# STABLY THICK SUBCATEGORIES OF MODULES OVER HOPF ALGEBRAS

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Abstract. We discuss a general method for classifying certain subcategories of the category of finite-dimensional modules over a finite-dimensional cocommutative Hopf algebra $B$. Our method is based on that of Benson-Carlson-Rickard [BCR96], who classify such subcategories when $B = kG$, the group ring of a finite group $G$ over an algebraically closed field $k$. We get a similar classification when $B$ is a finite sub-Hopf algebra of the mod 2 Steenrod algebra, with scalars extended to the algebraic closure of $\mathbf{F}_2$. Along the way, we prove a Quillen stratification theorem for cohomological varieties of modules over any $B$, in terms of quasi-elementary sub-Hopf algebras of $B$.

## Introduction

The ultimate goal of modular representation theory is to classify all finite-dimensional modules over the group ring $kG$ of a finite group, up to isomorphism. To begin such a project, one needs an invariant of such a module. The most powerful such invariant, introduced by Quillen almost thirty years ago [Qui71], is the cohomological variety of a module. Since then, cohomological varieties have been used extensively to study restricted Lie algebras as well as finite groups. Recent work of Benson-Carlson-Rickard [BCR96, BCR97] has determined precisely how faithful an invariant the cohomological variety is: when $G$ is a finite group and $k$ is an algebraically closed field, two finite-dimensional $kG$-modules have the same cohomological variety if and only if they belong to the same stably thick subcategory of $kG$-modules. Here a full subcategory is *stably thick* if it is closed under retracts and tensoring with any simple module, and also if two out of three modules in a short exact sequence are in the subcategory, so is the third. As a consequence of this, Benson-Carlson-Rickard are able to classify all stably thick subcategories of finite-dimensional modules.

The object of this paper is to extend the theory of cohomological varieties to general finite-dimensional cocommutative Hopf algebras. Naturally the theory becomes somewhat more complicated. The usual induction method from elementary abelian subgroups is replaced by induction from quasi-elementary sub-Hopf algebras; the precise statement of this induction is called the Quillen stratification theorem. Unfortunately, very little is known about quasi-elementary Hopf algebras, so we can go no further in general. However, in situations where the quasi-elementary sub-Hopf algebras are well understood, it should be possible to continue and classify all stably thick subcategories of finite-dimensional modules. We carry out this plan for finite sub-Hopf algebras of the mod 2 Steenrod algebra, with scalars extended to the algebraic closure of $\mathbf{F}_2$.

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In more detail, suppose that $B$ is a finite-dimensional cocommutative Hopf algebra over a field $k$ of characteristic $p > 0$. Given a finitely generated $B$-module $M$, define the ideal $I = I(M)$ in $\operatorname{Ext}_B^*(k,k)$ to be the kernel of $\operatorname{Ext}_B^*(k,k) \to \operatorname{Ext}_B^*(M,M)$. This ideal determines a subvariety of $\operatorname{Spec}\operatorname{Ext}_B^*(k,k)$, which we write $V_B(M)$ and call the *cohomological variety* of $M$. As mentioned above, in the cases when $B$ is a group algebra or restricted enveloping algebra, these varieties have been studied for some time, and they are known to have a number of nice properties in these cases; for example, there is the "tensor product theorem", $V_B(M \otimes N) = V_B(M) \cap V_B(N)$. Recently, in the case $B = kG$ where $k$ is an algebraically closed field and $G$ is a finite group, Benson, Carlson, and Rickard [BCR96] have extended the definition of cohomological variety to infinitely generated modules, and they have proved that these new varieties satisfy many of the same properties. The tensor product theorem for infinitely generated modules is one of the most important, because it leads to structural information about the category of $kG$-modules, such as a classification of the stably thick subcategories of finitely generated $kG$-modules [BCR97].

Benson, Carlson, and Rickard prove the tensor product theorem in several steps. First, they show that the variety for any $kG$-module $M$ can be recovered from the varieties over $kE$ of $M$ restricted to $E$, for all elementary abelian subgroups $E$ of $G$; they do this with analogues for these new varieties of Quillen and Avrunin-Scott stratification. In particular, they show that if one knows the tensor product property in the elementary abelian case, then one can conclude it in general. Second, in the elementary abelian case, they develop the theory of "rank varieties" for infinitely generated modules; they prove the tensor product theorem for rank varieties, and they show that rank varieties agree with cohomological varieties.

In our more general setting, we define varieties as they do; see Section 2. We are able to prove that we always have an analogue of Quillen stratification, for any finite-dimensional cocommutative Hopf algebra $B$. The role of elementary abelian groups is played by "quasi-elementary" Hopf algebras. When $B$ is graded and connected, we reduce the question of whether $B$ satisfies a sort of Avrunin-Scott stratification to whether the quasi-elementary sub-Hopf algebras of $B$ do. Still in the graded connected case, we are able to reduce the question of whether the tensor product theorem holds to the quasi-elementary case: we show that varieties over $B$ satisfy the desired tensor product formula as long as the varieties over the quasi-elementary sub-Hopf algebras of $B$ satisfy both Avrunin-Scott stratification and the tensor product formula.

Let $A$ be the mod 2 Steenrod algebra. In the setting of sub-Hopf algebras of $A$, quasi-elementary Hopf algebras are accessible: a sub-Hopf algebra $Q$ of $A$ is quasi-elementary if and only if it is isomorphic, as an ungraded algebra, to the mod 2 group algebra of an elementary abelian 2-group. Because of this, we can apply the theory of rank varieties. There are two main distinctions between a quasi-elementary sub-Hopf algebra $Q$ of $A$ and the group algebra of an elementary abelian 2-group: first, $Q$ is graded, so one needs to understand the effect of that on varieties. Second, the coproducts are not the same. This is important because the coproduct is used to define the module structure on a tensor product, so knowing that rank varieties for elementary abelian 2-groups satisfy the tensor product formula is not apparently relevant to these sub-Hopf algebras of $A$. It turns out that one can get around both of these problems; hence one can prove the tensor product theorem for graded $Q$-modules over an algebraically closed field of characteristic 2. One can also verify Avrunin-Scott stratification for these Hopf algebras $Q$, hence every finite-dimensional sub-Hopf algebra of $A$ satisfies the tensor product formula. As a result, if $B$ is a finite-dimensional sub-Hopf algebra of $A$, we have a classification of stably thick subcategories of finitely generated $B$-modules, a complete description of the Bousfield lattice in the category of all $B$-modules, and a proof of the telescope conjecture in the category of $B$-modules. Note that the proof relies on the case for elementary abelian 2-groups, and in particular requires us to work over an algebraically closed field, rather than the field $\mathbf{F}_2$.

For example, we have the following result. If $B$ is a finite-dimensional sub-Hopf algebra of the mod 2 Steenrod algebra, then after extending scalars by the algebraic closure $\overline{\mathbf{F}}_2$ of $\mathbf{F}_2$, the stably thick subcategories of finitely generated $B$-modules are in one-to-one correspondence with sets $T$ of bihomogeneous prime ideals of the ring $\operatorname{Ext}_B^{**}(\overline{\mathbf{F}}_2, \overline{\mathbf{F}}_2)$ which are closed under specialization: if $\mathfrak{p} \in T$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{q} \in T$. See Corollaries 3.7 and 8.6 for more details. The restriction that the field be algebraically closed has recently been removed by the authors [HP99], both in this case and in the group algebra case.

We point out that, when dealing with stably thick subcategories, it is most convenient to work in an appropriate triangulated category of $B$-modules. Two natural choices are the stable module category $\operatorname{StMod}(B)$ and the category $\mathcal{S}(B)$ of unbounded chain complexes of projective $B$-modules. See Section 1 for descriptions of these categories, as well as the relationship between thick subcategories of $\operatorname{StMod}(B)$ and stably thick subcategories of the abelian category of $B$-modules.

Many of the results of this paper are related to results in [NP98]. In that paper, though, the focus was on finitely generated modules, and in this paper, we need to study varieties for arbitrary modules. Also, the results in [NP98] were given in terms of homogeneous prime ideals of $\operatorname{Ext}_B^{**}(k, k)$. For our purposes, we need to use bihomogeneous prime ideals, and this turns out to be a not insignificant difference.

The paper is organized as follows. We start in Section 1 by recalling some basic facts and constructions regarding Hopf algebras and their modules; we also define the module categories with which we work for the remainder of the paper. We want to use many tools from stable homotopy theory here, so in the second section, we recall enough axiomatic stable homotopy theory for our needs. We also define cohomological varieties; they are defined in terms of certain "idempotent" modules. In the third section, we describe the tensor product property, and show that if it holds, it leads to various important structural information about the module categories under consideration, such as classifications of thick subcategories and Bousfield classes; Corollary 3.7 is one of the main results. In Section 4, we show how restriction to sub-Hopf algebras behaves on the idempotent modules used to define varieties. In Section 5, we prove the Quillen stratification theorem and discuss Avrunin-Scott stratification; we also relate these to the tensor product property—see Corollary 5.10, for example. In the sixth section, we discuss rank varieties and use them to prove the tensor product property in some special cases. One can view rank varieties as ad hoc tools for computing cohomological varieties; in Section 7 we present an outline of a general proof of the tensor product theorem, in an attempt to explain why rank varieties are easier to use in this setting than cohomological varieties. In the last section, we apply the theory to graded connected Hopf algebras, and in particular sub-Hopf algebras of the mod 2 Steenrod algebra.

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the tensor product theorem. Also, this paper arose from trying to understand the work of Benson, Carlson, and Rickard on varieties for infinitely generated modules, so we owe them our thanks both for their papers and for many discussions on the subject.

## 1. Recollections on Hopf algebras, modules, and categories

**Convention 1.1.** Unless otherwise indicated, all Hopf algebras in this paper are assumed to be graded, cocommutative, and defined and finite-dimensional over a field $k$; the characteristic of $k$ will be nonzero throughout. "Module" means "left module" throughout. All modules are graded modules, except in Section 6. The spectrum $\operatorname{Spec} R$ of a bigraded ring will always denote the space of bihomogeneous prime ideals in the Zariski topology, again except in Section 6, when we will consider primes which are only homogeneous with respect to one grading.

In particular, all Hopf algebras under consideration are self-injective—see [LS69, p. 85]. Hence projective and injective graded $B$-modules coincide. (Note also that we view a group algebra as being graded trivially: $kG$ is concentrated in degree zero.)

In addition, because our Hopf algebras are cocommutative, the tensor product over $k$ makes the category of $B$-modules into a symmetric monoidal category, with unit $k$.

We work with two categories in this paper. First, the *stable module category* $\operatorname{StMod}(B)$ is defined as follows. Its objects are $B$-modules, and the morphisms from $M$ to $N$ are written $\underline{\operatorname{Hom}}_B(M,N)$ and are defined to be

$$\underline{\operatorname{Hom}}_B(M,N) = \operatorname{Hom}_B(M,N)/\simeq,$$

where maps $f$ and $g$ are *stably equivalent*, written $f \simeq g$, if $f-g$ factors through a projective module. Because projectives and injectives are the same, this category is well-behaved; for example, the stable module category is a triangulated category [Mar83, Chapter 14]. This means, in particular, that there is a suspension functor $\Sigma\colon \operatorname{StMod}(B) \to \operatorname{StMod}(B)$, where $\Sigma M$ is the cokernel of an embedding of $M$ into an injective module. We refer to this as "external" suspension. The functor $\Sigma$ is an equivalence of categories. We can also suspend "internally", by changing the grading of $M$. We then have functors $\Sigma^{i,j}\colon \operatorname{StMod}(B) \to \operatorname{StMod}(B)$, where the first grading is the external suspension and the second grading is the internal suspension. Furthermore, $\underline{\operatorname{Hom}}_B(M, \Sigma^{i,j}N) \cong \operatorname{Ext}_B^{i,j}(M,N)$ when $i>0$. The tensor product descends to make $\operatorname{StMod}(B)$ a symmetric monoidal category with unit $k$.

The second category is written $\mathcal{S}(B)$, and it was defined in [HPS97, Section 9.5]. The objects of this category are unbounded chain complexes of projective $B$-modules, and the morphisms of $\mathcal{S}(B)$ are chain homotopy classes of chain maps. $\mathcal{S}(B)$ is also triangulated: the functor $\Sigma^{i,j}$ shifts the chain complex degree up by $i$ and the internal degree up by $j$, with the usual sign conventions. Writing $[X,Y]$ for chain homotopy classes from $X$ to $Y$, if $X$ and $Y$ are projective resolutions of modules $M$ and $N$, respectively, then we have $[X, \Sigma^{i,j}Y] \cong \operatorname{Ext}_B^{i,j}(M,N)$. The usual tensor product of chain complexes makes $\mathcal{S}(B)$ into a symmetric monoidal category; the unit $S$ is any projective resolution of $k$.

Whichever category we are working in, we tend to abuse notation and write "=" for isomorphism in that category; for instance, when working in StMod($B$), $P = 0$ for any projective module $P$.

We are concerned with certain sorts of subcategories of these. To define one of these, we need to recall that an object $F$ of a triangulated category $\mathcal{C}$ is called *small* if the natural map $\bigoplus_i \mathcal{C}(F, X_i) \to \mathcal{C}(F, \coprod_i X_i)$ is an isomorphism for all sets $\{X_i\}$ with a copoduct in $\mathcal{C}$.

**Definition 1.2.** A full subcategory of a (bigraded) triangulated category is called *thick* if it is (bigraded and) triangulated and closed under retracts. A thick subcategory of a symmetric monoidal triangulated category is called *tensor-closed* if it is closed under the symmetric monoidal product with any small object. A thick subcategory is called *localizing* if it is closed under coproducts as well.

In particular, from the stable homotopy theory perspective, it is natural to ask for a classification of the (tensor-closed) thick subcategories of small objects in either of the above categories: thick subcategories of finitely generated modules in StMod($B$), or thick subcategories of bounded below chain complexes whose homology is finite-dimensional in $\mathcal{S}(B)$. We point out that this is equivalent to a classification question in the abelian category of $B$-modules.

**Definition 1.3.** A full subcategory $\mathcal{C}$ of $B$-Mod is called *stably thick* if

(a) $\mathcal{C}$ is closed under summands.
(b) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence and two of $M_1$, $M_2$, and $M_3$ are in $\mathcal{C}$, then so is the third.
(c) If $M$ is in $\mathcal{C}$, then $S \otimes M$ is in $\mathcal{C}$ for every simple module $S$.

The empty category and the category containing only the object 0 are always stably thick, and we refer to them as *trivial* stably thick subcategories.

By induction on composition factors, if $M$ is in $\mathcal{C}$, then so is $M \otimes N$ for any finite-dimensional module $N$. In particular, $M \otimes B$ is in $\mathcal{C}$. But $M \otimes B$ is a free module; hence every non-trivial stably thick subcategory of $B$-Mod contains all projective modules.

Note also that for many of the examples of this paper, for example when $B$ is graded connected, the trivial module $k$ is the only simple module, so the third condition is automatically satisfied.

It may be of interest to consider subcategories that only satisfy the first two conditions in Definition 1.3, but we know nothing about these in general.

There is a functor $j\colon B\text{-Mod} \to \mathrm{StMod}(B)$ which is the identity on objects, and takes each map $f$ to its stable equivalence class. It provides a link between the stably thick subcategories of $B$-Mod and the thick subcategories of StMod($B$).

**Proposition 1.4.** *The functor $j\colon B\text{-Mod} \to \mathrm{StMod}(B)$ induces a bijection between nontrivial stably thick subcategories of finitely generated modules in $B$-*Mod *and nonempty tensor-closed thick subcategories of finitely generated modules in* StMod($B$).

## 2. Noetherian stable homotopy theory

In this section, we recall some axiomatic stable homotopy theory from [HPS97] and apply it to our situation.

The categories $\mathcal{C}$ we consider are all bigraded in the following sense: there are functors $\Sigma^{i,j}\colon \mathcal{C} \to \mathcal{C}$ for pairs of integers $(i, j)$, such that $\Sigma^{i,j} \circ \Sigma^{i',j'}$ is naturally, and coherently, isomorphic to $\Sigma^{i+i',j+j'}$, and $\Sigma^{0,0}$ is naturally isomorphic to the identity. We denote $\mathcal{C}(X, \Sigma^{i,j}Y)$ by $[X, Y]_{i,j}$, and the symbol $[X, Y]_{**}$ will stand for the entire bigraded set of morphisms.

Usually in stable homotopy theory, one writes $[X, Y]_i$ for $\mathcal{C}(\Sigma^i X, Y)$ rather than $\mathcal{C}(X, \Sigma^i Y) = \mathcal{C}(\Sigma^{-i}X, Y)$. We have it graded "backwards" because our homotopy groups are essentially Ext, in which the boundary homomorphism raises degrees. Because of this difference between Ext and ordinary homotopy groups, some part of the grading has to look non-standard to topologists, and we have chosen to make it this part.

The following definition is based on [HPS97, Definition 6.0.1], but we give a more general definition.

**Definition 2.1.** A *bigraded Noetherian stable homotopy category* is a bigraded category $\mathcal{C}$ which has all products and coproducts, together with a triangulation with $\Sigma^{1,0}$ as the suspension functor, and a closed symmetric monoidal structure compatible with the triangulation. In keeping with usual stable homotopy theoretic notation, we write $X \vee Y$ for the coproduct of $X$ and $Y$ and call it the "wedge" of $X$ and $Y$; we write $X \wedge Y$ for the symmetric monoidal structure, and call it the "smash product". We denote the unit of the symmetric monoidal structure by $S$. We also require that there be a *finite* set $\mathcal{G}$ of objects of $\mathcal{C}$ such that $S$ and $\mathcal{G}$ satisfy the following properties

(a) $S$ and every element of $\mathcal{G}$ are small. That is, the natural map
$$\bigoplus_i [A, X_i] \to [A, \bigvee_i X_i]$$
is an isomorphism for all $A \in \mathcal{G}$, or for $A = S$, and all sets $\{X_i\}$ of objects of $\mathcal{C}$.

(b) $\mathcal{G}$ is a set of bigraded weak generators. That is, $X$ is isomorphic to the 0 object of $\mathcal{C}$ if and only if $[A, X]_{**} = 0$ for all $A \in \mathcal{G}$.

(c) The ring $[S, S]_{**}$ is commutative and Noetherian as a bigraded ring. For any objects $Y$ and $Z$ of $\mathcal{C}$, $[Y, Z]_{**}$ is a module over $[S, S]_{**}$ via the smash product; when $Y$ and $Z$ are small, we require that $[Y, Z]_{**}$ be a finitely generated module over $[S, S]_{**}$.

When we can take $\mathcal{G} = \{S\}$, we say that $\mathcal{C}$ is *monogenic*. Monogenic Noetherian stable homotopy categories are studied in [HPS97, Section 6]. Straightforward analogues of the results proved there hold in any Noetherian stable homotopy category, with only trivial modifications to the statements and proofs. In general, the word "thick" in [HPS97, Section 6] needs to be replaced by "tensor-closed thick" wherever it appears, and the expression "$\pi_* X$" needs to be replaced by "$\bigoplus_{A\in\mathcal{G}}[A, X]_*$.

If $\mathcal{C}$ is a bigraded Noetherian stable homotopy category, we usually denote $[S, X]_{**}$ by $\pi_{**}X$, in analogy with stable homotopy theory. A small object of $\mathcal{C}$, as in part (a) of Definition 2.1, is also called *finite*.

**Example 2.2.** Let $B$ be a finite-dimensional graded cocommutative Hopf algebra over a field $k$. We follow the usual convention of denoting the conjugation by $\chi$ and the diagonal by $\Delta(b) = \sum b' \otimes b''$. Then the stable module category $\mathrm{StMod}(B)$ is almost a bigraded Noetherian stable homotopy category. Indeed, as mentioned above, StMod($B$) is triangulated. The monoidal structure takes $(M, N)$ to $M \otimes_k N$, given the diagonal $B$-action. This is symmetric monoidal since $B$ is cocommutative. The closed structure takes $(M, N)$ to $\mathrm{Hom}_k(M, N)$, where $(bf)(m) = \sum b' f(\chi(b'')m)$. The generating set $\mathcal{G}$ is the set of simple modules (or, more accurately, a set containing one element of each isomorphism class of simple modules). Every simple module is finite-dimensional, so is small. There are only finitely many of them, since every simple module occurs as a composition factor in $B$. The finite objects are the modules which are stably equivalent to finitely generated modules. However, $[k, k]_{**}$ is isomorphic to the Tate cohomology of $B$, and this may not be Noetherian.

For this and other reasons, we prefer to work with the bigraded Noetherian stable homotopy category $\mathcal{S}(B)$, described in Section 1. The triangulated structure comes from short exact sequences of chain complexes, and the symmetric monoidal structure is the usual tensor product (over $k$) of chain complexes. The unit $S$ is a projective resolution of $k$. A generating set $\mathcal{G}$ can be obtained by taking projective resolutions of the simple modules. In $\mathcal{S}(B)$ we have $[S, S]_{**} \cong \mathrm{Ext}_B^{**}(k, k)$, which is a bigraded commutative Noetherian ring (see [FS97] and [Wil81]). One can also find in [FS97] a proof that $\mathrm{Ext}_B^{**}(M, N)$ is finitely generated over $[S, S]_{**}$ for any finite-dimensional modules $M$ and $N$, so in particular for $M, N \in \mathcal{G}$. It follows by thick subcategory arguments that $[X, Y]_{**}$ is finitely generated over $[S, S]_{**}$ for all finite $X$ and $Y$.

One good way of studying stable homotopy categories is via "Bousfield localization"; this is also a good way of producing new categories from old ones. For a general discussion of localization, see [HPS97, Chapter 3]. In this paper, we will use one kind of Bousfield localization, called "finite localization".

**Theorem 2.3** (Theorem 3.3.3 in [HPS97])**.** *Let $\mathcal{C}$ be a bigraded Noetherian stable homotopy category, and let $\mathcal{A}$ be a tensor-closed thick subcategory of finite objects of $\mathcal{C}$. There is a functor $L_{\mathcal{A}}\colon \mathcal{C} \to \mathcal{C}$ with the following properties*:

(a) *$L_{\mathcal{A}}$ is exact.*
(b) *There is a natural transformation $1 \to L_{\mathcal{A}}$.*
(c) *$L_{\mathcal{A}}$ is idempotent—for any $X$, the map $L_{\mathcal{A}}X \to L_{\mathcal{A}}L_{\mathcal{A}}X$ induced by the natural transformation in* (b) *is an equivalence.*
(d) *For any $X$, $L_{\mathcal{A}}X = X \wedge L_{\mathcal{A}}S$. Hence by idempotence, $L_{\mathcal{A}}S \wedge L_{\mathcal{A}}S = L_{\mathcal{A}}S$.*
(e) *For any finite $X$, $L_{\mathcal{A}}X = 0$ if and only if $X \in \mathcal{A}$. For any $X$, $L_{\mathcal{A}}X = 0$ if and only if $X$ is in the localizing subcategory generated by $\mathcal{A}$.*

**Definition 2.4.** In the situation of the theorem, $L_{\mathcal{A}}$ is called *finite localization away from* $\mathcal{A}$, and $\mathcal{A}$ is the category of *finite acyclics for* $L_{\mathcal{A}}$. For each $X$, define $C_{\mathcal{A}}X$ by the exact triangle

$$C_{\mathcal{A}}X \to X \to L_{\mathcal{A}}X.$$

Then $C_{\mathcal{A}}$ is a functor; it is called the *acyclization functor* associated to $\mathcal{A}$ and $L_{\mathcal{A}}$.

If $L_{\mathcal{A}}$ is a finite localization functor on a category $\mathcal{C}$, one calls the full subcategory consisting of objects $\{L_{\mathcal{A}}X \mid X \in \mathcal{C}\}$ the "finite localization of $\mathcal{C}$ away from $\mathcal{A}$". The inclusion of that subcategory into $\mathcal{C}$ is right adjoint to $L_{\mathcal{A}}$.

**Example 2.5.** We will normally work with $\mathcal{S}(B)$ rather than StMod($B$), as it has better formal properties. In addition, StMod($B$) is equivalent to the full subcategory of $\mathcal{S}(B)$ consisting of chain complexes with no homology [HPS97, Section 9.6]. This subcategory is the finite localization of $\mathcal{S}(B)$ away from the tensor-closed thick subcategory generated by the chain complex $B$ concentrated in degree 0. We can thus recover results about $\mathrm{StMod}(B)$ from results about $\mathcal{S}(B)$, via finite localization.

Now suppose that $\mathcal{C}$ is a bigraded Noetherian stable homotopy category, and suppose that $\mathfrak{p}$ is a bihomogeneous prime ideal in $\pi_{**}S$. Then there is a finite localization functor $L_{\mathfrak{p}}$ on $\mathcal{C}$, which is called "$\mathfrak{p}$-localization" and is described in [HPS97, Proposition 6.0.7]. We often denote $L_{\mathfrak{p}}X = X_{\mathfrak{p}}$. For our purposes here, its key properties are the following:

- $\pi_{**}(X_{\mathfrak{p}}) = (\pi_{**}X)_{\mathfrak{p}}$.
- $X_{\mathfrak{p}} = X \wedge S_{\mathfrak{p}}$, and $S_{\mathfrak{p}} \wedge S_{\mathfrak{p}} = S_{\mathfrak{p}}$ (since $L_{\mathfrak{p}}$ is a finite localization functor).
- $X_{\mathfrak{p}} = 0$ if and only if $X_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \subseteq \mathfrak{p}$.

There is another finite localization functor $L_{<\mathfrak{p}}$, with finite acyclics

$$\{X \text{ finite} \mid X_{\mathfrak{q}} = 0 \text{ for all } \mathfrak{q} \subsetneqq \mathfrak{p}\}.$$

As for $L_{\mathfrak{p}}$, $L_{<\mathfrak{p}}X = X \wedge L_{<\mathfrak{p}}S$ for all $X$, and $L_{<\mathfrak{p}}S$ is smash-idempotent. There is a natural transformation $L_{\mathfrak{p}}X \to L_{<\mathfrak{p}}X$, and we denote the fiber by $M_{\mathfrak{p}}X$. We usually denote $M_{\mathfrak{p}}S$ by $M_{\mathfrak{p}}$. Note that $M_{\mathfrak{p}}X = X \wedge M_{\mathfrak{p}}$ for all $X$, and $M_{\mathfrak{p}}$ is smash-idempotent. $M_{\mathfrak{p}}$ can also be described as follows:

$$M_{\mathfrak{p}} = L_{\mathfrak{p}}S \wedge C_{<\mathfrak{p}}S.$$

**Definition 2.6.** Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category. Given an object $X$ of $\mathcal{C}$, its *support variety* $\mathcal{V}(X)$, also called its *cohomological variety*, is the collection of all bihomogeneous prime ideals $\mathfrak{p}$ of $\pi_{**}S$ such that $M_{\mathfrak{p}} \wedge X$ is nonzero. Given a full subcategory $\mathcal{D}$ of $\mathcal{C}$, we define $\mathcal{V}(\mathcal{D}) = \bigcup_{X\in\mathcal{D}} \mathcal{V}(X)$. If we want to emphasize the role of $\mathcal{C}$, we will denote $\mathcal{V}(X)$ by $\mathcal{V}_{\mathcal{C}}(X)$. In the case $\mathcal{C} = \mathcal{S}(B)$, we write $\mathcal{V}_B(X)$ for $\mathcal{V}_{\mathcal{S}(B)}(X)$.

The variety $\mathcal{V}(X)$ was written as $\mathrm{supp}(X)$ in [HPS97, 6.1.6]. Note that, if we apply this to $\mathcal{S}(kG)$, where $G$ is a finite $p$-group and $\mathrm{char}(k) = p$, we recover the definition of the cohomological variety given in [BCR96, Definition 10.2], by [BCR96, Lemma 10.3]. Their $\kappa(V)$ is equal to our $M_{\mathfrak{p}}$. We can also localize the chain complexes $M_{\mathfrak{p}}$ to get objects in the category $\mathrm{StMod}(B)$, as indicated in Example 2.5; hence, even though $\mathrm{StMod}(B)$ is not Noetherian, we can still discuss support varieties there.

In general, the support variety of an object $X \in \mathcal{C}$ can be an arbitrary collection of bihomogeneous prime ideals, since $\mathcal{V}(M_{\mathfrak{p}}) = \{\mathfrak{p}\}$. (This is proved by combining [HPS97, Proposition 6.1.7] and [HPS97, Theorem 6.1.8]). However, if $X$ is finite, then $\mathcal{V}(X)$ is a Zariski closed set [HPS97, Proposition 6.1.7]. (This requires that $\mathcal{G}$ be finite). Thus, if $\mathcal{D}$ is a collection of finite objects of $\mathcal{C}$, then $\mathcal{V}(\mathcal{D})$ is a union of Zariski closed sets, and so is closed under specialization. Recall this means that if $\mathfrak{p} \in \mathcal{V}(\mathcal{D})$ and $\mathfrak{q} \supseteq \mathfrak{p}$, then $\mathfrak{q} \in \mathcal{V}(\mathcal{D})$.

For a finite object $X$, the support variety has several interpretations. First,

$$\begin{aligned}\mathcal{V}(X) &= \{\mathfrak{p} : X_{\mathfrak{p}} \neq 0\} \\ &= \{\mathfrak{p} : ([A,X]_{**})_{\mathfrak{p}} \neq 0 \text{ for some } A \in \mathcal{G}\} \\ &= \{\mathfrak{p} : ([A,X]_{**})_{\mathfrak{p}} \neq 0 \text{ for some finite } A \in \mathcal{C}\},\end{aligned}$$

by [HPS97, Proposition 6.1.7]. That is, the support variety of $X$ is the union of the supports of the finitely generated $\pi_{**}S$-modules $[A,X]_{**}$ for $A \in \mathcal{G}$, or for all finite $A$. On the other hand, the support of a finitely generated module $M$ over a

Noetherian ring is always $V(\operatorname{ann} M)$, the set of primes containing the annihilator of $M$. This gives the following internal characterization of the support.

**Lemma 2.7.** *Suppose $X$ is a finite object of a bigraded Noetherian stable homotopy category $\mathcal{C}$. Let $I(X)$ be the ideal in $\pi_{**}S$ consisting of all $a \in \pi_{**}S$ such that the graded map $a \wedge 1\colon X \to \Sigma^{|a|}X$ is nilpotent. Then*

$$\mathcal{V}(X) = V(I(X)),$$

*the set of bihomogeneous primes containing $I(X)$.*

*Proof.* We have seen above that

$$\begin{aligned}\mathcal{V}(X) &= \bigcup_{A\in\mathcal{G}} V(\operatorname{ann}[A,X]_{**}) = V\Big(\bigcap_{A\in\mathcal{G}} \operatorname{ann}\pi_{**}(X\wedge DA)\Big)\\ &= V\Big(\bigcap_{A\in\mathcal{G}} \sqrt{\operatorname{ann}\pi_{**}(X\wedge DA)}\Big).\end{aligned}$$

Note that this equality depends on $\mathcal{G}$ being finite. We claim that

$$\bigcap_{A\in\mathcal{G}} \sqrt{\operatorname{ann}\pi_{**}(X\wedge DA)} = I(X).$$

It is clear that $I(X)$ is contained in this intersection, since if $a \in I(X)$, some power of $a$ acts trivially on $X$, so also on $X \wedge DA$ for any $A$. Conversely, suppose $a$ is in the intersection above, and consider the telescope $a^{-1}X$, the sequential (homotopy) colimit of the self-map $a$ induces on $X$. Then $\pi_{**}(a^{-1}X \wedge DA) = 0$ for all $A \in \mathcal{G}$, so $a^{-1}X = 0$. Since $X$ is small, this forces $a \in I(X)$. □

Given an invariant such as the support variety, there are two natural questions. First, can we compute the support variety? In the case of $\mathcal{S}(B)$, we will answer this question by proving the Quillen stratification theorem, which reduces the computation of $\mathcal{V}(X)$ to the computation of the support varieties of the restrictions of $X$ to the quasi-elementary sub-Hopf algebras of $B$. The second question is: what does $\mathcal{V}(X)$ tell us about $X$? We are unable to answer this question in full, but we provide more information in Section 3.

## 3. The tensor product property

In this section we investigate the thick subcategories of a bigraded Noetherian stable homotopy category $\mathcal{C}$, and in particular of $\mathcal{S}(B)$. The main point is that if a "tensor product theorem" holds for $\mathcal{C}$, then one has a great deal of global structural information about $\mathcal{C}$. Much of this is a recapitulation of results from [HPS97, Chapter 6].

**Definition 3.1.** Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category. We say that $\mathcal{C}$ has the *tensor product property* if, for all bihomogeneous prime ideals $\mathfrak{p}$ of $\pi_{**}S$ and all objects $X, Y \in \mathcal{C}$, then $M_{\mathfrak{p}} \wedge X \wedge Y = 0$ if and only if either $M_{\mathfrak{p}} \wedge X = 0$ or $M_{\mathfrak{p}} \wedge Y = 0$. We say that a Hopf algebra $B$ has the tensor product property if $\mathcal{S}(B)$ does.

In other words, if $\mathcal{C}$ has the tensor product property, then for all objects $X$ and $Y$ of $\mathcal{C}$, $\mathcal{V}(X \wedge Y) = \mathcal{V}(X) \cap \mathcal{V}(Y)$. This is called the "tensor product" property because it is modeled after the situation for $kG$-modules: in the cases where $\mathcal{C}$ is $\mathcal{S}(B)$ for a Hopf algebra $B$, the smash product is defined to be the tensor product.

The tensor product property, if it holds, has many important corollaries. To state them, it is convenient to have the following definition, originally due to Bousfield [Bou79]; see also [HPS97, Definition 3.6.1].

**Definition 3.2.** The *Bousfield class* $\langle X\rangle$ of an object $X$ in a stable homotopy category is the collection of all $Y$ such that $X\wedge Y=0$. We order Bousfield classes by reverse inclusion, so that $\langle 0\rangle$ is the least Bousfield class and the unit $\langle S\rangle$ of the smash product is the greatest.

The partially ordered set of Bousfield classes carries a lot of information about structure of the category $\mathcal{C}$, so one would like to understand as much as possible about it.

In any bigraded Noetherian stable homotopy category $\mathcal{C}$, we have the following equality of Bousfield classes, from Theorem 6.1.9 of [HPS97]:

$$\langle S\rangle = \langle \bigvee_{\mathfrak{p}} M_{\mathfrak{p}}\rangle,$$

as $\mathfrak{p}$ runs through the bihomogeneous prime ideals of $\pi_{**}S$. In other words, an object $X$ is isomorphic to zero if and only if $M_{\mathfrak{p}}\wedge X=0$ for all primes $\mathfrak{p}$. We also note that if $\mathfrak{p}\neq\mathfrak{q}$, then $M_{\mathfrak{p}}\wedge M_{\mathfrak{q}}=0$, by [HPS97, Proposition 6.1.7].

Hence if each $M_{\mathfrak{p}}$ were a skew field object (a ring object such that every module is free), and $\mathcal{G}=\{S\}$, then the tensor product property would follow—see [HPS97, Section 3.7]. The $M_{\mathfrak{p}}$ themselves are not even ring objects, however. They do have the same Bousfield class (see below) as a ring object, as explained in [HPS97, Section 6], but this ring is almost never a skew field.

**Corollary 3.3.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category with the tensor product property. Then the Bousfield class $\langle M_{\mathfrak{p}}\rangle$ is minimal in $\mathcal{C}$.*

*Proof.* Suppose $\langle E\rangle < \langle M_{\mathfrak{p}}\rangle$. Then there is an object $Y$ such that $E\wedge Y=0$ but $M_{\mathfrak{p}}\wedge Y$ is nonzero. Since $M_{\mathfrak{p}}\wedge E\wedge Y=0$, the tensor product property implies that $M_{\mathfrak{p}}\wedge E=0$. On the other hand, if $\mathfrak{q}\neq\mathfrak{p}$, then $M_{\mathfrak{p}}\wedge M_{\mathfrak{q}}=0$; hence $E\wedge M_{\mathfrak{q}}=0$ for all $\mathfrak{q}\in\operatorname{Spec}\pi_{**}S$. But then Theorem 6.1.9 of [HPS97] implies that $E=0$. □

**Corollary 3.4.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category with the tensor product property. Then the Bousfield lattice of $\mathcal{C}$ is isomorphic to the complete Boolean algebra on the atoms $M_{\mathfrak{p}}$.*

This is in contrast with the Bousfield lattice in ordinary stable homotopy theory, which is much more complicated. See [HP98] for more information.

*Proof.* Again using Theorem 6.1.9 of [HPS97], we find that

$$\langle X\rangle = \bigvee_{\mathfrak{p}} \langle X\wedge M_{\mathfrak{p}}\rangle.$$

By Corollary 3.3, $\langle X\wedge M_{\mathfrak{p}}\rangle$ is either $\langle 0\rangle$ or $\langle M_{\mathfrak{p}}\rangle$. Therefore, every Bousfield class is a wedge of various of the $\langle M_{\mathfrak{p}}\rangle$. Any two such wedges are distinct, since $M_{\mathfrak{p}}\wedge M_{\mathfrak{q}}=0$ when $\mathfrak{p}\neq\mathfrak{q}$. □

**Corollary 3.5.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category with the tensor product property. Then there is an isomorphism of partially ordered sets between nonempty tensor-closed thick subcategories of finite objects in $\mathcal{C}$ and subsets of $\operatorname{Spec}\pi_{**}S_{\mathcal{C}}$ that are closed under specialization.*

This corollary follows immediately from [HPS97, Theorem 6.2.3] and [HPS97, Proposition 9.6.8]. The isomorphism in question takes a tensor-closed thick subcategory $\mathcal{D}$ to $\mathcal{V}(\mathcal{D}) = \bigcup_{X \in \mathcal{D}} \mathcal{V}(X)$; the inverse takes a subset $V$ closed under specialization to the thick subcategory consisting of all finite $X$ such that $\mathcal{V}(X) \subseteq V$.

**Corollary 3.6.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category with the tensor product property. Then the telescope conjecture holds for $\mathcal{C}$. That is, every smashing localization is a finite localization.*

This corollary follows immediately from [HPS97, Theorem 6.3.7].

In general, one hopes that the Bousfield lattice is (anti-)isomorphic to the lattice of tensor-closed localizing subcategories (tensor-closed thick subcategories also closed under coproducts); see [HPS97, Corollary 6.3.4]. Every Bousfield class is a tensor-closed localizing subcategory, but it is not known that every localizing subcategory arises in this way. In order to prove this, it would suffice to show that the tensor-closed localizing subcategory generated by $M_{\mathfrak{p}}$ is minimal. We do not know whether this holds, even in the cases of $\mathcal{S}(kG)$ and $\mathrm{StMod}(kG)$ for $G$ a finite $p$-group.

For the reader's convenience, we restate these results in terms of Hopf algebras.

**Corollary 3.7.** *Suppose that $B$ is a finite-dimensional graded cocommutative Hopf algebra over a field $k$. Assume that $B$ has the tensor product property.*

(a) *The Bousfield lattice of $\mathcal{S}(B)$ is isomorphic to the complete Boolean algebra on the atoms $M_{\mathfrak{p}}$.*
(b) *The Bousfield lattice of* $\mathrm{StMod}(B)$ *is isomorphic to the complete Boolean algebra on the atoms $M_{\mathfrak{p}}$ for $\mathfrak{p}$ not equal to the unique maximal ideal.*
(c) *There is a poset isomorphism between nonempty tensor-closed thick subcategories of finite objects in $\mathcal{S}(B)$ and subsets of* $\operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$ *that are closed under specialization.*
(d) *There is a poset isomorphism between nonempty tensor-closed thick subcategories of finitely generated modules in* $\mathrm{StMod}(B)$ *and nonempty subsets of* $\operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$ *that are closed under specialization.*
(e) *The telescope conjecture holds for $\mathcal{S}(B)$ and* $\mathrm{StMod}(B)$: *every smashing localization is a finite localization.*

*Proof.* Most of this is immediate from the Corollaries 3.3–3.6. Part (b) follows from the fact that $B \wedge M_{\mathfrak{q}} = 0$ for all non-maximal $\mathfrak{q}$, for dimensional reasons. Part (d) follows from part (c) and [HPS97, Proposition 9.6.8]. □

We can be a bit more precise about certain aspects of the bijections; for example, in parts (c) and (d), we have the following: the empty subset of $\operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$ corresponds to the thick subcategory of $\mathcal{S}(B)$ generated by the zero chain complex; the subset consisting only of the unique maximal ideal corresponds to the thick subcategory (both of $\mathcal{S}(B)$ and $\mathrm{StMod}(B)$) generated by the free module $B$; the subset consisting of all of Spec corresponds to the thick subcategory of all finite objects.

We will combine this with Corollary 5.10 and Corollary 6.13 to examine some specific examples in Section 8.

## 4. Certain finite localizations

Our goal for the next few sections is to reduce the question of whether a Hopf algebra $B$ has the tensor product property to whether certain of its sub-Hopf algebras have it. In order to understand this, we need to understand how varieties over $B$ behave under restriction to sub-Hopf algebras—this is the content of the Quillen stratification theorem, Theorem 5.2 below. To understand the restrictions of varieties, we must understand the restrictions of the objects $M_{\mathfrak{p}}$. This section is devoted to this problem. The results are similar to those of [BCR96, Section 8]; the main results are Theorem 4.11 and its corollary, Corollary 4.12.

We point out that it appears to be more difficult to understand how the $M_{\mathfrak{p}}$ behave under induction. This difficulty is related to the difficulty in proving the tensor product property in general, as we explain in Section 7.

Stated rather generally, our goal here is to understand what happens to finite localizations, and hence also to the $M_{\mathfrak{p}}$, under well-behaved functors of Noetherian stable homotopy categories. So first we discuss what sorts of functors to consider.

**Definition 4.1.** Recall from [HPS97, Definition 3.4.1] that a functor $F\colon \mathcal{C} \to \mathcal{D}$ between stable homotopy categories is called a *geometric morphism* if $F$ is a left adjoint and preserves exact triangles, the smash product, and the unit (up to coherent natural isomorphism). A geometric morphism $F$ is *finitary* if its right adjoint $G$ is also left adjoint to $F$ and sends finite objects of $\mathcal{D}$ to finite objects of $\mathcal{C}$.

**Example 4.2.** For example, if $Q$ is a sub-Hopf algebra of $B$, the restriction functor res: $\mathcal{S}(B) \to \mathcal{S}(Q)$, applied dimension-wise to a chain complex, is a finitary geometric morphism. The right adjoint of restriction is coinduction; it takes a chain complex $X$ of projective $Q$-modules to the chain complex $\operatorname{Hom}(B, X)$ of $B$-modules. Since the Hopf algebras we are considering are finite-dimensional and self-injective, hence self-dual, coinduction actually coincides with the induction functor ind, which takes $X$ to $B \otimes_Q X$. So induction is both the left and right adjoint of restriction; since $Q$ has finite index in $B$, induction will preserve finite objects.

Note that a geometric morphism $F$ induces a map of rings $F\colon \pi_{**}S_{\mathcal{C}} \to \pi_{**}S_{\mathcal{D}}$, and so also a map $F^*\colon \operatorname{Spec}\pi_{**}S_{\mathcal{D}} \to \operatorname{Spec}\pi_{**}S_{\mathcal{C}}$. If $\mathcal{C}$ is Noetherian and $F$ is finitary, then

$$\pi_{**}S_{\mathcal{D}} \cong [GS_{\mathcal{D}}, S_{\mathcal{C}}],$$

which is a finitely generated $\pi_{**}S_{\mathcal{C}}$-module by the definition of a Noetherian stable homotopy category, Definition 2.1. In particular, $F^*$ is a closed mapping in this case, by the going-up theorem [AM69, Theorem 5.11].

We now discuss how geometric morphisms interact with support varieties; our immediate goal is Corollary 4.7. We use the alternative definition of the support for finite objects given in Lemma 2.7.

**Lemma 4.3.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a geometric morphism of bigraded Noetherian stable homotopy categories, and $X$ is a finite object of $\mathcal{C}$. Then $\mathcal{V}_{\mathcal{D}}(FX) \subseteq (F^*)^{-1}\mathcal{V}_{\mathcal{C}}(X)$.*

*Proof.* Suppose $a \in I(X)$. Then the self-map $a \wedge 1$ of $X$ is nilpotent, so the self-map $Fa \wedge 1 = F(a \wedge 1)$ of $FX$ is nilpotent. Thus $F(I(X)) \subseteq I(FX)$. Now suppose $\mathfrak{q} \in \mathcal{V}_{\mathcal{D}}(FX)$. Then $\mathfrak{q} \supseteq I(FX) \supseteq F(I(X))$, by Lemma 2.7. Hence $F^*(\mathfrak{q}) \supseteq I(X)$, so $F^*(\mathfrak{q}) \in \mathcal{V}_{\mathcal{C}}(X)$, by Lemma 2.7 again. $\square$

We now determine precisely the support variety of $GY$ when $F$ is a finitary geometric morphism with two-sided adjoint $G$. This is a special case of the following more general lemma. To state the lemma, let us denote the support of a module $M$ over a ring $A$ by $\mathcal{V}_A(M)$.

**Lemma 4.4.** *Suppose $f\colon A \to B$ is an integral homomorphism of commutative rings, and $M$ is a finitely generated $B$-module. Then $\mathcal{V}_A(M) = f^*\mathcal{V}_B(M)$, where $f^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the induced map.*

*Proof.* For any finitely generated $A$-module $N$, we have $\mathcal{V}_A(N) = V(\operatorname{ann}_A N)$. Hence

$$\mathcal{V}_A(M) = V(\operatorname{ann}_A N) = V(f^{-1} \operatorname{ann}_B M).$$

Similarly, $f^*\mathcal{V}_B(M) = f^*V(\operatorname{ann}_B M)$. By definition, any prime in $f^*V(\operatorname{ann}_B M)$ contains $f^{-1} \operatorname{ann}_B M$, so is in $V(f^{-1} \operatorname{ann}_B M)$. On the other hand, if $\mathfrak{p}$ is a prime ideal containing $f^{-1} \operatorname{ann}_B M$, then it contains $f^{-1}\sqrt{\operatorname{ann}_B M}$, so must contain $f^{-1}\mathfrak{q}$ for some prime ideal $\mathfrak{q}$ containing $\operatorname{ann}_B M$. Hence $\mathfrak{p}$ is in the closure of $f^*V(\operatorname{ann}_B M)$, but $f^*$ is a closed map since $f$ is integral. □

**Corollary 4.5.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a finitary geometric morphism of bigraded Noetherian stable homotopy categories, with left adjoint $G$. Then for all finite objects $Y$ of $\mathcal{D}$, we have $\mathcal{V}_{\mathcal{C}}(GY) \subseteq F^*\mathcal{V}_{\mathcal{D}}(Y)$. Equality holds if $\mathcal{C}$ is monogenic.*

*Proof.* We have

$$\mathcal{V}_{\mathcal{C}}(GY) = \bigcup_{A\in\mathcal{G}_{\mathcal{C}}} \mathcal{V}_{\pi_{**}S_{\mathcal{C}}}([A, GY]_{**}) = \bigcup_{A\in\mathcal{G}_{\mathcal{C}}} \mathcal{V}_{\pi_{**}S_{\mathcal{C}}}([FA, Y]_{**}),$$

since $G$ is also the right adjoint of $F$. The lemma implies that

$$\mathcal{V}_{\pi_{**}S_{\mathcal{C}}}([FA, Y]_{**}) = F^*\mathcal{V}_{\pi_{**}S_{\mathcal{D}}}([FA, Y]_{**}).$$

Since $\mathcal{V}_{\mathcal{D}}(Y)$ is the union of the $\mathcal{V}_{\pi_{**}S_{\mathcal{D}}}([B, Y]_{**})$ for all finite $B$, we find that $\mathcal{V}_{\mathcal{C}}(GY) \subseteq F^*\mathcal{V}_{\mathcal{D}}(Y)$, as required. If $\mathcal{C}$ is monogenic, then we need only consider $A = S$, so equality holds. □

We can now prove the desired result on the behavior of finite localizations under certain geometric morphisms.

**Theorem 4.6.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a finitary geometric morphism of bigraded Noetherian stable homotopy categories. Let $T \subseteq \operatorname{Spec} \pi_{**}S_{\mathcal{C}}$ be closed under specialization. Then $F$ takes the exact triangle*

$$C_T S \to S \to L_T S$$

*to the exact triangle*

$$C_{(F^*)^{-1}T} S \to S \to L_{(F^*)^{-1}T} S.$$

For any bigraded Noetherian stable homotopy category $\mathcal{C}$ and any subset $T$ of $\operatorname{Spec} \pi_{**}S_{\mathcal{C}}$ closed under specialization, let $\mathcal{A}_T$ denote the thick subcategory of all finite $X \in \mathcal{C}$ such that $\mathcal{V}(X) \subseteq T$, as in [HPS97, Section 6.2].

*Proof.* Recall that $C_T S$ is in the localizing subcategory generated by $\mathcal{A}_T$. Lemma 4.3 implies that $F$ takes $\mathcal{A}_T$ to $\mathcal{A}_{(F^*)^{-1}T}$. Therefore $F(C_T S)$ is in the localizing subcategory generated by $\mathcal{A}_{(F^*)^{-1}T}$. To complete the proof, it suffices to show that $F(L_T S)$ is local with respect to $L_{(F^*)^{-1}T}$. But if $Y \in \mathcal{A}_{(F^*)^{-1}T}$, we have

$\mathcal{D}(Y, F(L_T S))_* \cong \mathcal{C}(GY, L_T S)_*$, where $G$ denotes the left adjoint of $F$. Corollary 4.5 implies that $\mathcal{V}_{\mathcal{C}}(GY) \subseteq F^*\mathcal{V}_{\mathcal{D}}(Y) \subseteq T$, and so $\mathcal{C}(GY, L_T S)_* = 0$, as required. □

This theorem tells us something about what a geometric morphism does to each $M_{\mathfrak{p}}$. We need some more terminology.

Given a bihomogeneous prime ideal $\mathfrak{p}$ of $\pi_{**}S$, we have seen how to form $M_{\mathfrak{p}}$ as the fiber of a map of finite localizations. In fact, given an arbitrary bihomogeneous ideal $\mathfrak{a}$ in $\pi_{**}S$, we can form $M_{\mathfrak{a}}$ in a somewhat analogous fashion. Let $\mathfrak{p}_i$, for $i = 1$ to $k$, denote the minimal prime ideals associated to $\mathfrak{a}$, so that the $V(\mathfrak{p}_i)$ are the irreducible components of the closed subvariety $V(\mathfrak{a})$. Define $T(\mathfrak{a})$ to be the set of all prime ideals $\mathfrak{p}$ not contained in any $\mathfrak{p}_i$. Define $T'(\mathfrak{a})$ to be the set of all prime ideals not *properly* contained in any of the $\mathfrak{p}_i$, so that $T'(\mathfrak{a})$ contains the $\mathfrak{p}_i$ themselves. Both $T(\mathfrak{a})$ and $T'(\mathfrak{a})$ are closed under specialization, so give rise to thick subcategories $\mathcal{A}_{T(\mathfrak{a})}$ and $\mathcal{A}_{T'(\mathfrak{a})}$ of finite objects: $\mathcal{A}_{T(\mathfrak{a})}$ is the collection of all finite $X \in \mathcal{C}$ such that $\mathcal{V}(X) \subseteq T(\mathfrak{a})$, and similarly for $T'(\mathfrak{a})$. There are then associated finite localization functors $L_{T(\mathfrak{a})}$ and $L_{T'(\mathfrak{a})}$. We define $M_{\mathfrak{a}}$ to be the fiber of the map $L_{T(\mathfrak{a})}S \to L_{T'(\mathfrak{a})}S$. Equivalently, $M_{\mathfrak{a}} = C_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{a})}S$, where $C_{T'(\mathfrak{a})}$ is the corresponding acyclization functor.

**Corollary 4.7.** *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a finitary geometric morphism of bigraded Noetherian stable homotopy categories, and $\mathfrak{p}$ is a prime ideal in $\pi_{**}S_{\mathcal{C}}$. Define $\mathfrak{a}$ to be the intersection of all prime ideals $\mathfrak{q}_i$ of $\pi_{**}S_{\mathcal{D}}$ such that $F^*(\mathfrak{q}_i) = \mathfrak{p}$. Then the $V(\mathfrak{q}_i)$ are the irreducible components of $V(\mathfrak{a})$, and $FM_{\mathfrak{p}} = M_{\mathfrak{a}}$.*

*Proof.* In view of Theorem 4.6, it suffices to identify $T(\mathfrak{a})$ with $(F^*)^{-1}T(\mathfrak{p})$ and $T'(\mathfrak{a})$ with $(F^*)^{-1}T'(\mathfrak{p})$. It is easy to check that $\mathfrak{q} \notin T(\mathfrak{a})$ implies that $\mathfrak{q} \notin (F^*)^{-1}T(\mathfrak{p})$. Therefore $(F^*)^{-1}T(\mathfrak{p}) \subseteq T(\mathfrak{a})$, and similarly $(F^*)^{-1}T'(\mathfrak{p}) \subseteq T'(\mathfrak{a})$.

The rest of the corollary follows from the going-up theorem. In particular, the going-up theorem implies that any two ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ of $\pi_{**}S_{\mathcal{D}}$ with $F^*(\mathfrak{q}_1) = F^*(\mathfrak{q}_2)$ must be incomparable under the containment relation, so the $V(\mathfrak{q}_i)$ are the irreducible components of $V(\mathfrak{a})$.

Now suppose $\mathfrak{q} \notin (F^*)^{-1}T(\mathfrak{p})$, so that $F^*(\mathfrak{q}) \subseteq \mathfrak{p}$. Then the going-up theorem guarantees that there is a prime ideal $\mathfrak{q}_i$ such that $\mathfrak{q} \subseteq \mathfrak{q}_i$ and $F^*(\mathfrak{q}_i) = \mathfrak{p}$. Hence $\mathfrak{q} \notin T(\mathfrak{a})$. Thus $T(\mathfrak{a}) = (F^*)^{-1}T(\mathfrak{p})$, and the same argument shows $T'(\mathfrak{a}) = (F^*)^{-1}T'(\mathfrak{p})$. □

So, if $F\colon \mathcal{C} \to \mathcal{D}$ is a finitary geometric morphism between bigraded Noetherian stable homotopy categories, to understand $FM_{\mathfrak{p}}$, we want to understand the objects $M_{\mathfrak{a}}$ in terms of the $M_{\mathfrak{q}}$, for prime ideals $\mathfrak{q}$ in $\pi_{**}S_{\mathcal{D}}$. Our goal here is Corollary 4.10.

We start by computing the Bousfield class of a finite localization.

**Lemma 4.8.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category, and $T \subseteq \operatorname{Spec} \pi_{**}S$ is closed under specialization. Let $L_T$ denote the associated finite localization functor, and $C_T$ the associated acyclization functor. Then*

$$\langle L_T S \rangle = \bigvee_{\mathfrak{p} \notin T} \langle M_{\mathfrak{p}} \rangle, \qquad \langle C_T S \rangle = \bigvee_{\mathfrak{p} \in T} \langle M_{\mathfrak{p}} \rangle.$$

*Proof.* Combine [HPS97, Theorem 6.1.9] with [HPS97, Lemma 6.3.1]. □

So, for example, if $\mathfrak{p} \notin T$, then $M_{\mathfrak{p}}$ is $L_T$-local. We now prove an arithmetic square proposition for finite localizations.

**Proposition 4.9.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category, and $T, U \subseteq \operatorname{Spec} \pi_{**}S$ are closed under specialization. Then there is a cofiber sequence*

$$L_{T\cap U}S \to L_T S \vee L_U S \to L_{T\cup U} S$$

*where the second map is the difference of the two evident maps.*

*Proof.* Let $Y$ denote the cofiber of the map $L_{T\cap U}S \to L_T S \vee L_U S$. Then there is a map $Y \to L_{T\cup U}S$, induced by the difference of the two maps $L_T S \to L_{T\cup U}S$ and $L_U S \to L_{T\cup U}S$. We claim that this map is an isomorphism.

To prove this, we first show that $Y$ is $L_{T\cup U}$-local. Since $L_{T\cup U}$ is a finite, and hence smashing, localization, it suffices to show that $C_{T\cup U}S \wedge Y = 0$, where $C_{T\cup U}$ is the associated acyclization functor. By Lemma 4.8, it suffices to show that $M_{\mathfrak{p}} \wedge Y = 0$ for all $\mathfrak{p} \in T \cup U$. Now, if $\mathfrak{p} \in T \cap U$, then, again using Lemma 4.8, $L_{T\cap U} M_{\mathfrak{p}} = L_T M_{\mathfrak{p}} = L_U M_{\mathfrak{p}} = 0$, and so $M_{\mathfrak{p}} \wedge Y = 0$. On the other hand, if $\mathfrak{p}$ is in exactly one of $T$ and $U$, say $T$ without loss of generality, then $L_T M_{\mathfrak{p}} = 0$ but $L_{T\cap U} M_{\mathfrak{p}} = L_U M_{\mathfrak{p}} = M_{\mathfrak{p}}$. Hence we still have $M_{\mathfrak{p}} \wedge Y = 0$, as required. Therefore $Y$ is $L_{T\cup U}$-local.

Now, apply $L_{T\cup U}$ to the cofiber sequence defining $Y$. This is a further localization, so we find that $Y$ is the cofiber of the diagonal map $L_{T\cup U}S \to L_{T\cup U}S \vee L_{T\cup U}S$. This cofiber is clearly $L_{T\cup U}S$, as required. □

**Corollary 4.10.** *Suppose $\mathcal{C}$ is a bigraded Noetherian stable homotopy category, and $\mathfrak{a}$ is a bihomogeneous ideal in $\pi_{**}S$ with associated minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_k$. Then there is an isomorphism $M_{\mathfrak{a}} \xrightarrow{\cong} M_{\mathfrak{p}_1} \vee \cdots \vee M_{\mathfrak{p}_k}$.*

*Proof.* Let $\mathfrak{b} = \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_k$. By induction, we know that $M_{\mathfrak{b}} = M_{\mathfrak{p}_2} \vee \cdots \vee M_{\mathfrak{p}_k}$; we want to show that $M_{\mathfrak{a}} \xrightarrow{\cong} M_{\mathfrak{p}_1} \vee M_{\mathfrak{b}}$. Since $T(\mathfrak{a}) = T(\mathfrak{p}_1) \cap T(\mathfrak{b})$, then Proposition 4.9 gives a cofiber sequence

$$L_{T(\mathfrak{a})}S \to L_{T(\mathfrak{p}_1)}S \vee L_{T(\mathfrak{b})}S \to L_{T(\mathfrak{p}_1)\cup T(\mathfrak{b})}S.$$

Now we smash this cofiber sequence with $C_{T'(\mathfrak{a})}S$. Lemma 4.8 implies that

$$\langle C_{T'(\mathfrak{a})}S \rangle = \bigvee_{\mathfrak{p}\in T'(\mathfrak{a})} \langle M_{\mathfrak{p}} \rangle.$$

But if $\mathfrak{p} \in T'(\mathfrak{a})$, there are two possibilities. Suppose first that $\mathfrak{p} \neq \mathfrak{p}_1$. Then $\mathfrak{p}$ is not contained in $\mathfrak{p}_1$, and so $\mathfrak{p} \in T(\mathfrak{p}_1)$. Thus $M_{\mathfrak{p}} \wedge L_{T(\mathfrak{p}_1)\cup T(\mathfrak{b})}S = 0$. On the other hand, if $\mathfrak{p} = \mathfrak{p}_1$, then $\mathfrak{p}$ is not contained in $\mathfrak{b}$, and so the same argument shows that $M_{\mathfrak{p}} \wedge L_{T(\mathfrak{p}_1)\cup T(\mathfrak{b})}S = 0$.

Thus we have an isomorphism

$$M_{\mathfrak{a}} \to (C_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{p}_1)}S) \vee (C_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{b})}S).$$

We claim that $C_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{p}_1)}S = M_{\mathfrak{p}_1}$ and $C_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{b})}S = M_{\mathfrak{b}}$. To see this, smash the cofiber sequence

$$C_{T'(\mathfrak{a})}S \to S \to L_{T'(\mathfrak{a})}S$$

with $L_{T(\mathfrak{p}_1)}S$. It suffices to show that $L_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{p}_1)}S = L_{T'(\mathfrak{p}_1)}S$. But another Bousfield class computation shows that $L_{T'(\mathfrak{a})}S \wedge M_{\mathfrak{p}_1} = 0$. Thus

$$L_{T'(\mathfrak{a})}S \wedge L_{T(\mathfrak{p}_1)}S = L_{T'(\mathfrak{a})}S \wedge L_{T'(\mathfrak{p}_1)}S = L_{T'(\mathfrak{p}_1)}S,$$

as required. □

Combining Corollary 4.7 and Corollary 4.10, we obtain the following theorem, which is our main result for this section.

**Theorem 4.11.** *Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a finitary geometric morphism of bigraded Noetherian stable homotopy categories, and let $\mathfrak{p}$ be a prime ideal in $\pi_{**}S_{\mathcal{C}}$. Then*

$$FM_{\mathfrak{p}} = \bigvee_{\mathfrak{q} \;|\; F^*\mathfrak{q}=\mathfrak{p}} M_{\mathfrak{q}}.$$

**Corollary 4.12.** *In particular, suppose that $B$ is a finite-dimensional graded cocommutative Hopf algebra over a field $k$. If $Q$ is a sub-Hopf algebra of $B$ and $\mathfrak{p} \in \operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$, then*

$$\operatorname{res} M_{\mathfrak{p}} \cong \bigvee_{\mathfrak{q} \;|\; \operatorname{res}^*(\mathfrak{q})=\mathfrak{p}} M_{\mathfrak{q}}.$$

We point out that when $B$ and $Q$ are connected, then the function $\operatorname{res}^*$ is one-to-one by Lemma 8.1, so $\operatorname{res} M_{\mathfrak{p}}$ is either a single $M_{\mathfrak{q}}$ or zero, depending on whether $\mathfrak{p}$ is in the image of $\operatorname{res}^*$ or not.

## 5. Quillen stratification

In this section, we apply Theorem 4.11 to compute support varieties in a Noetherian stable homotopy category $\mathcal{C}$ in terms of geometric morphisms out of $\mathcal{C}$. The result is Theorem 5.2, an analogue of the Quillen stratification theorem. Then we try to strengthen this result to get "Avrunin-Scott stratification". We show that Avrunin-Scott stratification lets us reduce the question of whether $\mathcal{C}$ has the tensor product property to whether other, perhaps simpler, stable homotopy categories $\mathcal{C}_\alpha$ have it. In the Hopf algebra setting, we show that Avrunin-Scott stratification for certain sub-Hopf algebras of $B$ (the "quasi-elementary" ones) implies it for $B$ itself, and we conclude that if the quasi-elementary sub-Hopf algebras of $B$ have the tensor product property, so does $B$.

This first theorem says that if we can detect whether objects are nonzero by applying certain geometric morphisms, then we can describe the support for any object in terms of those geometric morphisms. This is a generalization of [BCR96, Theorem 10.6].

**Definition 5.1.** Let $\mathcal{C}$ be a bigraded Noetherian stable homotopy category. A set $\{F_\alpha\colon \mathcal{C} \to \mathcal{C}_\alpha\}$ of finitary geometric morphisms of bigraded Noetherian stable homotopy categories is said to *exhaust* $\mathcal{C}$ if, for all objects $X$ of $\mathcal{C}$, $X = 0$ if and only if $F_\alpha X = 0$ for all $\alpha$.

**Theorem 5.2** (Quillen stratification)**.** *Let $\mathcal{C}$ be a bigraded Noetherian stable homotopy category, and let $\{F_\alpha\colon \mathcal{C} \to \mathcal{C}_\alpha\}$ be a set of finitary geometric morphisms of bigraded Noetherian stable homotopy categories that exhaust $\mathcal{C}$. Then*

$$\mathcal{V}_{\mathcal{C}}(X) = \bigcup_\alpha F_\alpha^*(\mathcal{V}_{\mathcal{C}_\alpha}(F_\alpha X)).$$

*Proof.* Suppose $\mathfrak{p} \in \mathcal{V}_{\mathcal{C}}(X)$, so that $M_{\mathfrak{p}} \wedge X \neq 0$. Then for some $\alpha$,

$$F_\alpha M_{\mathfrak{p}} \wedge F_\alpha X = F_\alpha(M_{\mathfrak{p}} \wedge X) \neq 0.$$

By Theorem 4.11,

$$F_\alpha(M_\mathfrak{p}) = \bigvee_{\mathfrak{q} \,\mid\, F_\alpha^*(\mathfrak{q})=\mathfrak{p}} M_\mathfrak{q}.$$

Writing $S_\alpha$ for the sphere object in $\mathcal{C}_\alpha$, there must be a $\mathfrak{q}$ in $\operatorname{Spec} \pi_{**} S_\alpha$ such that $M_\mathfrak{q} \wedge F_\alpha X \neq 0$.

Conversely, if $M_\mathfrak{q} \wedge F_\alpha X \neq 0$ for some $\mathfrak{q} \subseteq \pi_{**} S_\alpha$, then Theorem 4.11 shows that $F_\alpha(M_\mathfrak{p} \wedge X) \neq 0$, and so $M_\mathfrak{p} \wedge X \neq 0$. □

In order to apply this to Hopf algebras, we need to recall the analog of Chouinard's theorem from [Pal97]. The definition of a quasi-elementary Hopf algebra can be found in [Pal97]; these Hopf algebras are the replacements for elementary abelian subgroups of $p$-groups. The definition itself is quite complicated, so we do not reproduce it here. We understand quasi-elementary Hopf algebras well in certain special cases; for example, a group algebra $kG$ over a field of characteristic $p$ is quasi-elementary if and only if $G$ is an elementary abelian $p$-group. We discuss more examples in Section 8.

**Theorem 5.3** (Chouinard's theorem). *Let $B$ be a finite-dimensional graded cocommutative Hopf algebra over a field $k$. Fix an object $X \in \mathcal{S}(B)$. Then $X \neq 0$ if and only if there is a quasi-elementary sub-Hopf algebra $Q$ of $B$ such that $\operatorname{res}_Q X \neq 0$.*

*Proof.* This follows from the corresponding result for $\operatorname{StMod}(B)$, [Pal97, Theorem 1.3]: suppose that $\operatorname{res} X = 0$ for all quasi-elementary sub-Hopf algebras $Q$ of $B$. Then certainly the homology of $X$ is 0, since (ordinary) homology does not change under restriction. By [HPS97, Lemma 9.6.6], this means that $X$ can be identified with an object of $\operatorname{StMod}(B)$. Restriction is compatible with this identification. Then [Pal97, Theorem 1.3] completes the proof. □

**Corollary 5.4.** *Suppose $B$ is a finite-dimensional graded cocommutative Hopf algebra over a field $k$. Fix $X \in \mathcal{S}(B)$. Then*

$$\mathcal{V}_B(X) = \bigcup_Q \operatorname{res}_Q^*(\mathcal{V}_Q(\operatorname{res}_Q X))$$

*as $Q$ runs over the quasi-elementary sub-Hopf algebras of $B$.*

Note that the Quillen stratification theorem determines $\mathcal{V}_\mathcal{C}(X)$ in terms of the $\mathcal{V}_{\mathcal{C}_\alpha}(F_\alpha X)$, but it does not determine $\mathcal{V}_{\mathcal{C}_\alpha}(F_\alpha X)$ in terms of $\mathcal{V}_\mathcal{C}(X)$. This is the subject of Avrunin-Scott stratification (cf. [BCR96, Theorem 10.7]).

**Definition 5.5.** Given a bigraded Noetherian stable homotopy category $\mathcal{C}$ and a family $\{F_\alpha \colon \mathcal{C} \to \mathcal{C}_\alpha\}$ of finitary geometric morphisms of bigraded Noetherian stable homotopy categories that exhaust $\mathcal{C}$, we say that $\mathcal{C}$ satisfies *Avrunin-Scott stratification* if, for all $X \in \mathcal{C}$ and all $\alpha$, we have

$$\mathcal{V}_{\mathcal{C}_\alpha}(F_\alpha X) = (F_\alpha^*)^{-1} \mathcal{V}_\mathcal{C}(X).$$

Note that Lemma 4.3 says that

$$\mathcal{V}_{\mathcal{C}_\alpha}(F_\alpha X) \subseteq (F_\alpha^*)^{-1} \mathcal{V}_\mathcal{C}(X)$$

for any geometric morphism $F_\alpha \colon \mathcal{C} \to \mathcal{C}_\alpha$. We are not able to prove the other containment in general, but in the case $\mathcal{C} = \mathcal{S}(B)$ for certain graded connected $B$, we can reduce it to the quasi-elementary case.

**Definition 5.6.** A quasi-elementary Hopf algebra $Q$ is *hereditary* if every sub-Hopf algebra of $Q$ is itself quasi-elementary.

Greg Henderson (private communication) has provided examples of quasi-elementary Hopf algebras which are not hereditary, unfortunately. The group algebra of an elementary abelian $p$-group is hereditary, as are the quasi-elementary sub-Hopf algebras of the mod 2 Steenrod algebra. We note that the hereditary assumption in the following may not be necessary, but it is certainly convenient.

**Proposition 5.7.** *Suppose that $B$ is a finite-dimensional graded connected cocommutative Hopf algebra such that every quasi-elementary sub-Hopf algebra $Q$ of $B$ is hereditary and satisfies Avrunin-Scott stratification. Then $B$ satisfies Avrunin-Scott stratification.*

Before proving this proposition, we need to recall the full power of the Quillen stratification theorem for $X = k$ from [NP98, Corollary 2.6].

**Theorem 5.8.** *Suppose that $B$ is a finite-dimensional graded connected cocommutative Hopf algebra. Then the space $\operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$ is a disjoint union of the subsets*

$$\operatorname{res}^* \operatorname{Spec}^+ \operatorname{Ext}_Q^{**}(k,k),$$

*as $Q$ runs through the quasi-elementary sub-Hopf algebras of $B$. Here*

$$\operatorname{Spec}^+ \operatorname{Ext}_Q^{**}(k,k) = \operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k) \setminus \bigcup \operatorname{res}^*_{Q,Q'} \operatorname{Spec}\operatorname{Ext}_{Q'}^{**}(k,k)$$

*as $Q'$ runs through all proper sub-Hopf algebras of $Q$.*

Note that the statement in [NP98] is about the maximal ideal spectrum, but since it is really a statement about a certain map being an F-isomorphism of rings, it will hold for the prime ideal spectrum as well.

Note as well that this is the first place in the paper where we need to place serious restrictions on the Hopf algebra $B$. A version of Theorem 5.8 for group algebras is one of the main results of [Qui71], and for general graded Hopf algebras, a version of Theorem 5.8 has been conjectured by the second author in [Pal97].

*Proof of Proposition* 5.7. Suppose that $M_{\mathfrak{p}} \wedge X \neq 0$, $Q$ is a quasi-elementary sub-Hopf algebra of $B$, and $\mathfrak{q} \in \operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k)$ has $\operatorname{res}^*(\mathfrak{q}) = \mathfrak{p}$. There must be some quasi-elementary sub-Hopf algebra $Q'$ of $B$ and prime ideal $\mathfrak{q}' \in \operatorname{Spec}\operatorname{Ext}_{Q'}(k,k)$ such that $\operatorname{res}^*_{Q'}(\mathfrak{q}') = \mathfrak{p}$ and $M_{\mathfrak{q}'} \wedge \operatorname{res}_{Q'} X \neq 0$, by the Quillen stratification theorem. By Theorem 5.8 applied to $Q'$, there is a unique (necessarily quasi-elementary) sub-Hopf algebra $Q''$ of $Q'$ and prime ideal $\mathfrak{q}'' \in \operatorname{Spec}^+ \operatorname{Ext}_{Q''}^{**}(k,k)$ such that $\operatorname{res}^*_{Q',Q''}(\mathfrak{q}'') = \mathfrak{q}'$. Applying Theorem 5.8 to $Q$ and $B$ as well, we find that $Q''$ must be a sub-Hopf algebra of $Q$, and that $\operatorname{res}^*_{Q,Q'}(\mathfrak{q}'') = \mathfrak{q}$. Furthermore, Avrunin-Scott stratification for $Q'$ implies that $M_{\mathfrak{q}''} \in \mathcal{V}_{Q''}(\operatorname{res}_{Q''} X)$. Hence the Quillen stratification theorem for $Q$ implies that $M_{\mathfrak{q}} \in \mathcal{V}_Q(\operatorname{res}_Q X)$, as required. □

Recall from Section 3 that we would like to know when a category $\mathcal{C}$ as in Definition 5.5 has the tensor product property. We finish this section by showing how Avrunin-Scott stratification reduces this to knowing that the categories $\mathcal{C}_\alpha$ have the tensor product property. Corollary 5.10 below is the application of this to the Hopf algebra situation.

**Proposition 5.9.** *Let $\mathcal{C}$ be a bigraded Noetherian stable homotopy category $\mathcal{C}$ and let $\{F_\alpha\colon \mathcal{C} \to \mathcal{C}_\alpha\}$ be a set of finitary geometric morphisms of bigraded Noetherian stable homotopy categories that exhaust $\mathcal{C}$ and satisfy Avrunin-Scott stratification. If each $\mathcal{C}_\alpha$ has the tensor product property, then so does $\mathcal{C}$.*

*Proof.* As before, we write $S_\alpha$ for the sphere object in $\mathcal{C}_\alpha$.

Suppose $X$ and $Y$ are objects of $\mathcal{C}$, and suppose $M_\mathfrak{p} \wedge X \neq 0$ and $M_\mathfrak{p} \wedge Y \neq 0$. Applying Theorem 5.2 to $X = S$, we find that there is an $\alpha$ and a prime ideal $\mathfrak{q}$ of $\pi_{**}S_\alpha$ so that $\mathfrak{p} = F_\alpha^*(\mathfrak{q})$. Since $\mathcal{C}$ satisfies Avrunin-Scott stratification, $M_\mathfrak{q} \wedge F_\alpha X \neq 0$ and $M_\mathfrak{q} \wedge F_\alpha Y \neq 0$. The tensor product property for $\mathcal{C}_\alpha$ then implies that

$$M_\mathfrak{q} \wedge F_\alpha(X \wedge Y) \cong M_\mathfrak{q} \wedge F_\alpha X \wedge F_\alpha Y \neq 0.$$

The Quillen stratification theorem 5.2 then implies that $M_\mathfrak{p} \wedge X \wedge Y \neq 0$, as required. □

**Corollary 5.10.** *Suppose $B$ is a finite-dimensional graded connected cocommutative Hopf algebra such that all quasi-elementary sub-Hopf algebras of $B$ are hereditary, have the tensor product property, and satisfy Avrunin-Scott stratification. Then $B$ has the tensor product property.*

## 6. Rank varieties and the tensor product property

To verify the tensor product property in specific cases, we use rank varieties. In this section, we define rank varieties for certain sorts of Hopf algebras, we verify the tensor product property for them, and we discuss the relationship between rank varieties and support varieties. Rank varieties have been used to study cohomological varieties in two settings: restricted Lie algebras in the work of Friedlander and Parshall, among others (see [FP86], for example), and elementary abelian $p$-groups in the work of Carlson and others (as in [BCR96]). For the most part we follow the group theory version, but both approaches are potentially useful.

In this section, we mainly work with ungraded modules over ungraded Hopf algebras. More precisely, in this section, we mainly work in the following categories: let $\mathrm{StMod}'(B)$ be the stable category of ungraded $B$-modules, and let $\mathcal{S}'(B)$ be the category with objects chain complexes of (ungraded) projective $B$-modules, and morphisms chain homotopy classes of maps.

Now, the chain complex $M_\mathfrak{p}$ described in Section 2 can be constructed in any Noetherian stable homotopy category—it need not be bigraded (the singly graded case is described in [HPS97]). To distinguish the ungraded case from the graded case, we write $M'_\mathfrak{p}$ for the object in $\mathcal{S}'(B)$. Since $\mathrm{StMod}'(B)$ is the finite localization of $\mathcal{S}'(B)$ (see Example 2.5), then one can localize $M'_\mathfrak{p}$ to get a $B$-module, also written $M'_\mathfrak{p}$, which is well-defined up to projective summands. Then we have support varieties $\mathcal{V}_{\mathcal{S}'(B)}(-)$ and $\mathcal{V}_{\mathrm{StMod}'(B)}(-)$, defined in terms of the $M'_\mathfrak{p}$. We will explain how to relate the varieties in the ungraded case to the varieties in the graded case in Corollary 6.13.

Now that we have defined support varieties in the ungraded case, we move on to rank varieties. From the Lie algebra perspective [FP86], we are led to this definition. (Similar definitions also appeared in [NP98].)

**Definition 6.1.** Let $k$ be a field of characteristic $p > 0$, and let $B$ be a finite-dimensional cocommutative Hopf algebra over $k$.

(a) A *rank structure* on $B$ is a restricted Lie algebra $\mathfrak{g} = \mathfrak{g}_B$ and an algebra isomorphism between $B$ and the restricted enveloping algebra $V(\mathfrak{g})$ of $\mathfrak{g}$. (Note that $B$ and $V(\mathfrak{g})$ need not be isomorphic as Hopf algebras.)

(b) Assume that $B$ has rank structure $\mathfrak{g}$. Given any $x \in \mathfrak{g}$, let $\mathrm{alg}(x)$ denote the subalgebra of $V(\mathfrak{g})$ generated by $x$. The *weak rank variety* of a $B$-module $M$ with respect to $\mathfrak{g}$ is this subset of $\mathfrak{g}$:

$$V^r_{B,\mathfrak{g}}(M) = \{x \in \mathfrak{g} \mid x \neq 0,\ x^{[p]} = 0,\ M \text{ is not projective over } \mathrm{alg}(x)\} \cup \{0\}.$$

This is called the "weak" rank variety because it may give incomplete information about the module $M$, at least when $M$ is not finitely generated; for example, Dade's lemma 6.3 may fail—see the discussion before Lemma 5.1 in [BCR96]. When $M$ is finitely generated, though, the work of Friedlander and Parshall [FP86] applies.

From the group theory point of view [BCR96], we are led to this definition.

**Definition 6.2.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $B$ be a finite-dimensional cocommutative Hopf algebra over $k$.

(a) [Wil81] $B$ is *elementary* if $B$ is isomorphic, as an algebra, to $k[X_1, \dots, X_n]/(X_i^p)$ for some $n$. In other words, $B$ is isomorphic, as an algebra, to the $k$-group algebra of an elementary abelian $p$-group.

(b) Suppose that $B$ is elementary, isomorphic as an algebra to $kE$ for some elementary abelian $p$-group $E$. For any $B$-module $M$, define the *rank variety* $\mathcal{V}^r_B(M)$ to be $\mathcal{V}^r_E(M)$, where $\mathcal{V}^r_E(M)$ is defined in [BCR96, Definition 5.4].

(c) The vector space of *indecomposables* of $B$ is $Q(B) = IB/(IB)^2$, where $IB$ is the augmentation ideal of $B$.

An elementary Hopf algebra $B \cong k[X_1, \dots, X_n]/(X_i^p)$ has a rank structure which is given by setting $\mathfrak{g} = \mathrm{Span}(X_1, \dots, X_n)$, with trivial bracket and restriction. Because the restriction is trivial, then $V^r_{B,\mathfrak{g}}(k)$ is the rank structure $\mathfrak{g}$. There are several choices for rank structure, but in this situation one can re-define it to be the indecomposables $Q(B)$—viewed this way, one no longer has a well-defined embedding of the rank structure into $B$, but one gains functoriality of rank varieties. (See also [Car83, Lemma 6.4] and [FSB97, Lemma 6.4] for results related to independence of the choice of embedding.)

Continuing with the case when $B$ is elementary, if the field $k$ is algebraically closed, then $\mathcal{V}^r_B(M)$ can be described as follows: let $K$ be a field extension of $k$ of transcendence degree at least $n = \dim_k \mathfrak{g}$. Then $\mathcal{V}^r_B(k)$ is the subset of $V^r_{K\otimes_k B, K\otimes_k \mathfrak{g}}(K) = K^n$ consisting of the generic points for the closed homogeneous irreducible subvarieties of $V^r_{B,\mathfrak{g}}(k)$; this is essentially independent of the field extension $K$—see Proposition 5.3, and the comments preceding it, in [BCR96]. Given $\mathcal{V}^r_B(k)$, then $\mathcal{V}^r_B(M)$ is described as follows:

$$\mathcal{V}^r_B(M) = \{x \in \mathcal{V}^r_B(k) \mid x \neq 0,\ K \otimes_k M \text{ is not projective over } \mathrm{alg}(x)\}.$$

(We have omitted the condition $x^p = 0$ since that is automatic for elementary Hopf algebras.) One could define $\mathcal{V}^r_{B,\mathfrak{g}}(M)$ this way for any Hopf algebra $B$ with rank structure $\mathfrak{g}$. We would guess that it is well-defined and well-behaved, but we have not verified it. Bendel has been working on related issues [Ben].

**Remark.** The above definitions of rank variety are incomplete in several ways.

(a) The weak rank variety is not obviously functorial (except when $B$ is elementary, by the sort of trick described above).

(b) There is a gap between the definitions of weak rank variety and rank variety; namely, rank varieties are not defined for every Hopf algebra with a rank structure. One would like to extend Benson, Carlson, and Rickard's definition to the general case, but this has not been done, to our knowledge. As it now stands, the notion of weak rank variety is, as the name implies, too weak for our purposes, in particular when dealing with infinitely generated modules.

(c) From a topologist's point of view, there is the flaw that in odd characteristics, we have only dealt with evenly-graded Hopf algebras. This flaw can likely be fixed, at least in the connected and commutative case: graded connected commutative cocommutative Hopf algebras at an odd prime split as a tensor product of an evenly-graded Hopf algebra and an exterior algebra generated by odd-dimensional classes, by [MM65, Proposition 7.21]. So one could define the rank variety to be the product of the rank varieties for two pieces, as in [NP98, Section 4]. Aside from some grading issues (as dealt with in Corollary 6.13), this is probably the right definition. Something similar might also work in the non-connected case.

(d) From an algebraist's point of view, we do not have a definition of weak rank variety for every finite-dimensional cocommutative Hopf algebra. Perhaps work of Friedlander and Suslin, as in the proof of [FS97, Theorem 1.1] in which they essentially write a finite group scheme as a semi-direct product of a finite group and an infinitesimal group scheme, will lead to a general definition of weak rank variety.

Note that both the weak rank variety $V^r_{B,\mathfrak{g}}(M)$ and the rank variety $\mathcal{V}^r_B(M)$ are independent of the coalgebra structure on the Hopf algebra $B$. As a consequence, many of the properties proved about these varieties for Lie algebras and elementary abelian groups carry over immediately to our situation. For example, we have the following results.

**Theorem 6.3** (Dade's lemma, Corollary 5.6 in [BCR96])**.** *Suppose that $k$ is an algebraically closed field, and let $B$ be a finite-dimensional elementary Hopf algebra over $k$. Then a $B$-module $M$ is projective if and only if $\mathcal{V}^r_B(M) = \emptyset$.*

According to [FP87, Proposition 1.5] (together with Proposition 6.4 below), we have this weaker result for a Hopf algebra $B$ with rank structure $\mathfrak{g}$, when working over an algebraically closed field: a finitely generated $B$-module $M$ is projective if and only if $V^r_{B,\mathfrak{g}}(M) = \{0\}$. The obvious conjecture is that, once one has defined $\mathcal{V}^r_{B,\mathfrak{g}}(M)$ appropriately, Theorem 6.3 will hold in general. Bendel has proved some related results [Ben].

The following was first proved for elementary abelian groups by Avrunin and Scott, and is reproduced as [BCR96, Proposition 6.1]; it was proved for restricted Lie algebras by Jantzen [Jan86].

**Proposition 6.4.** *Let $k$ be an algebraically closed field. Let $B$ be a finite-dimensional cocommutative Hopf algebra over $k$ with rank structure $\mathfrak{g}$. Then there is an $F$-isomorphism of varieties*

$$\beta^*\colon\ \operatorname{MaxSpec}\operatorname{Ext}_B(k,k) \to V^r_{B,\mathfrak{g}}(k).$$

For restricted Lie algebras, this map is usually called $\Phi$. We denote it $\beta^*$ following the group theory situation, in which $V^r_E(k)$ is dual to $H^1(E;k)$, the cohomological variety $V_E(k)$ is dual to the image of the Bockstein $\beta$ in $H^2(E;k)$, and the map is the dual of $\beta$, extended from $\mathbf{F}_p$ to $k$ via the Frobenius. See [BCR96, Section 6].

Looking at the closed homogeneous irreducible subvarieties of $V^r_{B,\mathfrak{g}}(k)$, i.e. looking at $\mathcal{V}^r_B(k)$, captures exactly the passage from MaxSpec to Spec, so we have the following. Recall that $\mathcal{V}_{\mathrm{StMod}'(B)}(-)$ is the support variety in the category $\mathrm{StMod}'(B)$; in particular $\mathcal{V}_{\mathrm{StMod}'(B)}(k)$ is the set of all homogeneous prime ideals in $\mathrm{Ext}^*_B(k,k)$, except for the unique maximal ideal consisting of all positive dimensional elements.

**Corollary 6.5.** *Let $k$ be an algebraically closed field, and let $B$ be an elementary Hopf algebra over $k$. Then there is an $F$-isomorphism of varieties*

$$\beta^*\colon \mathcal{V}_{\mathrm{StMod}'(B)}(k) \to \mathcal{V}^r_B(k).$$

The coproduct structure on $B$ comes into play when dealing with rank varieties of tensor products of $B$-modules. We claim that frequently, although the $B$-module structure on $M \otimes N$ depends on the coproduct on $B$, the (weak) rank variety for $M \otimes N$ is somewhat independent of the coproduct structure.

**Theorem 6.6.** *Let $B$ be a finite-dimensional Hopf algebra over a field $k$ with rank structure $\mathfrak{g}$, and suppose that the coproduct $\Delta\colon B \to B \otimes B$ induces the diagonal embedding*

$$\begin{aligned} V^r_{B,\mathfrak{g}}(k) &\hookrightarrow V^r_{B\otimes B,\mathfrak{g}\times\mathfrak{g}}(k) = V^r_{B,\mathfrak{g}}(k) \times V^r_{B,\mathfrak{g}}(k). \\ v &\mapsto (v,v) \end{aligned}$$

*Then for any $B$-modules $M$ and $N$,*

$$V^r_{B,\mathfrak{g}}(M\otimes N) = V^r_{B,\mathfrak{g}}(M) \cap V^r_{B,\mathfrak{g}}(N).$$

For example, if $B$ is the enveloping algebra $V(\mathfrak{g})$, then the coproduct is induced by the diagonal map $\mathfrak{g} \hookrightarrow \mathfrak{g} \times \mathfrak{g}$, and hence the hypotheses are satisfied. We show in Lemma 6.8 below that the hypotheses also hold for any elementary Hopf algebra (*regardless* of the coproduct).

We need a lemma to prove Theorem 6.6.

**Lemma 6.7.** *Let $B_1$ and $B_2$ be finite-dimensional Hopf algebras with rank structures $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. If $B_1 \hookrightarrow Q_2$ is an inclusion of Hopf algebras which induces an embedding $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2$, and if $M$ is a $B_2$-module, then*

$$V^r_{B_1,\mathfrak{g}_1}(M) = V^r_{B_1,\mathfrak{g}_1}(k) \cap V^r_{B_2,\mathfrak{g}_2}(M).$$

*Proof.* See [BCR96, Proposition 7.3] and [FP86, Proposition 2.1]. □

*Proof of Theorem* 6.6. We follow the proof of the tensor product theorem for rank varieties for elementary abelian groups in [BCR96, Section 7]. First of all, the weak rank variety for $M \otimes N$ as a $B \otimes B$-module is independent of the coproduct on $B$, so we have the following, by [BCR96, Theorem 7.2]:

$$V^r_{B\otimes B,\mathfrak{g}\times\mathfrak{g}}(M\otimes N) = V^r_{B,\mathfrak{g}}(M) \times V^r_{B,\mathfrak{g}}(N)$$

as subsets of $V^r_{B\otimes B,\mathfrak{g}\times\mathfrak{g}}(k) = V^r_{B,\mathfrak{g}}(k) \times V^r_{B,\mathfrak{g}}(k)$. Next, applying Lemma 6.7 to the inclusion $\Delta\colon B \hookrightarrow B \otimes B$ gives

$$\begin{aligned} V^r_{B,\mathfrak{g}}(M\otimes N) &= V^r_{\Delta(B),\Delta(\mathfrak{g})}(k) \cap V^r_{B\otimes B,\mathfrak{g}\times\mathfrak{g}}(M\otimes N) \\ &= V^r_{\Delta(B),\Delta(\mathfrak{g})}(k) \cap (V^r_{B,\mathfrak{g}}(M) \times V^r_{B,\mathfrak{g}}(N)) \\ &= V^r_{B,\mathfrak{g}}(M) \cap V^r_{B,\mathfrak{g}}(N), \end{aligned}$$

where the last equality follows because the embedding of weak rank varieties is the diagonal map. □

As mentioned above, the hypotheses of Theorem 6.6 are satisfied when $B$ is elementary.

**Lemma 6.8.** *Let $B$ be a finite-dimensional elementary Hopf algebra over a field $k$, with rank structure $\mathfrak{g}$. Writing $\Delta$ for the map on weak rank varieties induced by the coproduct on $B$, then for all $v \in V^r_{B,\mathfrak{g}}(k)$, we have $\Delta(v) = v \times v$.*

*Proof.* We view $V^r_Q(k)$ as the indecomposables $Q(B)$ of $B$. For any $v \in IB$, the coproduct on $v$ is of the form

$$\Delta(v) = v \otimes 1 + 1 \otimes v + \sum_i v_i \otimes w_i,$$

where $v_i$ and $w_i$ are in $IB$ for each $i$. In particular, $v_i \otimes w_i$ is decomposable in $B \otimes B$, so $\Delta\colon B \to B \otimes B$ induces this map on indecomposables: $\Delta(v) = v \otimes 1 + 1 \otimes v$. In other words, as a map $Q(B) \to Q(B) \times Q(B)$, we have

$$\Delta(v) = v \times 0 + 0 \times v = v \times v,$$

as desired. □

When $k$ is algebraically closed (so that $\mathcal{V}^r_B(M)$ is defined), then the discussion after Definition 6.2 applies, giving this corollary.

**Corollary 6.9.** *Let $B$ be a finite-dimensional elementary Hopf algebra over an algebraically closed field $k$. For any $B$-modules $M$ and $N$, we have*

$$\mathcal{V}^r_B(M \otimes N) = \mathcal{V}^r_B(M) \cap \mathcal{V}^r_B(N).$$

We conclude that, not only does the tensor product theorem hold for rank varieties, but the entire theory of rank varieties is independent of the coproduct on the elementary Hopf algebra $B$. For example, we have the following.

**Theorem 6.10** (Theorem 10.5 in [BCR96])**.** *Let $B$ be a finite-dimensional elementary Hopf algebra over an algebraically closed field $k$. For every $B$-module $M$, the $F$-isomorphism $\beta^*$ from Proposition 6.4 induces an $F$-isomorphism $\mathcal{V}_{\mathrm{StMod}'(B)}(M) \to \mathcal{V}^r_B(M)$.*

We have been working in the setting of ungraded modules; we need to convert our results from ungraded to graded, and also from modules to chain complexes. First, in the ungraded case, we have the following.

**Corollary 6.11.** *Let $B$ be an elementary Hopf algebra over an algebraically closed field. Then the category $\mathrm{StMod}'(B)$ has the tensor product property. That is, for all $B$-modules $M$ and $N$,*

$$\mathcal{V}_{\mathrm{StMod}'(B)}(M \otimes N) = \mathcal{V}_{\mathrm{StMod}'(B)}(M) \cap \mathcal{V}_{\mathrm{StMod}'(B)}(N).$$

First, we convert from $\mathrm{StMod}'(B)$ to $\mathcal{S}'(B)$. We recall from [HPS97, Definition 6.0.8] the construction of an object $K(\mathfrak{p})$ which is Bousfield equivalent to $M'_{\mathfrak{p}}$, by [HPS97, Theorem 6.1.8]: given a prime ideal $\mathfrak{p} \subseteq \mathrm{Ext}^*_B(k,k)$, choose generators $y_1, \ldots, y_n$ for $\mathfrak{p}$. For each $i$, let $S/y_i$ be the cofiber in $\mathcal{S}'(B)$ of the map $y_i\colon S \to S$. Then define $S/\mathfrak{p}$ by

$$S/\mathfrak{p} = S/y_1 \wedge \ldots \wedge S/y_n,$$

and let $K(\mathfrak{p}) = L_{\mathfrak{p}} S/\mathfrak{p}$. The objects $S/\mathfrak{p}$ and $K(\mathfrak{p})$ depend on the choice of generators, but their Bousfield classes $\langle S/\mathfrak{p}\rangle$ and $\langle K(\mathfrak{p})\rangle = \langle M'_{\mathfrak{p}}\rangle$ do not.

**Corollary 6.12.** *Let $B$ be an elementary Hopf algebra over an algebraically closed field. Then the category $\mathcal{S}'(B)$ has the tensor product property.*

*Proof.* We recall from [HPS97, Theorem 9.6.4] the relationship between $\mathcal{S}'(B)$ and $\mathrm{StMod}'(B)$: $\mathrm{StMod}'(B)$ may be obtained from $\mathcal{S}'(B)$ by finite localization $L_B$ away from the thick subcategory generated by the module $B$, viewed as a chain complex concentrated in degree zero. (See Theorem 2.3 and Definition 2.4 for information about finite localization.)

To verify the tensor product property for $\mathcal{S}'(B)$, we need to verify that for any homogeneous prime ideal $\mathfrak{p}$ in $\mathrm{Ext}^*_B(k,k)$ and chain complex $X$ with $X \wedge M'_{\mathfrak{p}} \neq 0$, then $\langle M'_{\mathfrak{p}} \wedge X\rangle = \langle M'_{\mathfrak{p}}\rangle$.

There are two cases: either $\mathfrak{p}$ is the unique maximal ideal $\mathfrak{m}$ consisting of all elements in positive degree, or not. If $\mathfrak{p} \neq \mathfrak{m}$, then $B \wedge M'_{\mathfrak{p}} = 0$; indeed, one can check that since $B$ is concentrated in degree zero and $\mathfrak{p}$ does not contain every element of positive degree, then both $L_{\mathfrak{p}} B$ and $L_{<\mathfrak{p}} B$ are zero. So by [HPS97, Lemma 9.6.6], $M'_{\mathfrak{p}}$ is $L_B$-local; hence $M'_{\mathfrak{p}} \wedge X$ is $L_B$-local for any $X$. Since the tensor product property holds in $\mathrm{StMod}'(B) = L_B\mathcal{S}'(B)$, and since $L_B$ is smashing (Theorem 2.3(d)), then we have

$$\langle L_B S \wedge M'_{\mathfrak{p}} \wedge X\rangle = \langle L_B S \wedge M'_{\mathfrak{p}}\rangle.$$

But since both $M'_{\mathfrak{p}}$ and $M'_{\mathfrak{p}} \wedge X$ are both $L_B$-local, we see that

$$\langle M'_{\mathfrak{p}} \wedge X\rangle = \langle M'_{\mathfrak{p}}\rangle,$$

as desired.

If $\mathfrak{p} = \mathfrak{m}$, then we claim that $\langle M'_{\mathfrak{m}}\rangle = \langle B\rangle$; since $B$ is a field object, if $X \wedge B \neq 0$, then $\langle X \wedge B\rangle = \langle B\rangle$. It remains to verify the equality $\langle M'_{\mathfrak{m}}\rangle = \langle B\rangle$, or equivalently, $\langle K(\mathfrak{m})\rangle = \langle B\rangle$. Note that $L_{\mathfrak{m}}$ is the identity functor, since it is defined by inverting the nonzero elements not in $\mathfrak{p}$, i.e., in $\mathrm{Ext}^0_B(k,k) = k$. Hence $K(\mathfrak{m}) = S/\mathfrak{m}$. By the proof of [HPS97, Lemma 6.0.9], we know that every element of $\mathfrak{m}$ acts nilpotently on $\pi_* S/\mathfrak{m}$, and so $\pi_* S/\mathfrak{m}$ is finite-dimensional as a vector space over $k$. Therefore it has a finite Postnikov tower, and so $\langle B\rangle \geq \langle S/\mathfrak{m}\rangle$.

Note also that $L_{<\mathfrak{m}} B = 0$ (since $L_{\mathfrak{p}} B = 0$ for all $\mathfrak{p}$ properly contained in $\mathfrak{m}$), so $M'_{\mathfrak{m}} \wedge B = L_{\mathfrak{m}} B = B$. Therefore, $\langle B\rangle \leq \langle M'_{\mathfrak{m}}\rangle$. □

Now we deal with the difference between the singly graded category $\mathcal{S}'(B)$ and the bigraded one $\mathcal{S}(B)$.

**Corollary 6.13.** *Every finite-dimensional graded elementary Hopf algebra over an algebraically closed field has the tensor product property.*

*Proof.* Let $U$ denote the forgetful functor from graded $B$-modules to $B$-modules; we also use $U$ for the induced functor $U\colon \mathcal{S}(B) \to \mathcal{S}'(B)$, and other related functors. For example, if $\mathfrak{p}$ is a bihomogeneous prime in $\mathrm{Ext}^{**}_B(k,k)$, then we write $U\mathfrak{p}$ for the corresponding homogeneous prime in $\mathrm{Ext}^*_B(k,k)$.

Note that if $\mathfrak{p}$ is a bihomogeneous ideal, then we may choose bihomogeneous generators, so we may assume that $U(S/\mathfrak{p}) = S/(U\mathfrak{p})$: when taking the quotient by $\mathfrak{p}$, it doesn't matter whether one pays attention to the grading or not. On the other hand, localization could be rather different: in $\mathcal{S}(B)$, the functor $L_{\mathfrak{p}}$ inverts all bihomogeneous elements not in $\mathfrak{p}$, whereas in $\mathcal{S}'(B)$, $L_{\mathfrak{p}}$ inverts more elements—all *homogeneous* elements not in $\mathfrak{p}$. So $UM_{\mathfrak{p}}$ will in general be different from $M'_{U\mathfrak{p}}$.

We claim, though, that if $X$ is an object in $\mathcal{S}(B)$, then

$$\mathcal{V}_{\mathcal{S}(B)}(X) = \mathcal{V}_{\mathcal{S}(B)}(S) \cap \mathcal{V}_{\mathcal{S}'(B)}(X).$$

In other words, $\mathcal{V}_{\mathcal{S}(B)}(X)$ is the set of all bihomogeneous primes contained in $\mathcal{V}_{\mathcal{S}'(B)}(X)$. Given this equality, the tensor product property for $\mathcal{S}'(B)$ implies it for $\mathcal{S}(B)$.

Since

$$\mathcal{V}_{\mathcal{S}(B)}(X) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is bihomogeneous, } X \wedge M_{\mathfrak{p}} \neq 0\}$$
$$\mathcal{V}_{\mathcal{S}'(B)}(UX) = \{\mathfrak{q} \mid \mathfrak{q} \text{ is homogeneous, } UX \wedge M'_{\mathfrak{q}} \neq 0\},$$

and since $M'_{\mathfrak{p}}$ involves inverting more elements than $M_{\mathfrak{p}}$, we clearly have the containment

$$\mathcal{V}_{\mathcal{S}(B)}(X) \supseteq \mathcal{V}_{\mathcal{S}(B)}(S) \cap \mathcal{V}_{\mathcal{S}'(B)}(UX).$$

We want to know that if $X$ is a graded object and $\mathfrak{p}$ is bihomogeneous prime ideal so that $UX \wedge M'_{U\mathfrak{p}} = 0$, then $X \wedge M_{\mathfrak{p}} = 0$.

Since $M_{\mathfrak{p}}$ is Bousfield equivalent to $K(\mathfrak{p})$, and since $S/\mathfrak{p}$ makes sense independently of the grading, we want to know that

$$L_{U\mathfrak{p}}(UX \wedge S/\mathfrak{p}) = 0 \Rightarrow L_{\mathfrak{p}}(X \wedge S/\mathfrak{p}) = 0.$$

Since the effect of $L_{\mathfrak{p}}$ is to localize homotopy groups at $\mathfrak{p}$, we can state this purely module-theoretically: we want to know that if $R$ is a graded ring, $\mathfrak{p}$ is a homogeneous prime ideal of $R$, and $N$ is a graded $R$-module, then

$$UN_{U\mathfrak{p}} = 0 \Rightarrow N_{\mathfrak{p}} = 0.$$

This is straightforward: if $UN_{U\mathfrak{p}} = 0$, then every element $x \in UN$ is annihilated by some $a \notin U\mathfrak{p}$. Hence every homogeneous element $x \in N$ is annihilated by some $a \notin U\mathfrak{p}$. Write $a$ as a sum of homogeneous elements: $a = a_1 + \cdots + a_n$ with the $a_i$ in distinct degrees. Then each $a_i x$ is in a different degree, so must be zero. Since $\sum a_i = a$ is not in $U\mathfrak{p}$, then some $a_i$ is not in $\mathfrak{p}$. Hence every homogeneous element in $n$ is annihilated by a homogeneous element not in $\mathfrak{p}$. □

## 7. Rank varieties–philosophy

In this short section, we try to motivate the rank varieties used in the previous section. To some extent, rank varieties are an ad hoc construction, but a stable homotopy theoretic point of view helps to explain some of their usefulness.

Rank varieties are used to compute support varieties, via results like Theorem 6.10; as we have noted, one of their main uses is to verify the tensor product property for support varieties. So in our search for more precise motivation for rank varieties, we examine the tensor product property. Ideally, we would like to prove it directly for support varieties, by imitating the proof for rank varieties in Theorem 6.6.

Let $\mathcal{C}$ be a bigraded Noetherian stable homotopy category. We point out in Section 3 that the tensor product property is equivalent to the following condition on the $M_{\mathfrak{p}}$, for objects $X$ and $Y$ of $\mathcal{C}$:

- For all prime ideals $\mathfrak{p}$ of $\pi_{**}S$, $M_{\mathfrak{p}} \wedge X \wedge Y = 0$ if and only if either $M_{\mathfrak{p}} \wedge X = 0$ or $M_{\mathfrak{p}} \wedge Y = 0$.

As remarked in Section 3, this would follow if the $M_{\mathfrak{p}}$ were (Bousfield equivalent to) skew field objects. Let $B$ be a Hopf algebra and let $\mathcal{C} = \mathcal{S}(B)$. Here are the steps involved in proving the tensor product theorem for $\mathcal{S}(B)$:

**Step 1.** The coproduct $\Delta\colon B \to B \otimes B$ induces a map $\operatorname{Ext}^{**}_{B\otimes B}(k,k) \to \operatorname{Ext}^{**}_{B}(k,k)$, and hence a map on varieties: $\Delta_*\colon \mathcal{V}_B(S) \to \mathcal{V}_{B\otimes B}(S)$. One can show that $\mathcal{V}_{B\otimes B}(S) = \mathcal{V}_B(S) \times \mathcal{V}_B(S)$, and one can compute the effect of the map $\Delta_*$:

$$\Delta_*\colon \mathcal{V}_B(S) \to \mathcal{V}_B(S) \times \mathcal{V}_B(S)$$
$$\mathfrak{p} \longmapsto (\mathfrak{p}, \mathfrak{p}).$$

For support varieties, this is easy to check: first, there is a Künneth isomorphism

$$\operatorname{Ext}^{**}_{B\otimes B}(k,k) \cong \operatorname{Ext}^{**}_{B}(k,k) \otimes \operatorname{Ext}^{**}_{B}(k,k),$$

which gives the identification of $\mathcal{V}_{B\otimes B}(S)$. Second, it is well known that $\Delta$ induces the usual Yoneda product on Ext, and it is easy to compute the effect of the product map on Spec.

For rank varieties, this is a bit harder. When $B$ is elementary, we complete Step 1 in Lemma 6.8. When $B$ is not elementary, though, we do not know how to verify this; hence we need to assume it in Theorem 6.6.

Given Step 1, the proof of Theorem 6.6 reduces us to the following. (In that proof, we apply this step to the Hopf algebra inclusion $\Delta\colon B \to B \otimes B$.)

**Step 2.** Let $i\colon B_1 \hookrightarrow B_2$ be an inclusion of finite-dimensional Hopf algebras for which the trivial module $k$ is the only simple module, and assume that the induced map $i_*\colon \operatorname{Spec}\operatorname{Ext}^{**}_{B_1}(k,k) \to \operatorname{Spec}\operatorname{Ext}^{**}_{B_2}(k,k)$ is an inclusion. Then for any object $X$ in $\mathcal{S}(B_2)$,

$$i_*\mathcal{V}_{B_1}(\operatorname{res} X) = \mathcal{V}_{B_2}(X) \cap i_*\mathcal{V}_{B_1}(S). \tag{7.1}$$

We state this for rank varieties in Lemma 6.7. Note also that in the graded connected case, if $B_1 \hookrightarrow B_2$ is an inclusion, then so is the induced map on Spec, by Lemma 8.1.

It turns out that to show the two inclusions $\subseteq$ and $\supseteq$ of Equation 7.1, one needs to understand the properties of the objects $M_{\mathfrak{p}}$ under restriction and induction, respectively. In particular, if one assumes that

$$\langle \operatorname{res} M_{i_*\mathfrak{q}} \rangle \geq \langle M_{\mathfrak{q}} \rangle$$

for all prime ideals $\mathfrak{q}$ in $\operatorname{Ext}^{**}_{B_1}(k,k)$, then the inclusion $\subseteq$ follows. (Since $i_*$ is an inclusion, then Corollary 4.12 implies an even stronger result, that $\operatorname{res} M_{i_*\mathfrak{q}} = M_{\mathfrak{q}}$.) Here are the details: we assume that $\mathfrak{p} \notin \mathcal{V}_{B_2}(X) \cap i_*(\mathcal{V}_{B_1}(S))$. Note that if $\mathfrak{p} \notin i_*(\mathcal{V}_{B_1}(S))$, then $\mathfrak{p}$ is certainly not in $i_*\mathcal{V}_{B_1}(\operatorname{res} X)$, so we assume that $\mathfrak{p} = i_*\mathfrak{q}$ for some $\mathfrak{q} \in \mathcal{V}_{B_1}(S)$. Since $\mathfrak{p} \notin \mathcal{V}_{B_2}(X)$, then $M_{\mathfrak{p}} \wedge X = 0$. Applying res, we find that $0 = \operatorname{res}(M_{\mathfrak{p}} \wedge X) = \operatorname{res} M_{\mathfrak{p}} \wedge \operatorname{res} X$. We are assuming that $\operatorname{res} M_{\mathfrak{p}}$ has a larger Bousfield class than $M_{\mathfrak{q}}$; hence $M_{\mathfrak{q}} \wedge \operatorname{res} X = 0$. In other words, $\mathfrak{q} \notin \mathcal{V}_{B_1}(\operatorname{res} X)$.

If one knows, in addition, that

$$\langle \operatorname{ind} M_{\mathfrak{q}} \rangle \geq \langle M_{i_*\mathfrak{q}} \rangle,$$

for all prime ideals $\mathfrak{q}$ in $\operatorname{Ext}^{**}_{B_1}(k,k)$, then the inclusion $\supseteq$ of Equation (7.1) follows. (Unfortunately, we do not know how to prove this inequality of Bousfield classes without using rank varieties. We discuss this below.) Here are the details of the verification of the inclusion $\supseteq$: assume that $\mathfrak{q} \notin \mathcal{V}_{B_1}(X)$, so that $M_{\mathfrak{q}} \wedge \operatorname{res} X = 0$. Since $\langle \operatorname{res} M_{i_* \mathfrak{q}} \rangle = \langle M_{\mathfrak{q}} \rangle$, then

$$0 = \operatorname{res} M_{i_* \mathfrak{q}} \wedge \operatorname{res} X = \operatorname{res}(M_{i_* \mathfrak{q}} \wedge X).$$

Applying ind gives $\operatorname{ind} \operatorname{res}(M_{i_* \mathfrak{q}} \wedge X)$. For modules, $\operatorname{ind} \operatorname{res} M \cong \operatorname{ind}(k) \otimes M$; we apply the corresponding result for chain complexes (twice) to get

$$\operatorname{ind} \operatorname{res}(M_{i_* \mathfrak{q}} \wedge X) = \operatorname{ind}(S) \wedge M_{i_* \mathfrak{q}} \wedge X = (\operatorname{ind} \operatorname{res} M_{i_* \mathfrak{q}}) \wedge X = \operatorname{ind} M_{\mathfrak{q}} \wedge X = 0.$$

Finally, since $\langle \operatorname{ind} M_{\mathfrak{q}} \rangle = \langle M_{i_* \mathfrak{q}} \rangle$, we see that

$$M_{i_* \mathfrak{q}} \wedge X = 0,$$

so $i_* \mathfrak{q} \notin \mathcal{V}_{B_2}(X)$.

(By the way, one can show, using the equalities $\operatorname{res} M_{i_* \mathfrak{q}} = M_{\mathfrak{q}}$ and $\operatorname{ind} \operatorname{res} X = \operatorname{ind}(S) \wedge X$, that

$$\langle \operatorname{ind} M_{\mathfrak{q}} \rangle \leq \langle M_{i_* \mathfrak{q}} \rangle,$$

so one wants to know that these are in fact Bousfield equivalent.)

The point of using rank varieties is that "they are well-behaved under induction". For each $x$ in the rank variety $\mathcal{V}_B^r(k)$, we construct an object $N_x$: first, consider the $B \otimes K$-module $H(x) = (B \otimes K) \otimes_{\operatorname{alg}(x)} K$. This has Ext groups

$$\operatorname{Ext}_{B \otimes K}(H(x), K) \cong \operatorname{Ext}_{\operatorname{alg}(x)}(K, K) \cong K[y]$$

for some $y$ in $\operatorname{Ext}^1$. $N_x$ is the chain complex obtained from $H(x)$ by inverting $y$, so its homotopy groups are $K[y, y^{-1}]$. Note that $N_x$ need not be a ring object, even though its homotopy groups make it look like it is trying to be a skew field in the category $\mathcal{S}(B)$.

It is easy to check that the objects $N_x$ are well-behaved under induction: if $B_1$ is a sub-Hopf algebra of $B_2$ so that $\mathcal{V}_{B_1}(S)$ includes into $\mathcal{V}_{B_2}(S)$, then since support varieties are the same as rank varieties by Corollary 6.5, we get a similar inclusion for rank varieties. Hence for $x \in \mathcal{V}_{B_1}^r(k)$, we have $\overline{x}$, the image of $x$ in $\mathcal{V}_{B_2}^r(k)$. By construction, $\operatorname{ind} H(x) = H(\overline{x})$; inverting $y$ in both of these $B_2$-chain complexes yields $\operatorname{ind} N_x = N_{\overline{x}}$, as desired.

Corollary 6.5 gives a bijection $\mathcal{V}_{\operatorname{StMod}'(B)}(k) \xrightarrow{\sim} \mathcal{V}_B^r(k)$. Using all of the results about both kinds of varieties, one can do better: one sees that if $\mathfrak{p} \in \mathcal{V}_{\operatorname{StMod}'(B)}(k)$ corresponds to $x \in \mathcal{V}_B^r(k)$, then $M_{\mathfrak{p}}$ and $N_x$ are Bousfield equivalent. Hence, one can complete Step 2 for support varieties, albeit in a roundabout fashion.

## 8. Examples

In this section, we apply our results to some particular Hopf algebras $B$—those whose quasi-elementary sub-Hopf algebras are elementary.

The class of quasi-elementary Hopf algebras, even the definition thereof [Pal97], is quite complicated. They are well understood in two cases: the group algebra $kG$ of a $p$-group $G$ is quasi-elementary if and only if $G$ is elementary abelian, by Serre's theorem about products of Bocksteins of elements in group cohomology [Ser65]. Of course in this setting, we know by the results in [BCR96] that $kG$ has all of the desired properties—Avrunin-Scott stratification and the tensor product property—and hence Corollary 3.7 holds for the group algebras of finite $p$-groups.

We are most interested in the second case, the finite sub-Hopf algebras of the mod 2 Steenrod algebra $A$, as these are useful in algebraic topology. Note that $A$ is defined over the Galois field $\mathbf{F}_2$, but we can consider $A$ over any field of characteristic 2 by a central extension of scalars. The quasi-elementary finite sub-Hopf algebras of $A$ were essentially classified in [Wil81, Theorem 6.4] (see also [NP98, Proposition 5.2]); they are all elementary Hopf algebras. Therefore, by Corollary 6.13, they all have the tensor product property. We will show that, since they are connected, this implies that they satisfy Avrunin-Scott stratification, so Corollary 5.10 implies that Corollary 3.7 holds in this case also—see Corollary 8.5 below.

Note that when $p > 2$, the quasi-elementary sub-Hopf algebras of the mod $p$ Steenrod algebra are classified in [NP98]; however, the proof given there has a gap which these authors do not know how to fill.

We start with the following observation.

**Lemma 8.1.** *Suppose that $B$ is a finite-dimensional graded connected cocommutative Hopf algebra over a field $k$ of characteristic $p$, and let $Q$ be a sub-Hopf algebra of $B$. Then the induced map*

$$\operatorname{res}_Q^*\colon \operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k) \to \operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$$

*is injective.*

*Proof.* This follows from [HS98, Theorem 4.13], which says that there is an $n$ so that for every element $y \in \operatorname{Ext}_Q^{**}(k,k)$, $y^{p^n}$ is in the image of

$$\operatorname{res}_Q\colon \operatorname{Ext}_B^{**}(k,k) \to \operatorname{Ext}_Q^{**}(k,k).$$

For a prime ideal $\mathfrak{p} \subseteq \operatorname{Ext}_Q^{**}(k,k)$, $\operatorname{res}_Q^*(\mathfrak{p})$ is defined to be $\operatorname{res}_Q^{-1}(\mathfrak{p})$. Assume that $\operatorname{res}_Q^*(\mathfrak{p}) = \operatorname{res}_Q^*(\mathfrak{q})$ for prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. For $y \in \mathfrak{p}$, then $y^{p^n} = \operatorname{res}_Q(x)$ for some $x$, so $x \in \operatorname{res}_Q^*(\mathfrak{p}) = \operatorname{res}_Q^*(\mathfrak{q})$, so $\operatorname{res}_Q(x) = y^{p^n}$ is in $\mathfrak{q}$. Since $\mathfrak{q}$ is prime, we conclude that $y \in \mathfrak{q}$. Therefore $\mathfrak{p} \subseteq \mathfrak{q}$. By the same argument, $\mathfrak{q} \subseteq \mathfrak{p}$. □

Whenever the maps induced on Spec by inclusions of quasi-elementary sub-Hopf algebras are injective, Avrunin-Scott stratification follows from the tensor product property. To see this, we calculate the acyclics for the restriction functor.

**Proposition 8.2.** *Suppose that $B$ is a finite-dimensional graded cocommutative Hopf algebra which has the tensor product property. Suppose as well that $Q$ is a sub-Hopf algebra of $B$, and $X \in \mathcal{S}(B)$. Then $\operatorname{res} X = 0$ if and only if $\mathcal{V}(X) \cap \operatorname{res}^*(\operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k)) = \emptyset$.*

*Proof.* Note that $\operatorname{res} X = 0$ if and only if $\mathcal{S}(Q)(S, \operatorname{res} X)_{**} = 0$. By adjointness, this holds if and only if $\mathcal{S}(B)(\operatorname{ind} S, X)_{**} = 0$. Now $\operatorname{ind} S$ is a finite object of $\mathcal{S}(B)$, since $B$ is finite-dimensional, so we can use Spanier-Whitehead duality (see [HPS97, Section 2.1]) to conclude that this holds if and only if $D(\operatorname{ind} S) \wedge X = 0$. Therefore, the acyclics for the restriction functor form the Bousfield class of $D(\operatorname{ind} S)$. But, since $B$ has the tensor product property, any Bousfield class is completely determined by the $M_{\mathfrak{p}}$ which belong to it by Corollary 3.4. But Corollary 4.12 implies that $\operatorname{res} M_{\mathfrak{p}} = 0$ if and only if $\mathfrak{p} \notin \operatorname{res}^* \operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k)$. □

This proposition yields the following theorem.

**Theorem 8.3.** *Suppose $B$ is a finite-dimensional graded cocommutative Hopf algebra which has the tensor product property. Suppose as well that, for every quasi-elementary sub-Hopf algebra $Q$ of $B$, the induced map*

$$\operatorname{res}_Q^*\colon \operatorname{Spec}\operatorname{Ext}_Q^{**}(k,k) \to \operatorname{Spec}\operatorname{Ext}_B^{**}(k,k)$$

*is injective. Then B satisfies Avrunin-Scott stratification.*

*Proof.* Suppose $X \in \mathcal{S}(B)$, $Q$ is a quasi-elementary sub-Hopf algebra of $B$, and $\mathfrak{q} \in (\mathrm{res}_Q^*)^{-1}\mathcal{V}_B(X)$. Let $\mathfrak{p} = \mathrm{res}_Q^* \mathfrak{q}$. Then $X \wedge M_\mathfrak{p}$ is nonzero by hypothesis. Proposition 8.2 implies that $\mathrm{res}_Q(X \wedge M_\mathfrak{p})$ is also nonzero. Since $\mathrm{res}_Q^*$ is injective, Theorem 4.11 then implies that $\mathrm{res}_Q X \wedge M_\mathfrak{q}$ is nonzero, as required. □

Hence we have the following corollaries.

**Corollary 8.4.** *Suppose $Q$ is a graded connected elementary Hopf algebra over an algebraically closed field. Then $Q$ satisfies Avrunin-Scott stratification.*

*Proof.* This follows from Corollary 6.13, Lemma 8.1, and Theorem 8.3. □

**Corollary 8.5.** *If $B$ is a finite-dimensional graded connected cocommutative Hopf algebra over an algebraically closed field, and if every quasi-elementary sub-Hopf algebra of $B$ is elementary, then $B$ has the tensor product property. Hence Corollary* 3.7 *applies.*

*Proof.* This follows from Corollaries 5.10, 6.13, and 8.4. □

Finally, since every quasi-elementary sub-Hopf algebra of the mod 2 Steenrod algebra is elementary, we have the following.

**Corollary 8.6.** *Let $A$ be the mod 2 Steenrod algebra, and let $\overline{\mathbf{F}}_2$ be the algebraic closure of $\mathbf{F}_2$. If $B$ is a finite-dimensional sub-Hopf algebra of $\overline{\mathbf{F}}_2 \otimes_{\mathbf{F}_2} A$, then $B$ has the tensor product property. Hence Corollary* 3.7 *applies.*

We point out that the usual classification of sub-Hopf algebras of $A$ [AD73] also classifies sub-Hopf algebras of $\overline{\mathbf{F}}_2 \otimes_{\mathbf{F}_2} A$; in particular, every finite-dimensional sub-Hopf algebra of $\overline{\mathbf{F}}_2 \otimes_{\mathbf{F}_2} A$ is of the form $\overline{\mathbf{F}}_2 \otimes_{\mathbf{F}_2} B$ for some finite-dimensional sub-Hopf algebra $B$ of $A$.

Also note that in [HP99], we have removed the requirement from Corollaries 8.5 and 8.6 that the field be algebraically closed.

As examples, we consider the sub-Hopf algebras $A(n)$ of the mod 2 Steenrod algebra, after extending scalars from $\mathbf{F}_2$ to $\overline{\mathbf{F}}_2$. Recall that $A$ is generated as an algebra by the elements $\{\mathrm{Sq}^k \mid k \geq 1\}$, and $A(n)$ is the subalgebra of $\overline{\mathbf{F}}_2 \otimes_{\mathbf{F}_2} A$ generated by $\{\mathrm{Sq}^k \mid 1 \leq k \leq 2^n\}$; it turns out that this is a finite-dimensional sub-Hopf algebra.

Let $k = \overline{\mathbf{F}}_2$. When $n = 0$, then $A(0)$ is isomorphic to the exterior algebra $k[\mathrm{Sq}^1]/(\mathrm{Sq}^1)^2$, with $\mathrm{Sq}^1$ primitive. It is easy to compute Ext: $\mathrm{Ext}_{A(0)}(k,k) = k[h_0]$, where $h_0$ is in bidegree $(1,1)$. The only bihomogeneous prime ideals are $(0)$ and $(h_0)$, and one can describe $M_{(0)}$ and $M_{(h_0)}$: first, $M_{(0)} = L_{(0)}S$, and this is a bigraded field object with $\pi_{**}M_{(0)} = k[h_0, h_0^{-1}]$. Second, $\pi_{**}M_{(h_0)}$ is isomorphic to a shifted copy of $k[h_0, h_0^{-1}]/k[h_0]$. (By the way, in [HPS97, Section 6], the authors construct an object $K(\mathfrak{p})$ which is Bousfield equivalent to $M_\mathfrak{p}$ for each prime $\mathfrak{p}$. In the case at hand, $K(\mathfrak{h}_\mathfrak{o})$ is, by definition, the cofiber $S/h_0$ of $h_0$, which is the chain complex consisting of the module $A(0)$ concentrated in degree 0. This object represents ordinary homology of chain complexes in the category $\mathcal{S}(A(0))$.)

Since there are three subsets of Spec which are closed under specialization—$\emptyset$, $\{(h_0)\}$, $\{(0),(h_0)\}$—then Corollary 3.7 tells us that there are three nonempty thick subcategories of finite objects in $\mathcal{S}(A(0))$: respectively, the thick subcategory consisting of the zero chain complex, the thick subcategory generated by $A(0)$, and the thick subcategory of all finite objects. Similarly, since there are two bihomogeneous prime ideals, the Bousfield lattice has four elements: $\langle 0\rangle$, $\langle M_{(0)}\rangle$, $\langle M_{(h_0)}\rangle$, and $\langle S\rangle = \langle M_{(0)} \vee M_{(h_0)}\rangle$.

Next, we consider $A(1)$, the sub-Hopf algebra of $k \otimes_{\mathbf{F}_2} A$ generated by $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$. This is 8-dimensional as a vector space. We have the following computation of Ext (as given in, say, [Wil81, p. 142]):

$$\mathrm{Ext}^{**}_{A(1)}(k,k) \cong k[h_0, h_1, w, v]/(h_0h_1, h_1^3, h_1w, w^2 - h_0^2v),$$

where $h_0$ has bidegree $(1,1)$, $h_1$ has bidegree $(1,2)$, $w$ has bidegree $(3,7)$, and $v$ has bidegree $(4,12)$. Note that there are four bihomogeneous prime ideals in $\mathrm{Ext}^{**}_{A(1)}(k,k)$; the nilradical $(h_1)$, the ideals $\mathfrak{p}_0 = (h_1, w, h_0)$ and $\mathfrak{p}_1 = (h_1, w, v)$, and the maximal ideal $(h_1, w, h_0, v)$. Hence there are six nonempty thick subcategories of finites and sixteen Bousfield classes in $\mathcal{S}(A(1))$. Write $M_0$ for a projective resolution of $A(1)/A(1)\,\mathrm{Sq}^1$, and write $M_1$ for a projective resolution of $A(0)$ with the apparent $A(1)$-module structure. One can check that the support variety of $M_0$ is the closure of $\{\mathfrak{p}_0\}$, and that the support variety of $M_1$ is the closure of $\{\mathfrak{p}_1\}$. The thick subcategories in $\mathcal{S}(A(1))$ are those generated by 0, $A(1)$, $M_0$, $M_1$, $M_0 \vee M_1$, and $S$.

The situation gets much more complicated with $A(n)$ for $n \geq 2$. For example, from the results of [Pal97] and [NP98], one knows that $\mathrm{Ext}^{**}_{A(2)}(k,k)$ is $F$-isomorphic to

$$k[h_0, v_1, v_2, h_{21}]/(h_0h_{21}),$$

where $|h_0| = (1,1)$, $|v_1| = (1,3)$, $|v_2| = (1,7)$, and $|h_{21}| = (1,6)$. Since an $F$-isomorphism of algebras induces a poset isomorphism on Spec, one concludes that there are infinitely many prime ideals in $\mathrm{Ext}^{**}_{A(2)}(k,k)$; for instance, one has the ideals $(\alpha v_1^3 + \beta v_0^2 v_2)$, for any scalars $\alpha$ and $\beta$. So there are infinitely many thick subcategories, and infinitely many Bousfield classes.

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