MAHOWALDEAN FAMILIES OF ELEMENTS IN STABLE HOMOTOPY GROUPS REVISITED

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1. Introduction

In the mid 1970's Mark Mahowald constructed a new infinite family of elements in the 2–component of the stable homotopy groups of spheres, $\eta_j \in \pi^S_{2^j}(S^0)_{(2)}$ [M]. Using standard Adams spectral sequence terminology (which will be recalled in §3 below), η_j is detected by $h_1 h_j \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$. Thus he had found an infinite family of elements all having the same Adams filtration (in this case, 2), thus dooming the so–called Doomsday Conjecture. His constructions were ingenious: his elements were constructed as composites of pairs of maps, with the intermediate spaces having, on one hand, a geometric origin coming from double loopspace theory, and, on the other hand, mod 2 cohomology making them amenable to Adams Spectral Sequence analysis and suggesting that they were related to the new discovered Brown–Gitler spectra [BG].

In the years that followed, various other related 2–primary infinite families were constructed, perhaps most notably (and correctly) R.Bruner's family detected by $h_2 h_{j^2} \in \operatorname{Ext}^{3,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$ [B]. An odd prime version was studied by R.Cohen [C], leading to a family in $\pi^S_*(S^0)_{(p)}$ detected by $h_0 b_j \in \operatorname{Ext}^{3,*}_{\mathcal{A}}(\mathbf{Z}/p, \mathbf{Z}/p)$, and a filtration 2 family in the stable homotopy groups of the odd prime Moore space. Cohen also initiated the development of odd primary Brown–Gitler spectra, completed in the mid 1980's, using a different approach, by P.Goerss [G], and given the ultimate "modern" treatment by Goerss, J.Lannes, and F.Morel in the 1993 paper [GLM]. Various papers in the late 1970's and early 1980's, e.g. [BP, C, BC], related some of these to loopspace constructions.

Our project originated with two goals. One was to see if any of the later work on Brown–Gitler spectra led to clarification of the original constructions. The other was to see if taking advantage of post Segal Conjecture knowledge of the stable cohomotopy of the classifying space $B\mathbf{Z}/p$ would help in constructing new families at odd primes, in particular a conjectural family detected by $h_0 h_j \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/p, \mathbf{Z}/p)$. (This followed a paper [K1] by one of us on 2 primary families from this point of view.)

What resulted, and what we do here, is the following. We isolate the two crucial results from the older literature (Proposition 2.1 and Proposition 2.2 below), and present these stripped of extraneous detours. We then reorganize how these results

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are used, together with the new idea that $B\mathbf{Z}/p$ should be an intermediate space in the constructions. This leads to streamlined proofs of the main theorems of [M, B, C]. In the odd prime case, improvements are most dramatic: generously counting, we need only 30 of the 75 pages making up Cohen's original proof.

With respect to our original goals, we can report the following.

Regarding Brown–Gitler spectra, in some logical sense, *nothing* subsequently learned about them helps simplify the construction of these sorts of homotopy classes. Indeed, one of our highlighted older results, Proposition 2.2, is precisely what is also essential when one reviews the literature relating Brown–Gitler spectra to pieces of double loop spaces. However, our proof of this key proposition does take advantage of some later observations: in particular, we use Carlsson modules [Ca], and the fact that these modules can be realized as the cohomology of certain mapping telescopes. In a companion paper [HK], we will discuss this more thoroughly.

Regarding odd primary classes, to recover the main theorems of [C], we find we need to construct one element in the cohomotopy of $B\mathbf{Z}/p$ (see Proposition 2.3). This we do using "elementary" methods and maps which would have all been available in the mid 1970's. We are not able to determine whether the elements h_0h_j are permanent cycles at odd primes. However our parallel development of the $p = 2$ and the odd prime cases suggest that they are *not*, and that a commonly held view, that the odd prime elements $h_0h_j \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/p, \mathbf{Z}/p)$ are analogous to $h_1h_j \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$, is perhaps misguided. We find that the elements $h_2h_j \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$, which are not permanent cycles, behave more similarly.

In §2, we quickly state three key propositions. Assuming these, the main theorems of [M, B, C], and related results, are then formally deduced in §3. The proof of each key proposition is then discussed in its own section. In a final short section, we deduce some related mod 2 results using elementary means (and note that odd prime analogues don't exist).

In the rest of the paper, we are working in the stable homotopy category. Spectra are completed at p, and $H^*(X)$ means cohomology with $\mathbf{Z}/p$ coefficients, where the prime p will be clear from the context.

Some of the results in this paper appeared in the first author's thesis [H].

2. Three key propositions

To state our first two propositions, we need to define certain finite spectra $T(n)$.

Recall that, if X is a path connected space, there is a stable decomposition [Sn]

$$\Omega^2\Sigma^2 X \simeq \bigvee_{m>0} D_{2,m}X.$$

Here $D_{2,m}X = F(\mathbf{R}^2, m)_+ \wedge_{\Sigma_m} X^{[m]}$, where $F(\mathbf{R}^2, m)$ is the configuration space of ordered m–tuples of distinct points in $\mathbf{R}^2$, Y_+ denotes a space Y with a disjoint basepoint, and the symmetric group Σ_m acts in the obvious way on both $F(\mathbf{R}^2, m)$ and $X^{[m]}$, the m–fold smash product of X with itself. (We also let $D_{2,0}X = S^0$.) Fixing a prime p, we then define $T(n)$ for $n \geq 0$ by S–duality:

$$T(2r + \epsilon) = \Sigma^{2pr+2\epsilon} \text{ Dual } (D_{2,pr+\epsilon}S^1)$$

for all $r \geq 0$, $\epsilon = 0, 1$.

Viewed as a module over the mod p Steenrod algebra $\mathcal{A}$, $H^*(T(n);\mathbf{Z}/p)$ is dual to an appropriate Brown–Gitler module. When $p=2$, this is [M, Thm.2.6]; when p is odd, this is the content of the 10 page second chapter of [C]. Our indexing has been chosen to be consistant with the modern literature (for example [Mi, §7]), so that $H^*(T(n);\mathbf{Z}/p)$ is the injective envelope of $H^*(S^n;\mathbf{Z}/p)$ in the category of unstable $\mathcal{A}$–modules[1]. In particular, $H^*(T(2p^j);\mathbf{Z}/p)$ has bottom classes in dimensions 1 and 2 linked by the Bockstein, and top class in dimension $2p^j$.

Let $C(g)$ denote the cofiber of a map g.

Proposition 2.1.

(1) *($p=2$) There exist maps* $g_j : S^{2^j} \to T(2^{j-3})$ *with* Sq^{2^j} *acting nonzero on* $H^1(C(g_j);\mathbf{Z}/2)$.
(2) *(p odd) There exist maps* $g_j : S^{2(p-1)p^j} \to T(2p^{j-1})$ *with* $\mathcal{P}^{p^j}$ *acting nonzero on* $H^1(C(g_j);\mathbf{Z}/p)$.

When $p=2$, this is [M, Thm.2(b)]. When p is odd, this is [C, Thm.IV.2.1]. (The maps in these references are S–dual to ours.)

Proposition 2.2.

(1) *($p=2$) There exist maps* $f_j : T(2^{j-3}) \to B\mathbf{Z}/2$ *with* $H_1(f_j;\mathbf{Z}/2) \neq 0$.
(2) *(p odd) There exist maps* $f_j : T(2p^{j-1}) \to B\mathbf{Z}/p$ *with* $H_1(f_j;\mathbf{Z}/p) \neq 0$.

When $p=2$, this was proved by Brown and Peterson in [BP, Lemma 4.1]. When p is odd, this is [C, Thm.III.2.2]. (Starting from these references, readers will have to use properties of S–duality of finite complexes and manifolds to read off the proposition as stated.)

Proposition 2.3.

(1) *($p=2$) Let* $M=\Sigma^{-3}\mathbf{R}P^2$. *There exists a map* $e : B\mathbf{Z}/2 \to M$ *with* Sq^4 *acting nonzero on* $H^i(C(e);\mathbf{Z}/2)$, $i=-2,-1$.
(2) *(p odd) Let* $M = S^{4-2p}\cup_p D^{5-2p}$. *There exists a map* $e : B\mathbf{Z}/p \to M$ *with* $\mathcal{P}^1$ *acting nonzero on* $H^i(C(e);\mathbf{Z}/p)$, $i=(4-2p),(5-2p)$.

When $p=2$, this was implicitly stated in [K1, paragraph before Thm.4.4].

3. The main theorems

Let $[X,Y]$ denote the stable homotopy classes of maps between two connective, p–complete, spectra X and Y. Recall (see, e.g. [R2]) that the classic Adams spectral sequence arises from a filtration of $[X,Y]$, where a map $f : X \to Y$ has filtration at least s if it can be written as a composite

$$X \xrightarrow{f_1} Y_1 \xrightarrow{f_2} \ldots \to Y_{(s-1)} \xrightarrow{f_s} Y$$

in which each $H^*(f_i;\mathbf{Z}/p)=0$. Intuitively, as the filtration increases, maps are harder to understand. The spectral sequence takes the form

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y;\mathbf{Z}/p),H^*(X;\mathbf{Z}/p)) = E_2^{s,t} \Rightarrow [\Sigma^{t-s}X,Y].$$

[1]It is miraculous that it is unstable.

Using standard notation, $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbf{Z}/2,\mathbf{Z}/2)$ is spanned by elements

$$h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2^j}(\mathbf{Z}/2,\mathbf{Z}/2), j \geq 0,$$

with h_j corresponding to the indecomposable $Sq^{2^j} \in \mathcal{A}$. Adams [A] showed that $d_2(h_j) = h_0 h_{j-1}^2$ and deduced that the permanent cyles in $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbf{Z}/2,\mathbf{Z}/2)$ are spanned by h_0, h_1, h_2, and h_3. These correspond to the classic Hopf maps $2 \in \pi_0^S(S^0)$, $\eta \in \pi_1^S(S^0)$, $\nu \in \pi_3^S(S^0)$, and $\sigma \in \pi_7^S(S^0)$, respectively.

For p odd, $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbf{Z}/p,\mathbf{Z}/p)$ is spanned by $a_0 \in \mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbf{Z}/p,\mathbf{Z}/p)$ and elements

$$h_j \in \mathrm{Ext}_{\mathcal{A}}^{1,2(p-1)p^j}(\mathbf{Z}/p,\mathbf{Z}/p), j \geq 0,$$

with a_0 and h_j respectively corresponding to β and $\mathcal{P}^{p^j} \in \mathcal{A}$. Then $d_2(h_j) = a_0 b_{j-1}$, where $b_{j-1} \in \mathrm{Ext}_{\mathcal{A}}^{2,2(p-1)p^j}(\mathbf{Z}/p,\mathbf{Z}/p)$ is the p–fold Massey product $\langle h_{j-1},\dots,h_{j-1}\rangle$ (see [Liu]). The filtration 1 permanent cycles are spanned by a_0 and h_0, which represent the elements $p \in \pi_0^S(S^0)$ and $\alpha \in \pi_{2p-3}^S(S^0)$.

Theorem 3.1.

(1) *($p = 2$)*
  (a) *The map g_j is represented by $i_{1*}(h_j) \in \mathrm{Ext}_{\mathcal{A}}^{1,*}(H^*(T(2^{j-3})),\mathbf{Z}/2)$.*
  (b) *The composite $S^{2^j} \xrightarrow{g_j} T(2^{j-3}) \xrightarrow{\pi} T(2^{j-3})/S^1$ is represented by $i_{2*}(h_{j-1}^2) \in \mathrm{Ext}_{\mathcal{A}}^{2,*}(H^*(T(2^{j-3})/S^1),\mathbf{Z}/2)$.*

(2) *(p odd)*
  (a) *The map g_j is represented by $i_{1*}(h_j) \in \mathrm{Ext}_{\mathcal{A}}^{1,*}(H^*(T(2p^{j-1})),\mathbf{Z}/2)$.*
  (b) *The composite $S^{2(p-1)p^j} \xrightarrow{g_j} T(2p^{j-1}) \xrightarrow{\pi} T(2p^{j-1})/S^1$ is represented by $i_{2*}(b_{j-1}) \in \mathrm{Ext}_{\mathcal{A}}^{2,*}(H^*(T(2p^{j-1})/S^1),\mathbf{Z}/p)$.*

Here $i_1 : S^1 \to T(2p^j)$ and $i_2 : S^2 \to T(2p^j)/S^1$ are inclusions of bottom cells.

Proof. Both parts (a) are just reformulations of Proposition 2.1, noting that, e.g. when p is odd, dimensions considerations show that the only primary operation that can connect $H^*(T(2p^{j-1});\mathbf{Z}/p)$ to $H^*(S^{2(p-1)p^j};\mathbf{Z}/p)$ in $H^*(C(g_j);\mathbf{Z}/p)$ is $\mathcal{P}^{p^j}$. Then both parts (b) follow from this, using the factorizations of Sq^{2^j} and $\mathcal{P}^{p^j}$ into a sum of primary operations composed with secondary operations (see [A, Liu]). Once again, dimension considerations show that the only secondary operations in this decomposition that can act nontrivially on $H^*(C(g_j);\mathbf{Z}/p)$ are the ones associated to h_{j-1}^2 (when $p = 2$) and b_{j-1} (when p is odd). □

Theorem 3.2.

(1) *($p = 2$)*
  (a) *The composite $f_j \circ g_j : S^{2^j} \to B\mathbf{Z}/2$ is represented by $i_{1*}(h_j) \in \mathrm{Ext}_{\mathcal{A}}^{1,*}(H^*(B\mathbf{Z}/2),\mathbf{Z}/2)$.*
  (b) *The composite $S^{2^j} \xrightarrow{f_j \circ g_j} B\mathbf{Z}/2 \xrightarrow{\pi} (B\mathbf{Z}/2)/S^1$ is represented by $i_{2*}(h_{j-1}^2) \in \mathrm{Ext}_{\mathcal{A}}^{2,*}(H^*((B\mathbf{Z}/2)/S^1),\mathbf{Z}/2)$.*

(2) *(p odd)*
  (a) *The composite $f_j \circ g_j : S^{2(p-1)p^j} \to B\mathbf{Z}/p$ is represented by $i_{1*}(h_j) \in \mathrm{Ext}_{\mathcal{A}}^{1,*}(H^*(B\mathbf{Z}/p),\mathbf{Z}/p)$.*

(b) *The composite* $S^{2(p-1)p^j} \xrightarrow{f_j \circ g_j} B\mathbf{Z}/p \xrightarrow{\pi} (B\mathbf{Z}/p)/S^1$ *is represented by* $i_{2*}(b_{j-1}) \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(H^*((B\mathbf{Z}/p)/S^1), \mathbf{Z}/p)$.

As before, $i_1 : S^1 \to B\mathbf{Z}/p$ and $i_2 : S^2 \to (B\mathbf{Z}/p)/S^1$ are inclusions of bottom cells.

Proof. Using naturality properties of the Adams spectral sequence with respect to composition, this is an immediate consequence of Theorem 3.1 and Proposition 2.2. □

Remarks 3.3.
(1) When $p = 2$, these last two theorems appear in work by Cohen, J.Jones, and Mahowald [CJM, Cor.4.7 and the argument on p.118], and they then go on to easily conclude that the composite

$$S^{2^j} \xrightarrow{f_j \circ g_j} B\mathbf{Z}/2 \to BSO(2)$$

is represented by $i_{2*}(h^2_{j-1}) \in \operatorname{Ext}^{2,*}_{\mathcal{A}}(H^*((BSO(2)), \mathbf{Z}/2)$, where $i_2 : S^2 \to BSO(2)$ is the inclusion of the bottom cell. (Since $\Sigma^2 MSO(2) \simeq BSO(2)$, this is then interpreted as a result about the Kervaire invariant of $2^j - 2$ dimensional oriented manifolds immersed in Euclidean space with codimension 2.) Though they proceed roughly as we have here, we have a couple of quibbles about their arguments. Firstly, they don't deduce part (1)(b) of Theorem 3.1 from part (1)(a) as we do. They use instead compatibility of the maps g_j with respect to various pairings, a method that fails at odd primes. Secondly, they use essentially circular reasoning in their argument for the existence of f_j: they argue that f_j as in Proposition 2.2 exists by using the fact that $T(2^j)$ was shown to be an appropriate S–dual of a Brown–Gitler spectrum by Brown and Peterson in [BP]. But this theorem of Brown and Peterson was itself proved by using the existence of maps f_j.
(2) When p is odd, Theorem 3.1(2)(b) essentially appears as [C, Cor.III.3.6], with an argument like ours. However, Cohen never combines Theorem 3.1(2) with Proposition 2.2(2) to deduce Theorem 3.2(2). Indeed, Proposition 2.2(2) is only used by him as a technical lemma enroute to proving that the family of spectra $T(2pr + 2), r \geq 0$, are S–dual to odd primary Brown–Gitler spectra (that he has constructed).
(3) When p is odd, the action of $\mathbf{Z}/(p-1) \simeq (\mathbf{Z}/p)^\times$ on $\mathbf{Z}/p$ induces the "eigenspectra" stable decomposition

$$B\mathbf{Z}/p \simeq Z(1) \vee \cdots \vee Z(p-1),$$

indexed so that $Z(j)$ has bottom cell in dimension $2j - 1$. ($Z(p-1)$ is the p–localization of $B\Sigma_p$.) Obviously, both Proposition 2.2(2) and Theorem 3.2(2) can be refined by replacing $B\mathbf{Z}/p$ by $Z(1)$.

If one is interested in constructing families of elements in the stable homotopy groups of spheres, Theorem 3.2 suggests hunting for elements in the cohomotopy of $B\mathbf{Z}/p$ and $(B\mathbf{Z}/p)/S^1$ that are tractable and nontrivial when respectively restricted to their bottom cells S^1 and S^2.

Certainly the simplest and best known element in $\pi^*_S(B\mathbf{Z}/p)$ is the Kahn–Priddy map $t : B\mathbf{Z}/p \to S^0$ defined as the composite $B\mathbf{Z}/p \to B\mathbf{Z}/p_+ \xrightarrow{t_{\mathbf{Z}/p}} S^0$, where $t_{\mathbf{Z}/p}$ is the $\mathbf{Z}/p$–transfer.

When $p = 2$, t restricted to S^1 is η, which is detected by $h_1 \in \mathrm{Ext}^{1,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$. One recovers Mahowald's original η_j family.

Theorem 3.4. *The composite $(t \circ f_j \circ g_j) \in \pi^S_{2^j}(S^0)$ is represented by $h_1 h_j \in \mathrm{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$.*

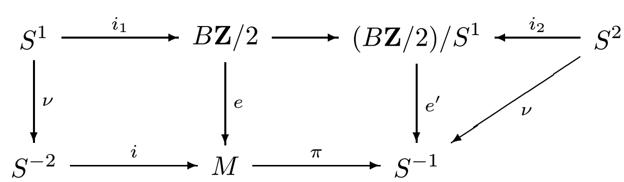
Proof. By dimension reasons, t has Adams filtration 1, and not 0. Thus t will be represented by an element $\tilde{h}_1 \in \mathrm{Ext}^{1,*}_{\mathcal{A}}(\mathbf{Z}/2, H^*(B\mathbf{Z}/2))$ such that $i^*(\tilde{h}_1) = h_1 \in \mathrm{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2, \mathbf{Z}/2)$. By Theorem 3.2(1)(a), $(f_j \circ g_j)$ is represented by $i_*(h_j) \in \mathrm{Ext}^{1,*}_{\mathcal{A}}(H^*(B\mathbf{Z}/2), \mathbf{Z}/2)$. Thus $(t \circ f_j \circ g_j)$ will be represented by $\tilde{h}_1 i_*(h_j) = i^*(\tilde{h}_1) h_j = h_1 h_j$. □

Now consider the same construction when p is odd. In contrast to the even case, t restricted to S^1 is null. Indeed, under the decomposition of $B\mathbf{Z}/p$ given in Remarks 3.3(3), t factors through the summand $Z(p-1) \simeq B\Sigma_p$. Thus, if the map f_j in Proposition 2.2 has been chosen to land in the $Z(1)$ summand, the composite $(t \circ f_j \circ g_j)$ will be 0.

The maps e of Proposition 2.3 amount to next simplest maps out of $B\mathbf{Z}/p$. The next proposition is simply a convenient reformulation of Proposition 2.3.

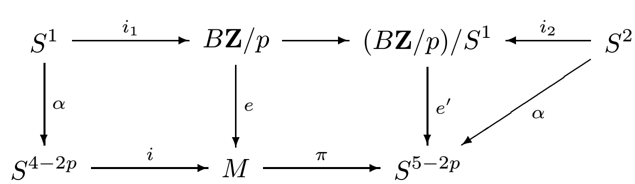
Proposition 3.5.

(1) *($p = 2$) Let $M = \Sigma^{-3}\mathbf{R}P^2$. There exists a diagram*



in which the left square commutes, e' is induced by e so that the middle square commutes, and the right triangle commutes up to multiplication by an odd integer.

(2) *(p odd) Let $M = S^{4-2p} \cup_p D^{5-2p}$. There exists a diagram*



in which the left square commutes, e' is induced by e so that the middle square commutes, and the right triangle commutes up to multiplication by an integer prime to p.

In each part of this proposition, the lower horizontal maps form the obvious cofibration sequence, and we note that, for all primes p, the projection map π has order p.

Once again, dimension reasons imply that the maps e and e' have Adams filtration 1, and not 0. Thus combined with Theorem 3.2, Proposition 3.5 implies the following theorem.

Theorem 3.6.

(1) *($p=2$) Let* $M=\Sigma^{-3}\mathbf{R}P^2$.
   (a) *The composite* $(e\circ f_j\circ g_j): S^{2^j}\to M$ *is represented by* $i_*(h_2h_j)\in \mathrm{Ext}^{2,*}_{\mathcal{A}}(H^*(M),\mathbf{Z}/2)$.
   (b) *The composite* $(\pi\circ e\circ f_j\circ g_j)\in\pi^S_{2^j+1}(S^0)$ *is an element of order* 2 *represented by* $h_2h_{j-1}^2\in\mathrm{Ext}^{3,*}_{\mathcal{A}}(\mathbf{Z}/2,\mathbf{Z}/2)$.

(2) *(p odd) Let* $M=S^{4-2p}\cup_p D^{5-2p}$.
   (a) *The composite* $(e\circ f_j\circ g_j): S^{2(p-1)p^j}\to M$ *is represented by* $i_*(h_0h_j)\in \mathrm{Ext}^{2,*}_{\mathcal{A}}(H^*(M),\mathbf{Z}/p)$.
   (b) *The composite* $(\pi\circ e\circ f_j\circ g_j)\in\pi^S_{2(p-1)p^j+2p-5}(S^0)$ *is an element of order p represented by* $h_0b_{j-1}\in\mathrm{Ext}^{3,*}_{\mathcal{A}}(\mathbf{Z}/p,\mathbf{Z}/p)$*, up to multiplication by an element in* $(\mathbf{Z}/p)^\times$.

Part (1)(b) of this theorem says that, at the prime 2, $h_2h_{j-1}^2$ is a permanent cycle representing an element of order 2 in $\pi^S_*(S^0)$. This is the main theorem of [B]. This proof of Bruner's theorem is roughly that of [CJM, Thm.4.12] and [K1, Thm.5.1], except that [CJM] don't recover that the elements have order 2, and [K1] more awkwardly shows this.

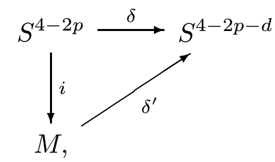
Part (2) of Theorem 3.6 says that, at an odd prime p, h_0b_{j-1} is a permanent cycle representing an element of order p in $\pi^S_*(S^0)$ and that $i_*(h_0h_j)$ is a permanent cycle in the Adams spectral computing the stable homotopy groups of a mod p Moore space. These are the two main theorems of [C], but our proof here is not Cohen's: as mentioned in the introduction, our argument is roughly 45 pages shorter than his.

Remark 3.7. It has been widely thought that the family of elements h_0h_j is the analogue, at odd primes, of the 2 primary family $h_1h_j\in\mathrm{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2,\mathbf{Z}/2)$. Thus, since h_1h_j is a permanent cycle when $p=2$, it has been conjectured that h_0h_j should be a permanent cycle at odd primes[2]. Our constructions indicate that the h_0h_j family would be better deemed analogous to the family $h_2h_j\in\mathrm{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/2,\mathbf{Z}/2)$. Since these are *not* permanent cycles, we conjecture that neither are $h_0h_j\in\mathrm{Ext}^{2,*}_{\mathcal{A}}(\mathbf{Z}/p,\mathbf{Z}/p)$.

There is an easy way to use parts (a) of Theorem 3.6 to construct more infinite families of permanent cycles in $\mathrm{Ext}^{*,*}_{\mathcal{A}}(\mathbf{Z}/p,\mathbf{Z}/p)$. Suppose given $\delta\in\pi^S_d(S^0)$ of

[2]Indeed the main theorem of [CG] asserts precisely this. Unfortunately, N.Minami has pointed out that the first sentence on p.186 of [CG] is incorrect in an essential way, and we ultimately learn nothing about the existance of odd primary η_j from these authors' efforts.

order p. Then, when p is odd, δ factors:



and there is an analogous diagram if $p = 2$.

If δ is represented by $d \in \mathrm{Ext}_{\mathcal{A}}^{s,d+s}(\mathbf{Z}/p, \mathbf{Z}/p)$, we might hope that the composite $(\delta' \circ e \circ f_j \circ g_j) \in \pi^S_{2(p-1)p^j+2p+d-4}(S^0)$ will be represented by dh_0h_j through Adams filtration $s+2$, so that dh_0h_j is a permanent cycle. (This would be dh_2h_j if $p = 2$.) This will be the case if the lift δ' can be chosen so that it also has Adams filtration s, and is represented by an element $d' \in \mathrm{Ext}_{\mathcal{A}}^{s,d-s}(\mathbf{Z}/p, H^*(M))$ satisfying $i^*(d') = d$. The following corollary, a strengthening and generalization to all primes of [K1, Thm.4.4], contains an easy to verify condition ensuring that this will be the case.

Corollary 3.8. *Suppose that $\delta \in \pi^S_d(S^0)$ is represented by $d \in \mathrm{Ext}_{\mathcal{A}}^{s,d+s}(\mathbf{Z}/p, \mathbf{Z}/p)$ and $p\delta = 0$. Suppose that $\mathrm{Ext}_{\mathcal{A}}^{s',d+s'+1}(\mathbf{Z}/p, \mathbf{Z}/p)$ contains only permanent cycles for $s' \leq s-1$.*

(1) *($p = 2$) Then $dh_2h_j \in \mathrm{Ext}_{\mathcal{A}}^{s+2,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ will be a permanent cycle, for all j.*
(2) *(p odd) Then $dh_0h_j \in \mathrm{Ext}_{\mathcal{A}}^{s+2,*}(\mathbf{Z}/p, \mathbf{Z}/p)$ will be a permanent cycle, for all j.*

The proof in [K1] works for all primes without change.

Example 3.9. Let p be odd. The element $\beta_2 \in \pi^S_{4p^2-2p-4}(S^0)$ has order p and is represented by $k_0 = \langle h_0, h_1, h_1 \rangle \in \mathrm{Ext}_{\mathcal{A}}^{2,2(p-1)(2p+1)}(\mathbf{Z}/p, \mathbf{Z}/p)$ [R2, p.205]. We conclude that $k_0h_0h_j \in \mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbf{Z}/p, \mathbf{Z}/p)$ will be a permanent cycle, for all j.

Remark 3.10. By construction, in the situation above, the composite $(\delta' \circ e \circ f_j \circ g_j)$ will be an element in the Toda bracket $\langle \delta, p, (\pi \circ e \circ f_j \circ g_j) \rangle$. Thus, even when the Ext condition of Corollary 3.8 fails, one still might be able to deduce the Adams spectral sequence name for this composite using work of Moss [Mo] (though checking his Ext conditions will require having some control over certain infinite families of Ext groups[3]).

4. An outline of the proof of Proposition 2.1

Here, for completeness, we briefly sketch the construction of the maps g_j of Proposition 2.1. In discussing this construction, and even more essentially, the construction of the maps f_j, it is useful to recall the following lemma from [CMM].

Lemma 4.1. $D_{2,m}\Sigma^2 X \simeq \Sigma^{2m} D_{2,m} X$.

[3]Checking Moss' conditions wasn't done in [L2, L3], but perhaps could be, completing Lin's arguments.

Recall that we defined $T(2r)$ to be Σ^{2pr} Dual $(D_{2,pr}S^1)$. Using the lemma, we can write

$$T(2r) \simeq \Sigma^{2(p-1)pr} \text{ Dual } (D_{2,pr}S^{(2p-3)}).$$

When p is odd, the construction of the maps g_j then goes as follows.

Let O be the infinite orthogonal group, and let Q_1S^0 the component of the identity map in $QS^0 = \operatorname{colim}_{n\to\infty} \Omega^n S^n$. Viewing $\alpha \in \pi^S_{2p-3}(S^0)$ as an unstable map $\alpha : S^{2p-3} \to Q_1S^0$, α factors through the J–homomorphism:

$$S^{2p-3} \xrightarrow{\alpha'} O \xrightarrow{J} Q_1S^0.$$

Using the infinite loopspace structure on O, α' has a canonical double loop extension

$$\alpha'' : \Omega^2 S^{2p-1} \to O,$$

and we let $\bar{\alpha} : (\Omega^2 S^{2p-1})_+ \to S^0$ be the *stable* map adjoint to the composite of based unstable maps

$$(\Omega^2 S^{2p-1})_+ \xrightarrow{\alpha''} O_+ \xrightarrow{J} QS^0.$$

The map g_j is then defined to be the $2(p-1)p^j$th suspension of the S–dual of the composite

$$D_{2,p^j} S^{2p-3} \hookrightarrow (\Omega^2 S^{2p-1})_+ \xrightarrow{\bar{\alpha}} S^0.$$

When $p = 2$ there is a similar construction, starting with a map $\sigma' : S^7 \to O$ lifting $\sigma \in \pi^S_7(S^0)$.

The assertion of Proposition 2.1, that appropriate cohomology operations act nontrivially in the cohomology of the cofibers, is proved in a couple of pages in [C, M] using characteristic class arguments. We know of no improvement upon these authors' arguments.

5. An outline of the proof of Proposition 2.2

In this section, we outline the proof of Proposition 2.2. Though this will basically be the same as in [BP, C], with the idea going back to [M], we will take advantage of work in the last 15 years on $\mathcal{U}$, the category of unstable $\mathcal{A}$–modules. This, we feel, greatly clarifies the presentation.

5.1. algebraic results. Let $J(n) = H^*(T(n); \mathbf{Z}/p)$. Then $J(n)$ is an unstable module, and for all $M \in \mathcal{U}$, we have a natural isomorphism:

$$\operatorname{Hom}_A(M, J(n)) \simeq (M_n)^*. \tag{5.1}$$

By Yoneda's lemma, given $a \in \mathcal{A}_d$, the natural transformation $a\cdot : M_n \to M_{n+d}$ will induce a map of $\mathcal{A}$–modules

$$\cdot a : J(n+d) \to J(n).$$

To give a unified discussion for all primes, we adopt the convention that if $p = 2$, $\mathcal{P}^n \in \mathcal{A}$ denotes Sq^{2n}. Then one has the so–called "Mahowald exact sequences".

Lemma 5.1. [S, Prop.2.2.3] *There is an exact sequence*

$$0 \to \Sigma J(pn-1) \to J(pn) \xrightarrow{\cdot \mathcal{P}^n} J(n) \to 0.$$

By (5.1) above, one sees that $J(n)$ is an injective object in $\mathcal{U}$. This is no longer true when one regards $J(n)$ in the category of all $\mathcal{A}$–modules, but it suggests that perhaps even in that category, one might have some control on an injective resolution of $J(n)$. The following critical lemma is a reflection of this.

Lemma 5.2. *In the category of $\mathcal{A}$–modules, there exist injective resolutions,*

$$0 \to J(n) \to I_0(n) \to I_1(n) \to I_2(n) \to \dots,$$

and chain maps under $\cdot\mathcal{P}^n : J(2pn) \to J(2n)$,

$$\gamma_n : I_*(2pn) \to I_*(n),$$

with the following property: if M is an unstable $\mathcal{A}$–module,

$$\gamma_n : \mathrm{Hom}_{\mathcal{A}}(M, \Sigma^t I_s(2pn)) \to \mathrm{Hom}_{\mathcal{A}}(M, \Sigma^t I_s(2n))$$

is zero for all $t - s < 0$.

This is "almost" in the literature. When $p = 2$, this is roughly [BC, Lemma 2.3(i)], though a form of it appears earlier in [M, proof of Lemma 5.6], and in Brown and Gitler's original article [BG, Lemma 2.8]. Cohen proves an odd prime version strong enough for our applications in [C, Cor.III.3.6]. See also [H, Prop.5.2.5]. In all of these references, explicit resolutions are constructed, using quotient subcomplexes of the Lamda algebra.

The lemma immediately implies

Theorem 5.3. *If M is an unstable $\mathcal{A}$–module,*

$$(\cdot\mathcal{P}^n)_* : \mathrm{Ext}^{s,t}_{\mathcal{A}}(M, J(2pn)) \to \mathrm{Ext}^{s,t}_{\mathcal{A}}(M, J(2n))$$

is zero for all $t - s < 0$.

Remark 5.4. When $s = 0$, the theorem reduces to the statement that, if M is unstable, then $\mathcal{P}^n : M_{2n-t} \to M_{2pn-t}$ is zero for all $t > 0$. This is, of course, (part of) the unstable condition.

We now continue as did G.Carlsson in [Ca]. Let $K(2n)$ be the unstable module defined as the inverse limit

$$K(2n) = \lim \{J(2n) \xleftarrow{\cdot\mathcal{P}^n} J(2pn) \xleftarrow{\cdot\mathcal{P}^{pn}} J(2p^2n) \xleftarrow{\cdot\mathcal{P}^{p^2n}} \dots\}.$$

By (5.1), for all $M \in \mathcal{U}$, there is a natural isomorphism

(5.2) $$\mathrm{Hom}_{\mathcal{A}}(M, K(2n)) \simeq (\mathrm{colim}\, \{M_{2n} \xrightarrow{\mathcal{P}^n} M_{2pn} \xrightarrow{\mathcal{P}^{pn}} M_{2p^2n} \xrightarrow{\mathcal{P}^{p^2n}} \dots\})^*.$$

Theorem 5.3 has the following consequence.

Corollary 5.5. *If M is an unstable $\mathcal{A}$–module,*

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(M, K(2n)) = 0, \text{ for all } t - s < 0.$$

5.2. topological results. The topological input we need is given by the next lemma.

Lemma 5.6. *There exists a map* $\Phi : T(2n) \to T(2pn)$ *such that* $\Phi^* = \cdot\mathcal{P}^n : J(2pn) \to J(2n)$.

When $p = 2$ this is proved in [CMM]. A similar method can be used at odd primes [H, Thm.5.1.1], and we will be outline the construction of Φ below. A odd prime version sufficient for our purposes appears as [C, Thm.III.4.1][4].

Now define spectra $\tilde{T}(2n)$ to be the telescopes

$$\tilde{T}(2n) = \text{hocolim } \{T(2n) \to T(2pn) \to T(2p^2n) \to \dots\}.$$

By construction, $H^*(\tilde{T}(2n); \mathbf{Z}/p) \simeq K(2n)$, as $\mathcal{A}$–modules.

Now suppose X is a space with $H^*(X; \mathbf{Z}/p)$ of finite type. Consider the Adams spectral sequence $\{E_r^{s,t}\}$ that computes maps from $\tilde{T}(2n)$ to $\Sigma^\infty X$. By Corollary 5.5, we see that

$$E_2^{s,t} = 0 \text{ if } t - s < 0.$$

Thus $E_2^{s,s}$ will consist of permanent cyles for all $s \geq 0$. By (5.2),

$$E_2^{0,0} \simeq \lim_j \ H_{2p^jn}(X; \mathbf{Z}/p).$$

We conclude

Theorem 5.7. *In this situation, the natural map*

$$[\tilde{T}(2n), \Sigma^\infty X] \to \lim_j \ H_{2p^jn}(X; \mathbf{Z}/p)$$

is onto.

Proof of Proposition 2.2. $H^*(B\mathbf{Z}/p; \mathbf{Z}/p) \simeq \Lambda(x) \otimes \mathbf{Z}/p[y]$, where x is one dimensional and $\beta(x) = y$. From this, one deduces that

$$\lim_j \ H_{2p^j}(B\mathbf{Z}/p; \mathbf{Z}/p) \simeq \mathbf{Z}/p.$$

Choosing a nonzero element of this inverse limit, Theorem 5.7 says that there exists a corresponding stable map $f : \tilde{T}(2) \to B\mathbf{Z}/p$. Such a map will be nonzero in mod p homology in dimensions 1 and 2. With ϵ equal to 1 if p is odd and 3 if $p = 2$, let f_j be the composite

$$T(2p^{j-\epsilon}) \to \tilde{T}(2) \xrightarrow{f} B\mathbf{Z}/p.$$

By construction, these maps have the needed property. □

We end this section with a sketch of the construction of the map Φ appearing in Lemma 5.6.

Recall that, if X is path connected, the Milgram–May model for $\Omega^2\Sigma^2 X$ comes equipped with a natural filtration, and if $F_m(\Omega^2\Sigma^2 X) \subset \Omega^2\Sigma^2 X$ denotes the mth stage of this, $D_{2,m}X = F_m(\Omega^2\Sigma^2 X)/F_{m-1}(\Omega^2\Sigma^2 X)$.

[4]However, Cohen's proof relies on [C, Prop.II.1.2] which, following tradition on this point, he incorrectly asserts is proved in [CLM].

Let $j : \Omega S^{2r+1} \to \Omega S^{2pr+1}$ be the pth Hopf invariant. Fixing n as in the lemma, if r is chosen sufficiently large, purely dimension reasons imply that

$$\Omega j : \Omega^2 S^{2r+1} \to \Omega^2 S^{2pr+1}$$

will carry $F_{p^2n}(\Omega^2 S^{2r+1})$ to $F_{pn}(\Omega^2 S^{2pr+1})$, and $F_{p^2n-1}(\Omega^2 S^{2r+1})$ to $F_{pn-1}(\Omega^2 S^{2pr+1})$. There is thus an induced map

$$D_{2,p^2n} S^{2r-1} \to D_{2,pn} S^{2pr-1}.$$

Recalling Lemma 4.1, the map Φ of Lemma 5.6 is defined to be the appropriate S–dual of this.

Remarks 5.8.
(1) Complete proofs of both Lemma 5.2 and Lemma 5.6 will appear in [HK].
(2) In the $p = 2$ case, generalizations of the telescopes $\tilde{T}(2n)$ are constructed in the second author's paper [K2], based on using S–duals of pieces of higher loopspaces. Algebraically there appear to be analogues of the maps f_j, with $B\mathbf{Z}/2$ replaced by higher Eilenberg–MacLane spaces $K(\mathbf{Z}/2, m)$. A heuristic argument is proposed, which, if it can be made rigorous, would give an alternative proof of Proposition 2.2 avoiding all Adams spectral sequence arguments and Brown–Gitler module technology.

6. The proof of Proposition 2.3

We begin this section with some notation.

If F is a finite complex with S–dual $D(F)$, we let $\Psi : S^0 \to F \wedge D(F)$ be the duality copairing.

We let M^n denote the mod p Moore spectrum $S^{n-1} \cup_p D^n$. Note that the S–dual of M^n is M^{1-n}.

With these conventions, we proceed to define the maps e of Proposition 2.3.

Definition 6.1. Let $n = 2$ when $p = 2$, and let $n = 2p - 4$ if p is odd. Then $e : B\mathbf{Z}/p \to M^{1-n}$ is defined to be the composite

$$B\mathbf{Z}/p \xrightarrow{1\wedge\Psi} B\mathbf{Z}/p \wedge M^n \wedge M^{1-n} \xrightarrow{\Theta\wedge 1} M^{1-n},$$

where $\Theta : B\mathbf{Z}/p \wedge M^n \to S^0$ is the composite

$$B\mathbf{Z}/p \wedge M^n \xrightarrow{i\wedge i} B(\mathbf{Z}/p \times \mathbf{Z}/p)_+ \xrightarrow{B(\text{add})} B\mathbf{Z}/p_+ \xrightarrow{t_{\mathbf{Z}/p}} S^0.$$

Here maps labelled "i" are inclusions. We remind the reader that the $2p-2$ skeleton of $B\mathbf{Z}/p$ is stably $M^2 \vee \cdots \vee M^{2p-4} \vee M^{2p-2}$. (At odd primes, these Moore spaces correspond to the bottom cells of the spectra $Z(j)$ of Remarks 3.3(3).)

To compute cohomology operations in $H^*(C(e); \mathbf{Z}/p)$ as needed in Proposition 2.3, we first observe that, by construction, there is a diagram of cofibration sequences:

$$
\begin{array}{ccccc}
M^{1-n} & \longrightarrow & C(e) & \longrightarrow & \Sigma B\mathbf{Z}/p \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \Sigma\Gamma} \\
M^{1-n} & \xrightarrow{i} & C(t_{\mathbf{Z}/p}) \wedge M^{1-n} & \xrightarrow{\pi} & \Sigma(B\mathbf{Z}/p_+) \wedge M^{1-n}
\end{array}
\tag{6.1}
$$

where the bottom sequence is just the cofibration sequence

$$S^0 \xrightarrow{i} C(t_{\mathbf{Z}/p}) \xrightarrow{\pi} \Sigma(B\mathbf{Z}/p_+)$$

smashed with M^{1-n}, and Γ is the composite

$$B\mathbf{Z}/p \xrightarrow{1\wedge\Psi} B\mathbf{Z}/p\wedge M^n\wedge M^{1-n} \xrightarrow{i\wedge i\wedge 1} B(\mathbf{Z}/p\times\mathbf{Z}/p)_+\wedge M^{1-n} \xrightarrow{B(\text{add})\wedge 1} B\mathbf{Z}/p_+\wedge M^{1-n}.$$

Having made this observation, the $\mathcal{A}$–module structure of $H^*(C(e);\mathbf{Z}/p)$ can be easily computed, as the $\mathcal{A}$–module structure of $H^*(C(t_{\mathbf{Z}/p});\mathbf{Z}/p)$ is well known, and it is routine to calulate Γ in cohomology. We sketch the details.

First assume $p = 2$. Then $H^*(C(t_{\mathbf{Z}/2});\mathbf{Z}/2)$ is the submodule of nonnegative degree elements in $\Sigma\mathbf{Z}/2[x, x^{-1}]$, and the Steenrod operations act by $Sq^i(x^k) = \binom{k}{i}x^{k+i}$. Let $m_i \in H^i(M^{-1};\mathbf{Z}/2), i = -2, -1$ be generators.

We wish to show that Sq^4 acts nonzero on $H^i(C(e);\mathbf{Z}/2)$, for $i = -2, -1$. By (6.1), it is equivalent to show that $\Gamma^*((\pi^*)^{-1}(Sq^4\sigma(x^{-1} \otimes m_i))) \neq 0$, where σ denotes suspension. Since $Sq^4(x^{-1} \otimes m_i) = x^3 \otimes m_i$, we just need to check that $\Gamma^*(x^3\otimes m_i) \neq 0$. By the next lemma, $\Gamma^*(x^3\otimes m_{-2}) = \binom{3}{2}x = x$, and $\Gamma^*(x^3\otimes m_{-1}) = \binom{3}{1}x^2 = x^2$.

Lemma 6.2. $\Gamma^*(x^k \otimes m_{-1}) = \binom{k}{1}x^{k-1}$ *and* $\Gamma^*(x^k \otimes m_{-2}) = \binom{k}{2}x^{k-2}$.

When p is odd, $H^*(C(t_{\mathbf{Z}/p});\mathbf{Z}/p)$ is the submodule of nonnegative degree elements in $\Sigma\Lambda(x)\otimes\mathbf{Z}/p[y, y^{-1}]$, and the Steenrod operations act by $\beta(y^i) = 0$, $\beta(x) = y$, and $\mathcal{P}^i(x^\epsilon y^k) = \binom{k}{i}x^\epsilon y^{k+(p-1)i}$. Let $m_i \in H^i(M^{5-2p};\mathbf{Z}/p)$, $i = 4-2p, 5-2p$ be generators chosen to be dual to xy^{p-3} and y^{p-2} in $H^*(M^{2p-4};\mathbf{Z}/p)$ respectively.

We wish to show that $\mathcal{P}^1$ acts nonzero on $H^i(C(e);\mathbf{Z}/2)$, for $i = 4-2p, 5-2p$. By (6.1), it is equivalent to show that $\Gamma^*((\pi^*)^{-1}(\mathcal{P}^1\sigma(xy^{-1} \otimes m_i))) \neq 0$, where σ denotes suspension. Since $\mathcal{P}^1(xy^{-1}\otimes m_i) = xy^{p-2}\otimes m_i$, we just need to check that $\Gamma^*(xy^{p-2} \otimes m_i) \neq 0$. By the next lemma, $\Gamma^*(xy^{p-2} \otimes m_{4-2p}) = \binom{p-2}{p-2}x = x$, and $\Gamma^*(xy^{p-2} \otimes m_{5-2p}) = -\binom{p-2}{p-3}y = 2y$.

Lemma 6.3. $\Gamma^*(xy^k\otimes m_{4-2p}) = \binom{k}{p-2}xy^{k+2-p}$, $\Gamma^*(xy^k\otimes m_{5-2p}) = -\binom{k}{p-3}y^{k+3-p}$, $\Gamma^*(y^k \otimes m_{4-2p}) = \binom{k}{p-2}y^{k+2-p}$, *and* $\Gamma^*(y^k \otimes m_{5-2p}) = 0$.

Remark 6.4. It is an exercise in the properties of the transfer to show that the composite

$$B(\mathbf{Z}/p \times \mathbf{Z}/p)_+ \xrightarrow{B(\text{add})} B\mathbf{Z}/p_+ \xrightarrow{t_{\mathbf{Z}/p}} S^0$$

agrees with the composite

$$B(\mathbf{Z}/p \times \mathbf{Z}/p)_+ \xrightarrow{t_\Delta} B\mathbf{Z}/p_+ \xrightarrow{c} S^0,$$

where c is the collapse map and t_Δ is the transfer associated to the diagonal inclusion $\Delta : \mathbf{Z}/p \to \mathbf{Z}/p \times \mathbf{Z}/p$. From this it follows that $e : B\mathbf{Z}/p \to M^{1-n}$ as defined here is the map $B\mathbf{Z}/p \xrightarrow{i} B\mathbf{Z}/p_+ \xrightarrow{s} D(B\mathbf{Z}/p_+) \xrightarrow{Di} D(M^n)$, where s is the map arising in the Segal conjecture: it is adjoint to $B(\mathbf{Z}/p \times \mathbf{Z}/p)_+ \xrightarrow{t_\Delta} B\mathbf{Z}/p_+ \xrightarrow{c} S^0$. Related to this, we note that, when p is odd, the S–dual of $Z(1)$ is $Z(p-2)$.

7. Two more maps out of $(B\mathbf{Z}/2)/S^1$

Theorem 3.2 suggests hunting for maps out of $B\mathbf{Z}/p$ which are nonzero when restricted to S^1, and hunting for maps out of $(B\mathbf{Z}/p)/S^1$ which are nonzero when restricted to S^2. Using the solution of the Segal Conjecture for the group $\mathbf{Z}/p$ (as in [L1], [R1]), one can systematically study such maps. When $p = 2$, this was done in [K1], and a similar analysis is possible when p is odd [H].

Proposition 2.3, of course, gives examples of such maps, constructed by "elementary" means. Here we note that there are two more 2 primary maps that can be constructed easily. Like $t : B\mathbf{Z}/2 \to S^0$, these have no odd prime analogues.

The first should be compared with Proposition 2.3(1) (or Proposition 3.5(1)).

Proposition 7.1. *There exists a map* $e_\mathbf{C} : (B\mathbf{Z}/2)/S^1 \to \Sigma^{-7}\mathbf{C}P^2$ *with* Sq^8 *acting nonzero on* $H^{-5}(C(e_\mathbf{C}); \mathbf{Z}/2)$*. Equivalently, there exists a commutative diagram*

$$\begin{array}{ccc} S^2 & \xrightarrow{i_2} & (B\mathbf{Z}/2)/S^1 \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle e_\mathbf{C}} \\ S^{-5} & \xrightarrow{i} & \Sigma^{-7}\mathbf{C}P^2. \end{array}$$

The map $e_\mathbf{C}$ is defined using the (desuspended) S^1–transfer $t_{S^1} : BS^1_+ \to S^{-1}$.

Definition 7.2. $e_\mathbf{C} : (B\mathbf{Z}/2)/S^1 \to \Sigma^{-7}\mathbf{C}P^2$ is defined to be the composite

$$(B\mathbf{Z}/2)/S^1 \xrightarrow{1\wedge\Psi} (B\mathbf{Z}/2)/S^1 \wedge \mathbf{C}P^2 \wedge \Sigma^{-7}\mathbf{C}P^2 \xrightarrow{\Theta\wedge 1} \Sigma^{-1}D\mathbf{C}P^2 = \Sigma^{-7}\mathbf{C}P^2,$$

where $\Theta : (B\mathbf{Z}/2)/S^1 \wedge \mathbf{C}P^2 \to S^{-1}$ is the composite

$$(B\mathbf{Z}/2)/S^1 \wedge \mathbf{C}P^2 \xrightarrow{j\wedge i} B(S^1 \times S^1)_+ \xrightarrow{B(\mathrm{add})} BS^1_+ \xrightarrow{t_{S^1}} S^{-1}.$$

Here i is the obvious inclusion, and $j : (B\mathbf{Z}/2)/S^1 \to BS^1_+$ is any extension of the composite of obvious maps $B\mathbf{Z}/2 \to BS^1 \to BS^1_+$.

To prove the proposition, one computes the action of Sq^8 on $H^{-5}(C(e_\mathbf{C}); \mathbf{Z}/2)$ using the method of the last section. With $i : S^2 \to \mathbf{C}P^2$ denoting the inclusion of the bottom cell, a consequence is

Theorem 7.3. *The composite* $(e_\mathbf{C} \circ \pi \circ f_j \circ g_j) \in \pi^S_{2^j+7}(\mathbf{C}P^2)$ *is represented by* $i_*(h_3 h_{j-1}^2) \in \mathrm{Ext}^{3,*}_\mathcal{A}(H^*(\mathbf{C}P^2), \mathbf{Z}/2)$.

This theorem is essentially due to W.H.Lin [L3, last paragraph of p.136], and leads to an easy to check criterion for deducing that a family of the form $dh_3h_{j-1}^2 \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ is a permanent cycle [K1, Thm.5.3].

Our last map arises in the following way. Let $\delta : (B\mathbf{Z}/p)/S^1 \to S^2$ be defined by the cofibration sequence

$$S^1 \xrightarrow{i_1} B\mathbf{Z}/p \to (B\mathbf{Z}/p)/S^1 \xrightarrow{\delta} S^2.$$

Then the composite $S^2 \xrightarrow{i_2} (B\mathbf{Z}/p)/S^1 \xrightarrow{\delta} S^2$ has degree p, and, thanks to Theorem 3.2, we can conclude that, when $p = 2$, $h_0h_{j-1}^2 \in \mathrm{Ext}_{\mathcal{A}}^{3,*}(\mathbf{Z}/2, \mathbf{Z}/2)$ is a permanent cycle, and, when p is odd, $a_0b_{j-1} \in \mathrm{Ext}_{\mathcal{A}}^{3,*}(\mathbf{Z}/p, \mathbf{Z}/p)$ is a permanent cycle. Due to the Hopf invariant one differential (see §3), neither of these facts is new. However, the next lemma shows that, when $p = 2$, δ lifts through $\pi : \Sigma^{-2}\mathbf{C}P^2 \to S^2$.

Lemma 7.4. *At the prime 2, $\eta \circ \delta = 0$. At odd primes, $\alpha \circ \delta \neq 0$.*

Proof. By diagram chasing, it is easily seen that $\eta \circ \delta = 0$ if and only if $\eta : S^1 \to S^0$ extends to a map $B\mathbf{Z}/2 \to S^0$, and it does. Similarly, at odd primes, $\alpha \circ \delta = 0$ if and only if $\alpha : S^1 \to S^{4-2p}$ extends to a map $B\mathbf{Z}/p \to S^{4-2p}$, and it doesn't. □

Thanks to the lemma, there is an interesting map $\delta' : (B\mathbf{Z}/2)/S^1 \to \Sigma^{-2}\mathbf{C}P^2$ (and no odd prime analogue). [K1, Thm.5.5] has a criterion for using the composites $\delta' \circ \pi \circ f_j \circ g_j : S^{2^j} \to \Sigma^{-2}\mathbf{C}P^2$, together with Toda bracket methods, to construct infinite families of permanent cycles in $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbf{Z}/2, \mathbf{Z}/2)$. For example, $(Ph_2)h_{j-1}^2 \in \mathrm{Ext}_{\mathcal{A}}^{7,2^j+16}(\mathbf{Z}/2, \mathbf{Z}/2)$ is a permanent cycle [K1, Ex.5.6].

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