ON TRIVIALITY OF DICKSON INVARIANTS IN THE HOMOLOGY OF THE STEENROD ALGEBRA

NGUYỄN H. V. HƯNG

Dedicated to Professor Nguyễn Duy Tiến on the occasion of his sixtieth birthday

ABSTRACT. Let $\mathcal{A}$ be the mod 2 Steenrod algebra and D_k the Dickson algebra of k variables. We study the Lannes-Zarati homomorphisms

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^*\,,$$

which correspond to an associated graded of the Hurewicz map $H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)$. An algebraic version of the long-standing conjecture on spherical classes predicts that $\varphi_k = 0$ in positive stems, for $k > 2$. That the conjecture is no longer valid for $k = 1$ and 2 is respectively an exposition of the existence of Hopf invariant one classes and Kervaire invariant one classes.

This conjecture has been proved for $k = 3$ in [9]. It has been shown that φ_k vanishes on decomposable elements for $k > 2$ in [14] and on the image of Singer's algebraic transfer for $k > 2$ in [9] and [12]. In this paper, we establish the conjecture for $k = 4$. To this end, our main tools include (1) an explicit chain-level representation of φ_k and (2) a squaring operation Sq^0 on $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$, which commutes with the classical Sq^0 on $Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati homomorphism.

1. Introduction and statement of results

Let $H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0)$ be the Hurewicz homomorphism of the basepoint component Q_0S^0 in the infinite loop space $QS^0 = \lim_n \Omega^n S^n$. Here and throughout the paper, homology and cohomology are taken with coefficients in $\mathbb{F}_2$, the field of two elements. The long-standing conjecture on spherical classes states as follows: *Only the classes of Hopf invariant one and those of Kervaire invariant one are detected by the Hurewicz homomorphism.* (See Curtis [6], Snaith and Tornehave [26] and Wellington [27] for a discussion.)

An algebraic version of this problem, which we are interested in, goes as follows. Let $P_k = \mathbb{F}_2[x_1, \dots, x_k]$ be the polynomial algebra on k generators $x_1, \dots, x_k$, each of degree 1. Let the general linear group $GL_k = GL(k, \mathbb{F}_2)$ and the mod 2 Steenrod algebra $\mathcal{A}$ both act on P_k in the usual way. The Dickson algebra of k variables, D_k, is the algebra of invariants

$$D_k := \mathbb{F}_2[x_1, \dots, x_k]^{GL_k}\,.$$

[1]The research was supported in part by Johns Hopkins University and the National Research Program, Grant $N^0$140801.

[2]2000 *Mathematics Subject Classification.* Primary 55P47, 55Q45, 55S10, 55T15.

[3]*Key words and phrases.* Spherical classes, Loop spaces, Adams spectral sequences, Steenrod algebra, lambda algebra, Invariant theory, Dickson algebra.

Since the action of $\mathcal{A}$ and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathcal{A}$. In [17], Lannes and Zarati construct homomorphisms

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^* \,,$$

which correspond to an associated graded of the Hurewicz map. The proof of this assertion is unpublished, but it is sketched by Lannes [16] and by Goerss [8]. The Hopf invariant one and the Kervaire invariant one classes are respectively represented by certain permanent cycles in $Ext_{\mathcal{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $Ext_{\mathcal{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$, on which φ_1 and φ_2 are non-zero (see Adams [1], Browder [5], Lannes-Zarati [17]). Therefore, we are led to the following conjecture.

Conjecture 1.1. $\varphi_k = 0$ in any positive stem i for $k > 2$.

The conjecture has been proved for $k = 3$ in [9] and for $k = 4$ in a range of stems in [14]. It has been shown that φ_k vanishes on decomposable elements for $k > 2$ in [14] and on the image of Singer's algebraic transfer $Tr_k : ((\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k})^* \to Ext_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$ in [9] and [12].

The following is the main result of the present paper.

Theorem 1.2. $\varphi_4 = 0$ *in positive stems.*

An ingredient in our proof of this theorem is the squaring operation Sq^0 on $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$, which is defined in our paper [9]. The key step in the proof is to show the following theorem.

Theorem 1.3. *The squaring operation Sq^0 on $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$ commutes with the classical squaring operation Sq^0 on $Ext_{\mathcal{A}}^{k}(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati homomorphism φ_k, for any k.*

Applying this theorem, we get a proof of Theorem 1.2 by combining the computation of $Ext_{\mathcal{A}}^{4}(\mathbb{F}_2, \mathbb{F}_2)$ by W. H. Lin [18] and that of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_4$ by the author and Peterson [13].

In order to prove Theorem 1.3, we need to exploit Singer's invariant-theoretic description of the lambda algebra [24]. According to Dickson [7], one has

$$D_k \cong \mathbb{F}_2[Q_{k,k-1}, ..., Q_{k,0}],$$

where $Q_{k,i}$ denotes the Dickson invariant of degree $2^k - 2^i$. Singer sets $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defines $\Gamma_k^\wedge$ to be a certain "not too large" submodule of Γ_k. He also equips $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ with a differential $\partial : \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ and a coproduct. Then, he shows that the differential coalgebra $\Gamma^\wedge$ is dual to the (opposite) lambda algebra of the six authors of [4]. Thus, $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. (Originally, Singer uses the notation Γ_k^+ to denote $\Gamma_k^\wedge$. However, by D_k^+, $\mathcal{A}^+$ we always mean the submodules of D_k and $\mathcal{A}$ respectively consisting of all elements of positive degrees, so Singer's notation Γ_k^+ would make a confusion in this paper. Therefore, we prefer the notation $\Gamma_k^\wedge$.)

The following result plays a key role in our proof of Theorem 1.3.

Theorem 1.4. ([11]) *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i \to Tor_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

By this theorem, Conjecture 1.1 is equivalent to our conjecture on the triviality of Dickson invariants in the homology of the Steenrod algebra:

Conjecture 1.5. ([10]) Let D_k^+ denote the submodule of all positive degree elements in D_k. If $q \in D_k^+$, then $[q] = 0$ in $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$.

Therefore, Theorem 1.2 can be restated as follows.

Theorem 1.6. *Every positive-degree Dickson invariant of four variables represents the* 0 *class in the homology,* $Tor_*^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$*, of the Steenrod algebra.*

Also, the theorem that φ_k vanishes on the image of the (Singer) algebraic transfer $Tr_k : ((\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k})^* \to Ext_{\mathcal{A}}^k(\mathbb{F}_2, \mathbb{F}_2)$ for $k > 2$ is restated as follows: *Every positive-degree Dickson invariant of* k *variables represents a class in the kernel of the algebraic transfer's dual* $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$ *for* $k > 2$ (see [10], [12]). It should be noted that the algebraic transfer is computationally showed to be highly nontrivial by Singer [25] and by Boardman [3].

The paper contains four sections. Section 2 is a recollection on modular invariant theory. Its goal is to make the paper self-contained by recalling Singer's invariant-theoretic description of the lambda algebra and our chain-level representation of the Lannes-Zarati dual map. Section 3 and Section 4 are respectively devoted to the proofs of Theorem 1.3 and Theorem 1.2.

Acknowledgment: The present paper was written during my visit to the Johns Hopkins University, Maryland (USA), in the Spring semester 2001. I would like to express my warmest thanks to Mike Boardman, Jean-Pierre Meyer, Jack Morava and Steve Wilson for their hospitality and for providing me with ideal working atmosphere and conditions.

2. Recollection on modular invariant theory

The purpose of this section is to make the paper self-contained. First, we summarize Singer's invariant-theoretic description of the lambda algebra.

Let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular $k \times k$-matrices with 1 on the main diagonal. The T_k-invariant ring, $M_k = P_k^{T_k}$, is called the Mùi algebra. In [22], Mùi shows that

$$P_k^{T_k} = \mathbb{F}_2[V_1, ..., V_k],$$

where

$$V_i = \prod_{c_j \in \mathbb{F}_2} (c_1 x_1 + \cdots + c_{i-1} x_{i-1} + x_i).$$

Then, the Dickson invariant $Q_{k,i}$ can inductively be defined by

$$Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i},$$

where, by convention, $Q_{k,k} = 1$ and $Q_{k,i} = 0$ for $i < 0$.

Let $S(k) \subset P_k$ be the multiplicative subset generated by all the non-zero linear forms in P_k. Let Φ_k be the localization: $\Phi_k = (P_k)_{S(k)}$. Using the results of Dickson [7] and Mùi [22], Singer notes in [24] that

$$\Delta_k := (\Phi_k)^{T_k} = \mathbb{F}_2[V_1^{\pm 1}, ..., V_k^{\pm 1}],$$

$$\Gamma_k := (\Phi_k)^{GL_k} = \mathbb{F}_2[Q_{k,k-1}, ..., Q_{k,1}, Q_{k,0}^{\pm 1}].$$

Further, he sets

$$v_1 = V_1, \quad v_k = V_k/V_1\cdots V_{k-1} \quad (k\geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}}\cdots v_{k-1}v_k \quad (k\geq 2).$$

Then, he obtains

$$\Delta_k = \mathbb{F}_2[v_1^{\pm 1},...,v_k^{\pm 1}],$$

with $\deg v_i = 1$ for every i.

Singer defines $\Gamma_k^\wedge$ to be the submodule of $\Gamma_k = D_k[Q_{k,0}^{-1}]$ spanned by all monomials $\gamma = Q_{k,k-1}^{i_{k-1}}\cdots Q_{k,0}^{i_0}$ with $i_{k-1},...,i_1 \geq 0, i_0\in\mathbb{Z}$, and $i_0 + \deg\gamma \geq 0$. He also shows in [24] that the homomorphism

$$\begin{aligned} \partial_k : \mathbb{F}_2[v_1^{\pm 1},...,v_k^{\pm 1}] &\to \mathbb{F}_2[v_1^{\pm 1},...,v_{k-1}^{\pm 1}],\\ \partial_k(v_1^{j_1}\cdots v_k^{j_k}) &:= \begin{cases} v_1^{j_1}\cdots v_{k-1}^{j_{k-1}}, & \text{if } j_k = -1,\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

maps $\Gamma_k^\wedge$ to $\Gamma_{k-1}^\wedge$. Moreover, it is a differential on $\Gamma^\wedge = \oplus_k\Gamma_k^\wedge$. This module is bigraded by putting $\text{bideg}(v_1^{j_1}\cdots v_k^{j_k}) = (k, k+\sum j_i)$.

Let Λ be the (opposite) lambda algebra, in which the product in lambda symbols is written in the order opposite to that used in [4]. It is also bigraded by putting (as in [23, p. 90]) $\text{bideg}(\lambda_i) = (1, 1+i)$. Singer proves in [24] that $\Gamma^\wedge$ is a differential bigraded coalgebra, which is dual to the differential bigraded lambda algebra Λ via the isomorphisms

2.1. $$\begin{aligned} \Gamma_k^\wedge &\to \Lambda_k^*,\\ v_1^{j_1}\cdots v_k^{j_k} &\mapsto (\lambda_{j_1}\cdots\lambda_{j_k})^*.\end{aligned}$$

Here the duality $*$ is taken with respect to the basis of admissible monomials of Λ. As a consequence, one gets an isomorphism of bigraded coalgebras

2.2. $$H_*(\Gamma^\wedge) \cong Tor_*^{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2).$$

As stated in Theorem 1.4, we prove in [11] that *the inclusion* $D_k \subset \Gamma_k^\wedge$ *is a chain-level representation of the Lannes–Zarati dual homomorphism*

$$\varphi_k^* : (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i \to Tor_{k,k+i}^{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2).$$

In the remaining part of this section, we recall definition of the classical squaring operation on $Ext_{\mathcal{A}}^*(\mathbb{F}_2,\mathbb{F}_2)$.

Liulevicius was perhaps the first person who noted in [20] that there are squaring operations $Sq^i : Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2,\mathbb{F}_2) \to Ext_{\mathcal{A}}^{k+i,2t}(\mathbb{F}_2,\mathbb{F}_2)$, which share most of the properties with Sq^i on the cohomology of spaces. In particular, $Sq^i(\alpha) = 0$ if $i > k$, $Sq^k(\alpha) = \alpha^2$ for $\alpha \in Ext_{\mathcal{A}}^{k,t}(\mathbb{F}_2,\mathbb{F}_2)$, and the Cartan formula holds for the Sq^i's. However, Sq^0 is not the identity. In fact, Sq^0 can be defined in terms of the lambda algebra as follows:

2.3. $$\begin{aligned} Sq^0 : \Lambda_k &\to \Lambda_k,\\ Sq^0(\lambda_{i_1}\cdots\lambda_{i_k}) &= \lambda_{2i_1+1}\cdots\lambda_{2i_k+1}.\end{aligned}$$

So, by dualizing, the following map

2.4. $$\begin{aligned} Sq_v^0 : \Gamma_k^\wedge &\to \Gamma_k^\wedge,\\ Sq_v^0(v_1^{j_1}\cdots v_k^{j_k}) &= \begin{cases} v_1^{\frac{j_1-1}{2}}\cdots v_k^{\frac{j_k-1}{2}}, & j_1,...,j_k \text{ odd},\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

is a chain-level representation of the dual squaring operation

$$Sq^0_* : Tor^{\mathcal{A}}_k(\mathbb{F}_2, \mathbb{F}_2) \to Tor^{\mathcal{A}}_k(\mathbb{F}_2, \mathbb{F}_2).$$

3. The squaring operations

Given a module M over the dual of the Steenrod algebra $\mathcal{A}_*$, let $P(M)$ denote the submodule of M spanned by all elements annihilated by any operations of positive degrees in $\mathcal{A}_*$.

Let $\mathbb{V}_k$ be an $\mathbb{F}_2$-vector space of dimension k. As is well known, $H^*(B\mathbb{V}_k) \cong P_k$. Then, it is easily seen that $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$ and $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ are respectively dual to $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} (P_k)^{GL_k}$ and $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$.

In [9], we have defined a squaring operation

$$Sq^0 : P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))\,,$$

which is derived from Kameko's squaring operation Sq^0 on $\mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$ (see [15], [3]). We also prove in [9, Proposition 4.2] that these two squaring operations commute with each other through the canonical homomorphism

$$j^*_k : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$$

induced by the identity map on $\mathbb{V}_k$.

The goal of this section is to show that the Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$ commutes with the classical squaring operation Sq^0 on $Ext^k_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ through the Lannes-Zarati map φ_k.

Now we recall the definitions of the above mentioned squaring operations.

As is well known, $H_*(B\mathbb{V}_k)$ is a divided power algebra

$$H_*(B\mathbb{V}_k) = \Gamma(a_1, \dots, a_k)$$

generated by $a_1, \dots, a_k$, each of degree 1, where a_i is dual to $x_i \in H^1(B\mathbb{V}_k)$. Here, the duality is taken with respect to the basis of $H^*(B\mathbb{V}_k)$ consisting of all monomials in $x_1, \dots, x_k$.

In [15] Kameko defines a GL_k-homomorphism

$$\begin{array}{rccc} Sq^0 : & H_*(B\mathbb{V}_k) & \to & H_*(B\mathbb{V}_k), \\ & a_1^{(i_1)} \cdots a_k^{(i_k)} & \mapsto & a_1^{(2i_1+1)} \cdots a_k^{(2i_k+1)}\,, \end{array}$$

where $a_1^{(i_1)} \cdots a_k^{(i_k)}$ is dual to $x_1^{i_1} \cdots x_k^{i_k}$. He shows that Sq^0 maps $PH_*(B\mathbb{V}_k)$ to itself. (See also [2].) The induced homomorphism, which is also denoted by Sq^0,

$$Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} PH_*(B\mathbb{V}_k)$$

is called Kameko's squaring operation.

In [9], we consider the homomorphism

$$Sq^0_D = 1 \underset{GL_k}{\otimes} Sq^0 : \mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k) \to \mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)$$

and show that it sends the primitive part $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$ to itself. The resulting homomorphism will be redenoted by Sq^0 for short:

$$Sq^0 : P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)) \to P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)).$$

The following theorem, which is a re-statement of Theorem 1.3, is the main result of this section.

Theorem 3.1. *For an arbitrary positive integer k, the squaring operation Sq^0 on $P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$ commutes with the classical Sq^0 on $Ext^k_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ through the Lannes-Zarati homomorphism φ_k. In other words, the following diagram commutes:*

$$\begin{array}{ccc} Ext^k_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2) & \xrightarrow{\varphi_k} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)) \\ \downarrow{\scriptstyle Sq^0} & & \downarrow{\scriptstyle Sq^0} \\ Ext^k_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2) & \xrightarrow{\varphi_k} & P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k)) . \end{array}$$

We will prove this theorem by showing its dual version. To this end, let us consider the dual homomorphism of Kameko's one:

$$\begin{aligned} Sq^0_x = Sq^0_* : \mathbb{F}_2[x_1,...,x_k] &\to \mathbb{F}_2[x_1,...,x_k], \\ Sq^0_x(x_1^{j_1}\cdots x_k^{j_k}) &= \begin{cases} x_1^{\frac{j_1-1}{2}}\cdots x_k^{\frac{j_k-1}{2}}, & j_1,...,j_k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

In order to explain the behavior of this homomorphism on modular invariants, we present a homomorphism:

$$\begin{aligned} Sq^0_v : \mathbb{F}_2[V_1,...,V_k] &\to \mathbb{F}_2[V_1,...,V_k], \\ Sq^0_v(v_1^{j_1}\cdots v_k^{j_k}) &= \begin{cases} v_1^{\frac{j_1-1}{2}}\cdots v_k^{\frac{j_k-1}{2}}, & j_1,...,j_k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Obviously, this map coincides with the map in 2.4 on the intersection of their domains.

The two homomorphisms Sq^0_x and Sq^0_v depend on k and, when necessary, will respectively be denoted by $Sq^0_{x,k}$ and $Sq^0_{v,k}$.

Technically, the following proposition is the key point in our proof of Theorem 3.1.

Proposition 3.2. *Sq^0_x coincides with Sq^0_v on $\mathbb{F}_2[V_1,...,V_k]$, for any k.*

This proposition will be shown by means of the following two lemmata, which directly come from the definitions of Sq^0_x and Sq^0_v given above.

Lemma 3.3. (i) *$Sq^0_{x,k}(ab^2) = Sq^0_{x,k}(a)b$, for any $a,b \in \mathbb{F}_2[x_1,...,x_k]$.*

(ii) *$Sq^0_{v,k}(AB^2) = Sq^0_{v,k}(A)B$, for any $A,B \in \mathbb{F}_2[V_1,...,V_k]$.*

Lemma 3.4. (i) *$Sq^0_{x,k}(ax_k) = Sq^0_{x,k-1}(a)$, for any $a \in \mathbb{F}_2[x_1,...,x_{k-1}]$.*

(ii) *$Sq^0_{v,k}(Av_k) = Sq^0_{v,k-1}(A)$, for any $A \in \mathbb{F}_2[V_1,...,V_{k-1}]$.*

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. The proof proceeds by induction on k.

For $k = 1$, since $x_1 = v_1$, we get obviously $Sq^0_{x,1} = Sq^0_{v,1}$.

Let $k > 1$ and suppose inductively that $Sq^0_{x,k-1} = Sq^0_{v,k-1}$. We need to show $Sq^0_{x,k} = Sq^0_{v,k}$. Let $V = V_1^{i_1} \cdots V_k^{i_k}$ be an arbitrary monomial in $M_k = \mathbb{F}_2[V_1, ..., V_k]$. We consider the following two cases.

Case 1: i_k is even. Recall that

$$V_k = Q_{k-1,0}x_k + Q_{k-1,1}x_k^2 + \cdots + Q_{k-1,k-1}x_k^{2^{k-1}}$$

(see Mùi [22, Appendix]). Since $Q_{k-1,0}, ..., Q_{k-1,k-1}, V_1, ..., V_{k-1}$ all do not depend on x_k, we have

$$V = V_1^{i_1} \cdots V_k^{i_k} = \sum_{j_k \text{ even}} x_1^{j_1} \cdots x_k^{j_k},$$

where j_k is even in every monomial of the sum. Therefore, by definition of $Sq^0_{x,k}$,

$$Sq^0_{x,k}(V) = \sum_{j_k \text{ even}} Sq^0_{x,k}(x_1^{j_1} \cdots x_k^{j_k}) = 0.$$

On the other hand, from the expansions of V_i's in terms of v_j's, we get

$$V = V_1^{i_1} \cdots V_k^{i_k} = v_1^{\ell_1} \cdots v_k^{\ell_k},$$

where $\ell_k = i_k$ is even. Hence, by definition of $Sq^0_{v,k}$,

$$Sq^0_{v,k}(V) = Sq^0_{v,k}(v_1^{\ell_1} \cdots v_k^{\ell_k}) = 0.$$

Case 2: $i_k = 2n + 1$. We have

$$\begin{aligned} V &= V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} V_k^{i_k} \\ &= V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} (Q_{k-1,0}x_k + Q_{k-1,1}x_k^2 + \cdots + Q_{k-1,k-1}x_k^{2^{k-1}}) V_k^{2n}. \end{aligned}$$

Since $V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0} x_k V_k^{2n}$ is the only term in the above expansion of V with power of x_k odd, we get

$$Sq^0_{x,k}(V) = Sq^0_{x,k}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0} x_k V_k^{2n}).$$

Note that $V_1, ..., V_{k-1}, Q_{k-1,0}$ all do not depend on x_k. Combining Lemma 3.3, Lemma 3.4 and the inductive hypothesis, we obtain

$$\begin{aligned} Sq^0_{x,k}(V) &= Sq^0_{x,k}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0} x_k) V_k^n && \text{(by Lemma 3.3)} \\ &= Sq^0_{x,k-1}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}) V_k^n && \text{(by Lemma 3.4)} \\ &= Sq^0_{v,k-1}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0}) V_k^n && \text{(by the inductive hypothesis)} \\ &= Sq^0_{v,k}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0} v_k) V_k^n && \text{(by Lemma 3.4)} \\ &= Sq^0_{v,k}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} Q_{k-1,0} v_k V_k^{2n}) && \text{(by Lemma 3.3)} \\ &= Sq^0_{v,k}(V_1^{i_1} \cdots V_{k-1}^{i_{k-1}} V_k^{2n+1}). \end{aligned}$$

The last equality comes from the expansions

$$Q_{k-1,0} v_k = V_1 \cdots V_{k-1} v_k = V_k.$$

The proposition is completely proved.

Now we come back to Theorem 3.1.

Proof of Theorem 3.1. We will show the commutativity of the dual diagram:

$$\begin{array}{ccc} \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k & \xrightarrow{\varphi_k^*} & Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \\ \downarrow{\scriptstyle Sq_*^0} & & \downarrow{\scriptstyle Sq_*^0} \\ \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k & \xrightarrow{\varphi_k^*} & Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)\ . \end{array}$$

This will be obtained from a commutative diagram of appropriate chain-level representations of the homomorphisms in questions.

Indeed, by definition of Sq^0 on $(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^* = P(\mathbb{F}_2 \underset{GL_k}{\otimes} H_*(B\mathbb{V}_k))$, the restriction of Sq_x^0 on D_k is a chain-level representation of $Sq_*^0 : \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k$. On the other hand, from 2.4, the map

$$\begin{aligned} Sq_v^0 : \Gamma_k^{\wedge} &\to \Gamma_k^{\wedge}, \\ Sq_v^0(v_1^{j_1} \cdots v_k^{j_k}) &= \begin{cases} v_1^{\frac{j_1-1}{2}} \cdots v_k^{\frac{j_k-1}{2}}, & j_1, ..., j_k \text{ odd}, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

is a chain-level representation of $Sq_*^0 : Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \to Tor_k^{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$. Now, since $D_k \subset M_k = \mathbb{F}_2[V_1, ..., V_k]$, Proposition 3.2 implies the commutativity of the diagram:

$$\begin{array}{ccc} D_k & \xrightarrow{\subset} & \Gamma_k^{\wedge} \\ \downarrow{\scriptstyle Sq_x^0} & & \downarrow{\scriptstyle Sq_v^0} \\ D_k & \xrightarrow{\subset} & \Gamma_k^{\wedge}\ . \end{array}$$

By Theorem 1.4, the inclusion $D_k \subset \Gamma_k^{\wedge}$ is a chain-level representation of the Lannes-Zarati's dual map φ_k^*. Therefore, the last commutative diagram shows the commutativity of the previous one.

Theorem 3.1 is proved.

4. The triviality of φ_4

The goal of this section is to prove Theorem 1.2, the main result of this paper.

To this end, we need to recall the computation of $Ext_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$ by W. H. Lin and that of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_4$ by F. P. Peterson and the author.

Theorem 4.1. (W. H. Lin [18], see also [19, Theorem2.2]). *The following classes form an $\mathbb{F}_2$-basis for the vector space of indecomposable elements in $Ext_{\mathcal{A}}^4(\mathbb{F}_2, \mathbb{F}_2)$:*

(1) $d_i = [(Sq^0)^i(\lambda_6\lambda_2\lambda_3^2 + \lambda_4^2\lambda_3^2 + \lambda_2\lambda_4\lambda_5\lambda_3 + \lambda_1\lambda_5\lambda_1\lambda_7)]$
$\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+1}}$, $i \geq 0$,

(2) $e_i = [(Sq^0)^i(\lambda_8\lambda_3^3 + \lambda_4(\lambda_5^2\lambda_3 + \lambda_7\lambda_3^2) + \lambda_2(\lambda_3\lambda_5\lambda_7 + \lambda_1\lambda_{11}\lambda_3))]$
$\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+2}+2^i}$, $i \geq 0$,

(3) $f_i = [(Sq^0)^i(\lambda_4\lambda_0\lambda_7^2 + \lambda_3(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_2^2\lambda_7^2)]$
$\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+2}+2^{i+1}}$, $i \geq 0$,

(4) $g_{i+1} = [(Sq^0)^i(\lambda_6\lambda_0\lambda_7^2 + \lambda_5(\lambda_9\lambda_3^2 + \lambda_3\lambda_5\lambda_7) + \lambda_3(\lambda_5\lambda_9\lambda_3 + \lambda_{11}\lambda_3^2))]$
$\in Ext_{\mathcal{A}}^{4,2^{i+4}+2^{i+3}}$, $i \geq 0$,

(5) $p_i = [(Sq^0)^i(\lambda_{14}\lambda_5\lambda_7^2 + \lambda_{10}\lambda_9\lambda_7^2 + \lambda_6\lambda_9\lambda_{11}\lambda_7)]$
$\in Ext_{\mathcal{A}}^{4,2^{i+5}+2^{i+2}+2^i}$, $i \geq 0$,

(6) $D_3(i) = [(Sq^0)^i(\lambda_{22}\lambda_1\lambda_7\lambda_{31} + \lambda_{16}\lambda_7^2\lambda_{31} + \lambda_{14}\lambda_9\lambda_7\lambda_{31} + \lambda_{12}\lambda_{11}\lambda_7\lambda_{31}]$
$\in Ext_{\mathcal{A}}^{4,2^{i+6}+2^i}$, $i \geq 0$,

(7) $p'_i = [(Sq^0)^i(\lambda_0\lambda_{39}\lambda_{15}^2 + \lambda_0\lambda_{15}\lambda_{23}\lambda_{31})]$
$\in Ext_{\mathcal{A}}^{4,2^{i+6}+2^{i+3}+2^i}$, $i \geq 0$.

To simplify notation, we will denote $Q_{4,3}^a Q_{4,2}^b Q_{4,1}^c Q_{4,0}^d$ by $Q(a,b,c,d)$ in the following theorem.

Theorem 4.2. (Hưng-Peterson [13]). *The following elements form an $\mathbb{F}_2$-basis for the vector space $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_4$:*

(1)	$Q(2^s-1,0,0,0)$,	$s \geq 0$,
(2)	$Q(2^r-2^s-1,2^s-1,1,0)$,	$r > s > 0$,
(3)	$Q(2^t-2^r-1,2^r-2^s-1,2^s-1,2)$,	$t > r > s > 1$,
(4)	$Q(2^r-2^{s+1}-2^s-1,2^s-1,2^s-1,2)$,	$r > s+1 > 2$.

They are of degrees $2^{s+3}-8$, $2^{r+3}+2^{s+2}-6$, $2^{t+3}+2^{r+2}+2^{s+1}-4$ and $2^{r+3}+2^{s+1}-4$, respectively.

Now we come back to prove Theorem 1.2.

Proof of Theorem 1.2.

In [14], F. Peterson and the author have proved that φ_k vanishes on any decomposable elements for $k > 2$ by showing that $\varphi_* = \oplus_k\varphi_k$ is a homomorphism of algebras and, more importantly, that the product of the algebra $\oplus_k(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$ is trivial, except for the case

$$(\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_1)^* \otimes (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_1)^* \to (\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_2)^*.$$

Therefore, we need only to show φ_4 vanishing on any indecomposable elements.

Let a_0 denote one of the seven generators

$$d_0, e_0, f_0, g_1, p_0, D_3(0), p'_0,$$

each of which is the element of lowest stem in its own family. Furthermore, set $a_i = (Sq^0)^i(a_0)$, for $i \geq 0$. From Theorem 3.1, we have

$$\varphi_4(a_i) = \varphi_4(Sq^0)^i(a_0) = (Sq^0)^i\varphi_4(a_0).$$

So, in order to prove that $\varphi_4(a_i) = 0$ for any i, it suffices to show $\varphi_4(a_0) = 0$. We will do this by checking that the stem of a_0 is different from degrees of all the generators of $\mathbb{F}_2 \underset{\mathcal{A}}{\otimes} D_4$ given in Theorem 4.2.

Now let us check it case by case.

Case 1: For d_0 of stem $2^4 + 2^1 - 4 = 14$,
$2^{s+3} = 8 + 14 = 22$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 14 = 16 + 4$, $r = 1, s = 0$, it does not satisfy $s > 0$;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 14 = 16 + 2$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 14 = 16 + 2$, $r = 1, s = 0$ it does not satisfy $r > s + 1 > 2$.

Case 2: For e_0 of stem $2^4 + 2^2 + 2^0 - 4 = 17$,
$2^{s+3} = 8 + 17 = 25$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 17 = 16 + 4 + 2 + 1$, no solution;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 17 = 16 + 4 + 1$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 17 = 16 + 4 + 1$, no solution.

Case 3: For f_0 of stem $2^4 + 2^2 + 2^1 - 4 = 18$,
$2^{s+3} = 8 + 18 = 26$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 18 = 16 + 8$, $r = 1, s = 1$ it does not satisfy $r > s$;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 18 = 16 + 4 + 2$, $t = 1, r = s = 0$ it does not satisfy $r > s > 1$;
$2^{r+3} + 2^{s+1} = 4 + 18 = 16 + 4 + 2$, no solution.

Case 4: For g_1 of stem $2^4 + 2^3 - 4 = 20$,
$2^{s+3} = 8 + 20 = 28$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 20 = 16 + 8 + 2$, no solution;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 20 = 16 + 8$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 20 = 16 + 8$, $r = 1, s = 2$, it does not satisfy $r > s + 1 > 2$.

Case 5: For p_0 of stem $2^5 + 2^2 + 2^0 - 4 = 33$,
$2^{s+3} = 8 + 33 = 41$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 33 = 32 + 4 + 2 + 1$, no solution;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 33 = 32 + 4 + 1$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 33 = 32 + 4 + 1$, no solution.

Case 6: For $D_3(0)$ of stem $2^6 + 2^0 - 4 = 61$,
$2^{s+3} = 8 + 61 = 69$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 61 = 64 + 2 + 1$, no solution;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 61 = 64 + 1$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 61 = 64 + 1$, no solution.

Case 7: For p_0' of stem $2^6 + 2^3 + 2^0 - 4 = 69$,
$2^{s+3} = 8 + 69 = 77$, no solution;
$2^{r+3} + 2^{s+2} = 6 + 69 = 64 + 8 + 2 + 1$, no solution;
$2^{t+3} + 2^{r+2} + 2^{s+1} = 4 + 69 = 64 + 8 + 1$, no solution;
$2^{r+3} + 2^{s+1} = 4 + 69 = 64 + 8 + 1$, no solution.

Therefore, φ_4 vanishes on any indecomposable elements.

In summary, $\varphi_4 = 0$ in positive stems. Theorem 1.2 is completely proved.

References

[1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. Math. **72** (1960), 20–104.

[2] M. A. Alghamdi, M. C. Crabb and J. R. Hubbuck, *Representations of the homology of BV and the Steenrod algebra I,* Adams Memorial Symposium on Algebraic Topology 2, N. Ray and G. Walker (ed.), London Math. Soc. Lect. Note Series **176** (1992), 217–234.

[3] J. M. Boardman, *Modular representations on the homology of powers of real projective space,* Algebraic Topology: Oaxtepec 1991, M. C. Tangora (ed.), Contemp. Math. **146** (1993), 49–70.
[4] A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector, J. W. Schlesinger, *The mod p lower central series and the Adams spectral sequence*, Topology **5** (1966), 331–342.
[5] W. Browder, *The Kervaire invariant of a framed manifold and its generalization,* Ann. Math. **90** (1969), 157–186.
[6] E. B. Curtis, *The Dyer–Lashof algebra and the lambda algebra*, Illinois Jour. Math. **18** (1975), 231–246.
[7] L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75–98.
[8] P. G. Goerss, *Unstable projectives and stable Ext: with applications*, Proc. London Math. Soc. **53** (1986), 539–561.
[9] N. H. V. Hưng, *Spherical classes and the algebraic transfer*, Trans. Amer. Math. Soc. **349** (1997), 3893–3910.
[10] N. H. V. Hưng, *The weak conjecture on spherical classes*, Math. Zeit. **231** (1999), 727–743.
[11] N. H. V. Hưng, *Spherical classes and the lambda algebra*, Trans. Amer. Math. Soc. **353** (2001), 4447–4460.
[12] N. H. V. Hưng and T. N. Nam, *The hit problem for the Dickson algebra*, Trans. Amer. Math. Soc. **353** (2001), 5029–5040.
[13] N. H. V. Hưng and F. P. Peterson, *$\mathcal{A}$–generators for the Dickson algebra*, Trans. Amer. Math. Soc. **347** (1995), 4687–4728.
[14] N. H. V. Hưng and F. P. Peterson, *Spherical classes and the Dickson algebra*, Math. Proc. Camb. Phil. Soc. **124** (1998), 253–264.
[15] M. Kameko, *Products of projective spaces as Steenrod modules*, Thesis, Johns Hopkins University 1990.
[16] J. Lannes, *Sur le n-dual du n-ème spectre de Brown-Gitler*, Math. Zeit. **199** (1988), 29–42.
[17] J. Lannes and S. Zarati, *Sur les foncteurs dérivés de la déstabilisation*, Math. Zeit. **194** (1987), 25–59.
[18] W. H. Lin, *Some differentials in Adams spectral sequence for spheres*, Trans. Amer. Math. Soc., to appear.
[19] W. H. Lin and M. Mahowald, *The Adams spectral sequence for Minami's theorem*, Contemp. Math. **220** (1998), 143–177.
[20] A. Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations*, Mem. Amer. Math. Soc. **42** (1962).
[21] I. Madsen, *On the action of the Dyer–Lashof algebra in $H_*(G)$*, Pacific Jour. Math. **60** (1975), 235–275.
[22] H. Mùi, *Modular invariant theory and cohomology algebras of symmetric groups*, Jour. Fac. Sci. Univ. Tokyo, **22** (1975), 310–369.
[23] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, London 1986.
[24] W. M. Singer, *Invariant theory and the lambda algebra*, Trans. Amer. Math. Soc. **280** (1983), 673–693.
[25] W. M. Singer, *The transfer in homological algebra*, Math. Zeit. **202** (1989), 493–523.
[26] V. Snaith and J. Tornehave, *On $\pi_*^S(BO)$ and the Arf invariant of framed manifolds*, Amer. Math. Soc. Contemporary Math. **12** (1982), 299–313.
[27] R. J. Wellington, *The unstable Adams spectral sequence of free iterated loop spaces*, Memoirs Amer. Math. Soc. **258** (1982).

Department of Mathematics, Johns Hopkins University
3400 N. Charles Street, Baltimore MD 21218 - 2689
E-mail address: nhvhung@math.jhu.edu

Permanent address:
Department of Mathematics, Vietnam National University, Hanoi
334 Nguyên Trai Street, Hanoi, Vietnam
E-mail address: nhvhung@vnu.edu.vn