

THE STRUCTURE OF THE BOUSFIELD LATTICE

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ABSTRACT. Using Ohkawa's theorem that the collection \mathbf{B} of Bousfield classes is a set, we perform a number of constructions with Bousfield classes. In particular, we describe a greatest lower bound operator; we also note that a certain subset \mathbf{DL} of \mathbf{B} is a frame, and we examine some consequences of this observation. We make several conjectures about the structure of \mathbf{B} and \mathbf{DL} .

1. INTRODUCTION

In [Bou79a] and [Bou79b], Bousfield introduced an equivalence relation on spectra that has turned out to be extremely important. Given a spectrum E , we define the *Bousfield class* $\langle E \rangle$ of E to be the collection of *E -acyclic* spectra X , where X is *E -acyclic* if and only if $E \wedge X = 0$. Then we say that E and F are *Bousfield equivalent* if and only if $\langle E \rangle = \langle F \rangle$. The notion of Bousfield equivalence, and hence Bousfield class, plays a major role in much of modern stable homotopy theory.

We can order the collection of Bousfield classes using reverse inclusion. We then have a partially ordered class associated to the stable homotopy category, and Bousfield and others have investigated properties of this partially ordered class. The nilpotence theorem of Devinatz, Hopkins, and Smith [DHS88], for example, is equivalent to the classification of Bousfield classes of finite spectra [HS]. We recently learned that Ohkawa has proved the surprising result that there is only a set of Bousfield classes [Ohk89]; see also [Str97]. He proves there are at most \beth_2 Bousfield classes, where $\beth_i = 2^{\beth_{i-1}}$ and $\beth_0 = \aleph_0$. In light of this result, the authors decided to re-examine the structure of the partially ordered set of Bousfield classes.

The goal of this paper is to provide some kind of global understanding of the partially ordered set \mathbf{B} of Bousfield classes. Using Ohkawa's result, we are able to perform certain constructions in \mathbf{B} , such as a greatest lower bound operation. We also bring to bear many methods and results from lattice theory; for instance, the sub-partially ordered set \mathbf{DL} of \mathbf{B} , which consists of all Bousfield classes $\langle X \rangle$ for which $\langle X \rangle = \langle X \wedge X \rangle$, is a very nice sort of distributive lattice known as a *frame*. This has some nice consequences, and it also leads to some interesting questions. Much of our understanding of the Bousfield lattice is only conjectural; we hope that the conjectures and their implications are interesting enough to prompt further study of this material. There are several questions we have not addressed. In particular, a frame such as \mathbf{DL} has an associated topological space. It would be interesting to understand something about this space, even conjecturally. Jack Morava has asked whether this space has a structure sheaf, probably of stable homotopy categories, associated to it. The stalk at $K(n)$, for example, might be

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the $K(n)$ -local category. There are also many frame-theoretic properties that \mathbf{DL} may or may not have, such as coherence.

Here is one of the conjectures that we do discuss. Call a Bousfield class $\langle X \rangle$ *strange* if $\langle X \rangle < \langle H\mathbf{F}_p \rangle$. For instance, the Brown-Comenetz dual of the p -local sphere has such a Bousfield class. By general lattice theory, the inclusion $\mathbf{DL} \hookrightarrow \mathbf{B}$ has a “right adjoint” $r: \mathbf{B} \rightarrow \mathbf{DL}$ which is a retraction onto \mathbf{DL} . One can see that r sends every strange Bousfield class to $\langle 0 \rangle$, and also that r induces a map $r': \mathbf{B}/(\text{strange}) \rightarrow \mathbf{DL}$, where $\mathbf{B}/(\text{strange})$ is the quotient lattice of \mathbf{B} by the ideal of strange Bousfield classes. Conjecture 3.12 states that r' is an isomorphism; this implies, for example, that $\langle E \wedge E \rangle = \langle E \wedge E \wedge E \rangle$ for all spectra E . Our other main conjectures are Conjectures 5.1, 6.12, 7.4–7.6, and 9.1.

Here is the structure of the paper. In Section 2, we define Bousfield classes and the basic operations one can perform on them: join, smash, meet, and complementation. Next we examine \mathbf{DL} and its relation to \mathbf{B} ; in particular, we note that \mathbf{DL} is a frame, and we construct a retraction from \mathbf{B} to \mathbf{DL} . We also give the conjectured description of this retraction in terms of strange Bousfield classes. We discuss more basic structure in Section 4: we discuss minimal and complemented Bousfield classes, and we recall some properties of \mathbf{BA} , the set of complemented Bousfield classes. For example, we recall Bousfield’s observation that \mathbf{BA} is a Boolean algebra. In Section 5 we examine spectra X for which there is a finite spectrum F with $X \wedge F = 0$; we give a conjectured classification of the Bousfield classes of such X . This provides some information about \mathbf{BA} . In Section 6, we return to the fact that \mathbf{DL} is a frame; this allows us to construct a complete Boolean algebra $\mathbf{cBA} \subseteq \mathbf{DL}$ which (properly) contains \mathbf{BA} , and we give a conjectured description of \mathbf{cBA} . Then in the next section, we examine Bousfield classes of spectra X for which $X \wedge F \neq 0$ for all finite F . This leads to a discussion of some properties of I , the Brown-Comenetz dual of the p -local sphere, as well as several conjectures about spectra with no finite acyclics. We show that these conjectures are all equivalent, and we discuss some of their consequences. Much of the paper to this point suggests that the set of strange Bousfield classes, those classes of p -local spectra X with $\langle X \rangle < \langle H\mathbf{F}_p \rangle$, is interesting; in Section 8 we examine some examples of such spectra. We end the paper in Section 9 with a discussion of the partially ordered class of localizing subcategories—recall that a subcategory is called *localizing* if it is thick and is closed under coproducts; the main conjecture is that every localizing subcategory is equal to the class of E -acyclics for some spectrum E . This conjecture has several equivalent formulations, and some deep structural consequences.

We work p -locally throughout the paper, except for Section 9, in which we work globally. As in all discussions of Bousfield classes of spectra, we work in the stable homotopy category of spectra, as described for example in [HPS97].

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2. BASIC STRUCTURE OF THE BOUSFIELD LATTICE: \vee , \wedge , \wedge , AND a

In this section we discuss the basic structure of the Bousfield lattice, including the wedge (a.k.a. the join) \vee , the smash product \wedge , the meet \wedge , and the complementation operator a .

We start with the definition of Bousfield equivalence and related ideas, due to Bousfield in [Bou79a] and [Bou79b].

Definition 2.1. Let E , F , X , and Z be spectra.

- (a) Z is E -acyclic if and only if $E \wedge Z = 0$.
- (b) The *Bousfield class* of E , written $\langle E \rangle$, is the collection of E -acyclic spectra.
- (c) The spectra E and F are *Bousfield equivalent* if and only if $\langle E \rangle = \langle F \rangle$.
- (d) The Bousfield classes are partially ordered by reverse inclusion: we write $\langle E \rangle \geq \langle F \rangle$ if and only if $E \wedge Z = 0 \Rightarrow F \wedge Z = 0$.
- (e) The *wedge* of $\langle E \rangle$ and $\langle F \rangle$ is defined to be $\langle E \rangle \vee \langle F \rangle = \langle E \vee F \rangle$. The wedge of an arbitrary set of Bousfield classes is defined the same way.
- (f) Similarly, the *smash product* of $\langle E \rangle$ and $\langle F \rangle$ is defined to be $\langle E \rangle \wedge \langle F \rangle = \langle E \wedge F \rangle$.
- (g) X is E -local if and only if $[Z, X] = 0$ for all E -acyclic spectra Z .

We denote the p -local sphere by S ; then $\langle S \rangle$ is the largest Bousfield class in this ordering, and $\langle 0 \rangle$ is the smallest. It is clear that $\langle E \rangle \vee \langle F \rangle$ is the least upper bound, or *join*, of $\langle E \rangle$ and $\langle F \rangle$; indeed, $\bigvee \langle E_i \rangle$ is the join of the set $\{\langle E_i \rangle\}$.

Now we recall Ohkawa's result.

Theorem 2.2 ([Ohk89]). *The class of Bousfield classes forms a set.*

We use \mathbf{B} to denote the set of Bousfield classes.

\mathbf{B} is a partially ordered set in which every subset has a least upper bound (i.e., \mathbf{B} is a *complete join semilattice*). Since there is a smallest element, then every subset also has a greatest lower bound, or *meet*, obtained by taking the join of all the lower bounds (we are using the fact there is a (nonempty) *set* of these lower bounds, so we can in fact take the join). Since \mathbf{B} has both finite joins and finite meets, it is a *lattice*; since it has arbitrary joins, it is a *complete lattice*. We denote the meet of $\langle X \rangle$ and $\langle Y \rangle$ by $\langle X \rangle \wedge \langle Y \rangle$. Unfortunately, this meet is not easily described. In particular, we do not know whether \mathbf{B} is distributive: in other words, is $\langle X \rangle \vee (\langle Y \rangle \wedge \langle Z \rangle) = (\langle X \rangle \vee \langle Y \rangle) \wedge (\langle X \rangle \vee \langle Z \rangle)$? The meet certainly does not distribute over infinite joins; see Example 7.3.

In contrast, the smash product of Bousfield classes distributes over infinite joins. $\langle X \rangle \wedge \langle Y \rangle$ is a lower bound for $\langle X \rangle$ and $\langle Y \rangle$, but it need not be the greatest lower bound; for example, if I is the Brown-Comenetz dual of the sphere, then $I \wedge I = 0$ (see [Bou79a, Lemma 2.5] and Lemma 3.8 below). In general, then, we have $\langle X \rangle \wedge \langle Y \rangle \leq \langle X \rangle \wedge \langle Y \rangle$.

In any complete lattice with an operation that distributes over infinite joins, we can define a complementation operator a : we define $a\langle X \rangle$ to be the join of all $\langle Y \rangle$ such that $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$. Here are some of the basic properties of a , most of which are due to Bousfield [Bou79b].

Lemma 2.3. *Let a be the complementation operator on the Bousfield lattice \mathbf{B} . Then a has the following properties.*

- (a) $\langle E \rangle \leq a\langle X \rangle$ if and only if $E \wedge X = 0$.
- (b) a is order-reversing: $\langle X \rangle \leq \langle Y \rangle$ if and only if $a\langle X \rangle \geq a\langle Y \rangle$.
- (c) $a^2\langle X \rangle = \langle X \rangle$.
- (d) $\langle X \rangle \wedge \langle Y \rangle = a(a\langle X \rangle \vee a\langle Y \rangle)$.
- (e) More generally, a converts arbitrary joins to meets and arbitrary meets to joins.

Proof. Part (a) holds since the smash product distributes over infinite joins, so $a\langle X \rangle \wedge \langle X \rangle = 0$. For the next part, suppose $\langle X \rangle \leq \langle Y \rangle$. Since $a\langle Y \rangle \wedge \langle Y \rangle = 0$,

then $a\langle Y \rangle \wedge \langle X \rangle = 0$. Hence $a\langle Y \rangle \leq a\langle X \rangle$, so a is order-reversing. The other half of part (b) follows from part (c).

For part (c), it is formal to verify that $\langle X \rangle \leq a^2\langle X \rangle$. Since a is order-reversing, it follows that $a\langle X \rangle \geq a^3\langle X \rangle$. Thus $a\langle X \rangle = a^3\langle X \rangle$. Now suppose $X \wedge Z = 0$. Then $\langle Z \rangle \leq a\langle X \rangle = a(a^2\langle X \rangle)$, so $a^2\langle X \rangle \wedge \langle Z \rangle = 0$. Thus $\langle X \rangle \geq a^2\langle X \rangle$, completing the proof of part (c). Parts (d) and (e) are formal consequences of the other parts, given Ohkawa's theorem. \square

Note that not all of these properties would hold if we tried to define a using the meet instead of the smash.

Bousfield's work predates Ohkawa's, so he had to work harder to construct the operator a . In particular, he constructs an operator at (or closer to) the spectrum level, and shows that it descends to give an operator on Bousfield classes. For any spectrum E , Bousfield shows in [Bou79b, Lemma 1.13] that the localizing subcategory of E -acyclic spectra is generated by a single spectrum aE . So for instance, a spectrum X is E -local if and only if $[aE, X]_* = 0$. The spectrum aE is not well-defined, but any other choice generates the same localizing subcategory, so in particular has the same Bousfield class. Thus $\langle aE \rangle$ is well-defined. In fact, $\langle aE \rangle = a\langle E \rangle$, since if $Z \wedge E = 0$, then Z is in the localizing subcategory generated by aE , so $\langle Z \rangle \leq \langle aE \rangle$.

As with the meet, it is rather difficult to compute the effect of the operator a . We will discuss it further and give some examples in Section 4.

Dan Christensen has pointed out that, just as one can define the meet operation \wedge by $\langle X \rangle \wedge \langle Y \rangle = a(a\langle X \rangle \vee a\langle Y \rangle)$, one can define an operation Υ by

$$\langle X \rangle \Upsilon \langle Y \rangle = a(a\langle X \rangle \wedge a\langle Y \rangle).$$

Then $\langle X \rangle \Upsilon \langle Y \rangle \geq \langle X \rangle \vee \langle Y \rangle$, and this inequality may be strict; for example, $a\langle I \rangle \Upsilon a\langle I \rangle = \langle S \rangle$, even though $a\langle I \rangle \neq \langle S \rangle$.

3. THE RETRACTION ONTO \mathbf{DL}

The Bousfield lattice \mathbf{B} is a complete lattice, but may not be distributive; we show in Example 7.3 that the meet does not distribute over infinite joins. In any case, the smash and the meet certainly do not coincide. To get around this problem, Bousfield introduced the sub-partially ordered set \mathbf{DL} of \mathbf{B} in [Bou79a]: \mathbf{DL} consists of the Bousfield classes $\langle E \rangle$ satisfying $\langle E \rangle = \langle E \rangle \wedge \langle E \rangle$. The goal of this section is to study \mathbf{DL} and its relationship to \mathbf{B} . In particular, we point out that there is a retraction $\mathbf{B} \rightarrow \mathbf{DL}$, and we make some conjectures about it.

Example 3.1. Bousfield observes in [Bou79a] that if E is a ring spectrum or a finite spectrum, then $\langle E \rangle$ is in \mathbf{DL} . On the other hand, I , the Brown-Comenetz dual of the p -local sphere is not: since $I \wedge I = 0$, then $\langle I \rangle > \langle I \rangle \wedge \langle I \rangle$.

We mention the following in passing.

Question 3.2. *Let E be a spectrum. Must the sequence*

$$\langle E \rangle \geq \langle E \rangle^{\wedge 2} \geq \langle E \rangle^{\wedge 3} \geq \dots$$

stabilize?

See Proposition 3.13(e) for a conjectured answer to this question.

Lemma 3.3. *Suppose $\langle E \rangle \in \mathbf{DL}$, $\langle E \rangle \leq \langle X \rangle$, and $\langle E \rangle \leq \langle Y \rangle$. Then $\langle E \rangle \leq \langle X \rangle \wedge \langle Y \rangle$.*

Proof. We have $\langle E \rangle = \langle E \rangle \wedge \langle E \rangle \leq \langle X \rangle \wedge \langle Y \rangle$. □

A *frame* is a complete lattice in which the meet distributes over infinite joins: $a \wedge \bigvee b_i = \bigvee (a \wedge b_i)$. For example, a topology on a space X has the structure of a frame, in which the open subsets of X are ordered by inclusion. Frames are also called *locales*, *complete Heyting algebras*, or *complete Brouwerian lattices*. They are used in categorical topology [Joh86], where a locale is viewed as a generalized topological space, lattice theory [Bir79], and logic [FS90].

Proposition 3.4. *\mathbf{DL} is a frame. In \mathbf{DL} , the join of $\{\langle X_i \rangle\}$ is $\bigvee \langle X_i \rangle$, and the meet of $\langle X \rangle$ and $\langle Y \rangle$ is $\langle X \rangle \wedge \langle Y \rangle$. The inclusion $i: \mathbf{DL} \rightarrow \mathbf{B}$ preserves arbitrary joins but does not preserve meets.*

Proof. Much of this is due to Bousfield [Bou79a], either explicitly or implicitly. We leave to the reader the straightforward check that $\bigvee \langle X_i \rangle$ and $\langle X \rangle \wedge \langle Y \rangle$ are in \mathbf{DL} if all $\langle X_i \rangle$, $\langle X \rangle$, and $\langle Y \rangle$ are in \mathbf{DL} . It follows from this that $\bigvee \langle X_i \rangle$ is the join of $\{\langle X_i \rangle\}$, that \mathbf{DL} is a complete lattice, and that the inclusion $i: \mathbf{DL} \rightarrow \mathbf{B}$ preserves joins. Lemma 3.3 implies that the meet in \mathbf{DL} is the smash product, and, since the smash product distributes over infinite joins, that \mathbf{DL} is a frame. To see that i does not preserve meets, note that both $\langle \mathbf{HF}_p \rangle$ and $\bigvee_n \langle K(n) \rangle$ are in \mathbf{DL} . Their smash product, and hence their meet in \mathbf{DL} , is 0, but their meet in \mathbf{B} is at least $\langle I \rangle$ by Proposition 7.2. □

We can think of a complete lattice, or indeed any partially ordered set, as a category with a unique map from x to y if and only if $x \leq y$. A complete lattice is just a partially ordered set that is complete and cocomplete as a category; the colimit of a functor to a lattice is the join of all of the objects in the image, and dually for the limit. From this point of view, an order-preserving map of partially ordered sets corresponds to a functor on the associated categories. A functor between complete lattices preserves colimits if and only if it preserves arbitrary joins. Obviously a left adjoint must have this property, and, for complete lattices, the converse is true as well. Note that, for maps of partially ordered sets f and g , g is right adjoint to f if and only if $fx \leq y$ is equivalent to $x \leq gy$.

Lemma 3.5. *Suppose $f: \mathbf{C} \rightarrow \mathbf{D}$ is an order-preserving map between complete lattices. Then f has a right adjoint if and only if f preserves arbitrary joins. In this case, the right adjoint of f is the map g defined by $gy = \bigvee \{x \mid fx \leq y\}$.*

Proof. One can easily verify that g is order-preserving and $fx \leq y$ implies $x \leq gy$. Conversely, if f preserves colimits, then $fgy = \bigvee \{fx \mid fx \leq y\} \leq y$, so $x \leq gy$ implies $fx \leq fgy \leq y$. □

Johnstone proves in [Joh86, Theorem I.4.2] the (equivalent) statement that a functor between complete lattices has a left adjoint if and only if it preserves arbitrary meets. Applying Lemma 3.5 to \mathbf{DL} , we get the following corollary, first pointed out to us by Neil Strickland.

Corollary 3.6. *The inclusion functor $\mathbf{DL} \rightarrow \mathbf{B}$ has a right adjoint $r: \mathbf{B} \rightarrow \mathbf{DL}$ defined by $r\langle X \rangle = \bigvee \{\langle Y \rangle \in \mathbf{DL} \mid \langle Y \rangle \leq \langle X \rangle\}$. The functor r preserves arbitrary meets, $r\langle X \rangle \leq \langle X \rangle$ for all X , and $r\langle X \rangle = \langle X \rangle$ if $\langle X \rangle \in \mathbf{DL}$.*

In fact r preserves the smash product as well.

Lemma 3.7. *The functor $r: \mathbf{B} \rightarrow \mathbf{DL}$ preserves the smash product: $r(\langle X \rangle \wedge \langle Y \rangle) = r\langle X \rangle \wedge r\langle Y \rangle$.*

Proof. Since $\langle X \rangle \wedge \langle Y \rangle$ is a lower bound for $\langle X \rangle$ and $\langle Y \rangle$, $r(\langle X \rangle \wedge \langle Y \rangle)$ is a lower bound for $r\langle X \rangle$ and $r\langle Y \rangle$, so $r(\langle X \rangle \wedge \langle Y \rangle) \leq r\langle X \rangle \wedge r\langle Y \rangle$. Conversely, $r\langle X \rangle \wedge r\langle Y \rangle \leq \langle X \rangle \wedge \langle Y \rangle$ and $r\langle X \rangle \wedge r\langle Y \rangle \in \mathbf{DL}$, so $r\langle X \rangle \wedge r\langle Y \rangle \leq r(\langle X \rangle \wedge \langle Y \rangle)$. \square

We would like to understand this map r more explicitly. We begin by pointing out that r does kill some Bousfield classes.

Lemma 3.8. *If $\langle E \rangle < \langle \mathbf{HF}_p \rangle$ and $\langle F \rangle \leq \langle \mathbf{HF}_p \rangle$, then $E \wedge F = 0$. In particular, $\langle E \rangle \wedge \langle E \rangle = 0$, so $r\langle E \rangle = 0$.*

Proof. We must have $E \wedge \mathbf{HF}_p = 0$, since otherwise $\langle E \rangle \geq \langle \mathbf{HF}_p \rangle$. Hence $E \wedge F = 0$. In particular $\langle E \rangle \wedge \langle E \rangle = 0$. Since $r\langle E \rangle \in \mathbf{DL}$, Lemma 3.3 implies that $r\langle E \rangle \leq \langle E \rangle \wedge \langle E \rangle = 0$, so $r\langle E \rangle = 0$. \square

By the argument in [Rav84, 2.6] (see also Lemma 7.1(c) below), I does have $\langle I \rangle < \langle \mathbf{HF}_p \rangle$, so there are nontrivial examples of such spectra.

Definition 3.9. We define a spectrum E to be *strange* if $\langle E \rangle < \langle \mathbf{HF}_p \rangle$.

Hence every strange spectrum is in the kernel of r . We will study some more examples of strange spectra in Section 8.

A subset J of a complete lattice \mathbf{C} is called a *complete ideal* if it is closed under arbitrary joins, and if $x \in J$ and $y \leq x$, then $y \in J$. Every complete ideal in a complete lattice is *principal*; if we let m be the join of all the elements of J , then $y \in J$ if and only if $y \leq m$. For the complete ideal of strange spectra, we can identify the maximal element m “explicitly.”

Lemma 3.10. *Let $D = a\mathbf{HF}_p \vee \mathbf{HF}_p$. Then the collection of strange Bousfield classes is the principal ideal generated by $a\langle D \rangle = a\langle \mathbf{HF}_p \rangle \wedge \langle \mathbf{HF}_p \rangle$.*

Proof. Note that $\langle E \rangle \leq a\langle D \rangle$ if and only if $E \wedge \mathbf{HF}_p = 0$ and $E \wedge a\mathbf{HF}_p = 0$. This second condition holds if and only if $\langle E \rangle \leq a^2\langle \mathbf{HF}_p \rangle = \langle \mathbf{HF}_p \rangle$. Hence $\langle E \rangle \leq a\langle D \rangle$ if and only if $\langle E \rangle \leq \langle \mathbf{HF}_p \rangle$ and $E \wedge \mathbf{HF}_p = 0$, that is, if and only if $\langle E \rangle < \langle \mathbf{HF}_p \rangle$. \square

Given a (complete) ideal J in a (complete) lattice \mathbf{C} , we can define $a \equiv b \pmod{J}$ if there is some $x \in J$ such that $a \vee x = b \vee x$. If J is principal, then $a \equiv b \pmod{J}$ if $a \vee m = b \vee m$, where m is the largest element in J . The equivalence classes under this congruence relation define a complete lattice \mathbf{C}/J (see [Bir79, II.4], and note that a complete join semilattice is a complete lattice). The obvious epimorphism $\mathbf{C} \rightarrow \mathbf{C}/J$ preserves arbitrary joins, and has kernel J . There are often other epimorphisms with kernel J ; hence given a poset map $\mathbf{C} \rightarrow \mathbf{D}$ with kernel containing J , there may not be an induced map $\mathbf{C}/J \rightarrow \mathbf{D}$.

Proposition 3.11. *Let J be the principal ideal of strange Bousfield classes. If $\langle X \rangle \equiv \langle Y \rangle \pmod{J}$, then $r\langle X \rangle = r\langle Y \rangle$.*

Proof. As before, we let $D = a\mathbf{HF}_p \vee \mathbf{HF}_p$. Since J is the principal ideal generated by $a\langle D \rangle$, we have $\langle X \rangle \equiv \langle Y \rangle \pmod{J}$ if and only if $\langle X \rangle \vee a\langle D \rangle = \langle Y \rangle \vee a\langle D \rangle$. It therefore suffices to show that $r(\langle X \rangle \vee a\langle D \rangle) = r\langle X \rangle$. So suppose $\langle Z \rangle \in \mathbf{DL}$ with $\langle Z \rangle \leq \langle X \rangle \vee a\langle D \rangle$. Then Lemma 3.3 implies that $\langle Z \rangle = (\langle Z \rangle \wedge \langle X \rangle) \vee (\langle Z \rangle \wedge a\langle D \rangle)$.

Now, if $Z \wedge H\mathbf{F}_p = 0$, then $Z \wedge aD = 0$, and so $\langle Z \rangle = \langle Z \rangle \wedge \langle X \rangle \leq \langle X \rangle$. On the other hand, if $Z \wedge H\mathbf{F}_p$ is nonzero, then $(X \vee aD) \wedge H\mathbf{F}_p$ is nonzero, so $X \wedge H\mathbf{F}_p$ is nonzero. Hence $\langle Z \rangle \wedge a\langle D \rangle \leq \langle H\mathbf{F}_p \rangle \leq \langle X \rangle$, so $\langle Z \rangle \leq \langle X \rangle$ in this case as well. Thus $r(\langle X \rangle \vee a\langle D \rangle) = r\langle X \rangle$ as required. \square

It follows from Proposition 3.11 that the epimorphism $r: \mathbf{B} \rightarrow \mathbf{DL}$ factors through an epimorphism $r': \mathbf{B}/J \rightarrow \mathbf{DL}$, where J is the ideal of strange spectra.

Conjecture 3.12. *The epimorphism $r': \mathbf{B}/J \rightarrow \mathbf{DL}$ is an isomorphism.*

This conjecture has two parts: that J is the kernel of r , and (since epimorphisms of lattices are not determined by their kernels) that the induced map is an isomorphism. The conjecture has several consequences.

Proposition 3.13. *Suppose Conjecture 3.12 holds. Then the following properties hold.*

- (a) $r\langle E \rangle = 0$ if and only if E is strange.
- (b) r preserves arbitrary joins.
- (c) If $E \wedge H\mathbf{F}_p \neq 0$, then $\langle E \rangle \in \mathbf{DL}$.
- (d) $r\langle E \rangle = \langle E \rangle \wedge \langle E \rangle$.
- (e) Hence $\langle E \rangle^{\wedge n} = \langle E \rangle^{\wedge(n+1)}$ when $n \geq 2$.

Proof. The first two parts are immediate. For part (c), note that Conjecture 3.12 implies that $\langle E \rangle \equiv r\langle E \rangle \pmod{J}$, so that $\langle E \rangle \vee a\langle D \rangle = r\langle E \rangle \vee a\langle D \rangle$, where $D = aH\mathbf{F}_p \vee H\mathbf{F}_p$ as usual. If $E \wedge H\mathbf{F}_p \neq 0$, then $\langle E \rangle \geq \langle H\mathbf{F}_p \rangle > a\langle D \rangle$, so $\langle E \rangle \vee a\langle D \rangle = \langle E \rangle$. Similarly, $r\langle E \rangle > a\langle D \rangle$, so $r\langle E \rangle \vee a\langle D \rangle = r\langle E \rangle$. Thus $\langle E \rangle = r\langle E \rangle$, and so $\langle E \rangle \in \mathbf{DL}$.

Part (d) is proved similarly. We can assume that $E \wedge H\mathbf{F}_p = 0$. Then

$$\begin{aligned} \langle E \rangle \wedge \langle E \rangle &= (\langle E \rangle \vee a\langle D \rangle) \wedge (\langle E \rangle \vee a\langle D \rangle) \\ &= (r\langle E \rangle \vee a\langle D \rangle) \wedge (r\langle E \rangle \vee a\langle D \rangle) \\ &= r\langle E \rangle \wedge r\langle E \rangle \\ &= r\langle E \rangle. \end{aligned}$$

Part (e) follows immediately. \square

Note that, if r preserves arbitrary joins, it must have a right adjoint $r^*: \mathbf{DL} \rightarrow \mathbf{B}$. This right adjoint must be defined by $r^*\langle E \rangle = \langle E \rangle \vee a\langle D \rangle$, where $D = aH\mathbf{F}_p \vee H\mathbf{F}_p$. We can define this map without knowing Conjecture 3.12, of course, but we do not know that it preserves arbitrary meets without Conjecture 3.12.

Another corollary of Conjecture 3.12 would be some understanding of the difference between the meet and the smash in \mathbf{B} . In particular, the meet and the smash are equivalent, modulo strange spectra.

Proposition 3.14. *Suppose Conjecture 3.12 holds. Let $D = aH\mathbf{F}_p \vee H\mathbf{F}_p$, so that $a\langle D \rangle$ is the maximum strange Bousfield class. Then if $\langle X \rangle$ and $\langle Y \rangle$ are arbitrary Bousfield classes, we have*

$$(\langle X \rangle \wedge \langle Y \rangle) \vee a\langle D \rangle = (\langle X \rangle \wedge \langle Y \rangle) \vee a\langle D \rangle.$$

Proof. Since r preserves both meets and the smash product, we have $r(\langle X \rangle \wedge \langle Y \rangle) = r(\langle X \rangle \wedge \langle Y \rangle)$. Conjecture 3.12 completes the proof. \square

4. MORE STRUCTURE OF \mathbf{B} : MINIMAL AND COMPLEMENTED CLASSES

In this section we discuss minimal, maximal, and complemented Bousfield classes.

We say that a nonzero Bousfield class $\langle E \rangle$ is *minimal* if there is no nonzero Bousfield class strictly less than $\langle E \rangle$. *Maximal* Bousfield classes are defined similarly.

Example 4.1. For $n \geq 0$, the n th Morava K -theory spectrum $K(n)$ has a minimal Bousfield class—see Section 5. We conjecture below (Conjecture 5.1) that $\langle A(n) \rangle$ is minimal when $n \geq 2$, where $A(n)$ is a spectrum that measures the failure of the telescope conjecture; we also conjecture (see Lemma 7.8) that $\langle I \rangle$ is minimal, where I is the Brown-Comenetz dual of the sphere.

It is natural to wonder whether a given Bousfield class can be written as the least upper bound of minimal ones, or dually, whether a class is the greatest lower bound of maximal ones. Since the least upper bound has a much more convenient description, we will focus on minimal Bousfield classes. Using the complementation operator a , one can easily check that $\langle X \rangle$ is minimal if and only if $a\langle X \rangle$ is maximal.

Although we have referred to a as the complementation operator, it is not always the case that $a\langle X \rangle \vee \langle X \rangle = \langle S \rangle$; when this happens, we say that $\langle X \rangle$ is *complemented*. One can easily check that if there is a Bousfield class $\langle Y \rangle$ so that $\langle X \rangle \vee \langle Y \rangle = \langle S \rangle$ and $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$, then $\langle Y \rangle = a\langle X \rangle$. This is the reason for the term “complemented.” We also define $\langle X \rangle$ to be \wedge -*complemented* if there is a Bousfield class $\langle Y \rangle$ so that $\langle X \rangle \wedge \langle Y \rangle = \langle 0 \rangle$ and $\langle X \rangle \vee \langle Y \rangle = \langle S \rangle$.

Now we note that we should only have made one definition.

Proposition 4.2. *$\langle X \rangle$ is \wedge -complemented if and only if $\langle X \rangle$ is complemented. If these conditions hold, then the \wedge -complement of $\langle X \rangle$ is $a\langle X \rangle$.*

Proof. Since $\langle X \rangle \wedge \langle Y \rangle \leq \langle X \rangle \wedge \langle Y \rangle$, we see that if $\langle X \rangle$ is \wedge -complemented, then $\langle X \rangle$ is complemented, with the same complement. Conversely, suppose that $a\langle X \rangle \vee \langle X \rangle = \langle S \rangle$. Then

$$a\langle X \rangle \wedge \langle X \rangle = a(a^2\langle X \rangle \vee a\langle X \rangle) = a\langle S \rangle = \langle 0 \rangle,$$

so $a\langle X \rangle$ is the \wedge -complement of $\langle X \rangle$. □

The collection of all complemented Bousfield classes is denoted \mathbf{BA} . Here are some of the basic properties of \mathbf{BA} ; these are all due to Bousfield [Bou79a].

Lemma 4.3. *Suppose that $\langle X \rangle$ and $\langle Y \rangle$ are in \mathbf{BA} , and $\langle E \rangle$ is an arbitrary Bousfield class. Then:*

- (a) $\langle E \rangle = (\langle E \rangle \wedge \langle X \rangle) \vee (\langle E \rangle \wedge a\langle X \rangle)$.
- (b) $\langle E \rangle \leq \langle X \rangle$ if and only if $\langle E \rangle = \langle E \rangle \wedge \langle X \rangle$.
- (c) $\langle X \rangle \wedge \langle Y \rangle = \langle X \rangle \wedge \langle Y \rangle$.
- (d) Hence $\mathbf{BA} \subseteq \mathbf{DL}$.
- (e) $\langle X \rangle \wedge \langle Y \rangle$ is in \mathbf{BA} , and $a(\langle X \rangle \wedge \langle Y \rangle) = a\langle X \rangle \vee a\langle Y \rangle$.
- (f) $\langle X \rangle \vee \langle Y \rangle$ is in \mathbf{BA} , and $a(\langle X \rangle \vee \langle Y \rangle) = a\langle X \rangle \wedge a\langle Y \rangle$.
- (g) \mathbf{BA} is a Boolean algebra.

(Recall that a *Boolean algebra* is a distributive lattice in which every element has a complement.)

Proof. For the first part, use the identity $\langle E \rangle = \langle E \rangle \wedge \langle S \rangle = \langle E \rangle \wedge (\langle X \rangle \vee a\langle X \rangle)$. The second part then follows immediately. For part (c), suppose $\langle E \rangle \leq \langle X \rangle$ and

$\langle E \rangle \leq \langle Y \rangle$. Then $\langle E \rangle = \langle E \rangle \wedge \langle X \rangle = (\langle E \rangle \wedge \langle Y \rangle) \wedge \langle X \rangle$. Hence $\langle E \rangle \leq \langle X \rangle \wedge \langle Y \rangle$. Part (d) is clear. For part (e), note that $a(\langle X \rangle \wedge \langle Y \rangle) = a(\langle X \rangle \wedge \langle Y \rangle) = a\langle X \rangle \vee a\langle Y \rangle$. Furthermore,

$$\begin{aligned} \langle S \rangle &= a\langle X \rangle \vee \langle X \rangle \\ &= a\langle X \rangle \vee (\langle X \rangle \wedge \langle Y \rangle) \vee (\langle X \rangle \wedge a\langle Y \rangle) \\ &\leq (\langle X \rangle \wedge \langle Y \rangle) \vee a\langle X \rangle \vee a\langle Y \rangle. \end{aligned}$$

Thus $\langle X \rangle \wedge \langle Y \rangle$ is complemented, as required. The proof of part (f) is similar, and part (g) follows immediately from the preceding parts. \square

Example 4.4. Bousfield shows in [Bou79a] that if F is a finite spectrum, then $\langle F \rangle$ is in \mathbf{BA} . He also notes that $\langle H\mathbf{Z} \rangle$ is not in \mathbf{BA} ; in particular, the inclusion $\mathbf{BA} \subset \mathbf{DL}$ is proper. We show in Section 5 that $\langle K(n) \rangle$ and $\langle A(n) \rangle$ are in \mathbf{BA} .

The structure theory of infinite Boolean algebras is considerably more complicated than the structure theory of finite Boolean algebras. In particular, \mathbf{BA} is not closed under infinite joins (see Corollary 7.10), and so is certainly not isomorphic to the complete Boolean algebra of all subsets of some infinite set. The simplest infinite Boolean algebra that is not complete is the Boolean algebra of all finite and cofinite subsets of an infinite set.

We have noted that every finite spectrum is complemented; some other examples of complemented spectra are provided by smashing localizations. Recall that every spectrum E determines a Bousfield localization functor L_E , as described in [Bou79b]. If E and F are Bousfield equivalent, then the functors L_E and L_F are equal—Bousfield localization only depends on the Bousfield class of the spectrum. We say that a Bousfield class $\langle E \rangle$ is *smashing* if the natural map $L_E S \wedge X \rightarrow L_E X$ is an equivalence. Ravenel proves the following in [Rav84, 1.31].

Proposition 4.5. *Every smashing Bousfield class $\langle E \rangle$ is complemented, with complement given by the fiber $A_E S$ of $S \rightarrow L_E S$.*

Proof. For a general Bousfield localization functor L_E , we have $\langle S \rangle = \langle L_E S \rangle \vee \langle A_E S \rangle$. Because L_E is smashing, we have $L_E S \wedge A_E S = L_E A_E S = 0$. \square

5. BOUSFIELD CLASSES WITH FINITE ACYCLICS

In this section we give a brief summary of what is known about Bousfield classes which contain finite spectra; this leads to information about the Boolean algebra \mathbf{BA} . Details can be found in [Hov95a].

As above, we denote the (p -local) sphere by S ; we write $M(p)$ for the mod p Moore spectrum. A generic finite spectrum of type n will be denoted by $F(n)$; then any choice for $F(n)$ generates the same thick subcategory \mathcal{C}_n , by the thick subcategory theorem of Hopkins-Smith [HS, Rav92a]. In particular, the Bousfield class of $F(n)$ is well-defined. Any $F(n)$ has an essentially unique v_n -self map whose cofiber is an $F(n+1)$ and whose telescope we will denote by $T(n)$. The Bousfield class of $T(n)$ is also well-defined.

By repeated use of [Rav84, 1.34], we have a Bousfield class decomposition

$$\langle S \rangle = \langle T(0) \rangle \vee \langle T(1) \rangle \vee \cdots \vee \langle T(n-1) \rangle \vee \langle F(n) \rangle.$$

Furthermore, $T(i) \wedge T(j) = 0$ unless $i = j$, and $T(i) \wedge F(n) = 0$ for $i < n$.

It follows that localization with respect to $T(0) \vee T(1) \vee \cdots \vee T(n-1)$, written L_{n-1}^f , is smashing and that its kernel is precisely the localizing subcategory generated by $F(n)$ —see [Mil92]. By the above decomposition (see also Proposition 4.5), $\langle F(n) \rangle$ is complemented with complement $\langle L_{n-1}^f S \rangle$; in other words, we have $\langle S \rangle = \langle L_{n-1}^f S \rangle \vee \langle F(n) \rangle$, and $F(n) \wedge L_{n-1}^f S = 0$.

Given a spectrum E , we say that E has a *finite acyclic* if there is a nontrivial finite spectrum X such that $E \wedge X = 0$. In this case, the thick subcategory theorem says that the collection of finite E -acyclics is \mathcal{C}_n for some finite n , and we have $\langle E \rangle \leq \langle L_{n-1}^f S \rangle$.

The Morava K -theory spectra $K(n)$ play an important role here. They are known to be *field spectra*, so that $K(n) \wedge E$ is a wedge of suspensions of $K(n)$ for any E . The telescope conjecture, recently proved to be false for $n = 2$ by Ravenel, asserts that $\langle T(n) \rangle = \langle K(n) \rangle$. If this were true, then for any E with a finite acyclic, we would have

$$\langle E \rangle = \bigvee_n \langle E \wedge K(n) \rangle = \bigvee_{\{n \mid E \wedge K(n) \neq 0\}} \langle K(n) \rangle.$$

The failure of the telescope conjecture is measured by the fiber $A(n)$ of the natural map $T(n) \rightarrow L_{K(n)} T(n)$. Once again, $A(n)$ is well-defined up to Bousfield class. With a little work, we have $\langle A(n) \rangle \vee \langle K(n) \rangle = \langle T(n) \rangle$; clearly $A(n) \wedge K(n) = 0$. It follows easily from this that $\langle K(n) \rangle$ and $\langle A(n) \rangle$ are both complemented, as of course is $\langle T(n) \rangle$. Since $K(n)$ is a complemented field spectrum, then $\langle K(n) \rangle$ is minimal, by [HPS97, 3.7.3].

The spectrum $A(n)$ is rather odd, as for example $\langle A(n) \rangle \wedge \langle A(n) \rangle = \langle A(n) \rangle$, yet $BP \wedge A(n) = 0$. So, for instance, $A(n)$ is not (Bousfield equivalent to) a nonzero ring spectrum. As far as detecting finite spectra goes, $A(n)$ behaves as $K(n)$ and $T(n)$ do:

$$\langle A(n) \rangle \wedge \langle F(i) \rangle = \begin{cases} \langle A(n) \rangle & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

Other than this, very little is known about $A(n)$. Since the telescope conjecture fails when $n = 2$, it seems likely that it fails for all $n \geq 2$, in which case $A(n)$ is nonzero when $n \geq 2$. We make the following conjectures. The first is a replacement, of sorts, for the telescope conjecture; it says that, although the telescope conjecture is false, the spectra $A(n)$ that measure its failure behave as well as possible.

Conjecture 5.1. *If $n \geq 2$, $\langle A(n) \rangle$ is a minimal nonzero Bousfield class. Hence, if E has a finite acyclic, then E is Bousfield equivalent to a finite wedge of spectra $K(n)$ and $A(n)$; in particular,*

$$\langle E \rangle = \bigvee_{\{n \mid E \wedge K(n) \neq 0\}} \langle K(n) \rangle \vee \bigvee_{\{n \mid E \wedge A(n) \neq 0\}} \langle A(n) \rangle.$$

Note that each of the wedges here is finite. This would mean that there are only countably many Bousfield classes with a finite acyclic. We also have the following proposition, whose proof is immediate.

Proposition 5.2. *Suppose Conjecture 5.1 holds. Then every Bousfield class with a finite acyclic is complemented.*

6. THE COMPLETE BOOLEAN ALGEBRA OF SPECTRA

We have seen that the sublattice \mathbf{DL} of the Bousfield lattice is a frame, and that the retraction map $r: \mathbf{B} \rightarrow \mathbf{DL}$ preserves arbitrary meets. We have conjectured that r preserves arbitrary joins. We have not discussed how r behaves with respect to complements, however, and we do so in this section. We also explore the relationship between \mathbf{DL} and its sub-poset \mathbf{BA} .

Definition 6.1. Define the *complement* operation $A: \mathbf{DL} \rightarrow \mathbf{DL}$ by

$$\begin{aligned} \mathbf{DL} &\xrightarrow{A} \mathbf{DL}, \\ \langle X \rangle &\mapsto r(a\langle X \rangle). \end{aligned}$$

Then we have the following straightforward lemma, whose proof we leave to the reader.

- Lemma 6.2.** (a) *If $\langle X \rangle$ and $\langle Y \rangle$ are in \mathbf{DL} , then $\langle Y \rangle \leq A\langle X \rangle$ if and only if $Y \wedge X = 0$. In other words, $A\langle X \rangle = \bigvee\{\langle Y \rangle \in \mathbf{DL} \mid \langle Y \rangle \wedge \langle X \rangle = 0\}$.*
 (b) *A is order-reversing: if $\langle X \rangle \leq \langle Y \rangle$ in \mathbf{DL} , then $A\langle X \rangle \geq A\langle Y \rangle$.*
 (c) *If $\langle X \rangle \in \mathbf{DL}$, then $\langle X \rangle \leq A^2\langle X \rangle$ and $A\langle X \rangle = A^3\langle X \rangle$.*
 (d) *A converts arbitrary joins to meets: if $\langle X_i \rangle$ is in \mathbf{DL} for all i , then $A(\bigvee\langle X_i \rangle)$ is the meet of the $A\langle X_i \rangle$.*

Note that this lemma actually holds in any frame, and the complement operator is well-known in the theory. See [Bir79, V.11], for example. We will recall some of this theory in the results below for the reader's convenience.

Also note that A does not convert meets to joins. For example, let $X = \bigvee_n K(n)$ and let $Y = \mathbf{HF}_p$. Then X and Y are both in \mathbf{DL} , and $X \wedge Y = 0$, and thus $A(\langle X \rangle \wedge \langle Y \rangle) = \langle S \rangle$. On the other hand, by the computations in Example 7.3, we have

$$A\langle X \rangle \vee A\langle Y \rangle \leq a\langle X \rangle \vee a\langle Y \rangle = a(\langle X \rangle \wedge \langle Y \rangle) \leq a\langle I \rangle < \langle S \rangle.$$

Of course, we do have $A(\langle X \rangle \wedge \langle Y \rangle) \geq A\langle X \rangle \vee A\langle Y \rangle$ for any $\langle X \rangle$ and $\langle Y \rangle$ in \mathbf{DL} .

This argument also implies that A^2 is not the identity—indeed, if A^2 were the identity, one can check that A would have to convert meets to joins. However, we do not know a specific spectrum X in \mathbf{DL} for which $A^2\langle X \rangle \neq \langle X \rangle$. Given Conjecture 3.12, $a\langle I \rangle$ is in \mathbf{DL} by Proposition 3.13(c), and $A(a\langle I \rangle) = r\langle I \rangle = 0$, so $A^2(a\langle I \rangle) = \langle S \rangle$.

Definition 6.3. A Bousfield class $\langle X \rangle$ is *closed* if $\langle X \rangle \in \mathbf{DL}$ and $A^2\langle X \rangle = \langle X \rangle$. The sub-partially ordered set of \mathbf{DL} consisting of the closed elements is denoted \mathbf{cBA} .

Note that every Bousfield class of the form $A\langle X \rangle$ is closed, by Lemma 6.2(c). We have the following theorem, which again holds in considerably more generality than we state it; see [Bir79, V.10–11] for the general approach.

Theorem 6.4. *The sub-poset \mathbf{cBA} of \mathbf{DL} is closed under arbitrary meets, and therefore is a complete lattice. The join in \mathbf{cBA} of $\{\langle X_i \rangle\}$ is $A^2(\bigvee\langle X_i \rangle)$. Every element in \mathbf{cBA} is complemented, so \mathbf{cBA} is in fact a complete Boolean algebra. The inclusion $\mathbf{cBA} \rightarrow \mathbf{DL}$ preserves arbitrary meets, and its left adjoint is given by $A^2: \mathbf{DL} \rightarrow \mathbf{cBA}$.*

We will write the join in \mathbf{cBA} as $\bigvee_{\mathbf{cBA}}$.

Proof. Note that A^2 is order-preserving. Thus, if we denote by $\bigwedge_i \langle X_i \rangle$ the meet in **DL** of $\{\langle X_i \rangle\}$, we have $\bigwedge_i \langle X_i \rangle \leq A^2(\bigwedge_i \langle X_i \rangle) \leq \bigwedge_i A^2 \langle X_i \rangle$. In particular, if each $\langle X_i \rangle$ is closed, so is $\bigwedge_i \langle X_i \rangle$. So **cBA** is closed under arbitrary meets, and hence is a complete lattice, with the join defined to be the meet of all upper bounds.

Now, certainly $A^2(\bigvee_i \langle X_i \rangle)$ is closed and is an upper bound for $\{\langle X_i \rangle\}$. If $\langle Z \rangle$ is closed and an upper bound for $\{\langle X_i \rangle\}$, we have $\langle Z \rangle = A^2 \langle Z \rangle \geq A^2(\bigvee_i \langle X_i \rangle)$, so the join in **cBA** is as claimed. One can easily check that A^2 is the left adjoint to the inclusion.

It remains to show that an arbitrary element $\langle X \rangle$ of **cBA** is complemented in **cBA**. To see this, note that $\langle X \rangle \wedge A \langle X \rangle = 0$, and

$$A^2(\langle X \rangle \vee A \langle X \rangle) = A(A \langle X \rangle \wedge A^2 \langle X \rangle) = A(0) = \langle S \rangle,$$

since A converts joins to meets. Thus $A \langle X \rangle$ is the complement of $\langle X \rangle$ in **cBA**, so **cBA** is a complete Boolean algebra. \square

This theorem explains our choice of symbol **cBA**. Note that a complete Boolean algebra need not be isomorphic to the lattice of subsets of a set.

Note that, if $\langle X \rangle$ is already complemented in the Bousfield lattice, so that $\langle X \rangle \in \mathbf{BA}$, then certainly $A^2 \langle X \rangle = \langle X \rangle$, so **BA** is a subBoolean algebra of **cBA**. Of course, the inclusion **BA** \subset **cBA** is proper, because **cBA** is complete and **BA** is not. Also, the lattice **cBA** is not a sublattice of the Bousfield lattice: the meets and joins are different in the two sets.

We now investigate how A and A^2 behave on meets. The following lemma appears in [Bir79, V.11]; we reproduce its proof for the reader's convenience.

Lemma 6.5. *Suppose $\langle X \rangle$ and $\langle Y \rangle$ are in **DL**. Then*

- (a) $A(\langle X \rangle \wedge \langle Y \rangle) = A(A^2 \langle X \rangle \wedge A^2 \langle Y \rangle)$.
- (b) A converts meets to joins in **cBA**: that is, $A(\langle X \rangle \wedge \langle Y \rangle) = A^2(A \langle X \rangle \vee A \langle Y \rangle)$.
- (c) A^2 preserves finite meets: that is, $A^2(\langle X \rangle \wedge \langle Y \rangle) = A^2 \langle X \rangle \wedge A^2 \langle Y \rangle$.

Proof. Certainly $A(\langle X \rangle \wedge \langle Y \rangle) \geq A(A^2 \langle X \rangle \wedge A^2 \langle Y \rangle)$. Conversely, suppose $\langle Z \rangle \leq A(\langle X \rangle \wedge \langle Y \rangle)$, so that $Z \wedge X \wedge Y = 0$. It suffices to show that

$$\langle Z' \rangle = \langle Z \rangle \wedge A^2 \langle X \rangle \wedge A^2 \langle Y \rangle = 0$$

as well. To see this, note that $\langle Z' \rangle \wedge \langle X \rangle \wedge \langle Y \rangle = 0$, so $\langle Z' \rangle \wedge \langle X \rangle \leq A \langle Y \rangle$. On the other hand, $\langle Z' \rangle \wedge \langle X \rangle \leq A^2 \langle Y \rangle$ by definition. Thus $\langle Z' \rangle \wedge \langle X \rangle \leq A \langle Y \rangle \wedge A^2 \langle Y \rangle = 0$. Hence $\langle Z' \rangle \leq A \langle X \rangle$. Since $\langle Z' \rangle \leq A^2 \langle X \rangle$ by definition, we have $\langle Z' \rangle \leq A \langle X \rangle \wedge A^2 \langle X \rangle = 0$.

Part (b) follows from part (a), since A converts joins to meets, so that

$$A^2(A \langle X \rangle \vee A \langle Y \rangle) = A(A^2 \langle X \rangle \wedge A^2 \langle Y \rangle).$$

Similarly, part (c) follows from part (b), since

$$A^2(\langle X \rangle \wedge \langle Y \rangle) = A^3(A \langle X \rangle \vee A \langle Y \rangle) = A(A \langle X \rangle \vee A \langle Y \rangle) = A^2 \langle X \rangle \wedge A^2 \langle Y \rangle. \quad \square$$

This lemma allows us to understand the map $A^2: \mathbf{DL} \rightarrow \mathbf{cBA}$.

Definition 6.6. A Bousfield class $\langle Z \rangle$ is said to be *dense* if $\langle Z \rangle \in \mathbf{DL}$ and $A^2 \langle Z \rangle = \langle S \rangle$.

The following theorem is a special case of Theorem V.26 of [Bir79], where it is attributed to Glivenko.

Theorem 6.7. For $\langle X \rangle$ and $\langle Y \rangle$ in \mathbf{DL} , $A^2\langle X \rangle = A^2\langle Y \rangle$ if and only if there is a dense Bousfield class $\langle Z \rangle$ such that $\langle X \rangle \wedge \langle Z \rangle = \langle Y \rangle \wedge \langle Z \rangle$.

Proof. First suppose there is a dense $\langle Z \rangle$ such that $\langle X \rangle \wedge \langle Z \rangle = \langle Y \rangle \wedge \langle Z \rangle$. Then $A^2(\langle X \rangle \wedge \langle Z \rangle) = A^2(\langle Y \rangle \wedge \langle Z \rangle)$. But since A^2 preserves finite meets, this means that $A^2\langle X \rangle \wedge A^2\langle Z \rangle = A^2\langle Y \rangle \wedge A^2\langle Z \rangle$. Since $A^2\langle Z \rangle = \langle S \rangle$, this means $A^2\langle X \rangle = A^2\langle Y \rangle$.

Conversely, suppose $A^2\langle X \rangle = A^2\langle Y \rangle$. Let $\langle Z \rangle = (\langle X \rangle \vee A\langle Y \rangle) \wedge (A\langle X \rangle \vee \langle Y \rangle)$. Then one can easily check that $\langle X \rangle \wedge \langle Z \rangle = \langle Y \rangle \wedge \langle Z \rangle$, so it remains to prove that $\langle Z \rangle$ is dense. To see this, note that $A^2: \mathbf{DL} \rightarrow \mathbf{cBA}$ preserves joins, so

$$A^2(\langle X \rangle \vee A\langle Y \rangle) = A^2(A^2\langle X \rangle \vee_{\mathbf{cBA}} A^3\langle Y \rangle) = A^2(A^2\langle Y \rangle \vee_{\mathbf{cBA}} A\langle Y \rangle) = \langle S \rangle.$$

as required. \square

Theorem 6.7 leads us to consider the dense Bousfield classes.

Lemma 6.8. Let $\langle D \rangle = a\langle \mathbf{HF}_p \rangle \vee \langle \mathbf{HF}_p \rangle$. If Z is in \mathbf{DL} and $\langle Z \rangle \geq \langle D \rangle$, then $\langle Z \rangle$ is dense. Conversely, if Conjecture 3.12 holds, then an arbitrary Bousfield class $\langle Z \rangle \in \mathbf{B}$ is dense if and only if $\langle Z \rangle \geq \langle D \rangle$.

Proof. If $\langle Z \rangle \geq \langle D \rangle$, then $A\langle Z \rangle = ra\langle Z \rangle \leq ra\langle D \rangle = 0$, since $a\langle D \rangle$ is the maximum strange Bousfield class. Hence $A^2\langle Z \rangle = \langle S \rangle$, so $\langle Z \rangle$ is dense. If Conjecture 3.12 holds, then any Z with $\langle Z \rangle \geq \langle D \rangle$ is automatically in \mathbf{DL} , so we can drop that hypothesis. Furthermore, if $\langle Z \rangle$ is dense, then $A\langle Z \rangle = A^3\langle Z \rangle = A\langle S \rangle = 0$, so $ra\langle Z \rangle = 0$. Given Conjecture 3.12, we can conclude that $a\langle Z \rangle$ is strange, and so $a\langle Z \rangle \leq a\langle D \rangle$. Thus $\langle Z \rangle \geq \langle D \rangle$. \square

The following corollary is an immediate consequence of Lemma 6.8 and Theorem 6.7.

Corollary 6.9. Let $\langle D \rangle = a\langle \mathbf{HF}_p \rangle \vee \langle \mathbf{HF}_p \rangle$. Suppose Conjecture 3.12 holds. Then for any $\langle X \rangle$ and $\langle Y \rangle$ in \mathbf{DL} , $A^2\langle X \rangle = A^2\langle Y \rangle$ if and only if $\langle X \rangle \wedge \langle D \rangle = \langle Y \rangle \wedge \langle D \rangle$.

This corollary suggests that a characterization of \mathbf{cBA} can be obtained from $\langle D \rangle = a\langle \mathbf{HF}_p \rangle \vee \langle \mathbf{HF}_p \rangle$. Let $L_D\mathbf{B}$ denote the sub-partially ordered set of \mathbf{B} consisting of all elements of the form $\langle X \rangle \wedge \langle D \rangle$. Then $L_D\mathbf{B}$ is closed under arbitrary joins, and so is a complete lattice. The inclusion $L_D\mathbf{B} \rightarrow \mathbf{B}$ preserves those arbitrary joins, so has a right adjoint $\mathbf{B} \rightarrow L_D\mathbf{B}$; this right adjoint takes $\langle X \rangle$ to

$$\bigvee \{ \langle Z \rangle \in L_D\mathbf{B} \mid \langle Z \rangle \leq \langle X \rangle \}.$$

If Conjecture 3.12 holds, then $D \in \mathbf{DL}$, so $\langle Y \rangle \wedge \langle D \rangle \leq \langle X \rangle$ implies that $\langle Y \rangle \wedge \langle D \rangle \leq \langle X \rangle \wedge \langle D \rangle$. Thus, assuming Conjecture 3.12, the right adjoint $\mathbf{B} \rightarrow L_D\mathbf{B}$ is just given by smashing with $\langle D \rangle$. Smashing with D preserves arbitrary joins, so has a right adjoint $L_D\mathbf{B} \rightarrow \mathbf{B}$ as well. This right adjoint takes $\langle X \rangle \in L_D\mathbf{B}$ to the largest $\langle Y \rangle$ such that $\langle Y \rangle \wedge \langle D \rangle = \langle X \rangle$.

Lemma 6.10. Suppose Conjecture 3.12 holds. Then $L_D\mathbf{B} \subseteq \mathbf{DL}$.

Proof. By Conjecture 3.12, we have $\langle X \rangle \vee a\langle D \rangle = r\langle X \rangle \vee a\langle D \rangle$. Thus

$$\langle D \rangle \wedge \langle X \rangle = \langle D \rangle \wedge (\langle X \rangle \vee a\langle D \rangle) = \langle D \rangle \wedge (r\langle X \rangle \vee a\langle D \rangle) = \langle D \rangle \wedge r\langle X \rangle.$$

Furthermore, we have $r(\langle D \rangle \wedge \langle X \rangle) = r\langle D \rangle \wedge r\langle X \rangle = \langle D \rangle \wedge r\langle X \rangle$, since Conjecture 3.12 also implies that D is in \mathbf{DL} . Thus $r(\langle D \rangle \wedge \langle X \rangle) = \langle D \rangle \wedge \langle X \rangle$, so $\langle D \rangle \wedge \langle X \rangle$ is in \mathbf{DL} for all $\langle X \rangle$. \square

Theorem 6.11. *Suppose Conjecture 3.12 holds. Then $A^2: \mathbf{DL} \rightarrow \mathbf{cBA}$ factors through the epimorphism $\langle D \rangle \wedge (-): \mathbf{DL} \rightarrow L_D \mathbf{B}$ to define an isomorphism $F: L_D \mathbf{B} \rightarrow \mathbf{cBA}$.*

Proof. We define $F(\langle D \rangle \wedge \langle X \rangle) = A^2 \langle X \rangle$. By Corollary 6.9, F is well-defined, injective, and order-preserving. On the other hand, F is obviously surjective since A^2 is. \square

Naturally, we would like a better description of $L_D \mathbf{B}$, in light of Theorem 6.11. See Conjecture 5.1 for a related result.

Conjecture 6.12. *We have*

$$\langle D \rangle = \bigvee_{n \geq 0} \langle K(n) \rangle \vee \bigvee_{n \geq 2} \langle A(n) \rangle \vee \langle \mathbf{HF}_p \rangle.$$

Note that $\langle K(n) \rangle \leq a \langle \mathbf{HF}_p \rangle$ and $\langle A(n) \rangle \leq a \langle \mathbf{HF}_p \rangle$ for all n , so the \geq half of the equality in Conjecture 6.12 holds.

By the definition of D and the computations in Section 5, the conjecture is equivalent to the following:

$$\langle D \rangle = a \langle \mathbf{HF}_p \rangle \vee \langle \mathbf{HF}_p \rangle = \bigvee_{n \geq 0} \langle T(n) \rangle \vee \langle \mathbf{HF}_p \rangle.$$

The following proposition completes our conjectural identification of \mathbf{cBA} up to isomorphism.

Proposition 6.13. *Suppose Conjectures 3.12, 5.1 and 6.12 hold. Then \mathbf{cBA} is isomorphic to the complete Boolean algebra generated by the atoms $\langle K(n) \rangle$ for $n \geq 0$, $\langle A(n) \rangle$ for $n \geq 2$, and $\langle \mathbf{HF}_p \rangle$.*

This isomorphism is given by applying A^2 , so to actually identify \mathbf{cBA} we need to understand the behavior of A^2 .

Proposition 6.14. *Suppose Conjecture 3.12, 5.1 and 6.12 hold. Then every subwedge of $\bigvee_{n \geq 0} \langle K(n) \rangle \vee \bigvee_{n \geq 2} \langle A(n) \rangle$ is closed. However, $A^2 \langle \mathbf{HF}_p \rangle \neq \langle \mathbf{HF}_p \rangle$.*

Proof. Let $\langle E \rangle$ denote an arbitrary subwedge of $\langle D \rangle$ such that $\langle E \rangle \wedge \langle \mathbf{HF}_p \rangle = \langle 0 \rangle$. Let $\langle E' \rangle$ denote the complementary subwedge of $\langle D \rangle$. We will show that $A \langle E' \rangle = \langle E \rangle$, so that $\langle E \rangle$ is closed. It is clear that $\langle E \rangle \leq A \langle E' \rangle$, since $\langle E \rangle \wedge \langle E' \rangle = \langle 0 \rangle$ and $\langle E \rangle \in \mathbf{DL}$. On the other hand, $\langle E' \rangle \geq \langle \mathbf{HF}_p \rangle$, so $A \langle E' \rangle \leq A \langle \mathbf{HF}_p \rangle \leq \langle D \rangle$. Since $A \langle E' \rangle \in \mathbf{DL}$, it follows that

$$A \langle E' \rangle = A \langle E' \rangle \wedge \langle D \rangle$$

and so $A \langle E' \rangle$ is a subwedge of $\langle D \rangle$. This subwedge cannot contain any term in $\langle E' \rangle$, so we must have $A \langle E' \rangle = \langle E \rangle$.

In particular, it follows that

$$A \langle \mathbf{HF}_p \rangle = \bigvee_{n \geq 0} \langle T(n) \rangle,$$

and hence

$$A^2 \langle \mathbf{HF}_p \rangle = A \left(\bigvee_{n \geq 0} \langle T(n) \rangle \right).$$

We now prove that this is strictly larger than $\langle \mathbf{HF}_p \rangle$, using [Rav84, Theorem 2.10]. Let $J = (p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n}, \dots)$ be an infinite regular sequence in BP_* . Then we can form a spectrum BPJ with $BPJ_* = BP_*/J$ in various ways; Ravenel uses the Bass-Sullivan construction. By [Rav84, Corollary 2.14], BPJ is a ring spectrum and hence is in **DL**. Since BPJ is built from BP , we have $BPJ \wedge A(n) = 0$ for all n . On the other hand, one can easily see that $BPJ \wedge K(n) = 0$ for all n , since a power of v_n is invariant modulo $(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ and this power has to act both invertibly and nilpotently on $K(n)_*BPJ$. Hence $\langle BPJ \rangle \leq A\left(\bigvee_{n \geq 0} \langle T(n) \rangle\right)$ for all infinite regular sequences J . On the other hand, Theorem 2.10 of [Rav84] implies that, for almost all such infinite regular sequences J , we have $\langle BPJ \rangle > \langle \mathbf{HF}_p \rangle$. \square

In light of these results, we would like to understand $A^2\langle \mathbf{HF}_p \rangle$. Given a regular sequence J as in the proof of Proposition 6.14, we can form a spectrum S/J by taking the sequential colimit of the partial quotients S/J_n . This spectrum may not be well-defined even up to Bousfield class, though each S/J_n is. The obvious conjecture is that $A^2\langle \mathbf{HF}_p \rangle$ should be the wedge of the $\langle S/J \rangle$ over all infinite regular sequences J and all representatives S/J .

7. BOUSFIELD CLASSES WITHOUT FINITE ACYCLICS

We have been discussing Bousfield classes with finite acyclics; in this section, we examine the rest of the Bousfield classes. No spectrum can have both a nonzero finite acyclic and a nonzero finite local; we conjecture that every spectrum has one or the other. In any case, we pay some attention to spectra with finite locals, and we discuss Brown-Comenetz duality and its relation to such spectra. We also show that a number of conjectures related to Bousfield classes without finite acyclics are equivalent.

Brown-Comenetz duality [BC76] is the main source of counterexamples in the theory of Bousfield classes. Given a spectrum X , we denote by IX its *Brown-Comenetz dual*, obtained by applying Brown representability to the cohomology theory $Y \mapsto \text{Hom}(\pi_0(X \wedge Y), \mathbf{Q}/\mathbf{Z}_{(p)})$. Let I denote the Brown-Comenetz dual of the sphere. Note that IX is the same as the function spectrum $F(X, I)$, and there is a natural map $X \rightarrow I^2X$ which is an isomorphism when the homotopy groups of X are finite. Also note that $IX = 0$ if and only if $X = 0$, since $\mathbf{Q}/\mathbf{Z}_{(p)}$ is an injective cogenerator of the category of p -local abelian groups. The spectrum I is the central example of this paper.

Recall the spectra $X(n)$ from [Rav84, Section 3], which interpolate between the Bousfield classes of $S = X(0)$ and $BP = X(\infty)$:

$$\langle S \rangle = \langle X(0) \rangle > \langle X(1) \rangle > \dots > \langle X(\infty) \rangle = \langle BP \rangle.$$

Some of the basic properties of I are as follows.

Lemma 7.1.

- (a) I is in the localizing subcategory generated by \mathbf{HF}_p ; hence $\langle \mathbf{HF}_p \rangle \geq \langle I \rangle$.
- (b) $X(1) \wedge I = 0$; hence $X(n) \wedge I = 0$ for all $n \geq 1$, and $BP \wedge I = 0$.
- (c) $\mathbf{HF}_p \wedge I = 0$; hence $\langle \mathbf{HF}_p \rangle > \langle I \rangle$, and $I \wedge I = 0$.
- (d) $T(n) \wedge I = 0$ for all n .
- (e) $\langle I \rangle \wedge \langle F(n) \rangle = \langle IF(n) \rangle = \langle I \rangle$ for all n .
- (f) The mod p Moore spectrum $M(p)$ (and every finite-dimensional torsion spectrum) is I -local.

Proof. Part (a) follows immediately from the fact that the homotopy of I is bounded-above and torsion, as in [Rav84, 2.6]. Part (b) follows from [Rav84, Lemma 3.2], where it is shown that $[X(1), M(p)] = 0$. Using the isomorphism $M(p) = F(IM(p), I)$ and adjointness, we find that $I(X(1) \wedge IM(p)) = 0$, so that $X(1) \wedge IM(p) = 0$. Since the homotopy groups of I are torsion, one can readily verify that $\langle IM(p) \rangle = \langle I \rangle$, so that $X(1) \wedge I = 0$. Since $\langle BP \rangle \geq \langle BP \wedge H\mathbf{F}_p \rangle = \langle H\mathbf{F}_p \rangle$, then part (c) follows from (a) and (b). Part (d) follows from part (a) and the well-known fact that $H\mathbf{F}_p \wedge T(n) = 0$ (because a v_n -self map must have positive Adams filtration). Part (e) follows from (d) and the Bousfield class decomposition of Section 5. It is proved in [HS97, Corollary B.13] that $M(p)$ is I -local, using the isomorphism $M(p) = I^2M(p)$. It follows from [HS97, Theorem B.6] that every finite-dimensional (defined in [HS97]) torsion spectrum is I -local. \square

Another useful property of I is that it detects when a spectrum has a finite local. We have already discussed spectra with a finite acyclic; similarly, we say that a spectrum E has a finite local if there is a nonzero finite spectrum X which is E -local. Note that no spectrum can have both a nonzero finite local and a nonzero finite acyclic: if M is a finite E -local and W is a finite E -acyclic, then $M \wedge W$ is both local and acyclic, and nonzero if both M and W are. In [Hov95a, Lemma 3.7], the first author shows that if E has a finite local, then every finite torsion spectrum is E -local. This was extended in [HS97, Theorem B.6] to all finite-dimensional torsion spectra.

Proposition 7.2. *The following are equivalent for a spectrum E :*

- (a) $M(p)$ is E -local.
- (b) E has a finite local.
- (c) $aE \wedge I = 0$.
- (d) $\langle E \rangle \geq \langle I \rangle$.

Proof. We have already noted that (a) and (b) are equivalent. To see that (c) and (d) are equivalent, note that $aE \wedge I = 0$ if and only if $a\langle E \rangle \leq a\langle I \rangle$. This holds if and only if $\langle E \rangle \geq \langle I \rangle$.

Since $M(p)$ is I -local, it follows that (d) \Rightarrow (a). To see that (a) \Rightarrow (c), suppose that $M(p)$ is E -local. Then $[aE, M(p)]_* = 0$. Using the isomorphism $M(p) = I^2M(p)$ and adjointness, we find that $I(aE \wedge IM(p)) = 0$. Thus $aE \wedge IM(p) = 0$. We have already seen in the proof of Lemma 7.1 that $\langle IM(p) \rangle = \langle I \rangle$, completing the proof. \square

Note that this proposition implies for example that every dissonant spectrum is I -acyclic, since finite spectra are harmonic.

Example 7.3. Since finite spectra are harmonic, $\langle I \rangle < \bigvee_n \langle K(n) \rangle$. In particular, $\langle I \rangle \wedge \bigvee_n \langle K(n) \rangle = \langle I \rangle$. But for each n , $\langle I \rangle \wedge \langle K(n) \rangle = \langle 0 \rangle$, since $\langle K(n) \rangle$ is minimal and $K(n) \wedge I = 0$. Thus the meet does not distribute over infinite joins in the Bousfield lattice.

We now consider three conjectures, which we will prove are equivalent. Note that for X finite, $X \wedge I = F(DX, I) = IDX$, where DX is the Spanier-Whitehead dual of X . In particular, $X \wedge I \neq 0$ for every finite X . This, combined with Lemma 7.1, suggests the following conjecture, first made in [HS97, Appendix B].

Conjecture 7.4. *If $E \wedge I \neq 0$, then $\langle E \rangle \geq \langle F(n) \rangle$ for some n .*

Note that the converse to Conjecture 7.4 is immediate from part (d) of Lemma 7.1. The following conjecture appeared in [Hov95a, Conjecture 3.10].

Conjecture 7.5 (The Dichotomy Conjecture). *Every spectrum has either a finite local or a finite acyclic.*

It was pointed out in [Hov95b] that the Dichotomy Conjecture is equivalent to the following conjecture.

Conjecture 7.6. *If E has no finite acyclics, then $\langle E \rangle \geq \langle I \rangle$.*

The converse to Conjecture 7.6 follows from Lemma 7.1(e).

Theorem 7.7. *The following are equivalent:*

- (a) *Conjecture 7.4.*
- (b) *The Dichotomy Conjecture 7.5.*
- (c) *Conjecture 7.6.*

Proof. We will prove that (a) \Leftrightarrow (b) and (b) \Leftrightarrow (c).

To see that (a) \Rightarrow (b), suppose that E has no finite locals. Then $aE \wedge I \neq 0$, by Proposition 7.2. Hence, by part (a), $\langle aE \rangle \geq \langle F(n) \rangle$ for some n . It follows that $\langle E \rangle \leq \langle aF(n) \rangle = \langle L_{n-1}^f S \rangle$, and so E has a finite acyclic.

To see that (b) \Rightarrow (a), suppose $E \wedge I \neq 0$. Then $a^2E \wedge I \neq 0$, so aE has no finite locals, again using Proposition 7.2. Hence aE must have a finite acyclic, by part (b), and so $\langle aE \rangle \leq \langle L_{n-1}^f S \rangle$ for some n . It follows that $\langle E \rangle \geq \langle F(n) \rangle$ for some n .

Proposition 7.2 shows that (b) \Rightarrow (c). To see that (c) \Rightarrow (b), suppose that E has no finite acyclics. Then part (c) implies $\langle E \rangle \geq \langle I \rangle$. Since $M(p)$ is I -local by part (e) of Lemma 7.1, it is also E -local. \square

The Dichotomy Conjecture has a few interesting consequences. The most obvious one is that it implies that $\langle I \rangle$ is minimal.

Lemma 7.8. *If E is a nontrivial spectrum with $\langle E \rangle < \langle I \rangle$, then E has no finite locals or finite acyclics. Hence, if the Dichotomy Conjecture holds, there are no such E , and $\langle I \rangle$ is a minimal Bousfield class.*

Proof. Proposition 7.2 implies that E has no finite locals. Since $\langle E \rangle < \langle I \rangle$, $E \wedge L_{n-1}^f S = 0$ for all n . Thus E can have no finite acyclics either, by the Bousfield class decomposition of Section 5. \square

The Dichotomy Conjecture also gives us a partial classification of complemented Bousfield classes, when combined with the following lemma.

Lemma 7.9.

- (a) *Suppose that K is a field spectrum. Then for any E , either $\langle E \rangle \geq \langle K \rangle$ or $\langle aE \rangle \geq \langle K \rangle$.*
- (b) *At least one of E and aE has a finite local.*
- (c) *If E is complemented and has a finite local, then $E \wedge I \neq 0$.*

Proof. (a): If $E \wedge K \neq 0$, then $\langle E \rangle \geq \langle E \wedge K \rangle = \langle K \rangle$, since $E \wedge K$ is a nontrivial wedge of suspensions of K . If $E \wedge K = 0$, then $\langle K \rangle \leq \langle aE \rangle$ by definition of $\langle aE \rangle$.

(b): Apply part (a) to HF_p .

(c): By Proposition 7.2, since E has a finite local, $aE \wedge I = 0$. Since E is complemented, aE must be its complement and $E \vee aE$ must detect every spectrum. Thus $E \wedge I \neq 0$. \square

Corollary 7.10. *None of the following spectra is complemented: $X(n)$, BP , $H\mathbf{F}_p$, $\bigvee_n K(n)$, $\bigvee_n T(n)$, and I . Furthermore, if the Dichotomy Conjecture holds and E is complemented, then either $\langle E \rangle \geq \langle F(n) \rangle$ for some n or $\langle E \rangle \leq \langle L_{n-1}^f S \rangle$ for some n .*

Proof. This follows from Lemma 7.1. \square

We have already seen that $K(n)$ is complemented for all n . Hence Corollary 7.10 shows that \mathbf{BA} is not closed under infinite joins.

By Proposition 5.2, Conjecture 5.1 implies the converse to the second half of the corollary: every E with $\langle E \rangle \geq \langle F(n) \rangle$ or $\langle E \rangle \leq \langle L_{n-1}^f S \rangle$ is complemented. We can restate this as the following corollary.

Corollary 7.11. *Suppose both the Dichotomy Conjecture and Conjecture 5.1 hold. Then the atoms of \mathbf{BA} are $\langle K(n) \rangle$ and, for $n \geq 2$, $\langle A(n) \rangle$. Every element of \mathbf{BA} can either be written as a finite join of atoms or the complement of a finite join of atoms, in a unique way. In particular, \mathbf{BA} is isomorphic to the Boolean algebra of finite and cofinite subsets of a countable set.*

8. STRANGE BOUSFIELD CLASSES

In this section, we investigate some strange Bousfield classes. We start with the following problem. As above, we write IX for the Brown-Comenetz dual of X .

Problem 8.1. *Classify the strange Bousfield classes. For instance, is every strange spectrum Bousfield equivalent to IA for some connective A with finitely generated homotopy groups? Or to IR for some connective ring spectrum R ?*

Note that $\langle IA \rangle \leq \langle H\mathbf{F}_p \rangle$ for any connective spectrum A with finitely generated homotopy groups, since then IA will have homotopy groups bounded-above and torsion, so will be in the localizing subcategory generated by $H\mathbf{F}_p$.

While the set of strange Bousfield classes may be more complicated than the guesses given in Problem 8.1, these guesses at least give us a starting place for the study of strange Bousfield classes. We find that when A is as above, IA is very much like I .

Lemma 8.2. *Suppose A is a connective spectrum with finitely generated homotopy groups. Then the following are equivalent for a spectrum E .*

- (a) $A \wedge M(p)$ is E -local.
- (b) $A \wedge X$ is E -local for some finite torsion spectrum X .
- (c) $aE \wedge IA = 0$.
- (d) $\langle E \rangle \geq \langle IA \rangle$.

The proof of this lemma is very similar to that of Proposition 7.2. We require A to have finitely generated homotopy groups so that $A \wedge X = I^2(A \wedge X)$ for all finite torsion X . We require that A be connective as well so that $\langle IA \rangle \leq \langle H\mathbf{F}_p \rangle$. This guarantees that $IA \wedge T(n) = 0$ for all n , and thus that $\langle I(A \wedge X) \rangle = \langle IA \rangle$ for all finite X . We leave the rest of the proof to the reader.

Similarly, we have the following analogue of Theorem 7.7.

Theorem 8.3. *Suppose A is connective and has finitely generated homotopy groups. Then the following are equivalent.*

- (a) *If $E \wedge IA \neq 0$, then $\langle E \rangle \geq \langle A \rangle \wedge \langle F(n) \rangle$ for some n .*
- (b) *For every E , either $A \wedge M(p)$ is E -local, or $E \wedge A \wedge F(n) = 0$ for some n .*
- (c) *If $E \wedge A$ has no finite acyclics, then $\langle E \rangle \geq \langle IA \rangle$.*

Again we leave the proof to the reader. The converses of parts (a) and (c) always hold, the key point being that $X \wedge IX$ is never zero unless X is. Indeed, there is a map $X \wedge IX \rightarrow I$ adjoint to the identity map of IX , and hence nontrivial.

We now examine some specific strange spectra. We introduced the spectra $X(n)$ in Section 7.

Theorem 8.4. *We have*

$$\langle I \rangle = \langle IX(0) \rangle < \langle IX(1) \rangle < \cdots < \langle IX(n) \rangle < \langle IX(n+1) \rangle < \cdots < \langle IBP \rangle < \langle HF_p \rangle.$$

Proof. We first show that $\langle IX(n) \rangle \leq \langle IX(n+1) \rangle$. By Lemma 8.2, this is equivalent to showing that $X(n) \wedge M(p)$ is $IX(n+1)$ -local. Because $X(n+1) \wedge M(p) = I^2(X(n+1) \wedge M(p))$, we can use the same argument as in the proof of Lemma 7.1(e) to find that $X(n+1) \wedge M(p)$ is $IX(n+1)$ -local. It therefore suffices to show that $X(n) \wedge M(p)$ is in the colocalizing subcategory generated by $X(n+1) \wedge M(p)$ (recall that a *colocalizing subcategory* is a thick subcategory closed under products). We use the $X(n+1)$ -based Adams tower. That is, we let $\overline{X(n+1)}$ be the fiber of the unit map of $X(n+1)$, we let $X_s = \overline{X(n+1)}^{\wedge s} \wedge X(n) \wedge M(p)$, and we let $K_s = X(n+1) \wedge X_s$. There are then cofiber sequences

$$X_{s+1} \rightarrow X_s \rightarrow K_s \rightarrow \Sigma X_{s+1},$$

and the homotopy inverse limit $\text{holim}(X_s)$ is trivial for connectivity reasons. We turn this around by letting X^s be the cofiber of the map $X_s \rightarrow X_0 = X(n) \wedge M(p)$. Then we have cofiber sequences

$$X^{s+1} \rightarrow X^s \rightarrow \Sigma K_s \rightarrow \Sigma X^{s+1},$$

and the homotopy inverse limit of X^s is $X(n) \wedge M(p)$. It therefore suffices to show that each K_s is in the colocalizing subcategory generated by $X(n+1) \wedge M(p)$.

To see this, we use [DHS88, Proposition 2.3], which shows that $X(n+1)_* X(k)$ is a free module over $X(n+1)_*$ for $k \leq n+1$. It follows that $X(n+1) \wedge X(n+1)$ and $X(n+1) \wedge X(n)$ are wedges of suspensions of $X(n+1)$. Then one can easily check that K_s is a wedge of suspensions of $X(n+1) \wedge M(p)$, and since everything is connective and locally finite, this wedge is also a product. Hence K_s is in the colocalizing subcategory generated by $X(n+1) \wedge M(p)$, and so $X(n) \wedge M(p)$ is as well.

A similar proof, using the fact that $BP_* X(n)$ is a free BP_* -module, shows that $X(n) \wedge M(p)$ is in the colocalizing subcategory generated by $BP \wedge M(p)$. Thus we have $\langle IX(n) \rangle \leq \langle IBP \rangle$. We have already seen that $\langle IX \rangle \leq \langle HF_p \rangle$ for any connective X with finitely generated homotopy groups.

It remains to show that all of the inequalities above are strict. For this we recall the method used by Ravenel in [Rav84, Sections 2 and 3]. He shows that there are no maps from $X(n+1)$ to $X(n) \wedge M(p)$, and that this is equivalent to the statement that

$$X(n+1) \wedge I(X(n) \wedge M(p)) = 0.$$

We have already seen that $\langle IX(n) \rangle = \langle I(X(n) \wedge M(p)) \rangle$. Hence $X(n+1) \wedge IX(n) = 0$, but $X(n+1) \wedge IX(n+1)$ is nonzero. Thus $\langle IX(n) \rangle < \langle IX(n+1) \rangle$. Similarly, Ravenel's proof that there are no maps from BP to $X(n) \wedge M(p)$ shows that $BP \wedge IX(n) = 0$. Since $BP \wedge IBP$ is nonzero, this shows that $\langle IX(n) \rangle < \langle IBP \rangle$. Finally, since there are no maps from $H\mathbf{F}_p$ to BP , then $H\mathbf{F}_p \wedge IBP = 0$, and so $\langle IBP \rangle < \langle H\mathbf{F}_p \rangle$. \square

There are probably more strange Bousfield classes than the ones described in Theorem 8.4. For example, Ravenel discusses spectra BPJ for infinite invariant regular sequences J in BP_* in [Rav84, Section 2]. We have already met these spectra in the proof of Proposition 6.14. He shows that $\langle BP \rangle > \langle BPJ \rangle > \langle H\mathbf{F}_p \rangle$ for $J \neq (p, v_1, \dots)$. Presumably the Brown-Comenetz duals of these spectra give other strange spectra. In addition, at $p = 2$, we have MSp as well. Ravenel sketched an argument to the first author once that $\langle MSp \rangle > \langle BP \rangle$, and presumably one would also have $\langle IMSp \rangle < \langle IBP \rangle$.

We do, however, make the following conjecture.

Conjecture 8.5. *The spectra $X(n)$ and $X(n+1)$ are adjacent in the Bousfield lattice. That is, if $\langle E \rangle > \langle X(n+1) \rangle$, then $\langle E \rangle \geq \langle X(n) \rangle$.*

Note that if Conjecture 8.5 holds, then $a\langle I \rangle = \langle X(1) \rangle$. Indeed, since $X(1) \wedge I = 0$, we have $a\langle I \rangle \geq \langle X(1) \rangle$. Similarly, we have seen above that $X(n+1) \wedge IX(n) = 0$, so $a\langle IX(n) \rangle \geq \langle X(n+1) \rangle$. But $X(n) \wedge IX(n)$ is nonzero, so we must have $a\langle IX(n) \rangle = \langle X(n+1) \rangle$ if Conjecture 8.5 holds. Thus Conjecture 8.5 also implies that $IX(n)$ and $IX(n+1)$ are adjacent in the Bousfield lattice.

Conjecture 8.5 also implies the following result.

Conjecture 8.6. $\langle X(n) \rangle \wedge \langle T(k) \rangle = \langle T(k) \rangle$ for all n and k .

Hopkins has proved Conjecture 8.6, but the authors have not seen a proof. To see that Conjecture 8.5 implies Conjecture 8.6, proceed by induction on n . We will only indicate the proof for $n = 1$. Conjecture 8.5 implies that $\langle X(1) \rangle \vee \langle T(k) \rangle = \langle X(1) \rangle$. By smashing with $T(k)$, we find that $\langle T(k) \rangle = \langle X(1) \rangle \wedge \langle T(k) \rangle$, as required.

We mention that Hopkins has proved the following, though again the authors do not know the proof.

Conjecture 8.7. $\langle S \rangle = \langle \mathbf{C}P^\infty \rangle$.

9. LOCALIZING AND COLOCALIZING SUBCATEGORIES

In this last section, we make a few remarks about general localizing and colocalizing subcategories. The outstanding question here is whether every localizing subcategory is the collection of E -acyclics for some E . As pointed out by Neil Strickland, Ohkawa's result [Ohk89] is relevant here.

Recall that a subcategory of the stable homotopy category is called *localizing* if it is thick and is closed under coproducts. The basic conjecture here is the following.

Conjecture 9.1. *Every localizing subcategory is the collection of E -acyclics for some E (and is therefore principal).*

There are several equivalent formulations of this conjecture. First we need some notation. Given a spectrum X , let $\text{loc}(X)$ denote the localizing subcategory generated by X .

Proposition 9.2. *The following are equivalent.*

- (a) *Conjecture 9.1 holds.*
- (b) *Every principal localizing subcategory $\text{loc}(X)$ is the collection of E -acyclics for some E .*
- (c) *For each X , $\text{loc}(X)$ is the collection of aX -acyclics.*
- (d) *$\langle X \rangle \leq \langle Y \rangle$ if and only if $X \in \text{loc}(Y)$.*

Proof. It is clear that (a) \Rightarrow (b). To see that (b) \Rightarrow (c), suppose $\text{loc}(X)$ is the E -acyclics for some E . Then $E \wedge X = 0$ so $\langle E \rangle \leq \langle aX \rangle$. On the other hand, if $E \wedge Z = 0$, then $Z \in \text{loc}(X)$, so $Z \wedge aX = 0$. Thus $\langle E \rangle = \langle aX \rangle$, so $\text{loc}(X)$ is also the collection of aX -acyclics.

To see that (c) \Rightarrow (d), note that $X \in \text{loc}(Y)$ implies that $\langle X \rangle \leq \langle Y \rangle$. Conversely, if $\langle X \rangle \leq \langle Y \rangle$, then $\langle aX \rangle \geq \langle aY \rangle$. In particular, X is an aY -acyclic. Thus, from part (c), we have $X \in \text{loc}(Y)$.

It remains to show that (d) \Rightarrow (a). We will first show (d) \Rightarrow (c). Indeed, suppose Y is aX -acyclic. Then $\langle Y \rangle \leq \langle aaX \rangle = \langle X \rangle$. By part (d), we have $Y \in \text{loc}(X)$. Hence $\text{loc}(X)$ is the collection of aX -acyclics, as required. It is clear that (c) \Rightarrow (b), so it remains to show that (b) \Rightarrow (a). We will do so by showing that, given part (b), every localizing subcategory is principal. Given a localizing subcategory \mathcal{C} , there is only a set of Bousfield classes represented by objects of \mathcal{C} by [Ohk89]. Since (b) \Rightarrow (d), this means there is only a set of principal localizing subcategories of \mathcal{C} . Choose a representative for each such principal localizing subcategory, and let X be the wedge of all of those representatives. Then $\text{loc}(X) = \mathcal{C}$, so \mathcal{C} is principal. \square

Note that Conjecture 9.1, together with Ohkawa's result, would imply that there is only a set of localizing subcategories. It would also imply that the cohomological localizations studied in [Hov95b] always exist, and are in fact homological localizations.

We would like a similar understanding of colocalizing subcategories (thick categories which are closed under products), but such an understanding has eluded us. The obvious conjecture is that there is a one-to-one correspondence between localizing subcategories and colocalizing subcategories, so that every colocalizing subcategory would be the collection of E -locals for some E , given Conjecture 9.1. One could also ask whether every colocalizing subcategory is principal. We do not know the answer, but we do have the following intriguing result.

Recall that a *coideal* is a thick subcategory \mathcal{C} with the additional property that if $X \in \mathcal{C}$ and Y is arbitrary, then $F(Y, X) \in \mathcal{C}$.

Proposition 9.3. *The colocalizing subcategory generated by I is the entire stable homotopy category, as is the coideal generated by I .*

Proof. We use the results of [CS]. Recall that they call a spectrum X *injective* if there are no phantom maps to it. They show in [CS, Proposition 3.9] that IX is injective for all X . They show in [CS, Proposition 4.15] that any X fits into a cofiber sequence $X \rightarrow I^2X \rightarrow K$, where K is injective. It follows from [CS, Lemma 4.14] that K is a retract of I^2K . Now, $IY = F(Y, I)$ is in the coideal generated by I for any Y , so both I^2X and K are as well. Hence X is too. \square

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