s. We start with a relatively simple result which describes the behaviour of the torsion-free classes:

Lemma 6.2. *The first non-trivial differential d_4 in the spectral sequence is non-vanishing on the class $\xi_k \cdot \gamma_3$ for every k, and satisfies:*

$$\begin{aligned} d_4(\xi_{qp^r-3+\alpha} \cdot \gamma_3) &= p^r.x \otimes \xi_{qp^r+\alpha} \quad (\alpha = 0, 1, 2) \\ &= x \otimes \xi_{qp^r+\alpha} \quad (2 < \alpha < p) \end{aligned}$$

(up to units); and $d_n(\xi_k) = 0$ for all n, k.
Also, $d_4(\xi_{qp^r} \cdot \iota) = 0$ (case $\alpha = 1$).

Proof. This is analogous to lemmas 4.1 and 4.2 (a). As before, we have that ξ_k is a permanent cycle; while $d_4(\gamma_3) = x \otimes \xi_3$ (the result was in fact proved over $\mathbf{Z}_p$). Hence, $d_4(\xi_k \cdot \gamma_3) = x \otimes \xi_3.\xi_k$. Once again, $\xi_3.\xi_k = (3, k)\xi_{k+3}$, which implies the formulae of the lemma as regards γ_3. In the case $\alpha = 1$, we have of course $d_4(\iota) = 0$ in the sequence of weight 1, from which the general case follows.

Accordingly, $\xi_k \cdot \gamma_3$ is killed in E_4, and fails to provide any generators for the p-local K-theory. For each $i > 0$, we have an infinite cycle $x^i \otimes \xi_k$ of order at most p^r; however, this is a priori an upper bound, since it is possible that some of these elements (for $k > 1$) could be d_{4k}-boundaries. ξ_k survives to E_∞, and there is a generator in $K^0(B\Gamma_k; \mathbf{Z}_p)$ which corresponds to it — it is of course again the unit in the K-theory of $B\Gamma_k$. Let us call it $\tilde{\xi}_k \in K^0(B\Gamma^k; \mathbf{Z}_p)$.
To clarify the situation further, let us consider $K^*(\Lambda^2 S^{2n+2} \times_{SO(3)} ESO(3); \mathbf{Z}_p)$ as a module over $S = K^*(BSO(3); \mathbf{Z}_p)$. The latter is, as we know [5] $R(SO(3))\hat{} \otimes \mathbf{Z}_p = R(Spin(3))\hat{} \otimes \mathbf{Z}_p$ (completed representation ring). Set θ equal to the basic representation of $Spin(3)$ (of rank 2) and let $\tilde{\theta} = \theta - 2$; then $S = \mathbf{Z}_p[[\tilde{\theta}]]$. We see from the spectral sequence that the S-module $S.\xi_k$ is filtered, with corresponding graded module coming from the component $(\mathbf{Z}_p[x]/(p^r x) \otimes \xi_k)/(\text{boundaries})$ of E_∞. Our crucial next step is to determine the structure of this module — (and, in passing, to eliminate the question of non-zero boundaries). We shall consider the case $\alpha = 1$ (which is special as usual) separately.

Proposition 6.1 *(i) Let $k = qp^r + \alpha$ as before, where $\alpha \neq 1$. Then the module $S.\xi_k$ is isomorphic to $K^*(BD_{k-\alpha}; \mathbf{Z}_p) = R(D_{k-\alpha})\hat{} \otimes \mathbf{Z}_p$ if $\alpha = 0, 2$, and is trivial (=$\mathbf{Z}_p.1$) if $2 < \alpha < p$.*
(ii) No non-zero element $p^s.x^i \otimes \xi_k$ is a boundary in the spectral sequence, for $i > 1$.
(iii) If $\alpha = 0, 2$, the annihilator of ξ_k in S is (the ideal generated by) a monic polynomial $F_r(\tilde{\theta})$ in $\tilde{\theta}$ of degree $\frac{p^r+1}{2}$ whose lowest non-vanishing term is $p^r.\tilde{\theta}$; if $2 < \alpha < p$, it is just $\tilde{\theta}$.
(iv) $K^*(B\Gamma^k; \mathbf{Z}_p)$ is an algebra over the ring $S.\xi_k$ as described in (i), (iii); in particular, the polynomial F_r annihilates every element of $K^*(B\Gamma^k; \mathbf{Z}_p)$ for $\alpha = 0, 2$, while if $2 < \alpha < p$, every element of $K^*(B\Gamma^k; \mathbf{Z}_p)$ is annihilated by $\tilde{\theta}$.

Proof. By theorem 3.1 (ii), we have a map j^* of spectral sequences from $E_r(\mathbf{Z}_p)$ to the spectral sequence of the fibration

(33) $$SO(3)/C(q(k)) \to SO(3)/C(q(k)) \times_{SO(3)} ESO(3) \to BSO(3)$$

which converges to $K^*(BC(q(k));\mathbf{Z}_p)$. Now, if for some least $i>0$ there is a relation $d_{4i}(u)=p^s.x^i\otimes\xi_k$, $(u\in K^1(F_k(S^2);\mathbf{Z}_p))$, then $d_{4i}(j^*(u))=p^s.x^i\otimes 1$ in the spectral sequence of $BC(q(k))$. However, $j^*(u)$ is necessarily 0, so p^s must be. This proves (ii). Hence, the submodule over $H^*(BSO(3);\mathbf{Z}_p)$ which ξ_k generates in E_∞ is exactly $\mathbf{Z}_p[x]/(p^r x)\otimes\xi_k$.
We see therefore that the map of spectral sequences which j^* defines is an isomorphism not only on the part of E_2 generated by $\xi_k, \xi_{k-3}\cdot\gamma_3$ (the 'A_0-component' of section 3), but on the corresponding part of E_∞ as well. Hence, the map of S-modules from $S.\xi_k$ to $K^*(BC(q(k));\mathbf{Z}_p)$ is an isomorphism on the associated graded modules, and so an isomorphism. Since $C(q(k))=D_{k-\alpha}$ for $\alpha=0,2$ and $=1$ in the other cases, this proves part (i).
For part (iii), we note that first by a standard argument (cf. [3]) the p-adic part of $K^*(BD_{qp^r})$ is isomorphic to $K^*(BD_{p^r})$; so, in considering the p-local theory, we need only consider the latter. If we consider the group extension spectral sequence $H^*(BC(2);K^*(BC(p^r);\mathbf{Z}_p)\Rightarrow K^*(BD_{p^r};\mathbf{Z}_p)$, we see that $K^*(BD_{p^r};\mathbf{Z}_p)$ is naturally identified with the invariants of the involution on $K^*(BC(p^r);\mathbf{Z}_p)$. The latter is generated by an element $\tilde{\phi}=\phi-1$ where ϕ is a character of order p^r, and can be described [3] as $\mathbf{Z}_p[[\tilde{\phi}]]/((1+\tilde{\phi})^{p^r}-1)$; while the involution takes ϕ to ϕ^{-1}.
Let $\psi=\phi+\phi^{-1}$, so $\tilde{\psi}=\psi-2\in K^0(BD_{p^r};\mathbf{Z}_p)$. It is now easy to see that ψ generates the relevant part of the representation ring of D_{p^r}, so that $K^*(BD_{p^r};\mathbf{Z}_p)$ is a quotient of $\mathbf{Z}_p[[\tilde{\psi}]]$. The sum $\psi_t=\phi^t+\phi^{-t}$ for $t\geq 1$ can be expressed as a polynomial of degree t in ψ_1, which is related to the usual expression for $\cos(tx)$. However, by considering their expressions in terms of ϕ,ϕ^{-1}, we can see that $\psi_t=\psi_{p^r-t}$. Hence, while $1,\psi_1,\ldots,\psi_{\frac{p^r-1}{2}}$ are linearly independent, $\psi_{\frac{p^r+1}{2}}=\psi_{\frac{p^r-1}{2}}$; and when translated as a function of ψ or $\tilde{\psi}$, this is a polynomial relation of minimum degree.
We can express this more explicitly as a function of $\tilde{\psi}$ (which is what we want) as follows. Put $\psi=2\cos(x)$, then (a) $\psi_t=2\cos(tx)$ and (b) $\tilde{\psi}=-4\sin^2(x/2)$. Our relation becomes

$$\cos(\frac{p^r+1}{2}x)-\cos(\frac{p^r-1}{2}x)=-2\sin(\frac{x}{2})\sin(\frac{p^r x}{2})=0$$

which can be written as a polynomial of degree $\frac{p^r+1}{2}$ in $\tilde{\psi}=-4\sin^2(\frac{x}{2})$, since p^r is odd. Since the relation is satisfied when $x=0$, the polynomial has no constant term. It is also clear (e. g. by differentiating) that the term of lowest degree is $p^r\tilde{\psi}$ (up to units). Finally, using $\sin(pu)=\frac{1}{2i}(e^{pu}-e^{-pu})$, we see that $\sin(pu)\equiv(2i)^{p-1}(\sin^p(u))\,(\mathrm{mod}\,p)$, so the polynomial is monic up to $\mathbf{Z}_p$-units (and all the coefficients apart from the leading one are zero mod p). Since the restriction from $R(SO(3))$ to $R(D_{p^r})$takes θ to ψ, this proves part (iii). Part (iv) is now an immediate consequence.

It should be noted that the result above is slightly unexpected. There is no reason to expect that the *map* $B\Gamma^k\to BSO(3)$ factors through $BC(q(k)$ to give a splitting of the inclusion j, even when p-completed. However, from the viewpoint of K-theory it does.

We next consider the slightly different case where $\alpha=1$; and here it is best to modify the approach, principally because $R(SO(3)_p\hat{}$ no longer maps onto $R(C(q(k)))_p\hat{}$, as we shall see. Instead, when $k = qp^r + 1$, we consider directly a map from $B\Gamma^k$ to $BSO(2)$. This will involve something of a detour.

Proposition 6.2. *(i) The map* $u : B\Gamma^{qp^r+1} \to BSO(3)_p$ *factors through the inclusion* $Bi : BSO(2)_p \to BSO(3)_p$ *via a map* $v : B\Gamma^{qp^r+1} \to BSO(2)_p$*:* $u = Bi \circ v$*;*
(ii) If $\tilde{\tau}$ *is the class of the inclusion* $BSO(2)_p = BU(1)_p \to BU_p \in K^0(BSO(2)_p; \mathbf{Z}_p)$*,* $v^*(\tilde{\tau})$ *restricts to the class* $\xi_{qp^r} \cdot \iota \in K^0(F_{qp^r+1}(S^2); \mathbf{Z}_p)$*.*

Proof. Let $\Gamma^{qp^r,1} \subset \Gamma^{qp^r+1}$ be the subgroup of diffeomorphisms leaving one point fixed (which we may take to be the north pole). Clearly (a) $\Gamma^{qp^r,1}$ is a subgroup of index $qp^r + 1$ in Γ^{qp^r+1}, so that $B\Gamma^{qp^r,1} \xrightarrow{\pi} B\Gamma^{qp^r+1}$ is a (qp^r+1)-fold covering; (b) the map $u \circ \pi$ factors through $v' : B\Gamma^{qp^r,1} \to BSO(2)_p$. (This is because we can interpret u as coming from the 'inclusion' of Γ^{qp^r+1} in $Diff^+(S^2) \simeq SO(3)$.) We thus have a commutative diagram:

$$
\begin{array}{ccc}
B\Gamma^{qp^r,1} & \xrightarrow{v'} & BSO(2)_p \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle Bi}\downarrow \\
B\Gamma^{qp^r+1} & \xrightarrow{v} & BSO(3)_p
\end{array} \tag{34}
$$

Since $BSO(2)_p$ classifies $H^2(\ \ ;\mathbf{Z}_p)$, v' can be regarded as an element of the group $H^2(B\Gamma^{qp^r,1}; \mathbf{Z}_p)$, and part (i) of the proposition will follow from:

Lemma 6.3. *The map* π *in the diagram (34) induces an isomorphism on* $H^2(\ \ ;\mathbf{Z}_p)$*.*

Proof. The transfer of the covering on H^2, π_*, satisfies $\pi_* \circ \pi^* = qp^r + 1$ (degree of the covering). Since this is a unit in $\mathbf{Z}_p$, π^* is an injection onto a direct summand. We know that $H^2(B\Gamma^{qp^r+1}; \mathbf{Z}_p) = \mathbf{Z}/p^r$; so if we can show that the same is true of $H^2(B\Gamma^{qp^r,1}; \mathbf{Z}_p)$, we are done.
For this, consider the fibration

$$F_{qp^r}(\mathbf{R}^2) \to B\Gamma^{qp^r,1} \to BSO(2) \tag{35}$$

which follows from the preceding descriptions. The spaces $F_n(\mathbf{R}^2)$ stand in the same relation to $\Omega^2 S^{2N+2}$ as the $F_n(S^2)$'s do to $\Lambda^2 S^{2N+2}$ (they are the braid group classifying spaces, cf. [9]). Now $H^*(\Omega^2 S^{2N+2}; \mathbf{Z}_p)$ contains a divided polynomial algebra on generators z_i of weight i corresponding to the ξ_i's in K-theory; and also a generator y of degree 1 and weight 2 corresponding to η_0. Other generators are of degree $\geq 2p-1$. It is easy to see that $d_2(y) = z_2 \otimes x$ in the spectral sequence of (34), where x generates $H^2(BSO(2); \mathbf{Z})$. Hence, the terms of weight n which affect the calculation of H^2 come from $z_{n-2}y, z_n \otimes x$, and $d_2(z_{n-2}y) = z_{n-2}z_2 \otimes x = (n-2, 2) z_n \otimes x$. If we substitute $qp^r + 1$ for n, and evaluate in p-local cohomology, we obtain the required result, since $(qp^r - 1, 2) = p^r$ up to units. If we define $v = \frac{1}{qp^r+1}\pi_*[v']$, we have the required factoring exactly.

But now, from the cohomology spectral sequence of $F_{qp^r+1}(S^2) \to B\Gamma^{qp^r+1} \to BSO(3)$, the restriction from $H^2(B\Gamma^{qp^r+1}; \mathbf{Z}_p)$ to $H^2(F_{qp^r+1}(S^2); \mathbf{Z}_p)$ is an isomorphism. So $v^*(\tilde{\tau})$ restricts to the image of this generator under the map $H^2(F_{qp^r+1}(S^2); \mathbf{Z}_p) \to$

$K^0(F_{qp^r+1}(S^2);\mathbf{Z}_p)$. By the remarks following proposition 2.4, this is adequate as a characterization of $\xi_{qp^r}\cdot\iota$, which proves part (ii).

To obtain the more delicate result which we want parallel to proposition 6.1, we consider $K^*(B\Gamma^{qp^r+1};\mathbf{Z}_p)$ as a module over $S'=K^*(BSO(2);\mathbf{Z}_p)$ via v^*.

Proposition 6.3. *(i) The S'-module $S'.\xi_{qp^r+1}$ is isomorphic to $K^*(BC_{qp^r};\mathbf{Z}_p)$.*
(ii) The elements $\xi_{qp^r}\cdot\iota$ are infinite cycles in the spectral sequence of $B\Gamma^{qp^r+1}$, and no non-zero element $p^s.x^i\otimes\xi_{qp^r+1}$ or $p^s.x^i\otimes\xi_{qp^r}\cdot\iota$ is a boundary.
(iii) The annihilator of ξ_{qp^r+1} in S' is generated by the polynomial $'F_r(\tilde\tau)=(1+\tilde\tau)^{p^r}-1$; this has degree p^r, and its lowest non-vanishing term is $p^r\tilde\tau$. And $\tilde\tau\cdot\xi_{qp^r+1}$ restricts to $\xi_{qp^r}\cdot\iota$ in $K^0(F^{qp^r+1}(S^2);\mathbf{Z}_p)$.
(iv) $K^*(B\Gamma^{qp^r+1};\mathbf{Z}_p)$ is an algebra over the ring $S'.\xi_{qp^r+1}$; in particular, the polynomial $'F_r$ annihilates every element of $K^*(B\Gamma^{qp^r+1};\mathbf{Z}_p)$

Proof. This is a matter of sorting out how much needs to be added to the proof of proposition 6.1. Here we note first that the spectral sequence of the fibration (33) in this case is different, but no harder to evaluate; $K^0(SO(3)/C(q(k));\mathbf{Z}_p)$ is in this case generated by 1 and a class z say, which is the restriction of $\xi_{q.p^r}\cdot\iota$, and which is clearly an infinite cycle. Furthermore, $\xi_{qp^r}\cdot\iota$ is the restriction of the class defined by $[v]$, and so is in turn an infinite cycle which restricts to z. Next, $K^1(SO(3)/C(q(k));\mathbf{Z}_p)$ is again killed by d_4. The same argument as before therefore shows that it is impossible for the elements $p^s.x^i\otimes\xi_{qp^r+1}$, $p^s.x^i\otimes\xi_{qp^r}\cdot\iota$ to be boundaries, which proves part (ii).

For part (i), note that $v\in H^2(B\Gamma^{qp^r+1};\mathbf{Z}_p)$ is of order p^r, so is the image under the Bockstein of an element of $H^1(B\Gamma^{qp^r+1};\mathbf{Z}/p^r)=[B\Gamma^{qp^r+1},BC_{p^r}]$, say $v=u\circ v_1$, where $v_1:B\Gamma^{qp^r+1}\to BC_{p^r}$ and $u:BC_{p^r}\to BSO(2)$ classifies the inclusion. Also, since $j:BC_{qp^r}\to B\Gamma^{qp^r+1}$ is an isomorphism on $H^1(\quad;\mathbf{Z}/p^r)$, $v_1\circ j:BC_{qp^r}\to BC_{p^r}$ is a p-local equivalence. From the second statement we can deduce that $j^*v_1^*$ is an isomorphism, so v_1^* is injective; from the first, $v^*=v_1^*u^*$. So $\mathrm{Ker}(v^*)=\mathrm{Ker}(u^*)$, which proves (i). The first part of (iii) follows from this, together with (iv), by the methods used in proposition 6.1; we need only note that $K^*(BC_{p^r};\mathbf{Z}_p)=R(C_{p^r})\hat{}\otimes\mathbf{Z}_p$ is well known to be $\mathbf{Z}_p[\tau]/(\tau^{p^r}-1)$ completed with repect to the topology generated by $\tilde\tau$. As for the second part of part (ii), this is proposition 6.2 (ii).

Our major step is to prove that the elements λ are also cycles.

Proposition 6.4. *(i) The element λ_{qp^r} is a permanent cycle in the spectral sequence $\{E_n(\mathbf{Z}_p)\}$ for all q,r, and so defines an element which we shall call $\tilde\lambda_{qp^r}\in K^1(B\Gamma^{2qp^r};\mathbf{Z}_p)$.*
(ii) Let $G_r(\tilde\theta)=F_r(\tilde\theta)/\tilde\theta$. Then the module $S.\tilde\lambda_{qp^r}$ is a quotient of $S/(G_r(\tilde\theta))$.
(iii) The $H^*(BSO(3))$-submodule of E_∞ generated by λ_{qp^r} gives rise to a submodule $S.\tilde\lambda_{qp^r}$ of K^1 which is profinite as an abelian group, and whose rank over $\mathbf{Z}_p\hat{}$ is at most $\frac{p^r-1}{2}$.

Note. Once again we have a problem arising from the possible existence of boundaries among the elements $p^sx^i\otimes\lambda_{qp^r}$; and again this will be eliminated at a later stage.

The method of proof will, as usual, be by induction on the integer qp^r; we shall use the induction hypothesis to derive information about the structure of the spectral sequence, and its consequences for the K-theory of $B\Gamma^{2qp^r}$, and deduce that any non-vanishing differential on $\lambda_{q.p^r}$ would give us a wrong p-adic rank.
First, let us show that (i) implies (ii) and (iii). Suppose, then, that we have shown that λ_{qp^r} is a permanent cycle. Then so is $x^i \otimes \lambda_{qp^r}$ for any k. Hence, in E_∞, λ_{qp^r} generates a graded module over $H^*(BSO(3);\mathbf{Z}_p)$ equal to the quotient by boundaries of

$$M_{qp^r} = (1 \otimes \lambda_{qp^r}).H^*(BSO(3);\mathbf{Z}_p)/(p^r)$$

corresponding to at most $\mathbf{Z}/p^r$ in each degree $4k \geq 0$. This implies in particular that $p^r.\tilde{\lambda}_{qp^r} \equiv 0 \pmod{\tilde{\theta}}$. Hence, there is a polynomial $G(\tilde{\theta})$ (of minimum degree) with constant term p^r such that $G(\tilde{\theta}).\tilde{\lambda}_{qp^r} = 0$. Since $F_r(\tilde{\theta}).\tilde{\lambda}_{qp^r} = 0$ by proposition 2.1(iii), $G(\tilde{\theta})$ divides $F_r(\tilde{\theta})$, and since its term of lowest degree is p^r, it must in fact divide $F_r(\tilde{\theta})/\tilde{\theta} = G_r(\tilde{\theta})$.
We therefore have an epimorphism from $S/(G_r(\tilde{\theta}))$ to $S.\tilde{\lambda}_{q.p^r}$, (which will induce an epimorphism of the corresponding graded modules over $H^*(BSO(3);\mathbf{Z}_p)$), as stated in (ii).Now (iii) follows from what we know of the structure of the augmentation ideal $I(D_{qp^r})\hat{}$.
Part (iii) will be crucial in the inductive proof of part (i) which follows. In fact, the upper bound for the module generated by $\tilde{\lambda}$ as measured by part (ii) is exactly right in terms of what we know of the p-adic part of $K^*(B\Gamma^{2qp^r})$. We shall show (in consequence) that if there are any non-zero differentials on λ_{qp^r}, then the corresponding number of p-adic summands must be strictly less than it should be.
We next deduce a simple consequence of the inductive hypothesis, the first stage in an eventual computation of the whole spectral sequence:

Lemma 6.4. *Assume that λ_{qp^r} is a permanent cycle in the spectral sequence. Then,*
(i) the product $\xi_{q'p^s+\alpha} \cdot \lambda_{q.p^r}$ is also a permanent cycle, for all q', s;
(ii) if $s \geq r$ and $\alpha = 0, 2$, then $d_4(\xi_{q'p^s+\alpha-3} \cdot \gamma_3 \cdot \lambda_{qp^r}) = 0$;
(iii) if $s < r$, and $\alpha = 0, 2$, then $d_4(\xi_{q'.p^s+\alpha-3} \cdot \gamma_3.\lambda_{qp^r}) = p^s.x \otimes \xi_{q'p^s+\alpha} \cdot \lambda_{qp^r}$.
(iv) if $2 < \alpha < p$, then $d_4(\xi_{q'p^s+\alpha-3} \cdot \gamma_3 \cdot \lambda_{qp^r}) = x \otimes \xi_{q'p^s+\alpha} \cdot \lambda_{qp^r}$.

Proof. These relations are all easy consequences of lemma 6.2 and of the Leibniz rule. Part (i) is immediate; parts (iii) and (iv) follow using lemma 6.2; while part (ii) is the same result as part (iii) once we have taken account of the fact that p^s annihilates λ_{qp^r} under the given conditions.
Lemma 6.4 enables us now to take care of the cases where there are no p-adic summands — which is important, if irrelevant to the induction argument.

Lemma 6.5. *Suppose that λ_j is a permanent cycle in the spectral sequence for all $j \leq qp^r$. Then the group $\tilde{K}^*(B\Gamma^{mp+\alpha};\mathbf{Z}_p)$ where $2 < \alpha < p$ and $m \leq 2qp^{r-1} + 1$ is (a) entirely torsion and (b) entirely in K^1. Its generators are all the elements $\tilde{\xi}_{(m-2s)p+\alpha}.\tilde{\lambda}_{sp}$, $s = 1 \ldots [m/2]$; and $\tilde{\xi}_{(m-2s)p+\alpha}.\tilde{\lambda}_{sp}$ generates a torsion summand of order $p^{v_p(s)+1}$, where as usual v_p denotes the p-value. There are accordingly $[m/2]$ such summands.*

Proof. From lemma 6.1 we have a description of the generators for $K^*(F_{mp+\alpha}(S^2);\mathbf{Z}_p)$ and of their torsion orders under the given conditions. From lemma 6.4, the differential d_4 in the spectral sequence kills almost everything, leaving only the identity element $\xi_{mp+\alpha}$ and the classes $\xi_{(m-2s)p+\alpha}\cdot\lambda_{sp}$, which are of the order stated and which correspond to the elements $\tilde{\xi}_{(m-2s)p+\alpha} cdot \tilde{\lambda}_{sp}$. The description of their order in E_∞ prima facie only implies that for each s, the product $p^{v_p(s)+1}.\tilde{\xi}_{(m-2s)p+\alpha}\cdot\tilde{\lambda}_{sp}$ is zero modulo powers of $\tilde{\theta}\in S$. However, by proposition 6.1 (iv), this implies that it is actually zero, and the lemma follows.

We now embark on the inductive proof of Proposition 6.4 (i). Accordingly, we are interested in the spectral sequence for $B\Gamma^{2qp^r}$. Let us suppose from now on (until the proof has been completed) that the proposition has been proved for λ_{mp}, $m = 1,\ldots,qp^{r-1}-1$. Note first that by lemma 6.4 (iii) and the induction hypothesis, if $s<t$ and ap^s, bp^t are less than qp^r, $p^s.x\otimes\xi_{ap^s}\cdot\lambda_{bp^t}$ is a d_4-boundary, while $p^{t-s}.\xi_{ap^s-3}\cdot\gamma_3\cdot\lambda_{bp^t}$ generates a summand of the d_4-cycles, of order p^s. These two elements contribute to E_5 as follows:
1) $\xi_{ap^s}\cdot\lambda_{bp^t}$ generates an $H^*(BSO(3);\mathbf{Z}_p)$-module which is isomorphic to $\mathbf{Z}/p^t.1\oplus\mathbf{Z}/p^s.\{x,x^2,\ldots\}$;
2) $p^{t-s}\xi_{ap^s-3}\cdot\gamma_3\cdot\lambda_{bp^t}$ generates an $H^*(BSO(3);\mathbf{Z}_p)$-module which is isomorphic to $\mathbf{Z}/p^s.\{1,x,x^2,\ldots\}$.
We now can deduce two important upper bounds:

Lemma 6.6. *Suppose that $s<t$ and ap^s, bp^t are less than qp^r. Then:*
(i) the generator $\tilde{\xi}_{ap^s}\cdot\tilde{\lambda}_{bp^t}$ corresponding to $\xi_{ap^s}\cdot\lambda_{bp^t}$ contributes a submodule to $K^1(B\Gamma^{ap^s+2bp^t};\mathbf{Z}_p)$ which is isomorphic to a quotient of $S/(F_s(\tilde{\theta}),G_t(\tilde{\theta}))$. This ring is (additively) the direct sum of $\mathbf{Z}/p^{t-s}$ and $\frac{p^s-1}{2}$ copies of $\mathbf{Z}_p\hat{}$.
(ii) If the generator $p^{t-s}\xi_{ap^s-3} cdot \gamma_3\cdot\lambda_{b.p^t}$ survives to E_∞, it contributes a submodule to $\tilde{K}^0(B\Gamma^{ap^s+2bp^t};\mathbf{Z}_p)$ which is isomorphic to a quotient of $S/(G_s(\tilde{\theta}))$, i.e. of the sum of $\frac{p^s-1}{2}$ copies of $\mathbf{Z}_p\hat{}$.

Proof. Clearly both F_s and G_t annihilate the product $u=\tilde{\xi}_{ap^s}\cdot\tilde{\lambda}_{bp^t}$. It is clear (cf the description in terms of sin functions above) that G_s divides G_t, so that $G_t = H(t,s)G_s$; and $H(t,s)$ is a monic polynomial whose constant term is p^{t-s}. Hence, (since $F_s = x.G_s$), $G_t\equiv p^{t-s}G_s \pmod{F_s}$. So u is annihilated by $F_s=\tilde{\theta}.G_s$ and by $p^{t-s}.G_s$.
Now, from the exact sequence

$$0\to\mathbf{Z}_p\cdot G_s\to S/(F_s)\to S/(G_s)\to 0$$

we can deduce that G_s generates a $\mathbf{Z}_p$-summand in $S/(F_s)$, and from this that additively $S/(F_s,p^{t-s}G_s)$ is isomorphic to the sum of $\mathbf{Z}/p^{t-s}$ and $S/(G_s)$. The latter is exactly the sum of $\frac{p^s-1}{2}$ copies of $\mathbf{Z}_p\hat{}$ as claimed. Hence, the contribution of u to K^1 is at most a quotient of this.
In the case of $p^{t-s}\xi_{ap^s-3}\cdot\gamma_3\cdot\lambda_{bp^t}$, we know that the corresponding generator of K^1 generates a module over $S/(F_s)$, since p^s is the highest power of p dividing ap^s+2bp^t (proposition 6.1 (iii)). However, from the fact that its contribution to $E_5^{0,0}$ is a $\mathbf{Z}/p^s$

(see (2) above), we can use the argument used above for proposition 6.1 (iii) and (iv) to show that this module is in fact a quotient of $S/(G_s)$, which is a p-adic sum as claimed. The corresponding case to lemma 6.6 when $s \geq t$ is different. In fact, by lemma 6.4 (ii), the differentials d_4 in this case are always zero, and the upper bounds are as follows:

Lemma 6.7. *(i) If $s \geq t$ and ap^s, bp^t are less than qp^r, then the generator $\tilde{\xi}_{ap^s} \cdot \tilde{\lambda}_{bp^t}$ which corresponds to $\xi_{ap^s} \cdot \lambda_{bp^t}$ contributes a submodule to $K^1(B\Gamma^{ap^s+2bp^t}; \mathbf{Z}_p)$ which is isomorphic to a quotient of $S/(G_t(\tilde{\theta}))$. This is (additively) the direct sum of $\frac{p^t-1}{2}$ copies of $\mathbf{Z}_p\hat{}$.*
(ii) If $s > t$, and the generator $\xi_{ap^s-3} \cdot \gamma_3 cdot \lambda_{bp^t}$ survives to E_∞, it contributes a submodule to $\tilde{K}^0(B\Gamma^{ap^s+2bp^t}; \mathbf{Z}_p)$ which is isomorphic to a quotient of $S/(G_t(\tilde{\theta}))$, i.e. of the sum of $\frac{p^t-1}{2}$ copies of $\mathbf{Z}_p\hat{}$.

The proof for part (i) is similar to that of lemma 6.6 with the appropriate changes, and without the problems caused by the need to compute a complicated quotient of S. For part (ii), suppose that $s > t$; then the p-value of $ap^s + 2bp^t$ is t, so that all elements are annihilated by $G_t(\tilde{\theta})$ as required. Note that this argument does not work if $t = s$, since it is possible that the p-value of $(a + 2b)p^s$ is greater than s. This case will be dealt with later (see proposition 6.7).

We should now look at the spectral sequence for $B\Gamma^{2qp^r}$. However, for greater generality, we extend the scope of the inquiry by taking together the two cases of $\{E_n(B\Gamma^{kp^r})\}$ where $k = 2q, 2q + 1$.
For $k = 2q$, the E_2-term is the module over $H^*(BSO(3); \mathbf{Z}_p)$ generated by:
(A) λ_{qp^r}, of order p^r;
(B) All products $\xi_{2mp} \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order p^s, where s is the p-value of $qp^r - mp$;
(C) All products $\xi_{2mp-3} \cdot \gamma_3 \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order as in (B);
(D) ξ_{2qp^r}, of infinite order;
(E) $\xi_{2qp^r-3} \cdot \gamma_3$, of infinite order.
For $k = 2q + 1$, it is the module over $H^*(BSO(3); \mathbf{Z}_p)$ generated by:
(B) All products $\xi_{(2m+1)p} \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order p^s, where s is the p-value of $qp^r - mp$;
(C) All products $\xi_{(2m+1)p-3} \cdot \gamma_3 \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order as in (B);
(D) $\xi_{(2q+1)p^r}$, of infinite order;
(E) $\xi_{(2q+1)p^r-3} \cdot \gamma_3$, of infinite order.
We have calculated the E_4 term and found various implications about the S-module structure; we shall now use the upper bounds above to find upper bounds for the size of $K^1(B\Gamma^{kp^r}; \mathbf{Z}_p)$ (odd part of the spectral sequence).

Lemma 6.8. *Suppose that, in the spectral sequence*

$$\{E_n(\mathbf{Z}_p)\} : H^*(BSO(3); K^*(F_{kp^r}(S^2); \mathbf{Z}_p) \Rightarrow K^*(B\Gamma^{kp^r}; \mathbf{Z}_p))$$

all differentials vanish after d_4. Then the rank of $\tilde{K}^1(B\Gamma^{kp^r}; \mathbf{Z}_p)$ as a module over $\mathbf{Z}_p\hat{}$

is at most equal to

$$(36)\qquad \frac{1}{2}\sum_{i=1}^{r}\phi(p^i)(qp^{r-i})$$

if $k = 2q$, and at most equal to

$$(37)\qquad \frac{1}{2}\sum_{i=1}^{r}\phi(p^i)\Big[\frac{(2q+1)p^{r-i}}{2}\Big]$$

if $k = 2q+1$.

Proof. First consider the even case $k = 2q$. The terms (A) give a summand of rank at most $\frac{p^r-1}{2}$, by proposition 6.2 (ii); while the terms (E) are killed by d_4.
The remaining odd terms are those of type (B). We count them according to the p-value of $2mp$ (i.e. of the ξ part). We have the following estimates:
(i) If the p-value of $2mp$ is $s < r$, then so is that of $qp^r - mp$ and they are equal; so lemma 6.7 applies. Hence, the corresponding (B) term contributes a module of rank at most $\frac{p^s-1}{2}$. The number of contributions for this value of s is the number of integers $np^s < qp^r$ which are not divisible by p^{s+1}. This is $q.\phi(p^{r-s})$. The total is $n_s = q.\phi(p^{r-s})\frac{p^s-1}{2}$.
(ii) If the p-value of $2mp$ is $s \geq r$, then I claim that the (B) term contributes a module of rank at most $\frac{p^r-1}{2}$. First, if $s > r$, the p-value of $qp^r - mp$ is exactly r, and the result follows from lemma 6.7. If $s = r$, then the p-value of $qp^r - mp$ is $\geq r$, and either lemma 6.6 or lemma 6.7 implies the same bound. The number of such contributions (i.e. of integers $2mp < 2qp^r$ of p-value $\geq r$) is the number of multiples of p^r which are $< qp^r$, i.e. $q-1$, and the total is $n_r = (q-1)\frac{p^r-1}{2}$.
We find for the upper bound on the rank:

$$\frac{p^r-1}{2} + \sum_{i=1}^{r} q.\phi(p^i)\frac{p^{r-i}-1}{2}, +(q-1).\frac{p^r-1}{2}$$

(setting $i = r - s$). This is not exactly the formula of equation (36), but it is not far different. In fact, the difference is equal to

$$\begin{aligned}&\frac{1}{2}q.\big(p^r - 1 - \sum_{i=1}^{r}\phi(p^i)\big)\\ &= \frac{1}{2}q.(p^r - 1 - (p-1)(1+p+\ldots+p^{r-1}))\\ &= 0\end{aligned}$$

The case where k is odd is slightly more complicated. The point to notice is that the number of integers $np^s < (2q+1)p^r$ which are odd and not divisible by p^{s+1} is

$(q+\frac{1}{2}).\phi(p^{r-s})$ for $s<r$, while the number of odd integers $np^r<(2q+1)p^r$ is exactly q. We deduce the bound

$$(q+\frac{1}{2})\sum_{i=1}^{r}\phi(p^i)\frac{p^{r-i}-1}{2}+q\frac{p^r-1}{2} \tag{38}$$

as before. Using the fact that

$$\Big[\frac{(2q+1)p^{r-i}}{2}\Big]=(q+\frac{1}{2})p^{r-i}-\frac{1}{2}$$

the difference between (37) and (38) is

$$q\frac{p^r-1}{2}+\frac{1}{4}\sum_{i=1}^{r}\phi(p^i)-\frac{1}{2}(q+\frac{1}{2})\sum_{i=1}^{r}\phi(p^i)$$

which is zero as before. This completes the proof of lemma 6.8.
The point of the lemma is that the count of p-rank which it provides is exactly the rank of the p-adic part of $\tilde{K}^*(B\Gamma^{kp^r};\mathbf{Z}_p)$ as computed in [7]. The key remaining result for Proposition 6.4 is now:

Lemma 6.9. *If there is a non-zero differential d_i on the generator λ_{qp^r} in the spectral sequence for $i>4$, then the rank of $\tilde{K}^1(B\Gamma^{2qp^r};\mathbf{Z}_p)$ must be strictly less than the upper bound computed by lemma 6.8.*

From this, Proposition 6.4 follows since if the rank is less than that given by formula (36), we have a result in contradiction with [7].
It remains to prove lemma 6.9. Suppose then that for some $v\in K^0(F_{2qp^r}(S^2);\mathbf{Z}_p)$ and $n>1$, we have $d_{4n}(\lambda_{qp^r})=x^n\otimes v$. The only possibilities for v are elements of type (C), (D) in the list above. Now, if v is of order p^s ($s\le r$), $p^s\lambda_{qp^r}$ is a d_{4n}-cycle; and for $k>0$, $d_{4n}(x^k\otimes\lambda_{qp^r})=x^{n+k}\otimes v$; $d_{4n}(x^k\otimes p^s\lambda_{qp^r})=0$.
It is possible that yet other differentials will fail to vanish on multiples of λ_{qp^r}; however, for some smallest s we will have a module (over $H^*(BSO(3);\mathbf{Z}_p)$) of infinite cycles which is generated by $p^s.\lambda_{qp^r}$ and its products with $x,x^2,\ldots$, modulo boundaries if any. (If $s=r$, this is zero.) Call this module M. Note that since we know by the inductive hypothesis that the other odd generators are infinite cycles, it is impossible that any element of form $x^k\otimes p^{s'}.\lambda_{qp^r}$ for $s'<s$ should become a cycle 'accidentally', i.e. because its boundary is the boundary of something else.
Now I claim that $M\subset E_\infty$ gives rise to a module in K^1 whose rank over $\mathbf{Z}_p\hat{}$ is strictly less than $\frac{p^r-1}{2}$. From this, lemma 6.9 clearly follows. To justify the claim, note that the cycle $p^s\lambda_{qp^r}$ which survives to E_∞ gives rise to an element in K^1, say $\tilde{u}$; and $\tilde{u}$ generates a filtered S-module N whose associated graded module is just M. Next, N is the entire part of $K^1(B\Gamma^{2qp^r};\mathbf{Z}_p)$ which comes from the (A) terms above. To find the size of N, we note that $p^{r-s}.u\equiv 0 \pmod{\tilde{\theta}}$ by considering the E_∞ term, so $\tilde{u}$ is annihilated by some polynomial $f(\tilde{\theta})$ whose constant term is p^{r-s}. However, $\tilde{u}$ is also

annihilated by G_r, a monic polynomial whose constant term is p^r. We can divide f by G_r if necessary and get a remainder, a polynomial whose degree d is less than $\frac{p^r-1}{2}$, and which also annihilates $\tilde{u}$. We deduce that N is generated over $\mathbf{Z}_p\hat{}$ by $\tilde{u}$ and its products with $\tilde{\theta}, \ldots, \tilde{\theta}^{d-1}$. This completes the proof.

We have now completed the induction, and can deduce that all products of ξ's with λ's (elements of type (B) or (D) in our list) are infinite cycles. However, we can do better. It does not follow from proposition 6.4 that the even generators of type (C) are infinite cycles; in fact, they cannot easily be decomposed into products, if at all. We prove now that this too is the case — i.e. that the spectral sequence is trivial from E_5 on.

Lemma 6.10. *If any differential d_i is non-vanishing on one of the generators (C) above, for $i > 4$, then the rank of $\tilde{K}^1(B\Gamma^{sp^r}; \mathbf{Z}_p)$ must be strictly less than the upper bound computed by lemma 6.8.*

Proof. Similar to lemma 6.9. Suppose that $d_{4n}(u) \neq 0$, where u is a generator of type (C). Then $d_{4n}(u) = p^t x^n \otimes v$, where v is of type (A) or (B). Furthermore, if $1 \otimes v$ is of order p^s in E_5, (so that the generator $\tilde{v}$ corresponding to v in K^1 is annihilated by F_s), then $t < s$ for the relation to be non-trivial. From the boundary relation on $x^n \otimes v$, we can deduce that $\tilde{v}$ satisfies a relation of form

$$p^t \tilde{\theta}^n.\tilde{v} = a_1 \tilde{\theta}^{n+1}.\tilde{v} + \ldots$$

i.e., $\tilde{v}$ is annihilated by a polynomial $f(\tilde{\theta})$ whose term of lowest degree is $p^t\tilde{\theta}^n$.
I now claim that we can choose f to have degree $< n + \frac{p^s-1}{2}$. In fact, from $F_s(\tilde{\theta}).\tilde{v} = 0$, we deduce that $\tilde{\theta}^{\frac{p^s+1}{2}}.\tilde{v}$ is a linear combination of $\tilde{\theta}.\tilde{v}, \ldots, \tilde{\theta}^{\frac{p^s-1}{2}}.\tilde{v}$. Hence we can divide f by F_s, leaving a remainder of degree $< n + \frac{p^s-1}{2}$ which annihilates $\tilde{u}$. Since F_s does not divide f, whose lowest degree term is $p^t < p^s$, this remainder is non-trivial. This gives a linear relation between $\tilde{\theta}^n.\tilde{v}, \ldots, \tilde{\theta}^{n+\frac{p^s-3}{2}}.\tilde{v}$.
However, the estimates in lemmas 6.6, 6.7 require that $\tilde{v}, \tilde{\theta}.\tilde{v}, \ldots, \tilde{\theta}^{\frac{p^s-3}{2}}.\tilde{v}$ should be linearly independent over $\mathbf{Z}_p\hat{}$. From this it follows necessarily that the same is true of their products with $\tilde{\theta}^n$. Since we have shown that if $p^t x^n \otimes v$ is a boundary, these products are linearly dependent, the existence of such a boundary must make the rank of K^1 strictly less than the upper bound.

We can now apply the previous argument. If there is a non-zero boundary, the rank of K^1 is less than the upper bound, which contradicts [7]. Hence, there are no such boundaries. Putting the previous results together (and noting that the case $\alpha = 2$ is obtained from that for $\alpha = 0$ simply by multiplying by ξ_2) we have:

Proposition 6.5. *If $\alpha = 0, 2$, all differentials after d_4 vanish in the spectral sequence for $K^*(B\Gamma^{mp+\alpha}; \mathbf{Z}_p)$.*

Proof. For the generators of type (A), (B), this follows from proposition 6.4; for those of type (C) from lemma 6.10; while those of type (D) are taken care of by lemma 6.2. The generators (E) have, of course, already been killed by d_4.

Corollary 6.1. *The S-module generated by any element $\xi_{mp} \cdot \lambda_{np}$ is exactly that given by lemmas 6.6, 6.7.*

In fact, we have seen that there can be no further relations on the odd generators beyond the ones we have already specified as minimal.
The structure of the spectral sequence is now determined. From lemma 6.6 we can also deduce the location and order of the torsion. In fact, we see that for each integer $2qp^s < kp^r$ such that $s > r$, we have a torsion summand of order p^{s-r} in K^1 coming from a relation on the generator $\tilde{\xi}_{kp^r-2qp^s} \cdot \tilde{\lambda}_{qp^s}$. Hence, the total number of torsion summands in $K^*(B\Gamma^{qp^s}; \mathbf{Z}_p)$ is at least equal to the number of multiples of $2p^{s+1}$ which are less than qp^s, i.e. to $\left[\frac{q}{2p}\right]$. However, this is exactly the number of torsion summands given by theorem 5.1, and so there can be no further torsion. We have:

Proposition 6.6. *(i) The torsion in $K^*(B\Gamma^{kp^r}; \mathbf{Z}_p)$ is all in K^1.*
(ii) For each integer $2qp^s < kp^r$, where $s > r$, there is a torsion summand of order p^{s-r}, and these are all the summands.

This result, together with lemma 6.5 (which is now true 'unconditionally', since we know that the λ's are permanent cycles), establishes theorem 1.1 except in the case $\alpha = 1$. Before proceeding to this case, we clear up the remaining questions about the S-module structure.
We know (since there is no torsion) the additive structure of $\tilde{K}^0$; it is a direct sum of copies of $\mathbf{Z}_p\hat{}$, the number of copies being given by the formula of [7]. It seems reasonable to deduce (since they add to the right number) that the generator which comes from (a multiple of) $\xi_{qp^s-3}\gamma_3\lambda_{qp^r}$ gives rise to an S-module of the right size. This is in fact true, but there are a few details to be taken care of.

Proposition 6.7. *Let $\tilde{u}(q'p^r, qp^s) \in \tilde{K}^0(B\Gamma^{q'p^r+2qp^s}; \mathbf{Z}_p)$ be a generator corresponding to (a multiple of) $\xi_{q'p^r-3} \cdot \gamma_3 \cdot \lambda_{qp^s}$. Then the annihilator of $\tilde{u}(q'p^r, qp^s)$ in S is precisely the ideal (G_s) if $s > r$ and (G_r) if $s \leq r$.*

Proof. What, essentially, we need to prove is that for all values of $q'pr$, qp^s, $\tilde{u}(q'p^r, qp^s)$ is annihilated by the polynomial in question. Then as before, having deduced an upper bound on the size of the S-module generated by $\tilde{u}(q'p^r, qp^s)$, the computations of the rank of $\tilde{K}^0(B\Gamma^{q'p^r+2qp^s}; \mathbf{Z}_p)$ ensure that this is a lower bound also, so that there is no larger annihilator.
In most cases, this is straightforward. Specifically, if $r > s$ resp. $r < s$, as we have seen in lemmas 6.6 and 6.7, the element is annihilated by G_s resp. G_r, simply using the p-value of the sum $q'p^r + 2qp^s$. However, if $r = s$ and $v_p(q' + 2q) > 0$, we cannot immediately assert that $\tilde{u}(q'p^r, qp^r)$ is annihilated by G_r. We shall prove this for all q', q, r by an induction on r. Specifically, we assert:

Lemma 6.10. *(i) If $\tilde{u}(ap^s, bp^s)$ is annihilated by G_s for all q', q and all $s < r$, then every generator $\tilde{u}(ap^s, bp^t)$ in every $\tilde{K}^0(B\Gamma^{qp^r}; \mathbf{Z}_p)$ is annihilated by precisely the right polynomial;*
(ii) Under the hypotheses of (i), every $\tilde{u}(ap^r, bp^r)$ is annihilated by G_r.

To prove part (i), note that if $qp^r = m + n$, then either $v_p(m), v_p(n)$ are equal and less than r (in which case the induction hypothesis applies); or at least one of the two p-values is equal to r (and then we know that G_r annihilates the elements of $\tilde{K}^0(B\Gamma^{qp^r}; \mathbf{Z}_p)$).
To prove (ii), consider $\tilde{u}(ap^r, bp^r) \in \tilde{K}^0(B\Gamma^{(a+2b)p^r}; \mathbf{Z}_p)$. If $t = v_p((a+2b)p^r)$, then certainly $G_t(\tilde{\theta}).\tilde{u}(ap^r, bp^r) = 0$. If $t = r$ (i.e. $a + 2b$ is prime to p), we are done, so suppose $t > r$. Then we still know that the order of $\xi_{ap^r-3} \cdot \gamma_3 \cdot \lambda_{bp^r}$ in E_∞ is p^r; whence, $\tilde{u}(ap^r, bp^r)$ is annihilated by some polynomial $f(\tilde{\theta})$ whose constant term is p^r. However, if we now consider $\tilde{\xi}_{p^r} \cdot \tilde{u}(ap^r, bp^r) = \tilde{u}((a+1)p^r, bp^r)$, this has weight $(a + 2b + 1)p^r$, whose p-value is exactly r. Hence, it is annihilated both by G_r and by f. If G_r is not a multiple of f, we can divide to find that $\tilde{\xi}_{p^r} \cdot \tilde{u}(ap^r, bp^r)$ is annihilated by a polynomial f' of lower degree than G_r. This contradicts the conclusion of part (i); so f is a factor of G_r. Furthermore, since the constant terms of f and of G_r are both p^r, $G_r = f.h$ where the constant term of h is 1.
We now use the fact that the irreducible factors of G_r are known; they are in fact just G_1, G_2/G_1,. . . ,G_r/G_{r-1}. This is an easy consequence of Galois theory. In fact, the roots of G_r/G_{r-1} correspond to the angles $\frac{2\pi k}{p^r}$ where k is prime to p (in our description above, they are $\cos\frac{2\pi k}{p^r} - 1 = -2\sin^2\frac{\pi k}{p^r}$). Let β be a generator of $(\mathbf{Z}/p^r)^*$. Then the automorphism of the cyclotomic field which takes z to z^β permutes the roots of G_r/G_{r-1} cyclically. Hence, each factor is irreducible. The constant term of each factor is p, so G_r has no non-trivial factors with constant term 1. We can conclude that $f = G_r$. This implies lemma 6.10 and hence proposition 6.7.

Finally, with waning enthusiasm (are you still there, reader?), we must discuss the case $\alpha = 1$. Accordingly, we consider the spectral sequence

$$\{E_{kp^r+1}(\mathbf{Z}_p)\} : H^i(BSO(3); K^j(F_{kp^r+1}(S^2); \mathbf{Z}_p)) \Rightarrow K^*(B\Gamma^{sp^r+1}; \mathbf{Z}_p)$$

We have a description of $K^*(F_{kp^r+1}(S^2); \mathbf{Z}_p))$ from proposition 2.6, and we also can use the substantial information already derived on the spectral sequence when $\alpha = 0$. Finally, we can use the S'-algebra structure derived from proposition 6.3 for our description as we used the S-algebra structure in the other cases. Parallel to our description of generators for the K-theory of $F_{kp^r}(S^2)$, we have:
For $k = 2q$, the E_2-term is the module over $H^*(BSO(3); \mathbf{Z}_p)$ generated by:
(A) $\xi_1 \cdot \lambda_{qp^r}$, of order p^r;
(B) All products $\xi_{2mp+1} \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order p^s, where s is the p-value of $qp^r - mp$;
(C) All products $\xi_{2mp-2} \cdot \gamma_3 \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order as in (B);
(D) ξ_{2qp^r+1}, of infinite order;
(E) $\xi_{2qp^r-2} \cdot \gamma_3$, of infinite order;
(F) $\iota \cdot \xi_{2qp^r}$, of order p^r;
(G) $\iota \cdot \lambda_{qp^r}$, of order p^r;
(H) All products $p^{m(r,s)}\eta_0 \cdot \xi_{2mp-1} \cdot \lambda_{qp^r-mp}$ for m as in (B), where $m(r, s) = \max(r - s, 0)$, and s is the p-value of $qp^r - mp$;

(I) All products $\iota \cdot \xi_{2mp} \cdot \lambda_{qp^r-mp}$ for m as in (B), where s is the p-value of $qp^r - mp$; the products in (H), (I) having order $p^{\min(r,s)}$.

For $k = 2q+1$, it is the module over $H^*(BSO(3); \mathbf{Z}_p)$ generated by:
(B) All products $\xi_{(2m+p^{r-1})p+1} \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order p^s, where s is the p-value of $qp^r - mp$;
(C) All products $\xi_{(2m+p^{r-1})p-2} \cdot \gamma_3 \cdot \lambda_{qp^r-mp}$ for $m = 1, \ldots, qp^{r-1} - 1$, of order as in (B);
(D) $\xi_{(2q+1)p^r+1}$, of infinite order;
(E) $\xi_{(2q+1)p^r-2} \cdot \gamma_3$, of infinite order.
(F) $\iota \cdot \xi_{(2q+1)p^r}$, of order p^r;
(H) All products $p^{m(r,s)}\eta_0 \cdot \xi_{(2m+p^{r-1})p-1} \cdot \lambda_{qp^r-mp}$ for m as in (B), where $m(r,s) = \max(r-s, 0)$, and s is the p-value of $qp^r - mp$;
(I) All products $\iota \cdot \xi_{(2m+p^{r-1})p} \cdot \lambda_{qp^r-mp}$ for m as in (B), where s is the p-value of $qp^r - mp$; the products in (H), (I) having order $p^{\min(r,s)}$.
The prospect of dealing with the associated spectral sequence is not so forbidding (given all these generators) as it at first appears. In fact, from lemma 6.1 and propositions 6.4 and 6.5, all differentials after d_4 vanish on all generators except (perhaps) the generators (H). In fact, even d_4 is only non-vanishing on the (C)'s and (E)'s. We can therefore use the same strategy as before; knowing that the (H)'s have even degree — so that their boundaries, if any, are odd — we can (a) find an upper bound for the rank of $K^1(B\Gamma^{kp^r+1}; \mathbf{Z}_p)$, assuming all differentials are zero; (b) show that this is exactly the rank we should have; and (c) deduce that if any differentials are non-vanishing on the (H)'s, the rank must be strictly less.

To combine as far as possible brevity with honesty, this is carried out as follows. First, we note that the odd generators in case $k = 2q$ are those of type (A), (B), (E), (G), (I); and as before, the (E)'s are knocked out by d_4. Next, in terms of the S'-module structure, the term (A) is paired with (G), giving a single summand in $\tilde{K}^1$ generated by $\tilde{\xi}_1 \cdot \tilde{\lambda}_{qp^r}$. By the argument used in proposition 6.4(ii) and (iii), this has rank at most $p^r - 1$ over $\mathbf{Z}_p\hat{}$ (and the same applies to the module generated by the product of $\tilde{\xi}_1 \cdot \tilde{\lambda}_{qp^r}$ with anything else). Now the important point to note is that although the orders of the terms (B), (I) which correspond are different in E_2, they become the same in E_4, i.e. $p^{\min(r,s)}$. We can deduce that the S'-modules which arise as summands have rank at most $p^{\min(r,s)} - 1$ in each case. By a calculation which exactly parallels that of lemma 6.8, we deduce:

Lemma 6.11. *Suppose that, in the spectral sequence*

$$\{E_n(\mathbf{Z}_p)\} : H^*(BSO(3); K^*(F_{kp^r+1}(S^2); \mathbf{Z}_p) \Rightarrow K^*(B\Gamma^{kp^r+1}; \mathbf{Z}_p))$$

all differentials vanish after d_4. Then the rank of $\tilde{K}^1(B\Gamma^{kp^r+1}; \mathbf{Z}_p)$ as a module over $\mathbf{Z}_p\hat{}$ is at most equal to

$$\sum_{i=1}^{r} \phi(p^i)(qp^{r-i}) \tag{39}$$

if $k = 2q$, and at most equal to

$$\sum_{i=1}^{r} \phi(p^i) \Big[\frac{(2q+1)p^{r-i}}{2} \Big] \tag{40}$$

if $k = 2q+1$.

That is, it is twice the sum (in each case) computed by lemma 6.8. This is again the count for the rank given by [7].
We don't need lemma 6.11 to deduce that the odd generators are cycles — as has been pointed out, we already know that. However, we do need it to show that none of them are boundaries, in particular to show that the unusual generators of type (H) are also infinite cycles. This result, parallel to lemma 6.10, is proved in exactly the same way. Accordingly, we have:

Proposition 6.8. *All differentials after d_4 vanish in the spectral sequence for $K^*(B\Gamma^{mp+1}; \mathbf{Z}_p)$.*

Corollary 6.2. *The S'-module generated by an element $\tilde{\xi}_{ap^s+1} \cdot \tilde{\lambda}_{bp^r}$ has rank exactly $p^{\min(r,s)} - 1$ over $\mathbf{Z}_p\hat{}$.*

(Compare proposition 6.5 and corollary 6.1.) We therefore once again know the additive structure of $K^1(B\Gamma^{mp+1}; \mathbf{Z}_p)$, in particular the torsion. Here we have something of a surprise. The previous arguments show that there is torsion coming from generators of type (B) — a torsion summand of order p^{s-r} in K^1 resulting from a relation on the generator $\tilde{\xi}_{kp^r-2qp^s+1} \cdot \tilde{\lambda}_{qp^s}$ whenever $s > r$; these have the same order and number as before. However, the same argument does not apply to $\iota \cdot \tilde{\xi}_{kp^r-2qp^s} \cdot \tilde{\lambda}_{qp^s}$, since this already has order p^r, and so does not contribute torsion in the same way. Thankfully, we observe that the results of §5 imply that this is the correct count for the number of torsion summands (i.e. the same as for $\alpha = 0$); there can therefore be no more, and we can state:

Proposition 6.9. *(i) The torsion in $K^*(B\Gamma^{kp^r+1}; \mathbf{Z}_p)$ is all in K^1.*
(ii) For each integer $2qp^s < kp^r$, where $s > r$, there is a torsion summand of order p^{s-r}, and these are all the summands.

This completes the remaining case of the proof of theorem 1.1.
Finally, what about the S'-module structure of K^0? Here the same argument applies, but with a significant difference. In fact, the generators of the modules do not come from the terms (C) (as they did in the previous case), but from the (H)'s; this is because of the relation $\gamma_3 = \eta_0 \cdot \iota$, properly interpreted in the K-theory of $B\Gamma^{kp^r+1}$. We can all the same pursue the argument on the same lines, noting that by the time we reach E_4, the (H) term and the corresponding (B) term have the same order. If we write $\tilde{v}(ap^r, bp^s)$ for the p-local generator corresponding to an appropriate multiple of $\eta_0 \cdot \xi_{ap^s-1} \cdot \lambda_{bp^r}$, it is again easy to show that $\tilde{v}(ap^r, bp^s)$ is annihilated by the right polynomial, and so generates the right module, if $r \neq s$; while if they are equal, a special argument on the lines of lemma 6.10 will prove:

Proposition 6.9. *The annihilator of* $\tilde{v}(ap^r, bp^s)$ *in S' is precisely the ideal* $('G_r)$ *if* $s > r$ *and* $('G_r)$ *if* $s \leq r$ *where* $'G_r(\tilde{\tau}) = {}'F_r(\tilde{\tau})/\tilde{\tau}$.

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