THE WEAK CONJECTURE ON SPHERICAL CLASSES

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Abstract. Let $\mathcal{A}$ be the mod 2 Steenrod algebra. We construct a chain-level representation of the dual of Singer's algebraic transfer, $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \mathbf{F}_2[x_1, ..., x_k]$, which maps Singer's invariant-theoretic model of the dual of the Lambda algebra, $\Gamma_k^\wedge$, to $\mathbf{F}_2[x_1^{\pm 1}, ..., x_k^{\pm 1}]$ and is the inclusion of the Dickson algebra, $D_k \subset \Gamma_k^\wedge$, into $\mathbf{F}_2[x_1, ..., x_k]$. This chain-level representation allows us to confirm the weak conjecture on spherical classes (see [9]), assuming the truth of (1) either the conjecture that *the Dickson invariants of at least* $k = 3$ *variables are homologically zero in* $Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$, (2) or a conjecture on *$\mathcal{A}$-decomposability of the Dickson algebra in* $\Gamma_k^\wedge$. We prove the conjecture in item (1) for $k = 3$ and also show a weak form of the conjecture in item (2).

1. Introduction and statement of results

This paper continues our study of spherical classes that started in [9]. To make the paper self-contained, we first recall certain background of the classical conjecture on spherical classes, which has been given in the introduction of our paper [9].

We are interested in the following conjecture on spherical classes in Q_0S^0, i.e. elements belonging to the image of the Hurewicz homomorphism

$$H : \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0) .$$

Here and throughout the paper, the coefficient ring for homology and cohomology is always $\mathbf{F}_2$, the field of two elements.

Conjecture 1.1. (conjecture on spherical classes). *There are no spherical classes in Q_0S^0, except the Hopf invariant one and the Kervaire invariant one elements.*

Some topologists believe that the conjecture is due to I. Madsen, while some others say it is due to E. Curtis. (See Curtis [6] and Wellington [24] for a discussion.)

Let E^k be an elementary abelian 2-group of rank k. It is also viewed as a k-dimensional vector space over $\mathbf{F}_2$. So, the general linear group $GL_k = GL(k, \mathbf{F}_2)$ acts on E^k and therefore on $H^*(BE^k)$ in the usual way. Let D_k be the Dickson algebra of k variables, i.e. the algebra of invariants

$$D_k := H^*(BE^k)^{GL_k} \cong \mathbf{F}_2[x_1, \dots, x_k]^{GL_k} ,$$

where $P_k = \mathbf{F}_2[x_1, \dots, x_k]$ is the polynomial algebra on k generators $x_1, \dots, x_k$, each of dimension 1. As the action of the (mod 2) Steenrod algebra, $\mathcal{A}$, and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathcal{A}$.

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One way to attack Conjecture 1.1 is to study the Lannes–Zarati homomorphism

$$\varphi_k : Ext_{\mathcal{A}}^{k,k+i}(\mathbf{F}_2, \mathbf{F}_2) \to (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k)_i^* ,$$

which is compatible with the Hurewicz homomorphism (see [13], [14, p. 46]). The domain of φ_k is the E_2-term of the Adams spectral sequence converging to $\pi_*^s(S^0) \cong \pi_*(Q_0S^0)$. According to Madsen's theorem [16], which asserts that D_k is dual to the coalgebra of Dyer–Lashof operations of length k, the range of φ_k is a submodule of $H_*(Q_0S^0)$. By compatibility of φ_k and the Hurewicz homomorphism we mean φ_k is a "lifting" of the latter from the "E_∞-level" to the "E_2-level".

The Hopf invariant one and the Kervaire invariant one elements are respectively represented by certain permanent cycles in $Ext_{\mathcal{A}}^{1,*}(\mathbf{F}_2, \mathbf{F}_2)$ and $Ext_{\mathcal{A}}^{2,*}(\mathbf{F}_2, \mathbf{F}_2)$, on which φ_1 and φ_2 are non-zero (see Adams [1], Browder [5], Lannes–Zarati [14]).

Therefore, Conjecture 1.1 is a consequence of the following one.

Conjecture 1.2. *$\varphi_k = 0$ in any positive stem i for $k > 2$.*

It is well known that the Ext group has intensively been studied, but remains very mysterious. In order to avoid the shortage of our knowledge of the Ext group, we want to restrict φ_k to a certain subgroup of the Ext group which (1) is large enough and worthwhile to pursue and (2) could be handled more easily than the Ext group itself. To this end, we combine the above data with Singer's algebraic transfer.

Singer defined in [22] the algebraic transfer

$$Tr_k : \mathbf{F}_2 \underset{GL_k}{\otimes} PH_i(BE^k) \to Ext_{\mathcal{A}}^{k,k+i}(\mathbf{F}_2, \mathbf{F}_2) ,$$

where $PH_*(BE^k)$ denotes the submodule consisting of all $\mathcal{A}$-annihilated elements in $H_*(BE^k)$. It is shown to be an isomorphism for $k \le 2$ by Singer [22] and for $k = 3$ by Boardman [3]. Singer also proved in [22] that it is *not* an isomorphism for $k = 5$, and conjectured that Tr_k is a *monomorphism* for any k.

Restricting φ_k to the image of Tr_k, we stated in [9] the following conjecture.

Conjecture 1.3. (weak conjecture on spherical classes).

$$\varphi_k \cdot Tr_k : \mathbf{F}_2 \underset{GL_k}{\otimes} PH_*(BE^k) \to P(\mathbf{F}_2 \underset{GL_k}{\otimes} H_*(BE^k)) := (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k)^*$$

is zero in positive dimensions for $k > 2$.

In other words, there are no spherical classes in Q_0S^0, which can be detected by the algebraic transfer, except the Hopf invariant one and the Kervaire invariant one elements.

In [9], we have proved that *the inclusion of D_k into P_k is a chain-level representation of $Tr_k^* \cdot \varphi_k^* : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$.* So, we get the following result.

Theorem 1.4. *The weak conjecture on spherical classes is equivalent to the conjecture that the homomorphism*

$$j_k : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} (P_k^{GL_k}) \to (\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k)^{GL_k}$$

induced by the identity map on P_k is zero in positive dimensions for $k > 2$.

Let $D_k^+, \mathcal{A}^+$ be respectively the submodules of D_k and $\mathcal{A}$ consisting of all elements of positive dimensions. Then the above conjecture on j_k can equivalently be stated as follows.

Conjecture 1.5. *If $k > 2$, then*

$$D_k^+ \subset \mathcal{A}^+ \cdot P_k.$$

In [9], we have got a proof of this conjecture for $k = 3$.

To introduce a new approach, we need to summarize Singer's invariant-theoretic description of the lambda algebra [21]. According to Dickson [7], one has

$$D_k \cong \mathbf{F}_2[Q_{k,k-1}, ..., Q_{k,0}],$$

where $Q_{k,i}$ denotes the Dickson invariant of dimension $2^k - 2^i$. Singer set $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defined $\Gamma_k^\wedge$ to be a certain "not too large" submodule of Γ_k. He also equiped $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ with a differential $\partial : \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ and a coproduct. Then, he showed that the differential coalgebra $\Gamma^\wedge$ is dual to the lambda algebra of the six authors of [4]. Thus, $H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$. (Originally, Singer used the notation Γ_k^+ to denote $\Gamma_k^\wedge$. However, by D_k^+, $\mathcal{A}^+$ we always mean the submodules of D_k and $\mathcal{A}$ respectively consisting of all elements of positive dimensions, so Singer's notation Γ_k^+ would make a confusion in this paper. Therefore, we prefer the notation $\Gamma_k^\wedge$ to Γ_k^+.)

One of the main results of this paper is to construct a homomorphism $\mathcal{T}_k : \Gamma_k^\wedge \to \mathbf{F}_2[x_1^{\pm 1}, ..., x_k^{\pm 1}]$ with the following properties.

Theorem 3.2 *The homomorphism $\mathcal{T}_k : \Gamma_k^\wedge \to \mathbf{F}_2[x_1^{\pm 1}, ..., x_k^{\pm 1}]$ maps the submodule of all cycles in $\Gamma_k^\wedge$ to P_k and is a chain-level representation of $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$. Moreover, its restriction to $D_k \subset \Gamma_k^\wedge$ is the inclusion of D_k into P_k.*

Note that every $q \in D_k^+$ is a cycle in the differential module $\Gamma_k^\wedge$. By means of this theorem, the weak conjecture on spherical classes is an immediate consequence of the following conjecture.

Conjecture 1.6. *If $q \in D_k^+$, then $[q] = 0$ in $Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ for $k > 2$.*

This is a corollary of the following conjecture. (See Proposition 4.2 for a proof.)

Conjecture 4.1 *Let $Ker\partial_k$ be the submodule of all cycles in $\Gamma_k^\wedge$. Then, for $k > 2$,*

$$D_k^+ \subset \mathcal{A}^+ \cdot Ker\partial_k.$$

We get the following result, which is weaker than the above conjecture.

Theorem 4.3 *For $k > 2$,*

$$D_k^+ \subset \mathcal{A}^+ \cdot \Gamma_k^\wedge.$$

We also have the following theorem.

Theorem 4.8 *Conjecture 1.6 is true for $k = 3$.*

Conjecture 1.5 is related to the difficult problem of determination of $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$. This problem has first been studied by F. Peterson [18], R. Wood [26], W. Singer [22], S. Priddy [19]... who show its relationships to several classical problems in Homotopy Theory. $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$ has explicitly been computed for $k \leq 3$. The cases $k = 1$ and

2 are not difficult, while the case $k = 3$ is very complicated and was solved by M. Kameko [12]. There is also another approach, the qualitative one, to the problem. By this we mean giving conditions on elements of P_k to show that they go to zero in $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$, i. e. belong to $\mathcal{A}^+ \cdot P_k$. Peterson's conjecture, which was established by Wood [26], claims that $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k = 0$ in dimension d such that $\alpha(d+k) > k$. Here $\alpha(n)$ denotes the number of ones in the dyadic expansion of n. Recently, W. Singer, K. Monks, J. Silverman... have refined the method of R. Wood to show that many more monomials in P_k are in $\mathcal{A}^+ \cdot P_k$. (See Silverman [20] and its references.) *Conjecture 1.5 presents a large family, whose elements are predicted to be in* $\mathcal{A}^+ \cdot P_k$.

The paper is organized as follows. Section 2 is a preliminary on invariant theory and Singer's algebraic transfer. Sections 3 and 4 are respectively devoted to prove Theorems 3.2, 4.3 and 4.8. Finally, in Section 5, we propose a way to prove the classical conjecture on spherical classes.

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2. Preliminary

This section begins with a few words on invariant theory and ends with a brief sketch of Singer's definition of the algebraic transfer.

Let T_k be the Sylow 2-subgroup of GL_k consisting of all upper triangular $k \times k$-matrices with 1 on the main diagonal. The T_k-invariant ring, $M_k = P_k^{T_k}$, is called the Mùi algebra. In [17], Mùi showed that

$$P_k^{T_k} = \mathbf{F}_2[V_1, ..., V_k],$$

where

$$V_i = \prod_{\lambda_j \in \mathbf{F}_2} (\lambda_1 x_1 + \cdots + \lambda_{i-1} x_{i-1} + x_i).$$

Then, the Dickson invariant $Q_{k,i}$ can inductively be defined by

$$Q_{k,i} = Q_{k-1,i-1}^2 + V_k \cdot Q_{k-1,i},$$

where, by convention, $Q_{k,k} = 1$ and $Q_{k,i} = 0$ for $i < 0$.

Let $S(k) \subset P_k$ be the multiplicative subset generated by all the non-zero linear forms in P_k. Let Φ_k be the localization: $\Phi_k = (P_k)_{S(k)}$. Using the results of Dickson [7] and Mùi [17], Singer noted in [21] that

$$\Delta_k := (\Phi_k)^{T_k} = \mathbf{F}_2[V_1^{\pm 1}, ..., V_k^{\pm 1}],$$

$$\Gamma_k := (\Phi_k)^{GL_k} = \mathbf{F}_2[Q_{k,k-1}, ..., Q_{k,1}, Q_{k,0}^{\pm 1}].$$

Further, he set

$$v_1 = V_1, \quad v_k = V_k / V_1 \cdots V_{k-1} \quad (k \geq 2),$$

so that

$$V_k = v_1^{2^{k-2}} v_2^{2^{k-3}} \cdots v_{k-1} v_k \quad (k \geq 2).$$

Then, he obtained

$$\Delta_k = \mathbf{F}_2[v_1^{\pm 1}, ..., v_k^{\pm 1}],$$

with $\dim v_i = 1$ for every i.

Singer defined $\Gamma_k^\wedge$ to be the submodule of $\Gamma_k = D_k[Q_{k,0}^{-1}]$ spanned by all monomials $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$ with $i_{k-1}, ..., i_1 \geq 0, i_0 \in \mathbf{Z}$, and $i_0 + \dim \gamma \geq 0$.

In the remaining part of this section, we briefly sketch Singer's definition of Tr_k^*.

Let $P_1 = \mathbf{F}_2[x]$ with $|x| = 1$. Let $\hat{P} \subset \mathbf{F}_2[x, x^{-1}]$ be the submodule spanned by all powers x^i with $i \geq -1$. The canonical $\mathcal{A}$-action on P_1 is extended to an $\mathcal{A}$-action on $\mathbf{F}_2[x, x^{-1}]$ (see Adams [2], Wilkerson [25]). $\hat{P}$ is an $\mathcal{A}$-submodule of $\mathbf{F}_2[x, x^{-1}]$. We have a short-exact sequence of $\mathcal{A}$-modules

2.1. $$0 \to P_1 \xrightarrow{\iota} \hat{P} \xrightarrow{\pi} \mathbf{F}_2 \to 0\ ,$$

where ι is the inclusion and π is given by $\pi(x^i) = 0$ if $i \neq -1$ and $\pi(x^{-1}) = 1$. Let e_1 be the corresponding element in $Ext^1_{\mathcal{A}}(\mathbf{F}_2, P_1)$. Then, Singer defined

2.2. $$e_k := e_1 \otimes \cdots \otimes e_1 \in Ext^k_{\mathcal{A}}(\mathbf{F}_2, P_k) \qquad (k\ times).$$

Now, $Tr_k^* : Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2) \to Tor_0^{\mathcal{A}}(\mathbf{F}_2, P_k) \equiv \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$ is defined by

2.3. $$Tr_k^*(z) := e_k \cap z\ .$$

Note that, since π raises internal degree by one, Tr_k^* lowers it by k.

We need to relate Tr_k^* with connecting homomorphisms. Let N, P, Q, R be (left) $\mathcal{A}$-modules. From MacLane [15, p. 229], one has

$$f \otimes g = (f \otimes R) \circ (N \otimes g)$$

for $f \in Ext^*_{\mathcal{A}}(N, P), g \in Ext^*_{\mathcal{A}}(Q, R)$. Hence

2.4. $$e_k = (e_1 \otimes P_{k-1}) \circ \cdots \circ (e_1 \otimes P_1) \circ e_1\ .$$

Cap and Yoneda products are related by the formula

$$(h \circ f) \cap z = h \cap (f \cap z)$$

for $z \in Tor_*^{\mathcal{A}}(M, N), f \in Ext^*_{\mathcal{A}}(N, P), h \in Ext^*_{\mathcal{A}}(P, Q)$.

Suppose $f \in Ext^1_{\mathcal{A}}(N_3, N_1)$ is represented by the short-exact sequence of left $\mathcal{A}$-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Let $\Delta(f) : Tor_s^{\mathcal{A}}(M, N_3) \to Tor_{s-1}^{\mathcal{A}}(M, N_1)$ be the connecting homomorphism associated with this short-exact sequence, for any right $\mathcal{A}$-module M. Then one verifies easily

$$\Delta(f)(z) = f \cap z$$

for any $z \in Tor_s^{\mathcal{A}}(M, N_3)$.

Consequently, we get

2.5. $$Tr_k^* = \Delta(e_1 \otimes P_{k-1}) \circ \cdots \circ \Delta(e_1 \otimes P_1) \circ \Delta e_1\ .$$

(See Singer [22, p. 498].)

Now we prepare to construct a chain-level representation of Tr_k^*.

For every left $\mathcal{A}$-module N, Singer defined in [21] a chain complex $\Gamma^\wedge N$: It is given in homological dimension k by $(\Gamma^\wedge N)_k = \Gamma_k^\wedge \otimes N$ as an $\mathbf{F}_2$-vector space and its differential $\partial : (\Gamma^\wedge N)_k \to (\Gamma^\wedge N)_{k-1}$ is defined by

$$\partial(v_1^{a_1} \cdots v_k^{a_k} \otimes y) := (v_1^{a_1} \cdots v_{k-1}^{a_{k-1}} \otimes Sq^{a_k+1}(y))\ ,$$

for $y \in N$. Then he proved that

$$H_k(\Gamma^\wedge N) \cong Tor_k^{\mathcal{A}}(\mathbf{F}_2, N).$$

This isomorphism is natural in N. Further, if $(f) :\ 0 \to N_1 \to N_2 \to N_3 \to 0$ is any short-exact sequence of left $\mathcal{A}$-modules, then $0 \to \Gamma^\wedge N_1 \to \Gamma^\wedge N_2 \to \Gamma^\wedge N_3 \to 0$ is a short-exact sequence of chain complexes, from which a chain-level representation of the connecting homomorphism $\Delta(f) : Tor_k^{\mathcal{A}}(\mathbf{F}_2, N_3) \to Tor_{k-1}^{\mathcal{A}}(\mathbf{F}_2, N_1)$ can be computed.

3. A chain-level representation of the dual of the algebraic transfer

Following Singer [21], the homomorphism

$$\partial_k : \mathbf{F}_2[v_1^{\pm 1},...,v_k^{\pm 1}] \to \mathbf{F}_2[v_1^{\pm 1},...,v_{k-1}^{\pm 1}]$$

$$\partial_k(v_1^{a_1}\cdots v_k^{a_k}) := \begin{cases} v_1^{a_1}\cdots v_{k-1}^{a_{k-1}} & \text{if } a_k = -1, \\ 0 & \text{otherwise} \end{cases}$$

maps $\Gamma_k^\wedge$ to $\Gamma_{k-1}^\wedge$. Moreover, it is a differential on $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ with $H_*(\Gamma^\wedge) \cong Tor_*^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$. This is an isomorphism of bigraded modules. Here the bidegree of $\Gamma^\wedge$ is given by

$$bideg(v_1^{a_1}\cdots v_k^{a_k}) := (k, k+\sum a_i).$$

Note that, for every $q \in D_k$, its expansion

$$q = \sum_{(a_1,...,a_k)} v_1^{a_1}\cdots v_k^{a_k}$$

always has $a_k \geq 0$ in any term of the sum. Therefore, $\partial_k(q) = 0$. This means that every Dickson invariant is a cycle in the complex $\Gamma^\wedge$, whose homology is $Tor_*^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$.

Now we construct a homomorphism, denoted $\mathcal{T}_k$, as follows.

Definition 3.1. *The homomorphism* $\mathcal{T}_k : \mathbf{F}_2[v_1^{\pm 1},...,v_k^{\pm 1}] \to \mathbf{F}_2[x_1^{\pm 1},...,x_k^{\pm 1}]$ *is defined by*

$$\mathcal{T}_k(v_1^{a_1}\cdots v_k^{a_k}) := Sq^{a_1+1}\Big(x_1^{-1}\cdots Sq^{a_{k-1}+1}(x_{k-1}^{-1}Sq^{a_k+1}(x_k^{-1}))\cdots\Big)\ ,$$

where $a_1,...,a_k$ *are arbitrary integers. Here, we mean* $Sq^i = 0$ *for* $i < 0$.

Let $Ker\partial_k$ be the submodule of all cycles in $\Gamma_k^\wedge$. Then, the goal of this section is to prove the following theorem.

Theorem 3.2. *The homomorphism* $\mathcal{T}_k$ *satisfies the following two properties:*

(i) $\mathcal{T}_k(Ker\partial_k) \subset P_k$. *Furthermore,* $\mathcal{T}_k|_{\Gamma_k^\wedge} : \Gamma_k^\wedge \to \mathbf{F}_2[x_1^{\pm 1},...,x_k^{\pm 1}]$ *is a chain-level representation of*

$$Tr_k^* : H_k(\Gamma^\wedge) \cong Tor_k^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2) \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k\ .$$

(ii) $\mathcal{T}_k|_{D_k}$ *is the inclusion of* D_k *into* P_k.

The first part of Theorem 3.2 is shown by the following lemma.

Lemma 3.3. $\mathcal{T}_k$ *maps* $Ker\partial_k$ *to* P_k *and is a chain-level representation of* Tr_k^*.

Proof. In order to construct a chain-level representation for Tr_k^*, following 2.5, we first construct a chain-level representation for

$$\Delta(e_1) : Tor_k^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2) \to Tor_{k-1}^{\mathcal{A}}(\mathbf{F}_2, P_1),$$

the connecting homomorphism associated with the short-exact sequence 2.1. Let us consider the induced short-exact sequence

$$0 \to \Gamma^\wedge P_1 \xrightarrow{\iota} \Gamma^\wedge \hat{P} \xrightarrow{\pi} \Gamma^\wedge \mathbf{F}_2 \to 0\ .$$

A lifting of a cycle $z \in \Gamma^\wedge = \Gamma^\wedge\mathbf{F}_2$ over the chain map π is given by the chain $z \otimes x_k^{-1} \in \Gamma^\wedge\hat{P}$, where we are writing $P_1 = \mathbf{F}_2[x_k]$, $\hat{P} = \text{Span}\{x_k^i | i \geq -1\}$. The boundary $\partial(z\otimes x_k^{-1})$ pulls back $z\otimes x_k^{-1}$ under ι to a cycle in $\Gamma^\wedge P_1$, which represents $\Delta(e_1)(z)$ in $Tor^{\mathcal{A}}_{k-1}(\mathbf{F}_2, P_1)$. If $z = \sum v_1^{a_1}\cdots v_k^{a_k}$, then by (4.1) of [21],

$$\partial(z\otimes x_k^{-1}) = \partial(\sum v_1^{a_1}\cdots v_k^{a_k}\otimes x_k^{-1}) = \sum v_1^{a_1}\cdots v_{k-1}^{a_{k-1}}\otimes Sq^{a_k+1}(x_k^{-1})\ .$$

This means that the map given by

$$v_1^{a_1}\cdots v_k^{a_k}\mapsto v_1^{a_1}\cdots v_{k-1}^{a_{k-1}}\otimes Sq^{a_k+1}(x_k^{-1})$$

is a chain-level representation of $\Delta(e_1)$.

We similarly construct a chain-level representation for the homomorphism $\Delta(e_1\otimes P_1) : Tor^{\mathcal{A}}_{k-1}(\mathbf{F}_2, P_1)\to Tor^{\mathcal{A}}_{k-2}(\mathbf{F}_2, P_2)$, which is the connecting homomorphism associated with the short-exact sequence of chain complexes

$$0\to\Gamma^\wedge P_2\overset{\iota\otimes P_1}{\to}\Gamma^\wedge(\hat{P}\otimes P_1)\overset{\pi\otimes P_1}{\to}\Gamma^\wedge P_1\to 0\ .$$

Here we are writing $P_1 = \mathbf{F}_2[x_k]$, $P_2 = \mathbf{F}_2[x_{k-1}, x_k]$ and $\hat{P} = \mathrm{Span}\{x_{k-1}^i | i\geq -1\}$. By the argument similar to the one given above, the map

$$v_1^{a_1}\cdots v_{k-1}^{a_{k-1}}\otimes x_k^i\mapsto\partial(v_1^{a_1}\cdots v_{k-1}^{a_{k-1}}\otimes x_{k-1}^{-1}x_k^i) = v_1^{a_1}\cdots v_{k-2}^{a_{k-2}}\otimes Sq^{a_{k-1}+1}(x_{k-1}^{-1}x_k^i)$$

is a chain-level representation of $\Delta(e_1\otimes P_1)$. So the composite map

$$v_1^{a_1}\cdots v_k^{a_k}\mapsto v_1^{a_1}\cdots v_{k-2}^{a_{k-2}}\otimes Sq^{a_{k-1}+1}\Big(x_{k-1}^{-1}Sq^{a_k+1}(x_k^{-1})\Big)$$

sends a cycle in $\Gamma^\wedge P_1$ to a cycle in $\Gamma^\wedge P_2$ and it is a chain-level representation of the composite homomorphism $\Delta(e_1\otimes P_1)\circ\Delta(e_1)$.

By repeating the above argument, it is easy to see that the map $\mathcal{T}_k$ given by Definition 3.1 is a chain-level representation of $Tr_k^* = \Delta(e_1\otimes P_{k-1})\circ\cdots\circ\Delta(e_1\otimes P_1)\circ\Delta(e_1)$. In particular, we have

$$\mathcal{T}_k(Ker\partial_k)\subset\Gamma_0^\wedge P_k = P_k.$$

For the convenience of the latter use, $\mathcal{T}_k$ is thought of as a morphism from $\mathbf{F}_2[v_1^{\pm 1}, ..., v_k^{\pm 1}]$ to $\mathbf{F}_2[x_1^{\pm 1}, ..., x_k^{\pm 1}]$.

The lemma follows.

Remark 3.4. Since the chain-level representation $\mathcal{T}_k$ has been extended to a homomorphism, whose domain is $\mathbf{F}_2[v_1^{\pm 1}, ..., v_k^{\pm 1}]$, the computation of $\mathcal{T}_k(V_i)$ makes sense as one can see in Lemmas 3.7 and 3.8 below.

The second part of Theorem 3.2 is proved by a number of lemmata.

Let M be a graded $\mathcal{A}$-module, which is concentrated in non-negative dimensions. We are concerned with the Steenrod homomorphism $d^*P : M\to\mathbf{F}_2[x^{\pm 1}]\otimes M$ given by

$$d^*P(u) = d^*P_x(u) = \sum_{i=0}^{|u|} x^{|u|-i}\otimes Sq^i(u)\ ,$$

where $u\in M$, x is of dimension 1 and $\mathbf{F}_2[x^{\pm 1}]$ supports the canonical $\mathcal{A}$-action as discussed in Section 2. Actually, Steenrod defined his Sq^i by using d^*P, which is constructed by means of cohomology of the weath products (see [23, Ch. VII]).

Suppose additionally that M is an $\mathcal{A}$-algebra. Then, by the Cartan formula, d^*P is a homomorphism of algebras.

Lemma 3.5. *Let M be a unstable $\mathcal{A}$-algebra. Suppose $\alpha, \beta \in M$, and $a \geq |\alpha|, b \geq |\beta|$. Then*

$$Sq^{a+b+1}(x^{-1} \otimes \alpha\beta) = Sq^{a+1}(x^{-1} \otimes \alpha) Sq^{b+1}(x^{-1} \otimes \beta).$$

Proof. Recall that $Sq^i(x^{-1}) = x^{i-1}$ for every integer i. From the unstability of M and $a \geq |\alpha|$, one gets

$$Sq^{a+1}(x^{-1} \otimes \alpha) = \sum_{i=0}^{|\alpha|} x^{a-i} \otimes Sq^i(\alpha) = x^{a-|\alpha|} d^*P(\alpha).$$

As d^*P is an algebra homomorphism, the lemma follows.

Applying repeatedly this lemma to $M = P_k$, we obtain

Lemma 3.6. *Suppose $v = v_1^{a_1} \cdots v_k^{a_k}$ and $v' = v_1^{b_1} \cdots v_k^{b_k}$ satisfy the conditions:*

$$a_i \geq a_{i+1} + \cdots + a_k \geq 0, \qquad b_i \geq b_{i+1} + \cdots + b_k \geq 0,$$

for $1 \leq i \leq k$. Then

$$\mathcal{T}_k(v \cdot v') = \mathcal{T}_k(v) \cdot \mathcal{T}_k(v') \ .$$

Furthermore, the both sides are elements of P_k.

Proof. Set $\tau_i(v) = Sq^{a_i+1}(x_i^{-1} \cdots Sq^{a_k+1}(x_k^{-1}))$. It is easy to see that $\tau_k(vv') = \tau_k(v)\tau_k(v')$ in P_k. Suppose inductively that $\tau_{i+1}(vv') = \tau_{i+1}(v)\tau_{i+1}(v')$ in P_k. Then, applying Lemma 3.5 with $a = a_i, b = b_i$, $\alpha = \tau_{i+1}(v), \beta = \tau_{i+1}(v')$, we get $\tau_i(vv') = \tau_i(v)\tau_i(v')$ in P_k.

Thus, the lemma follows as $\mathcal{T}_k = \tau_1$.

Lemma 3.7. *The restriction of $\mathcal{T}_k$ to the Mùi algebra $M_k = \mathbf{F}_2[V_1, ..., V_k]$ is a homomorphism of algebras.*

Proof. Recall that $V_i = v_1^{2^{i-2}} v_2^{2^{i-3}} \cdots v_{i-1} v_i$. So, one easily verifies that every element $v \in M_k$ is a sum of monomials $v_1^{a_1} \cdots v_k^{a_k}$, which satisfy the conditions of Lemma 3.6. The lemma follows from the previous one.

Lemma 3.8.

$$\mathcal{T}_k(V_i) = V_i \qquad (1 \leq i \leq k).$$

Therefore, $\mathcal{T}_k|_{M_k}$ is the inclusion of M_k into P_k.

Proof. This is proved by induction on k. It is easy for $k = 1$. Suppose inductively that it has been shown for $k - 1$. Then, using the expansion $V_i = v_1^{2^{i-2}} v_2^{2^{i-3}} \cdots v_{i-1} v_i$, one has

$$\begin{aligned} \mathcal{T}_k(V_i) &= \partial\Big(v_1^{2^{i-2}} \otimes x_1^{-1} \mathcal{T}_{k-1}(V_{i-1}(x_2, ..., x_i))\Big) \\ &= Sq^{2^{i-2}+1}\Big(x_1^{-1} V_{i-1}(x_2, ..., x_i)\Big) \quad \text{(by inductive hypothesis)} \\ &= d^*P_{x_1}\Big(V_{i-1}(x_2, ..., x_i)\Big) \quad \text{(since } |V_{i-1}| = 2^{i-2}) \\ &= V_i(x_1, x_2, ..., x_i). \end{aligned}$$

The last equality is showed in Mùi [17, Lemma 5.3], (see also [8]).

The lemma is proved.

Since $D_k \subset M_k$, Lemmas 3.7 and 3.8 show that $\mathcal{T}_k|_{D_k}$ is the inclusion of D_k into P_k.

Theorem 3.2 is completly proved.

4. Homological classes induced by Dickson invariants

According to Singer [21], $\Gamma_k^\wedge$ supports a canonical $\mathcal{A}$-action as follows. The usual action of $\mathcal{A}$ on P_k commutes with the action of GL_k. So, it induces an action of $\mathcal{A}$ on $D_k = P_k^{GL_k} = \mathbf{F}_2[Q_{k,k-1}, ..., Q_{k,0}]$. This is extended to an action of $\mathcal{A}$ on $\Gamma_k = D_k[Q_{k,0}^{-1}]$. By definition [21], $\Gamma_k^\wedge$ is the submodule of Γ_k spanned by all monomials $\gamma = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$ with $-\infty < i_0 < +\infty, 0 \le i_1, ..., i_{k-1}$ and $0 \le i_0 + \dim\gamma$. By means of the Hai–Hưng formula for the $\mathcal{A}$-action on D_k (see [8] or 4.4 below), it is easy to verify that $\Gamma_k^\wedge$ is an $\mathcal{A}$-submodule of Γ_k.

From Singer [21], the $\mathcal{A}$-action on $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ commutes with the differential of $\Gamma^\wedge$.

Now we show that the weak conjecture on spherical classes follows from the following one.

Conjecture 4.1. *Let $Ker\partial_k$ be the submodule of all cycles in $\Gamma_k^\wedge$. Then,*

$$D_k^+ \subset \mathcal{A}^+ \cdot Ker\partial_k,$$

for $k > 2$.

Proposition 4.2. *Conjecture 4.1 implies Conjecture 1.6, which in turn implies the weak conjecture on spherical classes.*

Proof. By Conjecture 4.1, for a given $q \in D_k^+$, we have $q = \sum_i Sq^i(\gamma_i)$ with some $i > 0$ and $\gamma_i \in Ker\partial_k$.

On the other hand, from Singer [21, Th. 1.3], the induced $\mathcal{A}$-action on $H_k(\Gamma^\wedge) = Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$ is trivial. So $[q] = \sum_i Sq^i[\gamma_i] = 0$ in $Tor_k^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$. By Theorem 3.2, $[q] = Tr^*[q] = 0$ in $\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$ for every $q \in D_k^+$, or equivalently, $D_k^+ \subset \mathcal{A}^+ \cdot P_k$, for $k > 2$. The proof is complete.

Here is an alternative proof for the fact that Conjecture 4.1 implies the weak conjecture on spherical classes.

By Theorem 3.2, $\mathcal{T}_k(Ker\partial_k) \subset P_k$, $\mathcal{T}_k(D_k) = D_k$.

Since $\mathcal{T}_k$ is an $\mathcal{A}$-homomorphism, Conjecture 4.1 implies

$$D_k^+ \subset \mathcal{A}^+ \cdot P_k,$$

for $k > 2$. That means Conjecture 1.5 holds. Hence, the weak conjecture on spherical classes is proved.

The following theorem is a weak form of Conjecture 4.1.

Theorem 4.3. *If $k > 2$, then*

$$D_k^+ \subset \mathcal{A}^+ \cdot \Gamma_k^\wedge .$$

In order to prove the theorem, we need the Hai–Hưng formula for the action of $\mathcal{A}$ on D_k (see [8]):

4.4.
$$Sq^i(Q_{k,j}) = \begin{cases} Q_{k,r} & i = 2^j - 2^r,\ r \le j, \\ Q_{k,t}Q_{k,r} & i = 2^k - 2^t + 2^j - 2^r,\ r \le j < t, \\ Q_{k,j}^2 & i = 2^k - 2^j, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$Sq^{2^{s+j-1}}(Q_{k,j}^{2^s}) = Q_{k,j-1}^{2^s},$$

for $0 < j < k$.

The following remark is an immediate consequence of 4.4 and the Cartan formula.

Remark 4.5. (on jump steps). If $Sq^i(Q_{k,j}^{2^s}) \neq 0$, then either $i = 0$ or $i \geq 2^{s+j-1}$. Furthermore, if $Sq^i X = Q_{k,j-1}^{2^s}$ for some $X \in D_k$, then either $i = 0$ or $i \geq 2^{s+j-1}$.

Proof of Theorem 4.3. We use techniques of the proof in [11] for the fact that

$$\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} \Gamma_k = 0 ,$$

for $k > 1$.

We follow the notations of [10]. For a given non-negative integer i, we denote by $s(i)$ the number with $2^{s(i)}$ being the first missing 2-power in the dyadic expansion of i. In other words, $i \equiv 2^{s(i)} - 1 \pmod{2^{s(i)+1}}$.

We prove the theorem by a downward induction.

Let $Q^I = Q_{k,k-1}^{i_{k-1}} \cdots Q_{k,0}^{i_0}$, $I = (i_{k-1}, \ldots, i_0)$ with $i_j \geq 0$ if $j \geq 1$ and $i_0 \in \mathbf{Z}$, be a monomial in Γ_k. Set $f_j(I) = s(i_j) + j - 1$ for $j \geq 1$ and $f(I) = \min\{f_j(I) |\ j \geq 1\}$.

Suppose $f_j(I) = f(I)$ with $j \geq 2$, then $s(i_{j-1}) + j - 2 = f_{j-1}(I) \geq f_j(I) = s(i_j) + j - 1$, so $s(i_{j-1}) \geq s(i_j) + 1$. In particular, $i_{j-1} \geq 2^{s(i_j)}$. Hence, applying 4.4 we get

4.6. $$Sq^{2^{f(I)}} Q^{(i_{k-1}, \ldots, i_j + 2^{s(i_j)}, i_{j-1} - 2^{s(i_j)}, \ldots, i_0)} = Q^I + \Sigma Q^L.$$

We show that $s(\ell_{j-1}) \leq s(i_j)$ and thus $f_{j-1}(L) = s(\ell_{j-1}) + j - 2 < s(i_j) + j - 1 = f_j(I) = f(I)$. Hence $f(L) < f(I)$ for any L in the sum. Let $\alpha_m(a)$ denote the coefficient of 2^m in the dyadic expansion of a, and $s = s(i_j)$. There are two cases. In the first case, $\alpha_s(\ell_{j-1}) = \alpha_s(i_{j-1}) = 1$. Then, using the remark on jump steps, we easily verify that $Sq^{2^{f(I)}}$ acts only to send $Q_{k,j}^{2^s}$ to $Q_{k,j-1}^{2^s}$. So $Q^L = Q^I$, but not an extra term. In the second case, $\alpha_s(\ell_{j-1}) = \alpha_s(i_{j-1} - 2^s) = 0$. Then, by definition, $s(\ell_{j-1}) \leq s = s(i_j)$.

It should be noted that if $Q^I \in D_k$, then the "killer"

$$Q^J = Q^{(i_{k-1}, \ldots, i_j + 2^{s(i_j)}, i_{j-1} - 2^{s(i_j)}, \ldots, i_0)}$$

and thus any extra term Q^L is in D_k.

Next we consider the case $f_1(I) = f(I)$ and $f_j(I) > f(I)$ if $1 < j < k$. Similarly as in 4.6, we have

4.7. $$Sq^{2^{f(I)}} Q^{(i_{k-1}, \ldots, i_1 + 2^{s(i_1)}, i_0 - 2^{s(i_1)})} = Q^I + \Sigma Q^L.$$

We show that, for any L in the sum, there exists j $(1 \leq j < k)$ such that $f_j(L) < f_1(I)$, therefore $f(L) < f(I)$. Suppose the contrary that $f_j(L) \geq f_1(I)$ or equivalently $s(\ell_j) + j - 1 \geq s(i_1)$ for $1 \leq j < k$. As $s(\ell_1) \geq s(i_1)$, then $\ell_1 = i_1 + 2^{s(i_1)}$. Otherwise, by the remark on jump steps, $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$ acts only to send $Q_{k,1}^{2^{s(i_1)}}$ to $Q_{k,0}^{2^{s(i_1)}}$, thus $Q^L = Q^I$, but not an extra term. Suppose inductively that $\ell_1 = i_1 + 2^{s(i_1)}, \ell_2 = i_2, \ldots, \ell_{j-1} = i_{j-1}$. We show $\ell_j = i_j$ $(1 \leq j < k)$. Indeed, if $\ell_j \neq i_j$, then combining $s(\ell_j) + j - 1 \geq s(i_1), s(i_j) + j - 1 > s(i_1)$ with the remark on jump steps, we easily verify that actually $s(\ell_j) + j - 1 = s(i_1)$ and $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$ acts only to send $Q_{k,j}^{2^{s(\ell_j)}}$ to $Q_{k,j-1}^{2^{s(\ell_j)}}$. This contradicts the hypothesis $\ell_{j-1} = i_{j-1}$ (or the hypothesis $\ell_1 = i_1 + 2^{s(i_1)}$ if $j = 2$). Consequently, we get $\ell_1 = i_1 + 2^{s(i_1)}, \ell_2 = i_2, ..., \ell_{k-1} = i_{k-1}$. So $Sq^{2^{f(I)}} = Sq^{2^{s(i_1)}}$ acts only to increase the power of $Q_{k,0}$. However, $dimQ_{k,0} = 2^k - 1 \not| 2^{s(i_1)}$ if $k > 1$. This is a contradiction.

Now we must show that if $Q^I \in D_k$, then the "killer"

$$Q^J = Q^{(i_{k-1},\cdots,i_1+2^{s(i_1)},i_0-2^{s(i_1)})}$$

and every "killer" needed in the procedure of using 4.6 and 4.7 to kill extra terms Q^L or to kill extra terms of extra terms ... are all in $\Gamma_k^\wedge$. Suppose Q^K is such a "killer". Let $s = s(i_1)$. As $f = f(I)$ decreases in the procedure, then

$$k_0 \geq i_0 - 2^s - 2^{s-1} - \cdots - 1 \ .$$

On the other hand, combining $f(I) = s$ with the two facts that the dimension of any extra term equals to $dimQ^I$ and $f(\text{extra term}) < f(I)$, we have

$$dimQ^K \geq dimQ^I - 2^{f(I)} = dimQ^I - 2^s.$$

Thus

$$\begin{aligned} k_0 + dimQ^K &\geq (i_0 - 2^s - 2^{s-1} - \cdots - 1) + (dimQ^I - 2^s) \\ &= (i_0 - 2^{s+1} + 1) + (dimQ^I - 2^s) \\ &\geq dimQ^I - 3 \cdot 2^s + 1 \ . \end{aligned}$$

From $s(i_1) = s$, it implies $i_1 \geq 2^s - 1$. Then we get

$$dimQ^I \geq dim(Q_{k,1}^{i_1}) \geq (2^s - 1)(2^k - 2) \ .$$

Hence

$$\begin{aligned} k_0 + dimQ^K &\geq (2^s - 1)(2^k - 2) - 3 \cdot 2^s + 1 \\ &= (2^s - 1)(2^k - 2) - 3 \cdot (2^s - 1) - 3 + 1 \\ &= (2^s - 1)(2^k - 5) - 2 \\ &\geq 2^k - 7 > 0 \quad (\text{as } k > 2) \ , \end{aligned}$$

except for $s = 0$. (This case is handled by the next step.) Consequently, $Q^K \in \Gamma_k^\wedge$.

To start the induction, assume $Q^I \in D_k$ with $f(I) = 0$. Then $f_1(I) = f(I) = 0$ as, by definition, $f_j(I) > 0$ for $1 < j < k$. Then $s = s(i_1) = 0$, and $i_1 \equiv 0 \pmod 2$. Thus

$$Sq^1 Q^{(i_{k-1},\cdots,i_1+1,i_0-1)} = Q^I.$$

The "killer" $Q^{(i_{k-1},\cdots,i_1+1,i_0-1)}$ is in $\Gamma_k^\wedge$, because

$$(i_0 - 1) + dimQ^{(i_{k-1},\cdots,i_1+1,i_0-1)} = i_0 + dimQ^I - 2 \geq 2^{k-1} - 2 > 0,$$

for $k > 2$. The first inequality holds as there exists at least one $i_j \neq 0$ and $dim(Q_{k,j}) \geq 2^{k-1}$ for $0 \leq j < k$.

The theorem is proved.

The following theorem establishes Conjecture 1.6 for $k = 3$.

Theorem 4.8. *Let $k = 3$. Then, for every $q \in D_3^+$, $[q] = 0$ in $H_3(\Gamma^\wedge) \cong Tor_3^{\mathcal{A}}(\mathbf{F}_2, \mathbf{F}_2)$.*

Proof. Let $\langle\cdot,\cdot\rangle$ denote the dual pairing between $Tor_3^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$ and $Ext_{\mathcal{A}}^3(\mathbf{F}_2,\mathbf{F}_2)$, and also the one between $(\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_3)^{GL_3}$ and $\mathbf{F}_2 \underset{GL_3}{\otimes} PH_*(BE^3)$. Here, by $PH_*(BE^3)$ we mean the submodule consisting of all $\mathcal{A}$-annihilated elements of $H_*(BE^3)$.

By Boardman [3], $Tr_3 : \mathbf{F}_2 \underset{GL_3}{\otimes} PH_*(BE^3) \to Ext_{\mathcal{A}}^3(\mathbf{F}_2,\mathbf{F}_2)$ is an isomorphism. In particular, we have

$$\begin{aligned}\langle [q], Ext_{\mathcal{A}}^3(\mathbf{F}_2,\mathbf{F}_2)\rangle &= \langle [q], Tr_3(\mathbf{F}_2 \underset{GL_3}{\otimes} PH_*(BE^3))\rangle \\ &= \langle Tr_3^*[q], \mathbf{F}_2 \underset{GL_3}{\otimes} PH_*(BE^3)\rangle \\ &= \langle [q], \mathbf{F}_2 \underset{GL_3}{\otimes} PH_*(BE^3)\rangle \quad \text{(by Theorem 3.2)} \\ &= 0,\end{aligned}$$

because $[q]=0$ in $(\mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_3)^{GL_3}$, by Theorem 3.2 of our paper [9].

Hence $[q]=0$ in $Tor_3^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$, for every $q \in D_3^+$. The theorem is proved.

5. Final remarks

Remark 5.1. The weak conjecture on spherical classes is actually equivalent to the fact that for every $q \in D_k^+$ and any $k > 2$, $[q] \in Tor_k^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$ vanishes on the image of Singer's k-th transfer, $Tr_k(\mathbf{F}_2 \underset{GL_k}{\otimes} PH_*(BE^k)) \subset Ext_{\mathcal{A}}^k(\mathbf{F}_2,\mathbf{F}_2)$. This observation can be read off from the proof of Theorem 4.8.

Remark 5.2. Conjecture 1.6, and therefore Conjecture 4.1, is false when $k=1$ or 2. Indeed, $[Q_{1,0}^{2^i-1}]$ and $[Q_{2,1}^{2^i-1}]$ are non-zero in $Tor_*^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$. They are respectively dual to the Adams element $h_i \in Ext_{\mathcal{A}}^1(\mathbf{F}_2,\mathbf{F}_2)$ and its square $h_i^2 \in Ext_{\mathcal{A}}^2(\mathbf{F}_2,\mathbf{F}_2)$. One easily verifies this assertion by combining Theorem 3.2 with Theorem 2.1 of [9] and Proposition 5.3 of [14]. The only Hopf invariant one elements are represented by h_i for $i=1,2,3$ (see Adams [1]). Furthermore, the only Kervaire invariant one elements are represented by h_i^2, wherever h_i^2 is a permanent cycle in the Adams spectral sequence for spheres (see Browder [5]).

Let $\varphi_k^* : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to Tor_k^{\mathcal{A}}(\mathbf{F}_2,\mathbf{F}_2)$ be the dual of the Lannes–Zarati homomorphism, which is compatible with the Hurewicz one $H : \pi_*(Q_0S^0) \to H_*(Q_0S^0)$. In [9] we have proved that the inclusion $D_k \subset P_k$ is a chain-level representation of $Tr_k^* \cdot \varphi_k^* : \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} D_k \to \mathbf{F}_2 \underset{\mathcal{A}}{\otimes} P_k$. This together with Theorem 3.2 lead us to the following conjecture.

Conjecture 5.3. *The inclusion $D_k \subset \Gamma_k^\wedge$ is a chain-level representation of φ_k^*.*

This conjecture and Conjecture 1.6 imply the classical conjecture on spherical classes.

References

[1] Adams, J. F.: *On the non-existence of elements of Hopf invariant one*, Ann. Math. **72**, 20–104 (1960)

[2] Adams, J. F.: *Operations of the n'th kind in K-theory and what we don't know about* $\mathbf{RP}^\infty$, New developments in topology. In: G. Segal (ed.), Lond. Math. Soc. Lect. Note Series **11**, 1–9 (1974)

[3] Boardman, J. M.: *Modular representations on the homology of powers of real projective space,* Algebraic Topology: Oaxtepec 1991. In: M. C. Tangora (ed.), Contemp. Math. **146**, 49–70 (1993)
[4] Bousfield, A. K., Curtis, E. B., Kan, D. M., Quillen, D. G., Rector, D. L., Schlesinger, J. W.: *The mod p lower central series and the Adams spectral sequence*, Topology **5**, 331–342 (1966)
[5] Browder, W.: *The Kervaire invariant of a framed manifold and its generalization,* Ann. Math. **90**, 157–186 (1969)
[6] Curtis, E. B.: *The Dyer–Lashof algebra and the Lambda algebra*, Illinois Jour. Math. **18**, 231–246 (1975)
[7] Dickson, L. E.: *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12**, 75–98 (1911)
[8] Hưng, N. H. V.: *The action of the Steenrod squares on the modular invariants of linear groups,* Proc. Amer. Math. Soc. **113**, 1097–1104 (1991)
[9] Hưng, N. H. V.: *Spherical classes and the algebraic transfer*, Trans. Amer. Math. Soc. **349**, 3893–3910 (1997)
[10] Hưng, N. H. V., Peterson, F. P.: *$\mathcal{A}$-generators for the Dickson algebra*, Trans. Amer. Math. Soc. **347**, 4687–4728 (1995)
[11] Hưng, N. H. V., Peterson, F. P.: *Spherical classes and the Dickson algebra*, Math. Proc. Camb. Phil. Soc. **124**, 253– 264 (1998)
[12] Kameko, M.: *Products of projective spaces as Steenrod modules*, Thesis, Johns Hopkins University 1990
[13] Lannes, J., Zarati, S.: *Invariants de Hopf d'ordre supérieur et suite spectrale d'Adams*, C. R. Acad. Sci. **296**, 695–698 (1983)
[14] Lannes, J., Zarati, S.: *Sur les foncteurs dérivés de la déstabilisation*, Math. Zeit. **194**, 25–59 (1987)
[15] MacLane, S.: *Homology*, Die Grundlehren der Math. Wissenschaften, Band **114**, Academic Press, Springer-Verlag, Berlin and New York 1963
[16] Madsen, I.: *On the action of the Dyer–Lashof algebra in $H_*(G)$*, Pacific Jour. Math. **60**, 235–275 (1975)
[17] Mùi, H.: *Modular invariant theory and cohomology algebras of symmetric groups*, Jour. Fac. Sci. Univ. Tokyo, **22**, 310–369 (1975)
[18] Peterson, F. P.: *Generators of $H^*(\mathbf{RP}^\infty \wedge \mathbf{RP}^\infty)$ as a module over the Steenrod algebra,* Abstracts Amer. Math. Soc., No **833**, April 1987
[19] Priddy, S.: *On characterizing summands in the classifying space of a group, I*, Amer. Jour. Math. **112**, 737–748 (1990)
[20] Silverman, J. H.: *Hit polynomials and the canonical antiautomorphism of the Steenrod algebra*, Proc. Amer. Math. Soc. **123**, 627–637 (1995)
[21] Singer, W. M.: *Invariant theory and the lambda algebra*, Trans. Amer. Math. Soc. **280**, 673–693 (1983)
[22] Singer, W. M.: *The transfer in homological algebra*, Math. Zeit. **202**, 493–523 (1989)
[23] Steenrod N. E., Epstein, D. B. A.: *Cohomology operations*, Ann. of Math. Studies, No. **50**, Princeton Univ. Press 1962
[24] Wellington, R. J.: *The unstable Adams spectral sequence of free iterated loop spaces*, Memoirs Amer. Math. Soc. **258** (1982)
[25] Wilkerson, C.: *Classifying spaces, Steenrod operations and algebraic closure*, Topology **16**, 227–237 (1977)
[26] Wood, R. M. W.: *Steenrod squares of polynomials and the Peterson conjecture*, Math. Proc. Camb. Phil. Soc. **105**, 307–309 (1989)

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Note added in proof: Recently, we have established Conjecture 5.3.