A nilpotence theorem for modules over the mod 2 Steenrod algebra

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Abstract

We prove that the mod 2 Steenrod algebra A satisfies the "detection" property; i.e., every non-nilpotent element of $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$ can be detected by restricting to an exterior sub-Hopf algebra of A.

1 Introduction and results

Let A be the mod 2 Steenrod algebra. In this paper we prove Theorem 1.1, a conjecture of Adams, which describes how to detect all non-nilpotent elements in $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$. One can view this result in two ways: it is a generalization of results of Lin [5] and Wilkerson [11] about Ext over certain sub-Hopf algebras of A (and hence is analogous to results of Quillen and others on group cohomology); and it is a Steenrod algebra version of Nishida's theorem [8], a special case of the nilpotence theorem of Devinatz, Hopkins, and Smith [1].

We need one definition in order to state our result: fix a prime p and a cocommutative $\mathbf{F}_p$ Hopf algebra A. An *elementary* sub-Hopf algebra B of A is a bicommutative sub-Hopf algebra with $b^p = 0$ for all $b \in IB$ (IB is the augmentation ideal). For instance when $p = 2$, then the elementary sub-Hopf algebras are the sub-Hopf algebras which are exterior algebras. Let $\iota_B : B \hookrightarrow A$ denote the inclusion, so ι_B^* is the restriction map on Ext.

Theorem 1.1 *Let A be a sub-Hopf algebra of the mod 2 Steenrod algebra; fix $z \in \mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$. If $\iota_E^*(z) = 0$ for every elementary sub-Hopf algebra $\iota_E : E \hookrightarrow A$, then z is nilpotent.*

Theorem 1.1 was first conjectured by Adams, as reported by Lin in [5].

We view Theorem 1.1 as a first step in proving structure theorems for S-teenrod algebra modules analogous to those for spectra given in [3] and [4]; for instance, one has the following conjecture (analogous to the nilpotence theorem):

Conjecture 1.2 *Let A be a sub-Hopf algebra of the mod 2 Steenrod algebra; let C be a bounded below coalgebra over A. Given $z \in \mathrm{Ext}_A^{**}(C, \mathbf{F}_2)$, if $\iota_B^*(z) = 0$ for every elementary sub-Hopf algebra $B \subset A$, then z is nilpotent.*

This is the "ring spectrum" version of the conjecture; one can make a similar conjecture about $\mathrm{Ext}_A^{**}(M, M)$ for any finite A-module M. If one could prove this, then one should be able to work as in [3] to determine the thick subcategories of the category of finite A-modules, and hence to prove an appropriate "periodicity" theorem.

Theorem 1.1 raises other questions; for instance, given A, can we find all of the non-nilpotent elements in $\mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$? One approach would be to investigate the image of ι_E^* for each E. Assume that E is normal; then this image lies in the set of generators for $\mathrm{Ext}_E^{**}(\mathbf{F}_2, \mathbf{F}_2)$ as an $A//E$-module (since ι_E^* is an edge homomorphism in the spectral sequence associated to the extension $E \to A \to A//E$); hence, the first step should be determining this set of generators. When A is the full Steenrod algebra, this is difficult already for the case $E = E(2) = (\mathbf{F}_2[\xi_2, \xi_3, \ldots]/(\xi_i^4))^*$, the maximal elementary sub-Hopf algebra of A containing P_2^1.

At odd primes, Wilkerson found a finite sub-Hopf algebra of the Steenrod algebra for which the odd primary version of Theorem 1.1 fails. A weakened version could still be true—perhaps all non-nilpotent elements in $\mathrm{Ext}_A^{**}(\mathbf{F}_p, \mathbf{F}_p)$ are detected by restricting to two-stage extensions of elementary sub-Hopf algebras [6].

In Section 2 we prove Theorem 1.1, and at the end of that section we discuss some reasons that our proof doesn't work for an arbitrary coalgebra C.

There is also an appendix in which we give a brief description of Eisen's calculation of certain localized Ext groups.

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2 Proof of Theorem 1.1

In this section we prove the main theorem. The proof is analogous to that for the nilpotence theorem for spectra (see [1] or [3]).

We prove the theorem in the case where A is the mod 2 Steenrod algebra; the proof easily generalizes to any sub-Hopf algebra. We fix some notation: A is dual to $A_* = \mathbf{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$; we dualize with respect to the monomial basis in A_*, and set $P_t^s = (\xi_t^{2^s})^*$. The maximal exterior sub-Hopf algebras of A are $E(n) = E[P_t^s : t \geq n, 0 \leq s \leq n-1]$, for $n \geq 1$ (see [5], for example). For $n \geq 1$ let $Y(n)$ be the sub-Hopf algebra dual to $\mathbf{F}_2[\xi_n, \xi_{n+1}, \ldots]$ (so we have $A = Y(1) \supset Y(2) \supset Y(3) \supset \cdots$).

Let $z \in \mathrm{Ext}_A^{**}(\mathbf{F}_2, \mathbf{F}_2)$; we will also use z to denote the restriction $\iota_{Y(n)}^*(z) \in \mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$. Assume that z is "not detected" by any exterior algebra $E \subset A$ (i.e., the restriction $\iota_E^*(z) = 0$ for all E). We will show that $z \in \mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$ is nilpotent by downward induction on n.

First, since $\mathrm{Ext}_{Y(n)}^{s,t}(\mathbf{F}_2, \mathbf{F}_2) = 0$ if $(2^n - 1)s > t$, then for $n \gg 0$, z restricts to 0 over $Y(n)$; this starts the induction. The inductive step is somewhat more involved.

Assume that z restricts to zero in $\mathrm{Ext}_{Y(n+1)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$. We want to show that z is nilpotent when restricted to $\mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$.

Note that $Y(n)//Y(n+1) \cong E[P_n^s : s \geq 0]$. Define a module G_k over this exterior algebra by $G_k = E[P_n^s : k-1 \geq s \geq 0]$; let $G_0 = \mathbf{F}_2$. Note also that for each s, P_n^s is indecomposable in $Y(n)$, so that the polynomial generators of $\mathrm{Ext}_{Y(n)//Y(n+1)}^{**}(\mathbf{F}_2, \mathbf{F}_2) = \mathbf{F}_2[h_{ns} : s \geq 0]$ map nontrivially to $\mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$. We also use h_{ns} to denote their images in $\mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2)$.

We will show the following:

Lemma 2.1 *For each s, there exist integers i and j so that $h_{ns}^{2^i} z^j = 0$.*

Lemma 2.2 *For some $k>0$, there is an integer N so that $z^N \otimes 1_{G_k} = 0$ in $\mathrm{Ext}^{**}_{Y(n)}(G_k, G_k)$.*

Lemma 2.3 *If for some $k>0$ we have $z \otimes 1_{G_k} = 0$, then there is an integer N' so that $z^{N'} \otimes 1_{G_{k-1}} = 0$ in $\mathrm{Ext}^{**}_{Y(n)}(G_{k-1}, G_{k-1})$.*

Lemmas 2.2 and 2.3 give us a downward induction on m to show that $z \otimes 1_{G_m}$ is nilpotent in $\mathrm{Ext}^{**}_{Y(n)}(G_m, G_m)$; since $G_0 = \mathbf{F}_2$, this is good enough. Lemma 2.1 is used to prove 2.3.

Proof of Lemma 2.1: This is in two parts: if $s \geq n$, then h_{ns} is nilpotent in $\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$ (see [5], [7]). Otherwise, z restricts to zero in $\mathrm{Ext}^{**}_{E(n)}(\mathbf{F}_2, \mathbf{F}_2)$; so z goes to zero in $h_{n0}^{-1}\mathrm{Ext}^{**}_{E(n)}(\mathbf{F}_2, \mathbf{F}_2)$. But by Eisen's calculation (see [2], or Theorem A.1 in the appendix), $h_{n0}^{-1}\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$ embeds in $h_{n0}^{-1}\mathrm{Ext}^{**}_{E(n)}(\mathbf{F}_2, \mathbf{F}_2)$, so z is zero in $h_{n0}^{-1}\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$. Hence in $\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$ we have $h_{n0}^{2^i} z = 0$ for some i. Let $|z| = m$, and choose i so that $2^i > 2^{n-1}m$. Then applying Sq^0 s times to the previous equation gives $h_{ns}^{2^i} z^{2^s} = 0$ for all $s \leq n-1$. $\square$

Proof of Lemma 2.2: Fix a finite module M. We will show by induction on the dimension of M that for $k \gg 0$ and for any $\alpha \in \mathrm{Ext}^{**}_{Y(n)}(G_k, M)$, some power of $z \otimes 1_{G_k}$ annihilates α. We will apply this to $M = G_k$ and $\alpha = 1_{G_k}$.

We start with $M = \mathbf{F}_2$. We have a normal algebra extension

$$Y(n+1) \to Y(n) \to Y(n)//Y(n+1).$$

Let $D = Y(n)//Y(n+1)$; as noted above, $D \cong E[P_n^s : s \geq 0]$. Note that for any k, G_k has a D-resolution

$$G_k \leftarrow D \otimes \mathbf{F}_2[h_{ns}, s \geq k],$$

where $|h_{ns}|$ has bidegree $(1, 2^s(2^n-1))$. Let $c = 2^n - 1$. Then for any bounded above D-module N, $\mathrm{Ext}^{**}_D(G_k, N)$ has a vanishing line of slope $2^k c$.

We use a Cartan-Eilenberg spectral sequence associated to this extension:

$$E_2 \cong \mathrm{Ext}^{p,*}_D(G_k, \mathrm{Ext}^{q,-*}_{Y(n+1)}(\mathbf{F}_2, \mathbf{F}_2)) \Rightarrow \mathrm{Ext}^{p+q,*}_{Y(n)}(G_k, \mathbf{F}_2).$$

$\mathrm{Ext}^{**}_{Y(n+1)}(\mathbf{F}_2, \mathbf{F}_2)$ has a vanishing line of slope $2c-1$, so the E_2-term has a vanishing plane: $E_2^{p,q,r} = 0$ if $r < 2^k cp + (2c-1)q$. Of course, we have another such spectral sequence which computes $\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$, and the action of $\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2)$ on $\mathrm{Ext}^{**}_{Y(n)}(G_k, \mathbf{F}_2)$ manifests itself as a pairing of the two spectral sequences. We are interested in the z-action, so we want to find the permanent cycle $\tilde{z}$ in the $\mathbf{F}_2$-spectral sequence that corresponds to z. So assume that $\tilde{z} \in E_2^{p_0,q_0,r_0}$. Can $p_0 = 0$? No, because $z \mapsto 0$ under the restriction $\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2) \to \mathrm{Ext}^{**}_{Y(n+1)}(\mathbf{F}_2, \mathbf{F}_2)$, and this map is the edge homomorphism in the spectral sequence. Hence $p_0 > 0$. This is enough: now we choose k large enough so that $2^k c > p_0$; then multiplication by a high enough power of $\tilde{z}$ in E_2 for G_k lands above the vanishing plane, and hence is zero. So for each $\alpha \in \mathrm{Ext}^{**}_{Y(n)}(G_k, \mathbf{F}_2)$, some power of z kills α.

Assume this is true for all $\alpha \in \mathrm{Ext}^{**}_{Y(n)}(G_k, N)$, as long as $\dim N < m$. Let M be any module of dimension m. We can always find a short exact sequence of $Y(n)$-modules (up to suspension)

$$0 \to \mathbf{F}_2 \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0,$$

with $\dim N = m-1$. Applying $\mathrm{Ext}^{**}_{Y(n)}(G_k, -)$ gives a long exact sequence

$$\cdots \to \mathrm{Ext}^{**}_{Y(n)}(G_k, \mathbf{F}_2) \xrightarrow{\varphi_*} \mathrm{Ext}^{**}_{Y(n)}(G_k, M) \xrightarrow{\psi_*} \mathrm{Ext}^{**}_{Y(n)}(G_k, N) \to \cdots.$$

Given any $\alpha \in \mathrm{Ext}^{**}_{Y(n)}(G_k, M)$, we can find i so that $\psi_*(z^i\alpha) = 0$, by induction. Then $z^i\alpha \in \mathrm{im}\ \varphi_*$, say $\varphi_*(\beta) = z^i\alpha$. But we can find j so that $z^j\beta = 0$, so $0 = \varphi_*(z^k\beta) = z^{i+j}\alpha$. $\square$

Proof of Lemma 2.3: For each k there is a short exact sequence

$$0 \to \Sigma^{2^k c} G_{k-1} \to G_k \to G_{k-1} \to 0$$

(where, as above, $c = 2^n - 1$), which gives $y \in \mathrm{Ext}^{**}_{Y(n)}(G_{k-1}, G_{k-1})$. One can check that this element is the image of h_{nk} under the map

$$\mathrm{Ext}^{**}_{Y(n)}(\mathbf{F}_2, \mathbf{F}_2) \xrightarrow{-\otimes G_{k-1}} \mathrm{Ext}^{**}_{Y(n)}(G_{k-1}, G_{k-1});$$

i.e., $y = h_{nk} \otimes 1_{G_{k-1}}$. For brevity, let $\mathrm{Ext}(M)$ denote $\mathrm{Ext}^{**}_{Y(n)}(M, \mathbf{F}_2)$. The short exact sequence above gives a long exact sequence in Ext:

$$\cdots \to \mathrm{Ext}(G_{k-1}) \xrightarrow{h_{nk}\otimes 1} \mathrm{Ext}(G_{k-1}) \to \mathrm{Ext}(G_k) \to \cdots.$$

We may assume (by taking powers) that $z \otimes 1_{G_k} = 0$; we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \to & \mathrm{Ext}(G_{k-1}) & \xrightarrow{h_{nk}\otimes 1} & \mathrm{Ext}(G_{k-1}) & \to & \mathrm{Ext}(G_k) & \to & \cdots \\
 & & \downarrow{\scriptstyle z\otimes 1} & \swarrow{\scriptstyle \overline{z}} & \downarrow{\scriptstyle z\otimes 1} & & \downarrow{\scriptstyle z\otimes 1 = 0} & & \\
\cdots & \to & \mathrm{Ext}(G_{k-1}) & \xrightarrow[h_{nk}\otimes 1]{} & \mathrm{Ext}(G_{k-1}) & \to & \mathrm{Ext}(G_k) & \to & \cdots
\end{array}$$

Since $z \otimes 1_{G_k} : \mathrm{Ext}(G_k) \to \mathrm{Ext}(G_k)$ is zero, we have a factorization $z \otimes 1_{G_{k-1}} = (h_{nk} \otimes 1) \circ \overline{z} : \mathrm{Ext}(G_{k-1}) \to \mathrm{Ext}(G_{k-1})$. A simple diagram chase then shows that $(z \otimes 1_{G_{k-1}})^j = (h_{nk}^j \otimes 1) \circ \overline{z}^j$ for all j. Thus for any i, $(z \otimes 1)^{i+j} = (h_{nk}^{i+j} z^i \otimes 1) \circ \overline{z}^j$; by choosing i and j large enough, we have (by Lemma 2.1) $h_{nk}^{i+j} z^i = 0$. Hence $z^{i+j} \otimes 1_{G_{k-1}} = 0$, as desired. □

This completes the proof of Theorem 1.1. □

Remark 2.4 *There are (at least) two obstacles to applying the method in this section to study non-nilpotence in* $\mathrm{Ext}_A^{**}(C, \mathbf{F}_2)$*, for* C *a bounded below coalgebra: the first is that we don't have a calculation like Eisen's for the appropriate localized Ext groups. In the proof of Theorem A.1, we can still embed the* E_2*-term of the* $Y(n)$ *spectral sequence in the* E_2*-term for* $E(n)$*, but in this case there is no reason for either spectral sequence to collapse. The second problem is that if* C *is not co-commutative, then we don't have Steenrod operations acting on* $\mathrm{Ext}_{Y(n)}^{**}(C, \mathbf{F}_2)$*, so knowing that some power of* h_{n0} *kills* z *doesn't necessarily tell us anything about* h_{n1} *acting on* z^2*.*

A Appendix: Eisen's calculation

In his thesis, Eisen proves the following result (with notation as above):

Theorem A.1

$$h_{n0}^{-1}\mathrm{Ext}_{Y(n)}^{**}(\mathbf{F}_2, \mathbf{F}_2) \cong \mathbf{F}_2[h_{n0}, h_{n0}^{-1}, h_{ts} : \left\{ \begin{array}{l} \text{if } s = 0, \text{ then } 2n > t > n \\ \text{if } n > s \geq 1, \text{ then } t \geq n \end{array} \right\}].$$

Since his work has never been published, we outline a proof.

First of all, for any $Y(n)$-module M, there is a spectral sequence, called the *Margolis Adams spectral sequence* (see [10] or [9]), with

$$E_2 = \mathrm{Ext}^{**}_{Y(n)^0_n}(H(M, P^0_n), \mathbf{F}_2) \otimes \mathbf{F}_2[h_{n0}, h_{n0}^{-1}] \Rightarrow h_{n0}^{-1}\mathrm{Ext}^{**}_{Y(n)}(M, \mathbf{F}_2),$$

where $Y(n)^0_n$ is the algebra of operations for P^0_n-homology. This spectral sequence is formed by making a "resolution" of M by direct sums of $Y(n)/Y(n)P^0_n$ and $Y(n)$ satisfying certain properties with respect to P^0_n-homology. For our purposes, we only need to know that $Y(n)^0_n$ is given by $Y(n)^0_n = H(Y(n)/Y(n)P^0_n, P^0_n)$, and that one can calculate without too much trouble that

$$Y(n)^0_n \cong E[P^s_t : s \text{ and } t \text{ as in A.1}].$$

So when $M = \mathbf{F}_2$ we have a spectral sequence with

$$E_2 \cong \mathbf{F}_2[h_{n0}, h_{n0}^{-1}, h_{ts} : s \text{ and } t \text{ as in A.1}];$$

we want to show that this spectral sequence collapses. To do this, we embed it in another Margolis Adams spectral sequence, this time for $E(n)$. For this one we have

$$\begin{aligned} E(n)^0_n &= H(E(n)/E(n)P^0_n, P^0_n) \\ &= E(n)/E(n)P^0_n, \end{aligned}$$

so

$$E_2 \cong \mathbf{F}_2[h_{n0}^{-1}, h_{ts} : t \geq n, n > s \geq 0].$$

Also, since $E(n)$ is an exterior algebra, we can see that the spectral sequence collapses. Lastly, we observe that the map $E(n) \to Y(n)$ induces an embedding of the E_2-term for the $Y(n)$-spectral sequence into that for $E(n)$, and hence the $Y(n)$ spectral sequence collapses as well. □

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