

# THE EILENBERG-WATTS THEOREM IN HOMOTOPICAL ALGEBRA

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ABSTRACT. The object of this paper is to prove that the standard categories in which homotopy theory is done, such as topological spaces, simplicial sets, chain complexes of abelian groups, and any of the various good models for spectra, are all **homotopically self-contained**. The left half of this statement essentially means that any functor that looks like it could be a tensor product (or product, or smash product) with a fixed object is in fact such a tensor product, up to homotopy. The right half says any functor that looks like it could be Hom into a fixed object is so, up to homotopy. More precisely, suppose we have a closed symmetric monoidal category (resp. Quillen model category)  $\mathcal{M}$ . Then the functor  $T_N: \mathcal{M} \rightarrow \mathcal{M}$  that takes  $M$  to  $M \otimes N$  is an  $\mathcal{M}$ -functor and a left adjoint. The same is true if  $N$  is an  $E$ - $E'$ -bimodule, where  $E$  and  $E'$  are monoids in  $\mathcal{M}$ , and  $T_N: \text{Mod-}E \rightarrow \text{Mod-}E'$  is defined by  $T_N(M) = M \otimes_E N$ . Define a closed symmetric monoidal category (resp. model category) to be **left self-contained** (resp. **homotopically left self-contained**) if every functor  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  that is an  $\mathcal{M}$ -functor and a left adjoint (resp. and a left Quillen functor) is naturally isomorphic (resp. naturally weakly equivalent) to  $T_N$  for some  $N$ . The classical Eilenberg-Watts theorem in algebra then just says that the category Ab of abelian groups is left self-contained, so we are generalizing that theorem.

## INTRODUCTION

The object of this paper is to extend the Eilenberg-Watts theorem [Eil60, Wat60] to situations such as topological spaces, chain complexes, or symmetric spectra, in which one is interested in objects and functors not up to isomorphism, but up to some notion of weak equivalence. We first recall the standard Eilenberg-Watts theorem.

**Theorem 0.1.** *Let  $R$  and  $S$  be rings. If  $F: \text{Mod-}R \rightarrow \text{Mod-}S$  is additive and a left adjoint, then  $FR$  is an  $R$ - $S$ -bimodule and there is a natural isomorphism*

$$X \otimes_R FR \rightarrow FX.$$

Here, and throughout the paper, all modules are right modules unless otherwise specified.

This is not the usual formulation of the Eilenberg-Watts theorem, which we now recall.

**Theorem 0.2.** *Let  $R$  and  $S$  be rings if  $F: \text{Mod-}R \rightarrow \text{Mod-}S$  is additive, right exact, and preserves direct sums, then  $FR$  is an  $R$ - $S$ -bimodule and there is a natural isomorphism*

$$X \otimes_R FR \rightarrow FX.$$

These two theorems are equivalent, however, since a right exact additive functor automatically preserves coequalizers, and then the adjoint functor theorem implies that any colimit-preserving functor between such nice categories is a left adjoint.

We take the position that if  $\mathcal{M}$  is a closed symmetric monoidal category,  $E$  and  $E'$  are monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a left adjoint and an  $\mathcal{M}$ -functor (the analogue of additive), then  $F$  is a sort of generalized  $E$ - $E'$ -bimodule.

We therefore make the following definition. Given monoids  $E$  and  $E'$  in a closed symmetric monoidal category  $\mathcal{M}$  and an  $E$ - $E'$ -bimodule  $N$ , define  $T_N: \text{Mod-}E \rightarrow \text{Mod-}E'$  by  $T_N(M) = M \otimes_E N$ . We will define this more precisely in the first section below. Similarly, for an  $E \otimes E'$ -module  $N$ , define  $S_N: \text{Mod-}E \rightarrow (\text{Mod-}E')^{\text{op}}$  by  $S_N(M) = \text{Hom}_E(M, N)$ .

**Definition 0.3.** Suppose  $\mathcal{M}$  is a closed symmetric monoidal category. We say that  $\mathcal{M}$  is **left self-contained** if, for every pair of monoids  $E, E'$  in  $\mathcal{M}$  and every  $\mathcal{M}$ -functor  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  that is a left adjoint, there is an  $E$ - $E'$ -bimodule  $N$  and a natural isomorphism  $T_N \cong F$ . Similarly,  $\mathcal{M}$  is **right self-contained** if, for every pair of monoids  $E, E'$  in  $\mathcal{M}$  and every  $\mathcal{M}$ -functor  $F: \text{Mod-}E \rightarrow (\text{Mod-}E')^{\text{op}}$  that is a left adjoint, there is an  $E \otimes E'$ -module  $N$  and a natural isomorphism  $F \cong S_N$ . We say that  $\mathcal{M}$  is **self-contained** if it is both left and right self-contained.

The Eilenberg-Watts theorem stated above is then the assertion that  $\text{Ab}$  is left self-contained. In fact, Eilenberg [Eil60] also proved that  $\text{Ab}$  is right self-contained. It would certainly be interesting to go on from here to try to find out when a closed symmetric monoidal category is self-contained, but the author is primarily interested in homotopy theory, so we go in a different direction.

Instead, we assume that we can also do homotopy theory in  $\mathcal{M}$ . So we assume that  $\mathcal{M}$  is a model category in the sense of Quillen [Qui67]. The key idea in model categories is that isomorphism is not the equivalence relation one cares about. Instead, there is a notion of weak equivalence. Formally inverting the weak equivalences gives the homotopy category  $\text{Ho } \mathcal{M}$ . We would like functors  $F$  on  $\mathcal{M}$  to induce functors  $LF$  on  $\text{Ho } \mathcal{M}$  in a natural way, but this is generally true only for **left Quillen** functors  $F$  (for this and other model category terminology, see [Hir03] or [Hov99]).

Thus we make the following definition.

**Definition 0.4.** Suppose  $\mathcal{M}$  is a closed symmetric monoidal model category. We say that  $\mathcal{M}$  is **homotopically left self-contained** if, for every pair of monoids  $E, E'$  in  $\mathcal{M}$  and every left Quillen  $\mathcal{M}$ -functor  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$ , there is an  $E$ - $E'$ -bimodule  $N$  and a natural isomorphism  $LT_N \rightarrow LF$  of functors on  $\text{Ho Mod-}E$ . Similarly,  $F$  is **homotopically right self-contained** if, for every pair of monoids  $E, E'$  in  $\mathcal{M}$  and every left Quillen  $\mathcal{M}$ -functor  $F: \text{Mod-}E \rightarrow (\text{Mod-}E')^{\text{op}}$ , there is an  $E \otimes E'$ -module  $N$  and a natural isomorphism  $LF \rightarrow LS_N$  of functors on  $\text{Ho Mod-}E$ . And  $\mathcal{M}$  is **homotopically self-contained** if it is both left and right homotopically self-contained.

For this definition to make precise sense, we need to assume enough about  $\mathcal{M}$  to be sure that  $\text{Mod-}E$  inherits a model structure from that of  $\mathcal{M}$  for all monoids  $E$ , where the weak equivalences (resp. fibrations) in  $\text{Ho Mod-}E$  are maps that are weak equivalences (resp. fibrations) when thought of as maps of  $\mathcal{M}$ .

The main result of this paper is then the following theorem, proved in Section 6.

**Theorem 0.5.** *The following closed symmetric monoidal model categories are homotopically self-contained.*

- (1) *(Compactly generated, weak Hausdorff) topological spaces.*
- (2) *Simplicial sets.*
- (3) *Chain complexes of abelian groups.*
- (4) *Symmetric spectra*
- (5) *Orthogonal spectra*
- (6)  *$S$ -modules.*

The last three are all models of stable homotopy theory.  $S$ -modules were introduced in [EKMM97], orthogonal spectra in [MMSS01], and symmetric spectra in [HSS00].

To approach Theorem 0.5, we recall the proof of the Eilenberg-Watts theorem. The first step is to prove that, if  $F: \text{Mod-}R \rightarrow \text{Mod-}S$  is an additive functor, then  $FR$  is an  $R$ - $S$ -bimodule, and there is a natural transformation

$$M \otimes_E FE \rightarrow FM.$$

This step is completely general. The following proposition is proved as Proposition 1.1, where we will define any unfamiliar terms.

**Proposition 0.6.** *Suppose  $\mathcal{M}$  is a closed symmetric monoidal category,  $E$  and  $E'$  are monoids in  $\mathcal{M}$ , and*

$$F: \text{Mod-}E \rightarrow \text{Mod-}E'$$

*is an  $\mathcal{M}$ -functor. Then  $FE$  is an  $E$ - $E'$ -bimodule, and there is a natural transformation*

$$\tau: X \otimes_E FE \rightarrow FX.$$

Given  $\tau$ , the proof of the usual Eilenberg-Watts theorem now proceeds noting that  $\tau$  is an isomorphism when  $M = R$ , both sides preserve colimits, and  $R$  generates  $\text{Mod-}R$  under colimits. There are partial generalizations of this to the general case. The nicest we have is Theorem 2.3, which asserts that  $\tau$  is a natural isomorphism when  $F$  is a strict  $\mathcal{M}$ -functor (defined in Section 2) and preserves coequalizers.

However, our main interest is in homotopical algebra, where it is a mistake to ask whether a natural transformation is an isomorphism for all  $X$ . Instead, we ask whether it is a weak equivalence for all cofibrant  $X$ ; equivalently, we ask when the derived natural transformation of  $\tau$  is an isomorphism of functors on  $\text{HoMod-}E$ . For this to make sense, we need to assume enough about  $\mathcal{M}$  so that we get model structures on  $\text{Mod-}E$  and  $\text{Mod-}E'$ , and we need to assume  $F$  is a left Quillen functor as mentioned above. We then need a general theorem about when the derived natural transformation  $L\tau$  of a natural transformation  $\tau$  of left Quillen functors is an isomorphism. The author thinks that such a theorem should have been proved before, but knows of no published reference. The following theorem is a combination of Theorem 4.1 and Theorem 4.4.

**Theorem 0.7.** *Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are model categories,  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  are left Quillen functors, and  $\tau: F \rightarrow G$  is a natural transformation. Suppose that one of the two following conditions hold.*

- (1)  *$\mathcal{C}$  and  $\mathcal{D}$  are stable, and there is a class  $\mathcal{G}$  of objects of  $\text{Ho}\mathcal{C}$  such that the localizing subcategory generated by  $\mathcal{G}$  is all of  $\text{Ho}\mathcal{C}$  and  $(L\tau)(X)$  is an isomorphism for all  $X \in \mathcal{G}$ .*

- (2)  $\mathcal{C}$  is cofibrantly generated such that the domains of the generating cofibrations of  $\mathcal{C}$  are cofibrant, and  $(L\tau)(X)$  is an isomorphism whenever  $X$  is a domain or codomain of one of the generating cofibrations.

Then  $L\tau$  is a natural isomorphism of functors on  $\text{Ho}\mathcal{C}$ .

We can then combine Theorem 0.7 with Proposition 0.6 to get versions of the Eilenberg-Watts theorem in homotopical algebra. Here are two of them, proved as Theorem 5.1 and Theorem 5.4. They use some terms we will define later.

**Theorem 0.8.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, symmetric monoidal model category. Let  $E$  and  $E'$  be monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  be a left Quillen  $\mathcal{M}$ -functor. Suppose one of the hypotheses below holds.*

- (1) *The domains of the generating cofibrations of  $\mathcal{M}$  are cofibrant, and the composite*

$$A \otimes FQE \rightarrow F(A \otimes QE) \rightarrow F(A \otimes E)$$

*is a weak equivalence when  $A$  is a domain or codomain of one of the generating cofibrations of  $\mathcal{M}$ ; or*

- (2)  *$\mathcal{M}$  is stable and monogenic with a cofibrant unit.*

Then the natural transformation

$$QX \otimes_E FQE \rightarrow X \otimes_E FE \rightarrow FQX$$

*is a natural isomorphism of functors on  $\text{Ho Mod-}E$ .*

Theorem 0.5, asserting that standard model categories are homotopically self-contained, now follows. Although we do not prove Theorem 0.5 in this way, a good way to think about it is that  $\tau$  is obviously an isomorphism for  $X = E$ , both the functors  $LT_N$  and  $LF$  preserve suspensions and homotopy colimits, and in the standard model categories,  $E$  generates all of  $\text{Ho Mod-}E$  under the operations of homotopy colimits and suspensions.

We point out that Keller's work on DG-categories implies in particular that chain complexes of abelian groups are homotopically self-contained [Kel94, Section 6.4], in a stronger form than the one we give. We can recover Keller's full result by our methods using the special features of the model structure on chain complexes of abelian groups.

Note that the usual Eilenberg-Watts theorem is closely related to Morita theory. After all, if you have an equivalence of additive categories  $F: \text{Mod-}R \rightarrow \text{Mod-}S$ , it satisfies the hypotheses of the Eilenberg-Watts theorem, so must be given by tensoring with a bimodule. This is the beginning of Morita theory. Similarly, our versions of the Eilenberg-Watts theorem are related to the Morita theory of ring spectra due to Schwede and Shipley [Sch04].

The original motivation for this work was the important, yet disturbing, paper of Christensen, Keller, and Neeman [CKN01], where they proved that not all homology theories on the derived category  $\mathcal{D}(R)$  of a sufficiently complicated ordinary ring  $R$  are representable. This followed work of Beligiannis [Bel00], who proved that not every morphism between representable homology functors on  $\mathcal{D}(R)$  is representable. Note that there is no problem with representability of cohomology functors and morphisms between them. This was rather a blow, and the author is not certain the field has adequately adjusted to it yet. This paper began by trying to find a

property of a homology functor  $h$  that would ensure that it is representable. This question is discussed in the last section of the paper.

The author owes a debt of thanks to his former student Manny Lopez, who first introduced him to the Eilenberg-Watts theorem. He also wishes to express his debt to Dan Christensen, Bernhard Keller, and Amnon Neeman, both for writing the paper [CKN01] and for their comments on an early draft.

We should also mention Neeman's paper [Nee98]. This paper considers representability for covariant exact functors on triangulated categories that preserve **products**. The author thinks this needs further investigation, even in the abelian category setting, although the situation is so different that such an investigation would not reasonably fit into this paper.

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## 1. $\mathcal{M}$ -FUNCTORS

In this section,  $\mathcal{M}$  is simply a closed symmetric monoidal category with all finite colimits and limits. We denote the monoidal product by  $\otimes$ , the closed structure by  $\text{Hom}$ , and the unit by  $S$ . The purpose of this section is to introduce enough background to prove the following proposition.

**Proposition 1.1.** *Suppose  $\mathcal{M}$  is a closed symmetric monoidal category with finite colimits and limits, and  $E$  and  $E'$  are monoids in  $\mathcal{M}$ . If  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is an  $\mathcal{M}$ -functor, then  $FE$  is an  $E$ - $E'$ -bimodule, and there is a natural transformation*

$$\tau: X \otimes_E FE \rightarrow FX$$

*of  $E'$ -modules that is an isomorphism when  $X = E$ . Similarly, if  $F$  is a contravariant  $\mathcal{M}$ -functor, then  $FE$  is an  $E \otimes E'$ -module, and there is a natural transformation*

$$\tau: FX \rightarrow \text{Hom}_E(X, FE)$$

*of  $E'$ -modules that is an isomorphism when  $X = E$ .*

For this proposition to make sense, we recall that a **monoid**  $E$  in a monoidal category  $\mathcal{M}$  is the object  $E$  equipped with a unit map  $S \rightarrow E$  and a multiplication map  $E \otimes E \rightarrow E$  that is unital and associative. In this case, a (right) module  $X$  over  $E$  is an object  $X$  equipped with an action map  $X \otimes E \rightarrow X$  that is associative and unital. The category  $\text{Mod-}E$  is the category of such right modules and module maps, which are of course maps in  $\mathcal{M}$  compatible with the actions.

The category  $\text{Mod-}E$  then becomes a closed (left) module category over  $\mathcal{M}$ , which means it is tensored, cotensored, and enriched over  $\mathcal{M}$ . In more detail, the tensor of  $K \in \mathcal{M}$  and  $X \in \text{Mod-}E$ , is the object  $K \otimes X \in \text{Mod-}E$ , where  $E$  acts only on  $X$ . The cotensor of  $K \in \mathcal{M}$  and  $X \in \text{Mod-}E$  is  $\text{Hom}(K, X) \in \text{Mod-}E$ . The action of  $E$  is defined by

$$\text{Hom}(K, X) \otimes E \cong \text{Hom}(K, X) \otimes \text{Hom}(S, E) \xrightarrow{\otimes} \text{Hom}(K, X \otimes E) \rightarrow \text{Hom}(K, X),$$

where the last map is induced by the action on  $X$  (recall that  $S$  is the unit). The enrichment  $\text{Hom}_E(X, Y) \in \mathcal{M}$  for  $X, Y \in \text{Mod-}E$  is defined as the equalizer of the evident two maps

$$\text{Hom}(X, Y) \rightarrow \text{Hom}(X \otimes E, Y),$$

one of which uses the action on  $X$  and the other of which uses the action on  $Y$ . We have the usual adjunction isomorphisms

$$\text{Mod-}E(K \otimes X, Y) \cong \text{Mod-}E(X, \text{Hom}(K, Y)) \cong \mathcal{M}(K, \text{Hom}_E(X, Y)).$$

Now, if  $X$  is a right  $E$ -module and  $Y$  is a left  $E$ -module, we can form the tensor product  $X \otimes_E Y \in \mathcal{M}$  as the coequalizer of the two maps

$$X \otimes E \otimes Y \rightarrow X \otimes Y.$$

We can also define  $E$ - $E'$ -bimodules in the usual way. In this case, the action map

$$X \otimes E' \rightarrow X$$

must be a map of left  $E$ -modules, which is equivalent to the left  $E$ -action being a map of right  $E'$ -modules. If  $Y$  is an  $E$ - $E'$ -bimodule (with the evident definition), then  $X \otimes_E Y$  is in  $\text{Mod-}E'$ , because the action map

$$X \otimes Y \otimes E' \rightarrow X \otimes Y$$

descends through the coequalizer. Similarly, if  $Y$  is both an  $E$ -module and an  $E'$ -module in compatible fashion, which is equivalent to  $Y$  being an  $E \otimes E'$ -module, then  $\text{Hom}_E(X, Y)$  is naturally an  $E'$ -module.

Finally, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a functor, we say that  $F$  is an  $\mathcal{M}$ -**functor** if  $F$  is compatible with (one of, and hence all of) the tensor, cotensor, and enrichment over  $\mathcal{M}$ . That is, if  $F$  is an  $\mathcal{M}$ -functor, then there are natural maps

$$\begin{aligned} K \otimes FX &\rightarrow F(K \otimes X), & F(\text{Hom}(K, X)) &\rightarrow \text{Hom}(K, FX) \\ &\text{and } \text{Hom}_E(X, Y) &\rightarrow \text{Hom}_{E'}(FX, FY) \end{aligned}$$

satisfying all the properties one would expect. If  $F$  is contravariant, then the roles of the tensor and cotensor are reversed, so  $F$  is an  $\mathcal{M}$ -functor when there are natural maps

$$\begin{aligned} F(K \otimes X) &\rightarrow \text{Hom}(K, FX), & K \otimes FX &\rightarrow F(\text{Hom}(K, X)) \\ &\text{and } \text{Hom}_E(X, Y) &\rightarrow \text{Hom}_{E'}(FY, FX) \end{aligned}$$

satisfying all the properties one would expect. As an example of how to get these maps from the enrichment

$$\text{Hom}_E(X, Y) \rightarrow \text{Hom}_{E'}(FY, FX),$$

the adjoint to the identity of  $K \otimes X$  is a map

$$K \rightarrow \text{Hom}_E(X, K \otimes X).$$

Composing this with the map

$$\mathrm{Hom}_E(X, K \otimes X) \rightarrow \mathrm{Hom}_{E'}(F(K \otimes X), FX)$$

and taking the adjoint gives us the map

$$F(K \otimes X) \rightarrow \mathrm{Hom}(K, FX).$$

With this background in hand, we can now prove Proposition 1.1.

*Proof.* We begin with the covariant case. For any monoid  $E$ , we have an isomorphism of monoids

$$E \cong \mathrm{Hom}(S, E) \cong \mathrm{Hom}_E(E, E).$$

If we had elements, we would say this takes  $x \in E$  to the map  $E \rightarrow E$  that is left multiplication by  $x$ . Using the fact that  $F$  is an  $\mathcal{M}$ -functor, we get a map

$$\mathrm{Hom}_E(E, E) \rightarrow \mathrm{Hom}_{E'}(FE, FE)$$

of monoids. The adjoint

$$E \otimes FE \rightarrow FE$$

in  $\mathrm{Mod}\text{-}E'$  to the composite map

$$E \rightarrow \mathrm{Hom}_{E'}(FE, FE)$$

maps  $FE$  an  $E$ - $E'$ -bimodule, though there are many details for the conscientious reader to check.

Similarly, the map

$$X \rightarrow \mathrm{Hom}_E(E, X) \rightarrow \mathrm{Hom}_{E'}(FE, FX)$$

has adjoint the desired natural transformation

$$\tau: X \otimes_E FE \rightarrow FX$$

of  $E'$ -modules. There are even more details to check here. In particular, *a priori* the adjoint to

$$\phi_X: X \rightarrow \mathrm{Hom}_{E'}(FE, FX)$$

is just a map

$$X \otimes FE \rightarrow FX$$

of right  $E'$ -modules. However,  $\phi_X$  is in fact a map of right  $E$ -modules, using the left  $E$ -module structure on  $FE$  to make the target of  $\phi_X$  a right  $E$ -module. This means that the adjoint descends through the relevant coequalizer diagram to give the desired map

$$\tau: X \otimes_E FE \rightarrow FX.$$

We now assume that  $F$  is contravariant. We get the right  $E$ -module structure on the  $E'$ -module  $FE$  via the adjoint to the map of monoids

$$E^{\mathrm{op}} \rightarrow \mathrm{Hom}_E(E, E)^{\mathrm{op}} \rightarrow \mathrm{Hom}_{E'}(FE, FE).$$

Here, if  $X$  is a monoid,  $X^{\mathrm{op}}$  is the monoid with the reversed multiplication. Similarly, we have the map

$$X \cong \mathrm{Hom}_E(E, X) \rightarrow \mathrm{Hom}_{E'}(FX, FE),$$

which is in fact a map of right  $E$ -modules. This has adjoint a map

$$FX \rightarrow \mathrm{Hom}(X, FE),$$

which in fact factors through  $\mathrm{Hom}_E(X, FE)$ , giving us the desired natural transformation.  $\square$

## 2. A STRICT ANALOG OF THE EILENBERG-WATTS THEOREM

We are most interested in analogues of the Eilenberg-Watts theorem that combine homotopy theory and algebra. But we can also prove a purely categorical version of the Eilenberg-Watts theorem, and we do so in this section.

We begin with a lemma about the structure of modules over a monoid  $E$  in a symmetric monoidal category  $\mathcal{M}$ . Define an  $E$ -module to be **extended** if it is isomorphic to one of the form  $X \otimes E$ , for some  $X \in \mathcal{M}$ .

**Lemma 2.1.** *Suppose  $\mathcal{M}$  is a symmetric monoidal category with all coequalizers, and  $E$  is a monoid in  $\mathcal{M}$ . Then any  $E$ -module is a coequalizer in  $\text{Mod-}E$  of a parallel pair of morphisms  $P_1 \rightrightarrows P_0$  between extended  $E$ -modules.*

*Proof.* Given an  $E$ -module  $X$ , the action map  $p: X \otimes E \rightarrow X$  is a map of  $E$ -modules when we give  $X \otimes E$  the free  $E$ -module structure. This map is an  $\mathcal{M}$ -split epimorphism by the map  $i: X \rightarrow X \otimes E$  induced by the unit of  $E$ . Hence  $X$  is the coequalizer in  $\mathcal{M}$  of  $ip: X \otimes E \rightarrow X \otimes E$  and the identity of  $X \otimes E$ . It follows that  $X$  is the coequalizer in  $\text{Mod-}E$  of

$$1 \otimes \mu: X \otimes E \otimes E \rightarrow X \otimes E,$$

where  $\mu$  denotes the multiplication in  $E$ , and the composite

$$X \otimes E \otimes E \xrightarrow{p \otimes 1} X \otimes E \xrightarrow{i \otimes 1} X \otimes E \otimes E \xrightarrow{1 \otimes \mu} X \otimes E.$$

This requires a little argument, using the fact that colimits in  $\text{Mod-}E$  can be calculated in  $\mathcal{M}$ .  $\square$

We then get a general version of the Eilenberg-Watts theorem, whose proof is straightforward enough, given Lemma 2.1, to leave to the reader.

**Theorem 2.2.** *Let  $\mathcal{M}$  be a closed symmetric monoidal category with all finite colimits and limits,  $E, E'$  be monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  be an  $\mathcal{M}$ -functor that preserves coequalizers. Suppose the natural map*

$$\tau_X: X \otimes_E FE \rightarrow FX$$

*is an isomorphism for all extended  $E$ -modules  $X$ . Then it is a natural isomorphism for all  $E$ -modules  $X$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant  $\mathcal{M}$ -functor that takes coequalizers to equalizers, and the natural map*

$$\tau_X: FX \rightarrow \text{Hom}_E(X, FE)$$

*is an isomorphism for all extended  $E$ -modules  $X$ , then it is an isomorphism for all  $E$ -modules  $X$ .*

Note that Theorem 2.2 does not imply the usual Eilenberg-Watts theorem directly, though the method of proof does do so. Indeed, when  $\mathcal{M}$  is abelian groups, every  $E$ -module is a coequalizer of a map of free  $E$ -modules (not just extended ones), so it suffices to know that  $F$  is right exact (which is equivalent to preserving coequalizers in this case) and  $\tau_X$  is an isomorphism on free  $E$ -modules. For this, we need  $F$  to preserve direct sums, and then we recover the usual Eilenberg-Watts theorem.

There is a special case in which  $\tau_X$  is automatically an isomorphism on extended  $E$ -modules. An  $\mathcal{M}$ -functor  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is called a **strict  $\mathcal{M}$ -functor** if the structure map

$$K \otimes FX \rightarrow F(K \otimes X)$$



is an isomorphism for all  $K \in \mathcal{M}$  and  $X \in \text{Mod-}E$ . In the contravariant case,  $F$  is a strict  $\mathcal{M}$ -functor if the structure map

$$F(K \otimes X) \rightarrow \text{Hom}(K, FX)$$

is an isomorphism for all  $K \in \mathcal{M}$  and  $X \in \text{Mod-}E$ .

**Theorem 2.3.** *Suppose  $\mathcal{M}$  is a closed symmetric monoidal category with all finite colimits and limits,  $E, E'$  are monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a strict  $\mathcal{M}$ -functor that preserves coequalizers. Then the natural map*

$$\tau_X: X \otimes_E FE \rightarrow FX$$

*is an isomorphism for all  $X$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant strict  $\mathcal{M}$ -functor that takes coequalizers to equalizers, then the natural map*

$$FX \rightarrow \text{Hom}_E(X, FE)$$

*is an isomorphism for all  $X$ .*

*Proof.* It suffices to check that  $\tau_X$  is an isomorphism on all extended  $E$ -modules. But if  $X \in \mathcal{M}$ ,

$$F(X \otimes E) \cong X \otimes FE \cong (X \otimes E) \otimes_E FE$$

since  $F$  is a strict  $\mathcal{M}$ -functor. The contravariant case is similar.  $\square$

### 3. MONOIDAL MODEL CATEGORIES

Since we are most interested in versions of the Eilenberg-Watts theorem “up to homotopy”, we now need to introduce the relevant homotopical algebra. For this, we will of course need to assume knowledge of model categories, for which see [Hir03, Part 2] or [Hov99].

Our base category  $\mathcal{M}$  will be both closed symmetric monoidal and have a model structure. Obviously, we will need to assume some compatibility between the model structure and the monoidal structure on  $\mathcal{M}$ . This is well understood in the theory of model categories, and we adopt the following definition,

**Definition 3.1.** (1) Suppose  $\mathcal{M}$  is a monoidal category, and  $f: A \rightarrow B$  and  $g: C \rightarrow D$  are maps in  $\mathcal{M}$ . The **pushout product** of  $f$  and  $g$ , written  $f \square g$ , is the map

$$(A \otimes D) \amalg_{A \otimes C} (B \otimes C) \rightarrow B \otimes D$$

from the pushout of  $A \otimes D$  and  $B \otimes C$  over  $A \otimes C$ , to  $B \otimes D$ .

(2) Now suppose  $\mathcal{M}$  is a symmetric monoidal category equipped with a model structure. We say that  $\mathcal{M}$  is a **symmetric monoidal model category** if the following conditions hold:

- (a) If  $f$  and  $g$  are cofibrations, then  $f \square g$  is a cofibration.
- (b) If  $f$  is a cofibration and  $g$  is a trivial cofibration, then  $f \square g$  is a trivial cofibration.

This definition is different from [Hov99, Definition 4.2.6], where a unit condition was added to ensure that the homotopy category of a symmetric monoidal model category has a unit. It automatically holds when the unit is cofibrant. This unit condition is not needed for some of our versions of the Eilenberg-Watts theorem, and when it is needed, it is much easier to assume the unit is cofibrant, so we omit it.

But we also need the category  $\text{Mod-}E$  of modules over a monoid  $E$  in  $\mathcal{M}$  to be a model category, in a way that is compatible with  $\mathcal{M}$ . We therefore make the following definition.

**Definition 3.2.** Suppose  $\mathcal{M}$  is a closed symmetric monoidal model category. We say that  $\mathcal{M}$  is **strongly cofibrantly generated** if there are sets  $I$  of cofibrations in  $\mathcal{M}$  and  $J$  of trivial cofibrations in  $\mathcal{M}$  such that, for every monoid  $E$  in  $\mathcal{M}$ , the sets  $I \otimes E$  and  $J \otimes E$  cofibrantly generate a model structure on  $\text{Mod-}E$  where the weak equivalences are maps of  $E$ -modules that are weak equivalences in  $\mathcal{M}$ .

This definition implies that the maps of  $I \otimes E$  are small with respect to  $(I \otimes E)$ -cell, and similarly for  $J \otimes E$ , as this is part of the definition of cofibrantly generated. Note also that a map  $p$  is a fibration (resp. trivial fibration) in  $\text{Mod-}E$  if and only if it has the right lifting property with respect to the maps of  $J \otimes E$  (resp.  $I \otimes E$ ), which is equivalent to  $p$  being a fibration (resp. trivial fibration) in  $\mathcal{M}$ . This definition also implies that the functor that takes  $X \in \mathcal{M}$  to  $X \otimes E \in \text{Mod-}E$  is a left Quillen functor.

We note that pretty much every closed symmetric monoidal model category that is commonly studied is strongly cofibrantly generated. The most useful theorem along these lines is the following, which is a paraphrase of [SS00, Theorem 4.1].

**Theorem 3.3** (Schwede-Shiplay). *Suppose  $\mathcal{M}$  is a cofibrantly generated, closed symmetric monoidal model category. If  $\mathcal{M}$  satisfies the monoid axiom of [SS00, Definition 3.3] and every object of  $\mathcal{M}$  is small, then  $\mathcal{M}$  is strongly cofibrantly generated.*

In fact, one does not need every object of  $\mathcal{M}$  to be small. Not every topological space is small, but the category of (compactly generated weak Hausdorff) topological spaces is still strongly cofibrantly generated.

One advantage of the strongly cofibrantly generated hypothesis is the following proposition.

**Proposition 3.4.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, closed symmetric monoidal model category, and  $E$  is a monoid in  $\mathcal{M}$ . If  $f$  is a cofibration in  $\mathcal{M}$  and  $g$  is a cofibration in  $\text{Mod-}E$ , then the pushout product  $f \square g$  is a cofibration in  $\text{Mod-}E$ , which is a trivial cofibration if either  $f$  or  $g$  is so. In particular, if  $A$  is cofibrant in  $\text{Mod-}E$ , and  $f$  is a cofibration in  $\mathcal{M}$ , then  $f \otimes A$  is a cofibration in  $\text{Mod-}E$ , and is a trivial cofibration if  $f$  is so.*

This proposition means that  $\text{Mod-}E$  is an  $\mathcal{M}$ -model category, in the language of [Hov99, Definition 4.2.18], except that we again have omitted a unit condition.

*Proof.* The last sentence follows from the rest of the proposition by taking  $g$  to be the map  $0 \rightarrow A$ . It suffices to check the statement about  $f \square g$  when  $f$  and  $g$  are generating cofibrations or trivial cofibrations [Hov99, Corollary 4.2.5]. In this case,  $g$  will be of the form  $h \otimes E$  for a map  $h$  in either  $I$  or  $J$ , and  $f$  will be in either  $I$  or  $J$ . But then

$$f \square (h \otimes E) \cong (f \square g) \otimes E$$

so the result follows from the fact that tensoring with  $E$  is a left Quillen functor.  $\square$

The reader might now reasonably expect us to assert that the tensor product

$$X \otimes_E Y: \text{Mod-}E \times (E\text{-}E'\text{-Bimod}) \rightarrow \text{Mod-}E'$$

is also a Quillen bifunctor, so that the pushout product of cofibrations is a cofibration and so on. However, this seems to require  $E$  to be cofibrant in  $\mathcal{M}$ , and in any case is not the most natural thing for us to consider since we know nothing about the structure of  $FE$  as a bimodule.

Instead, we have the following proposition.

**Proposition 3.5.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, closed symmetric monoidal model category,  $E$  and  $E'$  are monoids in  $\mathcal{M}$ , and  $A$  is an  $E$ - $E'$ -bimodule that is cofibrant as a right  $E'$ -module. Then the functor*

$$X \mapsto X \otimes_E A: \text{Mod-}E \rightarrow \text{Mod-}E'$$

*is a left Quillen functor, with right adjoint  $Y \mapsto \text{Hom}_{E'}(A, Y)$ . Similarly, if  $A$  is an  $E \otimes E'$ -module that is fibrant as an object of  $\mathcal{M}$ , then the functor*

$$X \mapsto \text{Hom}_E(X, A): \text{Mod-}E \rightarrow \text{Mod-}E'$$

*is a contravariant left Quillen functor, with right adjoint  $Y \mapsto \text{Hom}_{E'}(Y, A)$ .*

*Proof.* We begin with the covariant case, and leave to the reader the check that  $\text{Hom}_{E'}(A, -)$  is indeed right adjoint to our functor. Note that we use the left  $E$ -module structure on  $A$  to make  $\text{Hom}_{E'}(A, -)$  a right  $E$ -module.

We need to show that if  $f$  is a cofibration or trivial cofibration in  $\text{Mod-}E$ , then  $f \otimes_E A$  is a cofibration or trivial cofibration in  $\text{Mod-}E'$ . The proof is similar in both cases, so we just work with cofibrations. Let  $I$  be a set of generating cofibrations in  $\mathcal{M}$  so that  $I \otimes E$  is a set of generating cofibrations in  $\text{Mod-}E$ . Then any cofibration in  $\text{Mod-}E$  is a retract of a transfinite composition of pushouts of maps of  $I \otimes E$ . Since the tensor product is a left adjoint, it preserves retracts (of course), transfinite compositions, and pushouts. So it suffices to show that if  $f \in I$ , then

$$(f \otimes E) \otimes_E A$$

is a cofibration in  $\text{Mod-}E'$ . But of course

$$(f \otimes E) \otimes_E A \cong f \otimes A,$$

so this is ensured by the fact that  $A$  is cofibrant as a right  $E'$ -module and Proposition 3.4.

The contravariant case is mostly similar. Again, we leave to the reader the proof of the adjointness relation, where one must take into account the fact that the functors are contravariant. We must then show that if  $f$  is a cofibration (resp. trivial cofibration) in  $\text{Mod-}E$ , then  $\text{Hom}_E(f, A)$  is a fibration (resp. trivial fibration) in  $\text{Mod-}E'$ , or equivalently, in  $\mathcal{M}$ . As before, the two cases are similar, so we only do the cofibration case. Every cofibration in  $\text{Mod-}E$  is a retract of a transfinite composition of pushouts of maps of  $I \otimes E$ , where  $I$  is the set of generating cofibrations of  $\mathcal{M}$ . Since  $\text{Hom}_E(-, A)$  is a contravariant left adjoint, it preserves retracts, converts transfinite compositions to inverse transfinite compositions, and converts pushouts to pullbacks. Since retracts, inverse transfinite compositions, and pullbacks of fibrations are fibrations, it suffices to check that

$$\text{Hom}_E(g \otimes E, A) = \text{Hom}(g, A)$$

is a fibration for all  $g \in I$ . But this is true because  $A$  is fibrant in  $\mathcal{M}$ .  $\square$

It now follows easily that the natural transformation of Proposition 1.1 descends to the homotopy category, at least if we replace  $E$  by a cofibrant approximation

$QE$  before applying  $F$ . More precisely, recall that in a model category  $\mathcal{M}$ , we can choose a functor  $Q: \mathcal{M} \rightarrow \mathcal{M}$  such that  $QX$  is cofibrant for all  $X$ , and a natural trivial fibration  $QX \rightarrow X$ .

**Corollary 3.6.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, closed symmetric monoidal model category,  $E$  and  $E'$  are monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a left Quillen  $\mathcal{M}$ -functor. Then the natural transformation  $\tau$  of Proposition 1.1 has a derived natural transformation*

$$L\tau: QX \otimes_E FQE \rightarrow FQX = (LF)(X)$$

of functors on  $\text{Ho Mod-}E$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant left Quillen  $\mathcal{M}$ -functor, then the natural transformation  $\tau$  of Proposition 1.1 has a derived natural transformation

$$L\tau: (LF)(X) = FQX \rightarrow \text{Hom}_E(QX, FQE)$$

of functors on  $\text{Ho Mod-}E$ .

In applications, it is sometimes useful to note that this corollary only requires  $F$  to be left Quillen as a functor to  $\text{Mod-}E'$  with **some** model structure where the weak equivalences are the maps of  $\text{Mod-}E'$  which are weak equivalences in  $\mathcal{M}$ . Indeed, the model structure on  $\text{Mod-}E'$  is never used in the proof of this corollary except that  $\text{Ho Mod-}E'$  is the target of the functors, and  $\text{Ho Mod-}E'$  depends only on the weak equivalences of  $\text{Mod-}E'$ .

We also note that this corollary would remain true if we used  $QFE$  instead of  $FQE$ , but that would go against one of the basic principles of model category theory, to cofibrantly replace objects BEFORE applying left Quillen functors.

#### 4. NATURAL TRANSFORMATIONS OF QUILLEN FUNCTORS

The object of this section is to find conditions under which the derived natural transformation

$$L\tau: LF \rightarrow LG$$

of a natural transformation

$$\tau: F \rightarrow G$$

of left Quillen functors is a natural isomorphism.

**Theorem 4.1.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories,  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be left Quillen functors, and  $\tau: F \rightarrow G$  be a natural transformation. Suppose  $\mathcal{C}$  is cofibrantly generated so that the domains of the generating cofibrations are cofibrant, and  $\tau_X$  is a weak equivalence for  $X$  any domain or codomain of a generating cofibration. Then  $L\tau: LF \rightarrow LG$  is a natural isomorphism.*

*Proof.* It suffices to show that  $\tau_Y$  is a weak equivalence for all cofibrant  $Y$ . Denote the generating cofibrations of  $\mathcal{C}$  by  $I$ . Any cofibrant  $Y$  is a retract of an object  $X$  such that the map  $0 \rightarrow X$  is a transfinite composition of pushouts of maps of  $I$ . It then suffices to prove that  $\tau_X$  is a weak equivalence, which we do by transfinite induction. The base case is clear since  $F$  and  $G$  preserve the initial object. For the

successor ordinal step, we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X_\alpha & \longrightarrow & X_{\alpha+1} \end{array}$$

where  $f$  is a map of  $I$ , and  $\tau_{X_\alpha}$  is a weak equivalence. This gives us two pushout squares of cofibrant objects

$$\begin{array}{ccc} FA & \longrightarrow & FB \\ \downarrow & & \downarrow \\ FX_\alpha & \longrightarrow & FX_{\alpha+1} \end{array}$$

and

$$\begin{array}{ccc} GA & \longrightarrow & GB \\ \downarrow & & \downarrow \\ GX_\alpha & \longrightarrow & GX_{\alpha+1} \end{array}$$

in which the top horizontal map is a cofibration. The natural transformation  $\tau$  defines a map from the top square to the bottom one, which is a weak equivalence on every corner except possibly the right bottom square. The cube lemma [Hov99, Lemma 5.2.6] then implies  $\tau$  is a weak equivalence on the bottom right corner as well.

For the limit ordinal step of the induction, we have a transfinite composition  $X_\beta = \varprojlim_{\alpha < \beta} X_\alpha$  of cofibrant objects, where each map  $X_\alpha \rightarrow X_{\alpha+1}$  is a cofibration. This gives a diagram of cofibrant objects

$$\begin{array}{ccccccc} 0 & \longrightarrow & \cdots & \longrightarrow & FX_\alpha & \longrightarrow & FX_{\alpha+1} & \longrightarrow & \cdots \\ \downarrow & & & & \tau_{X_\alpha} \downarrow & & \downarrow \tau_{X_{\alpha+1}} & & \\ 0 & \longrightarrow & \cdots & \longrightarrow & GX_\alpha & \longrightarrow & GX_{\alpha+1} & \longrightarrow & \cdots \end{array}$$

in which each vertical arrow is a weak equivalence and each horizontal arrow is a cofibration. Then [Hir03, Proposition 15.10.12] implies  $\phi_{X_\beta}$  is also a weak equivalence, completing the induction. This is not quite accurate, because Hirshhorn states Proposition 15.10.12 only for ordinary sequences, but the same proof works for transfinite sequences.  $\square$

There is another version of Theorem 4.1 when  $\text{Mod-}E'$  is left proper. Recall that a model category is **left proper** when the pushout of a weak equivalence through a cofibration is again a weak equivalence.

**Theorem 4.2.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories,  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be left Quillen functors, and  $\tau: F \rightarrow G$  be a natural transformation. Suppose  $\mathcal{C}$  is cofibrantly generated,  $\mathcal{D}$  is left proper, and  $\tau_X$  is a weak equivalence for  $X$  any domain or codomain of a generating cofibration. Then  $L\tau: LF \rightarrow LG$  is a natural isomorphism.*

*Proof.* Use the same proof as that of Theorem 4.1, except replace the use of the cube lemma with [Hir03, Proposition 13.5.4], which requires that  $\mathcal{D}$  be left proper.  $\square$

Theorem 4.1 simplifies further in the stable situation. Recall that a model category  $\mathcal{C}$  is called **stable** if it is pointed and the suspension functor is an equivalence of  $\text{Ho}\mathcal{C}$ . This hypothesis makes  $\text{Ho}\mathcal{C}$  a triangulated category [Hov99, Ch.7].

**Corollary 4.3.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories,  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be left Quillen functors, and  $\tau: F \rightarrow G$  be a natural transformation. Suppose  $\mathcal{C}$  is cofibrantly generated,  $\mathcal{D}$  is stable, and  $\tau_X$  is a weak equivalence for  $X$  any cokernel of a generating cofibration of  $\mathcal{C}$ . Then  $L\tau: LF \rightarrow LG$  is a natural isomorphism.*

*Proof.* The proof is the same as the proof of Theorem 4.1 except in the successor ordinal case. Recall that in this case we have a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X_\alpha & \xrightarrow{g_\alpha} & X_{\alpha+1} \end{array}$$

where  $f$  is a generating cofibration. Let  $C$  denote the cokernel of  $f$ , so that  $g_\alpha$  is a cofibration of cofibrant objects with cokernel  $C$ . Then we have the diagram

$$\begin{array}{ccccccc} FX_\alpha & \longrightarrow & FX_{\alpha+1} & \longrightarrow & FC & \longrightarrow & \Sigma FX_\alpha \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ GX_\alpha & \longrightarrow & GX_{\alpha+1} & \longrightarrow & GC & \longrightarrow & \Sigma GX_\alpha \end{array}$$

of exact triangles in the triangulated category  $\text{Ho}\mathcal{D}$ , where the vertical maps are the natural transformation  $L\tau$ . These vertical maps are isomorphisms on  $FX_\alpha$  and on  $FC$ , and so must also be an isomorphism on  $FX_{\alpha+1}$ .  $\square$

We also have another version of Corollary 4.3 in the stable case. For this to make sense, we recall that a **localizing subcategory** in a triangulated category is a full triangulated subcategory closed under retracts and arbitrary coproducts. The intersection of all localizing subcategories containing a set  $\mathcal{G}$  is written  $\text{loc}\langle \mathcal{G} \rangle$ , and is the smallest localizing subcategory containing  $\mathcal{G}$ .

**Theorem 4.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be model categories,  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be left Quillen functors, and  $\tau: F \rightarrow G$  be a natural transformation. Suppose  $\mathcal{C}$  and  $\mathcal{D}$  are stable, and there is a class  $\mathcal{G}$  of objects of  $\text{Ho}\mathcal{C}$  such that  $\text{loc}\langle \mathcal{G} \rangle = \text{Ho}\mathcal{C}$  and  $(L\tau)_X$  is an isomorphism for  $X \in \mathcal{G}$ . Then  $L\tau: LF \rightarrow LG$  is a natural isomorphism.*

*Proof.* Because  $F$  and  $G$  are a left Quillen functors,  $LF$  and  $LG$  preserve exact triangles (see [Hov99, Section 6.4]), coproducts, and suspensions. Hence the collection of all  $X$  such that  $(L\tau)_X$  is an isomorphism is a localizing subcategory. Since it contains  $\mathcal{G}$ , it contains all of  $\text{Ho}\mathcal{C}$ .  $\square$

## 5. THE EILENBERG-WATTS THEOREM

We can now combine the results of the last two sections to prove homotopical versions of the Eilenberg-Watts theorem.

**Theorem 5.1.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, symmetric monoidal model category in which the domains of the generating cofibrations are cofibrant.*

Let  $E$  and  $E'$  be monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  be a left Quillen  $\mathcal{M}$ -functor. Suppose that the composite

$$A \otimes FQE \rightarrow F(A \otimes QE) \rightarrow F(A \otimes E)$$

is a weak equivalence when  $A$  is a domain or codomain of one of the generating cofibrations of  $\mathcal{M}$ . Then there is a natural isomorphism

$$QX \otimes_E FQE \rightarrow FQX = (LF)(X)$$

of functors on  $\text{Ho Mod-}E$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant left Quillen  $\mathcal{M}$ -functor such that the composite

$$F(A \otimes E) \rightarrow \text{Hom}(A, FE) \rightarrow \text{Hom}(A, FQE)$$

is a weak equivalence when  $A$  is a domain or codomain of one of the generating cofibrations of  $\mathcal{M}$ , then there is a natural isomorphism

$$(LF)(X) = FQX \rightarrow \text{Hom}_E(QX, FQE)$$

of functors on  $\text{Ho Mod-}E$ .

Note that in the composite

$$A \otimes FQE \rightarrow F(A \otimes QE) \rightarrow F(A \otimes E)$$

the first map is the structure map of the  $\mathcal{M}$ -functor  $F$ , and the second map is induced by the weak equivalence  $QE \rightarrow E$ . A similar remark holds in the contravariant case.

We also note that the model structure on  $\text{Mod-}E'$  that we use is again immaterial in the proof, as long as its weak equivalences are the maps that are weak equivalences in  $\mathcal{M}$ . In practice, this means that the condition that  $F$  be left Quillen is not as restrictive as it might appear at first glance. We also point out that in case  $F$  is a *strict*  $\mathcal{M}$ -functor, we should use Theorem 2.3 to analyze  $F$  rather than Theorem 5.1.

*Proof.* Begin with the covariant case. Proposition 3.5 tells us that

$$\tau: X \otimes_E FQE \rightarrow FX$$

is a natural transformation of left Quillen functors. Now apply Theorem 4.1. The contravariant case is similar, where we think of

$$\tau: FX \rightarrow \text{Hom}_E(X, FQE)$$

as a natural transformation of left Quillen functors from  $\text{Mod-}E$  to  $(\text{Mod-}E')^{\text{op}}$ .  $\square$

We get versions of Theorem 5.1 corresponding to each version of Theorem 4.1. Corresponding to Theorem 4.2, we have the following theorem.

**Theorem 5.2.** *Suppose the hypotheses of Theorem 5.1 hold true, except that rather than assuming the domains of the generating cofibrations of  $\mathcal{M}$  are cofibrant, we instead assume that  $\text{Mod-}E'$  is left proper in the covariant case and right proper in the contravariant case. Then the conclusions of Theorem 5.1 remain true.*

The reason for the right proper hypothesis in the contravariant case is that we have left Quillen functors from  $\text{Mod-}E$  to  $(\text{Mod-}E')^{\text{op}}$ , so we need  $(\text{Mod-}E')^{\text{op}}$  to be left proper.

Note that Theorem 5.2 will work with any left proper model structure on  $\text{Mod-}E'$  in which the weak equivalences are the maps that are weak equivalences in  $\mathcal{M}$ .

Also, the model structure on  $\text{Mod-}E'$  that we get from the fact that  $\mathcal{M}$  is strongly cofibrantly generated will be left proper if  $\mathcal{M}$  is so and  $E'$  is cofibrant in  $\mathcal{M}$ , for in this case every cofibration in  $\text{Mod-}E'$  is in particular a cofibration in  $\mathcal{M}$ . This model structure on  $\text{Mod-}E'$  is right proper whenever  $\mathcal{M}$  is, since weak equivalences and fibrations are detected in  $\mathcal{M}$ .

Corresponding to Corollary 4.3, we have the following theorem.

**Theorem 5.3.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, stable, symmetric monoidal model category where the unit is cofibrant. Let  $E$  and  $E'$  be monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  be a left Quillen  $\mathcal{M}$ -functor. Suppose that the structure map*

$$A \otimes FE \rightarrow F(A \otimes E)$$

*is a weak equivalence when  $A$  is a cokernel of one of the generating cofibrations of  $\mathcal{M}$ . Then there is a natural isomorphism*

$$QX \otimes_E FE \rightarrow FQX = (LF)(X)$$

*of functors on  $\text{Ho Mod-}E$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant left Quillen  $\mathcal{M}$ -functor such that the structure map*

$$F(A \otimes E) \rightarrow \text{Hom}(A, FE)$$

*is a weak equivalence when  $A$  is a cokernel of one of the generating cofibrations of  $\mathcal{M}$ , then there is a natural isomorphism*

$$(LF)(X) = FQX \rightarrow \text{Hom}_E(QX, FE)$$

*of functors on  $\text{Ho Mod-}E$ .*

*Proof.* Since the unit  $S$  is cofibrant in  $\mathcal{M}$ ,  $E$  is cofibrant in  $\text{Mod-}E$ . This is the reason we do not need  $QE$  anywhere. The main point of the proof is that if  $\mathcal{M}$  is stable with cofibrant unit, so is  $\text{Mod-}E'$ . Indeed, the suspension functor in  $\text{Ho } \mathcal{M}$  or on  $\text{Ho Mod-}E'$  is the total derived functor of  $S^1 \otimes -$ , where  $S^1$  is the suspension of the unit of  $\mathcal{M}$  (this is where we are assuming the unit is cofibrant). If the suspension functor is an equivalence on  $\text{Ho } \mathcal{M}$ , there is an object  $S^{-1} \in \text{Ho } \mathcal{M}$  so that tensoring with  $S^{-1}$  is the inverse of suspension. But then we can use  $S^{-1}$  to define an inverse of the suspension on  $\text{Ho Mod-}E'$  as well. Thus we can apply Corollary 4.3.  $\square$

Finally, we get the most direct analogue to the original Eilenberg-Watts theorem as a corollary to Theorem 4.4. Here we need to assume  $\mathcal{M}$  is stable with cofibrant unit and  $\text{Ho } \mathcal{M}$  is **monogenic**. This means that the unit  $S$  is a compact weak generator of the triangulated category  $\text{Ho } \mathcal{M}$ . Recall that an object  $X$  of a triangulated category  $\mathcal{T}$  is called **compact** if  $\mathcal{T}(X, -)_*$  preserves coproducts. The object  $X$  is called a **weak generator** if the functor  $\mathcal{T}(X, -)_*$  is faithful on objects, so that  $Y = 0$  in  $\mathcal{T}$  if and only if  $\mathcal{T}(X, Y)_* = 0$ . As we will discuss in the next section, many of the most common stable symmetric monoidal categories are monogenic. The relevance of the monogenic condition to Theorem 4.4 is that if  $X$  is a compact weak generator of a triangulated category  $\mathcal{T}$  with all coproducts, then  $\text{loc } \langle X \rangle = \mathcal{T}$  [HPS97, Theorem 2.3.2].

**Theorem 5.4.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, stable, monogenic, symmetric monoidal model category where the unit is cofibrant. Let  $E$  and  $E'$  be*



monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  be a left Quillen  $\mathcal{M}$ -functor. Then there is a natural isomorphism

$$QX \otimes_E FE \rightarrow FQX = (LF)(X)$$

of functors on  $\text{Ho Mod-}E$ . Similarly, if  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is a contravariant left Quillen  $\mathcal{M}$ -functor, then there is a natural isomorphism

$$(LF)(X) = FQX \rightarrow \text{Hom}_E(QX, FE)$$

of functors on  $\text{Ho Mod-}E$ .

*Proof.* As discussed in the proof of Theorem 5.3, the fact that  $\mathcal{M}$  is stable with cofibrant unit  $S$  means that  $\text{Mod-}E$  is stable for all monoids  $E$ . We claim that  $E$  is in fact a compact weak generator of the category  $\text{Ho Mod-}E$ . For this, let  $U$  denote the forgetful functor from  $\text{Mod-}E$  to  $\mathcal{M}$ , so that  $U$  preserves and detects weak equivalences. Since  $U$  preserves weak equivalences, its total right derived functor  $RU$  is just  $U$  itself; that is,  $(RU)X = UX$ . Hence  $RU$  will also preserve coproducts. Thus

$$\begin{aligned} \text{Ho Mod-}E(E, \coprod X_\alpha) &\cong \text{Ho } \mathcal{M}(S, (RU)(\coprod X_\alpha)) \cong \text{Ho } \mathcal{M}(S, \coprod UX_\alpha) \\ &\cong \bigoplus \text{Ho } \mathcal{M}(S, UX_\alpha) \cong \bigoplus \text{Ho Mod-}E(E, X_\alpha). \end{aligned}$$

Thus  $E$  is compact. Similarly, if  $\text{Ho Mod-}E(E, X)_* = 0$ , then  $\text{Ho } \mathcal{M}(S, UX)_* = 0$ , so  $UX$  is weakly equivalent to  $0 = U(0)$ . Since  $U$  detects weak equivalences,  $X$  is weakly equivalent to  $0$ , so  $E$  is a weak generator. As mentioned above, we then get that  $\text{loc } \langle E \rangle = \text{Ho Mod-}E$  by the proof of [HPS97, Theorem 2.3.2] (the statement of that theorem makes some irrelevant assumptions about a tensor product). Since  $(L\tau)_E$  is an isomorphism, Theorem 4.4 completes the proof.  $\square$

## 6. EXAMPLES

In this section, we prove Theorem 0.5 by applying our versions of the Eilenberg-Watts theorem to the standard model categories of symmetric spectra, chain complexes, simplicial sets, and topological spaces.

We begin with the symmetric spectra of [HSS00], based on simplicial sets. This is a symmetric monoidal model category (under the smash product) whose homotopy category is the standard stable homotopy category of algebraic topology, so it is stable and monogenic. It is cofibrantly generated, it satisfies the monoid axiom, and every object is small, so it is strongly cofibrantly generated by 3.3 of Schwede and Shipley. The unit  $S$  is cofibrant in symmetric spectra, making symmetric spectra easier for us to handle than the  $S$ -modules of [EKMM97]. A monoid in symmetric spectra is frequently called a **symmetric ring spectrum**, and the homotopy category of symmetric ring spectra is equivalent to any other homotopy category of  $A_\infty$  ring spectra [MMSS01]. A model category that is enriched over symmetric spectra is frequently called a **spectral model category**, and an enriched functor is called a **spectral functor**.

Theorem 5.4 then gives an Eilenberg-Watts theorem for symmetric ring spectra.

**Theorem 6.1.** *Symmetric spectra are homotopically self-contained.*

There are similar theorems for symmetric spectra based on topological spaces and for the orthogonal spectra of [MMSS01]. The only (slight) subtlety is that no nontrivial object of the category is small with respect to all maps (although even

that problem can be avoided by using the  $\Delta$ -generated spaces of Jeff Smith, which are supposed to be a locally presentable category), so one has to take a little care in proving that the model categories in question are strongly cofibrantly generated.

There is also a similar theorem for the  $S$ -modules of [EKMM97], even though the unit  $S$  is not cofibrant in that case. The main issue here is we need enough control over the unit to be sure that  $\text{Mod-}E$  is stable and that  $E$  is a small weak generator of  $\text{Ho Mod-}E$ , for any  $S$ -algebra  $E$ . One cannot point directly to a theorem in [EKMM97] that says this, but it does follow from the results Chapter III. Proposition III.1.3 is especially relevant.

We now turn to chain complexes. Here the base symmetric monoidal model category is the category of unbounded chain complexes of abelian groups  $\text{Ch}(\mathbb{Z})$ , with the projective model structure [Hov99, Section 2.3]. This is a cofibrantly generated model category satisfying the monoid axiom, in which every object is small. It is therefore strongly cofibrantly generated by Theorem 3.3. It is also stable and monogenic, and the unit is cofibrant. Thus Theorem 5.4 applies. A monoid in this category is a differential graded algebra and an enriched functor is often called a DG-functor.

**Theorem 6.2.**  *$\text{Ch}(\mathbb{Z})$  is homotopically self-contained.*

In fact, Theorem 6.2 is actually a special case of [Kel94, Section 6.4], where Keller proves that any DG-functor  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  that commutes with direct sums has  $QX \otimes_E FE \cong (LF)(X)$ . That is, he does not assume that  $F$  is left Quillen. In the special case when  $E'$  is an ordinary ring, we can recover Keller's result, and this is a worthwhile exercise, as it illustrates that one can often use the methods of Theorem 5.4 to get tighter results than one would at first expect. The proof below likely works for arbitrary DG-algebras as well.

We begin with a useful general lemma.

**Lemma 6.3.** *Suppose  $\mathcal{M}$  is a closed symmetric monoidal, strongly cofibrantly generated, model category in which the unit  $S$  is cofibrant,  $E, E'$  are monoids in  $\mathcal{M}$ , and  $F: \text{Mod-}E \rightarrow \text{Mod-}E'$  is an  $\mathcal{M}$ -functor. Then  $F$  preserves the homotopy relation on maps between cofibrant and fibrant objects, so preserves weak equivalences between cofibrant and fibrant objects.*

Note that this means we could define a derived functor  $DF$  for any  $\mathcal{M}$ -functor  $F$  via  $(DF)(X) = F(QRX)$ , where  $R$  denotes fibrant replacement. However, this would be neither a left nor a right derived functor in general, and does not seem to have good properties without further assumptions on  $F$ .

*Proof.* Let  $I$  be a cylinder object in  $\mathcal{M}$  for the unit  $S$ . Then  $I \otimes X$  is a cylinder object for any cofibrant  $E$ -module  $X$ . In particular, if  $Y$  is a fibrant  $E$ -module, and  $f, g: X \rightarrow Y$  are homotopic, then there is a homotopy  $I \otimes X \rightarrow Y$  between  $f$  and  $g$ . This corresponds to a map  $I \rightarrow \text{Hom}_E(X, Y)$  in  $\mathcal{M}$ . Since  $F$  is an  $\mathcal{M}$ -functor, we get an induced map  $I: \text{Hom}_{E'}(FX, FY)$ , which is a homotopy between  $Ff$  and  $Fg$ .  $\square$

**Theorem 6.4.** *[Keller] Suppose  $R$  and  $S$  are ordinary rings, and  $F: \text{Ch}(R) \rightarrow \text{Ch}(S)$  is a DG-functor that commutes with arbitrary coproducts. Then there is a natural isomorphism*

$$L\tau: QX \otimes_E FE \rightarrow FQX = (LF)(X)$$

of functors on  $\mathcal{D}(R)$ . Similarly, if  $F: Ch(R) \rightarrow Ch(S)$  is a contravariant DG-functor that converts coproducts to products, then there is a natural isomorphism

$$(LF)(X) = FQX \rightarrow \text{Hom}_E(QX, FE)$$

of functors on  $\mathcal{D}(R)$ .

*Proof.* Lemma 6.3 tells us that a DG-functor preserves chain homotopy. Because every object is fibrant, then, any DG-functor  $G$  has a left derived functor  $(LG)(X) = G(QX)$ , where  $QX$  is a cofibrant (DG-projective) replacement for  $X$  (since weak equivalences between DG-projective objects are chain homotopy equivalences). This left derived functor is automatically exact on  $\mathcal{D}(R)$ , as pointed out to the author by Keller. Indeed, we can use the injective model structure on  $\mathcal{D}(S)$  [Hov99, Theorem 2.3.13], in which cofibrations are degreewise split monomorphisms. As a functor to this model structure, a DG-functor like  $F$  automatically preserves cofibrations, and hence  $LF$  is exact. To see this, we show that  $F$  preserves degreewise split monomorphisms. Indeed, if  $f: X \rightarrow Y$  is a degreewise split monomorphism, there is a  $g \in \text{Hom}_R(Y, X)_0$  (which is, of course, not a cycle unless  $f$  is actually split) such that  $gf$  is the identity. Applying  $F$ , we see that  $Ff$  is also degreewise split. Altogether then,  $L\tau$  is a natural transformation between exact coproduct-preserving functors on  $\mathcal{D}(R)$  from that is an isomorphism on  $R$ . It is therefore an isomorphism on the localizing subcategory generated by  $R$ , which is  $\mathcal{D}(R)$ .  $\square$

We now consider simplicial sets, which are of course not stable. An excellent description of the model structure on the category  $\mathbf{SSet}$  of simplicial sets can be found in [GJ99]; there is also a description in [Hov99, Chapter 3]. We find that  $\mathbf{SSet}$  is a strongly cofibrantly generated, closed symmetric monoidal (under the product) model category in which every object is cofibrant. (The fact that every object is cofibrant makes the monoid axiom automatic, hence  $\mathbf{SSet}$  is strongly cofibrantly generated). We can therefore apply Theorem 5.1. The set  $I$  of generating cofibrations consists of the maps  $\partial\Delta[n] \rightarrow \Delta[n]$ . We note that the vertex  $n$  is a simplicial deformation retract of  $\Delta[n]$  [Hov99, Lemma 3.4.6], though no other vertex is.

**Theorem 6.5.** *Simplicial sets are homotopically self-contained.*

Pointed simplicial sets are also homotopically self-contained, and the proof is very similar.

*Proof.* We just prove the covariant case, as the contravariant case is similar. In view of Theorem 5.1, we have to show that

$$\phi_A: A \times FE \rightarrow F(A \times E)$$

is a weak equivalence for  $A = \partial\Delta[n]$  and  $A = \Delta[n]$ . For  $A = \Delta[n]$ , we have a commutative diagram

$$\begin{array}{ccc} \Delta[n] \times FE & \longrightarrow & F(\Delta[n] \times E) \\ \downarrow & & \downarrow \\ * \times FE & \longrightarrow & F(* \times E) \end{array}$$

where the vertical maps are induced by the simplicial homotopy equivalence  $\Delta[n] \rightarrow *$  that collapses  $\Delta[n]$  onto the vertex  $n$ . The same proof as that of Lemma 6.3 implies that  $F$  preserves simplicial homotopy equivalences. Thus the vertical maps

are simplicial homotopy equivalences, and the the bottom horizontal map is an isomorphism, so the top map is a weak equivalence as required.

We prove that  $\phi_{\partial\Delta[n]}$  is a weak equivalence by induction on  $n$ . The case  $n = 0$  is trivial, and the case  $n = 1$  is straightforward since  $\partial\Delta[1]$  is the coproduct of two copies of  $\Delta[0]$ , and  $F$  preserves coproducts. Now suppose that  $\phi_{\partial\Delta[n]}$  is a weak equivalence. We first show that  $\phi_{\Delta[n]/\partial\Delta[n]}$  is a weak equivalence using the cube lemma [Hov99, Lemma 5.2.6]. Indeed, we have two pushout squares

$$\begin{array}{ccc} \partial\Delta[n] \times FE & \longrightarrow & \Delta[n] \times FE \\ \downarrow & & \downarrow \\ * \times FE & \longrightarrow & \Delta[n]/\partial\Delta[n] \times FE \end{array}$$

and

$$\begin{array}{ccc} F(\partial\Delta[n] \times E) & \longrightarrow & F(\Delta[n] \times E) \\ \downarrow & & \downarrow \\ F(* \times E) & \longrightarrow & F(\Delta[n]/\partial\Delta[n] \times E) \end{array}$$

of cofibrant objects, where the top horizontal maps are cofibrations. The map  $\phi$  defines a map from the first pushout square to the second, which is a weak equivalence at every spot except the lower right corner. The cube lemma tells us that it is also a weak equivalence at the lower right corner.

Now, there is a map  $g: \partial\Delta[n+1] \rightarrow \Delta[n]/\partial\Delta[n]$  of simplicial sets, which is a weak equivalence. It is easier to explain this map geometrically. The geometric realization of  $\partial\Delta[n+1]$  is a triangulation of the  $n$ -sphere, and the geometric realization of  $\Delta[n]/\partial\Delta[n]$  is the usual CW description of an  $n$ -sphere, with one point and one  $n$ -cell. What we want to do is to take one face of  $\partial\Delta[n+1]$  and spread it out over the whole  $n$ -cell, sending everything else in  $\partial\Delta[n+1]$  to the basepoint. This is obviously a homotopy equivalence. We can realize it simplicially by sending each  $k$ -simplex of  $\partial\Delta[n+1]$  for  $k \leq n$  except  $123 \cdots n$  to the simplex represented by  $k$  0's, and sending  $123 \cdots n$  to the unique nondegenerate  $n$  simplex of  $\Delta[n]/\partial\Delta[n]$ .

We then get the commutative diagram below.

$$\begin{array}{ccc} \partial\Delta[n+1] \times FE & \longrightarrow & F(\partial\Delta[n+1] \times E) \\ g \times FE \downarrow & & \downarrow F(g \times E) \\ \Delta[n]/\partial\Delta[n] \times FE & \longrightarrow & F(\Delta[n]/\partial\Delta[n] \times E) \end{array}$$

The bottom horizontal map is a weak equivalence, as we have seen. The vertical maps are also weak equivalences, because the map  $g$  is a weak equivalence between cofibrant objects, and all functors involved preserve those (in particular,  $FE$  is cofibrant in  $\text{Mod-}E'$  since  $E$  is cofibrant in  $\text{Mod-}E$ , so the product with  $FE$  preserves such weak equivalences). Hence the top horizontal map is a weak equivalence, completing the induction step and the proof.  $\square$

We now consider the case when our base model category is topological spaces. Of course, we need a closed symmetric monoidal category of topological spaces; for definiteness, we choose the compactly generated weak Hausdorff spaces used in [EKMM97], and refer to this category as **Top**. The model structure on **Top** is described in [Hov99, Section 2.4], and it is strongly cofibrantly generated with

generating cofibrations  $S^{n-1} \rightarrow D^n$  for all  $n \geq 0$ . A **Top**-functor is usually called a continuous functor.

**Theorem 6.6.** *Topological spaces are homotopically self-contained.*

Again, we find similarly that pointed topological spaces are homotopically self-contained as well.

We note that of course the most obvious topological monoid is a topological group  $G$ , in which case we are talking about  $G$ -spaces. The model structure we are using on  $G$ -spaces is the one in which the weak equivalences are equivariant maps which are underlying weak equivalences. We would like to be using the complete model structure, where a map  $f$  is a weak equivalence if and only the induced map  $f^H$  on  $H$ -fixed points is a weak equivalence for all subgroups  $H$  (in some family, perhaps). Our methods would apply to this case, except for one point. We get the natural transformation

$$X \times_G FG \rightarrow FX$$

but it is not clear that the left hand functor would ever be a left Quillen functor.

*Proof.* The proof is precisely analogous to that of Theorem 6.5. Theorem 5.1 tells us that we have to show that

$$\phi_A: A \times FE \rightarrow F(A \times E)$$

is a weak equivalence for  $A = S^{n-1}$  and  $A = D^n$ . The space  $D^n$  is contractible, and continuous functors preserve homotopy by Lemma 6.3, so we can use the same argument as in Theorem 6.5 for  $A = D^n$ . We can use induction on  $n$  for  $A = S^{n-1}$ , just as in Theorem 6.5, and it is even easier, as  $D^n/S^{n-1}$  is homeomorphic to  $S^n$ .  $\square$

## 7. BROWN REPRESENTABILITY

In this section, we point out that our results are relevant to Brown representability of homology and cohomology theories. For simplicity, we stick to the stable case.

Recall, then, that if  $\mathcal{T}$  is a triangulated category, a **homology functor** is an exact, coproduct-preserving functor  $h: \mathcal{T} \rightarrow \mathcal{A}$  to some abelian category  $\mathcal{A}$ . We will refer to the graded version  $h_*$  of  $h$ , defined by  $h_n(X) = h(\Sigma^n X)$ , as the associated **homology theory**. When considering a homology theory, we need to consider the isomorphisms  $h_n(X) \cong h_{n+1}(\Sigma X)$  as part of the data. Similarly, a **cohomology functor** is an exact, contravariant functor  $\mathcal{T} \rightarrow \mathcal{A}$  that converts coproducts to products, and we have a similar induced cohomology theory. A cohomology functor  $h$  is **representable** if there is a natural isomorphism

$$h(X) \cong \mathcal{T}(X, Y)$$

for some object  $Y$  of  $\mathcal{T}$ , and we say that **Brown representability for cohomology functors** holds if every cohomology functor is representable. This is true in considerable generality; see [Nee01, Proposition 8.4.2], for example.

Representability for homology functors is much more complicated. Even understanding what it means for a homology functor to be representable is not obvious. Since we will be working with triangulated categories of the form  $\text{Ho Mod-}E$ , for  $E$  a monoid in a strongly cofibrantly generated, closed symmetric monoidal stable

monogenic model category  $\mathcal{M}$ , the natural definition for us is that a homology functor  $h$  is **representable** if there is a natural isomorphism

$$h(X) \cong \mathrm{Ho} \mathcal{M}(S, X \otimes_E^L Y)$$

for some left  $E$ -module  $Y$ , where  $X \otimes_E^L Y$  denotes the derived tensor product and  $S$  denotes the unit of  $\mathcal{M}$ . Note that this is much more subtle; for example, there is no reason to think that a morphism between representable homology theories must be itself representable by a map between the representing objects.

Of course, homology functors, and natural transformations between them, on the stable homotopy category are representable. The same is true for  $\mathcal{D}(R)$  for countable rings  $R$ . However, Christensen, Keller, and Neeman proved in [CKN01] that there are rings  $R$  for which not every homology theory on  $\mathcal{D}(R)$  is representable. Before that, Beligiannis [Bel00] had proved that natural transformations between representable homology functors on  $\mathcal{D}(R)$  need not be representable.

We can use our versions of the Eilenberg-Watts theorem to partially salvage Brown representability for homology theories.

**Definition 7.1.** Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, stable, monogenic, symmetric monoidal model category where the unit  $S$  is cofibrant. Let  $E$  be a monoid in  $\mathcal{M}$ , and  $h: \mathrm{Ho} \mathrm{Mod}\text{-}E \rightarrow \mathrm{Mod}\text{-}A$  be a homology (resp. cohomology) functor, where  $A$  is an ordinary ring. We say that  $h_*$  has a **strict model** if there is a monoid  $E'$  in  $\mathcal{M}$  with  $\mathrm{Ho} \mathcal{M}(S, E')_* \cong A_*$  as rings, a left Quillen  $\mathcal{M}$ -functor (resp. a contravariant left Quillen  $\mathcal{M}$ -functor)  $F: \mathrm{Mod}\text{-}E \rightarrow \mathrm{Mod}\text{-}E'$ , and a natural isomorphism

$$\rho: h(X) \cong \mathrm{Ho} \mathrm{Mod}\text{-}E'(E', FQX) \cong \mathrm{Ho} \mathcal{M}(S, FQX)$$

of  $A$ -modules

Here, then, is our version of Brown representability, which follows immediately from Theorem 5.4.

**Theorem 7.2.** *Suppose  $\mathcal{M}$  is a strongly cofibrantly generated, stable, monogenic, symmetric monoidal model category where the unit  $S$  is cofibrant. Let  $E$  be a monoid in  $\mathcal{M}$ , and  $h: \mathrm{Ho} \mathrm{Mod}\text{-}E \rightarrow \mathrm{Mod}\text{-}A$  be a homology (resp. cohomology) theory with a strict model  $F: \mathrm{Mod}\text{-}E \rightarrow \mathrm{Mod}\text{-}E'$ . Then  $h$  is representable. More precisely, in the homology case, we have a natural isomorphism*

$$h(X) \cong \mathrm{Ho} \mathrm{Mod}\text{-}E(E, QX \otimes_E FE),$$

and in the cohomology case, we have a natural isomorphism

$$h(X) \cong \mathrm{Ho} \mathrm{Mod}\text{-}E(QX, FE).$$

We point out that, although the hypotheses in Theorem 7.2 are much stronger than in usual forms of Brown representability, the conclusion is also stronger. Typically, Brown representability theorems just say that a cohomology functor  $h$  is representable by an object of  $\mathrm{Ho} \mathrm{Mod}\text{-}E$ , but this theorem says that the representing object is actually an  $E \otimes E'$ -module.

In practice, we usually do not need to assume quite so much to get a Brown representability theorem. We will illustrate this in the case of chain complexes. The main point is that every object is fibrant in  $\mathrm{Ch}(\mathbb{Z})$ . This means that, if  $F: \mathrm{Ch}(R) \rightarrow \mathrm{Ch}(S)$  is a DG-functor, then  $F$  preserves weak equivalences between cofibrant objects (see Lemma 6.3), and so has a left derived functor  $(LF)(X) = F(QX)$ . We

remind the reader that we can think of suspension in  $\mathcal{D}(R)$  as  $\Sigma X = S^1 \otimes QX$ , where  $S^1$  is the complex which is  $\mathbb{Z}$  in degree 1 and 0 elsewhere.

**Definition 7.3.** Suppose  $R$  and  $S$  are rings, and  $h_*: \mathcal{D}(R) \rightarrow \text{Mod-}S$  is a homology (resp. cohomology) theory. We say that  $h_*$  has a **chain model** if there is a DG-functor  $F: \text{Ch}(R) \rightarrow \text{Ch}(S)$  and a natural isomorphism

$$\rho: h_*(X) \cong H_*(FQX)$$

of  $S$ -modules that is compatible with the suspension. To explain this, we assume  $h_*$  is a homology theory and leave the evident modifications in the cohomology case to the reader. To say that  $\rho$  is compatible with the suspension means that the isomorphism  $h_n(X) \cong h_{n+1}(\Sigma X)$  corresponds to the composite

$$H_n(FQX) \cong H_{n+1}(S^1 \otimes FQX) \rightarrow H_{n+1}(F(S^1 \otimes QX)) \cong H_{n+1}(FQ\Sigma X),$$

where the last isomorphism comes from the fact that  $F$  preserves the homology isomorphism

$$Q\Sigma X = Q(S^1 \otimes QX) \rightarrow S^1 \otimes QX$$

between cofibrant objects. We remind the reader

**Theorem 7.4.** *Suppose  $R$  and  $S$  are rings and  $h_*: \mathcal{D}(R) \rightarrow (\text{Mod-}S)_*$  is a homology or cohomology theory with a chain model  $F: \text{Ch}(R) \rightarrow \text{Ch}(S)$ . Then  $h_*$  is representable. More precisely, in the homology case, there is a natural isomorphism*

$$h_*(X) \cong H_*(QX \otimes_E FE)$$

and in the cohomology case there is a natural isomorphism

$$h^*(X) \cong \mathcal{D}(R)(QX, FE)^*.$$

Again we remind the reader that the representing object for  $h_*$  in the above theorem is a complex of  $R$ - $S$ -bimodules, and for  $h^*$  it is a complex of  $R \otimes S$ -modules. The usual Brown representability theorems, when they apply, would just give an action of  $S$  on the representing object up to homotopy.

*Proof.* We assume  $h_*$  is a homology theory, and leave the modifications in the cohomology case to the reader. Since  $F$  is a DG-functor, we have a natural transformation (by Proposition 1.1)

$$\tau: X \otimes_E FE \rightarrow FX.$$

Since both the domain and target of this natural transformation preserve weak equivalences between cofibrant objects, there is a derived natural transformation

$$L\tau: QX \otimes_E FE \rightarrow (LF)(X) = F(QX).$$

This natural transformation is an isomorphism when  $X = S^0R$ . As in the proof of Theorem 5.4, the localizing subcategory generated by  $S^0R$  is all of  $\mathcal{D}(R)$ . If we knew  $F$  were a left Quillen functor, we could then use Theorem 4.4 to conclude that  $L\tau$  is an isomorphism for all  $X$ . Instead, we use the fact that  $h$  is a homology theory to conclude that  $LF$  is exact and preserves coproducts and suspensions, giving the desired result.

To see that  $LF$  preserves coproducts, we just note that the natural map

$$\coprod_i (LF)X_i \rightarrow (LF)\left(\coprod_i X_i\right)$$

becomes an isomorphism on applying  $H_*$ , and is therefore an isomorphism in  $\mathcal{D}(S)$ . Because  $F$  is a DG-functor, there is a natural map

$$S^1 \otimes QFQX \rightarrow F(S^1 \otimes QX),$$

and our hypothesis that the isomorphism

$$h_*(X) \cong H_*(FQX)$$

is compatible with the suspension guarantees that this is a homology isomorphism, so is an isomorphism in  $\mathcal{D}(S)$ . Hence  $LF$  commutes with the suspension.

To see that  $LF$  preserves exact triangles, we need to recall a little more about exact triangles in  $\mathcal{D}(R)$ . Any such exact triangle comes from a cofibration  $f: X \rightarrow Y$  of cofibrant objects and is isomorphic in  $\mathcal{D}(R)$  to the sequence

$$X \xrightarrow{f} Y \rightarrow C(f) \rightarrow \Sigma X$$

where  $\text{Cyl}(f)$  is the mapping cylinder of  $f$  and  $C(f)$  is the mapping cone. So  $\text{Cyl}(f)$  is the pushout of  $Y \xleftarrow{f} X \xrightarrow{i_1 \otimes X} I \otimes X$ , where  $I$  is the chain complex previously mentioned, with  $\mathbb{Z}$  in degree 1 and  $\mathbb{Z} \oplus \mathbb{Z}$  in degree 0. The map  $i_1$  hits the ‘‘right endpoint’’ copy of  $\mathbb{Z}$  in degree 0. There is a chain homotopy equivalence  $\text{Cyl}(f) \rightarrow Y$  corresponding to ‘‘stepping on the cylinder’’. Then  $Cf$  is the quotient of the composite

$$X \xrightarrow{i_0 \otimes 1} I \otimes X \rightarrow \text{Cyl}(f),$$

so that  $(Cf)_n = Y_n \oplus X_{n-1}$ , and the map  $Cf \rightarrow \Sigma X$  is the quotient of  $Y \rightarrow C(f)$ .

We will show that there is a commutative diagram

$$\begin{array}{ccccccc} FX & \longrightarrow & \text{Cyl}(Ff) & \longrightarrow & C(Ff) & \longrightarrow & \Sigma FX \\ & & \parallel & & \downarrow & & \downarrow \\ FX & \longrightarrow & F(\text{Cyl } f) & \longrightarrow & F(Cf) & \longrightarrow & F(\Sigma X) \end{array}$$

in which the vertical maps are homology isomorphisms. The rightmost vertical map comes from the fact that  $F$  is a  $\text{Ch}(\mathbb{Z})$ -functor, and we have already seen that this map is a homology isomorphism (for DG-projective  $X$ ). The map  $\text{Cyl}(Ff) \rightarrow F(\text{Cyl } f)$  also exists because  $F$  is a  $\text{Ch}(\mathbb{Z})$  functor; it is the map  $FY \rightarrow F(\text{Cyl } f)$  on  $FY$  and the map

$$I \otimes FX \rightarrow F(I \otimes X) \rightarrow FY$$

on  $I \otimes FX$ . Since  $\text{Cyl}(f) \rightarrow Y$  is a chain homotopy equivalence, so is  $F(\text{Cyl } f) \rightarrow FY$ . Of course,  $\text{Cyl}(Ff) \rightarrow FY$  is also a chain homotopy equivalence, from which we conclude that  $\text{Cyl}(Ff) \rightarrow F(\text{Cyl } f)$  is a homology isomorphism.

By taking quotients with some care, using the fact that  $F$  is a  $\text{Ch}(\mathbb{Z})$ -functor again, we get an induced map  $C(Ff) \rightarrow F(Cf)$  making the diagram above commute on both sides. Now, the top row has a long exact sequence in homology, and the fact that  $h_*$  is a homology theory means that the bottom row also has a long exact sequence in homology. The 5-lemma then implies that the map  $C(Ff) \rightarrow F(Cf)$  is a homology isomorphism, completing the proof that  $LF$  is a coproduct-preserving triangulated functor.  $\square$

We point out that there is a similar theorem to Theorem 7.2 for morphisms between homology and cohomology functors with a strict model. If such a morphism



is induced by an  $\mathcal{M}$ -natural transformation of the strict model, then it is representable as a morphism between the representing objects. There is also an analog of Theorem 7.4 for morphisms.

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