# ON THE FUNDAMENTAL GROUPS OF SYMPLECTICALLY ASPHERICAL MANIFOLDS

R. IBÁÑEZ, YU. RUDYAK, AND A. TRALLE

Abstract. In this paper we are interested in the fundamental groups of closed symplectically aspherical manifolds. Motivated by some results of Gompf, we introduce two classes of fundamental groups $\pi_1(M)$ of symplectically aspherical manifolds $M$ with $\pi_2(M) = 0$ and $\pi_2(M) \neq 0$. Relations between these classes are discussed. We show that several important classes of groups can be realized in both classes. Also, we notice that there are some dimensional phenomena in the realization problem.

## Introduction

Throughout the paper the term "symplectic manifold" means a closed symplectic manifold $(M, \omega)$ such that the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ lies in the integral lattice $H^2(M)/\text{tors}$. We say that the symplectic form $\omega$ is *symplectically aspherical* if

$$\int_{S^2} f^*\omega = 0,$$

for every map $f : S^2 \to M$. In cohomological terms, it means that

$$\langle [\omega], h(a) \rangle = 0$$

for every $a \in \pi_2(M)$, where $h : \pi_2(M) \to H_2(M)$ is the Hurewicz homomorphism. Frequently one writes the last equality as $[\omega]|_{\pi_2(M)} = 0$.

By the definition, a symplectically aspherical manifold is a symplectic manifold whose symplectic form is symplectically aspherical. The importance of symplectically aspherical manifolds in symplectic geometry and topology is well-known, see e.g. [F, H, LO, R2, RO, RT].

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Clearly, every symplectic manifold $M$ with $\pi_2(M) = 0$ is symplectically aspherical. On the other hand there are some reasons to know whether there are symplectically aspherical manifolds with $\pi_2(M) \neq 0$, see e.g. [G2]. Examples of such manifolds were given in [G2] as some 4-dimensional closed manifolds obtained as branched coverings. Here we use a theorem of Donaldson [D] on hyperplane sections of symplectic manifolds in order to give another construction and get other examples of 4-dimensional symplectically aspherical manifolds $M$ with non-trivial $\pi_2(M)$.

In fact, we are interesting in searching for fundamental groups of symplectically aspherical manifolds. It is well known that every finitely presented group can be realized as the fundamental group of a closed symplectic manifold, [G1]. However, not every such group can be realized as the fundamental group of a closed symplectically aspherical manifold. For example, the trivial (or, more generally, any finite) group cannot. In the sequel we call a group *symplectically aspherical* if it can be realized as a fundamental group of a closed symplectically aspherical manifold. According to what we have said above, it is interesting to compare the fundamental groups of symplectically aspherical manifolds having $\pi_2(M) = 0$ with these ones having $\pi_2(M) \neq 0$.

We always identify de Rham cohomology of a manifold $M$ with $H^*(M;\mathbb{R})$. As usual, we call a closed connected manifolod $M^{2n}$ *cohomologically symplectic* or, brielfy, *c-symplectic* if there exists a class $a \in H^2(M;\mathbb{R})$ with $a^n \neq 0$. Finally, we call a group $\pi$ c-symplectic if there exists a (closed) c-symplectic manifold $M$ which is $K(\pi, 1)$.

## 1. Preliminaries

**1.1. Theorem** (Hopf)**.** *Let $X$ be a connected CW-space with $\pi_1(X) = \pi$ and $\pi_i(X) = 0$ for $i < n$. Then there is an exact sequence*

$$\pi_n(X) \xrightarrow{h} H_n(X) \longrightarrow H_n(\pi) \longrightarrow 0.$$

*Proof.* See [B, Theorem II.5.2]. □

The following theorem is a symplectic analog of the Lefschetz Theorem on Hyperplane Sections.

**1.2. Theorem.** *Let $(M^{2n}, \omega)$ be a symplectic manifold, and let $h \in H^2(M)$ be an integral lift of $[\omega]$. Then for $N$ large enough the Poincaré dual of $Nh$, in $H_{2n-2}(M)$, can be realized by a symplectic submanifold $V^{2n-2}$ of $M^{2n}$. Moreover, we can choose $V$ such that the inclusion*

*$i : V \hookrightarrow M$ is an $(n-2)$-equivalence, i.e. the homomorphism $i_* : \pi_k(V) \to \pi_k(M)$ is an isomorphism for $k \le n-2$ and an epimorphism for $k = n-1$.*

*Proof.* See [D, Theorem 1 and Proposition 39]. □

We need also the following homotopic characterization of symplectically aspherical closed manifolds.

**1.3. Proposition.** *Let $(M, \omega)$ be a symplectic manifold, and let $K$ denote the Eilenberg–Mac Lane space $K(\pi_1(M), 1)$. The following three conditions are equivalent:*

(i) *$(M, \omega)$ is symplectically aspherical;*

(ii) *there exists a map $f : M \to K$ which induces isomorphism on the fundamental groups and such that*

$$[\omega] \in \mathrm{Im}\,\{f^* : H^2(K, \mathbb{R}) \to H^2(M, \mathbb{R})\};$$

(iii) *there exists a map $f : M \to K$ such that*

$$[\omega] \in \mathrm{Im}\,\{f^* : H^2(K; \mathbb{R}) \to H^2(M, \mathbb{R})\}.$$

*Proof.* See [RT, Corollary 2.2], cf. also [LO, Lemma 4.2]. □

**1.4. Proposition.** *If a group $\pi$ is symplectically aspherical and $\tau$ is a subgroup of a finite index of $\pi$, then $\tau$ is symplectically aspherical.*

*Proof.* This holds, because a finite covering space over a closed symplectically aspherical manifold is a closed symplectically aspherical manifold. □

**1.5. Corollary.** *No finite groups are symplectically aspherical.*

*Proof.* The trivial group is not symplectically aspherical because the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ is an isomorphism for every simply-connected space $M$. Now the result follows from Proposition 1.4. □

## 2. Dimension phenomena

**2.1. Proposition.** *Suppose that a group $\pi$ can be realized as the fundamental group of a symplectically aspherical manifold $(M^{2n}, \omega)$ with $n \ge 3$. Then $\pi$ can be realized as the fundamental group of a $(2n-2)$-dimensional symplectically aspherical manifold.*

*Proof.* Without loss of generality we can assume that the symplectic form $\omega$ is an integral form. (Indeed, we can a find a rational form $\eta$ which is $C^\infty$-closed to $\omega$, and therefore $\eta$ is a symplectic form. Now take a suitable multiple of $\eta$.) According to the Donaldson Theorem 1.2, there exists a symplectic submanifold $V^{2n-2}$ of $M$ such that the inclusion $i : V \hookrightarrow M$ is an $(n-2)$-equivalence. In particular, $\pi_1(V) = \pi_1(M)$. Clearly, $V$ is symplectically aspherical since $M$ is, and the result follows. □

**2.2. Corollary.** *Suppose that a group $\pi$ can be realized as the fundamental group of a symplectically aspherical manifold $(M^{2n}, \omega)$ with $n \geq 3$. Then $\pi$ can be realized as the fundamental group of a 4-dimensional symplectically aspherical manifold.* □

So, we can decrease the dimension of the symplecticaly aspherical manifold with a prescribed fundamental group. However, we are not always able to increase the dimension, as the following proposition shows.

**2.3. Proposition.** *Let $\pi$ be a group such that $H^i(\pi; \mathbb{R}) = 0$ for $i > k$. Suppose that $\pi$ can be realized as the fundamental group of a symplectically aspherical manifold $M^{2n}$. Then $2n \leq k$.*

*Proof.* Because of 1.3, there exists a map $f : M \to K(\pi, 1)$ such that

$$[\omega] \in \operatorname{Im}\{f^* : H^2(\pi; \mathbb{R}) = H^*(K(\pi, 1); \mathbb{R}) \to H^2(M; \mathbb{R})\}.$$

Since $[\omega]^{2n} \neq 0$, we conclude that $2n \leq k$. □

**2.4. Corollary.** *The group $\mathbb{Z}^m$ cannot be realized as the fundamental group of a symplectically aspherical manifold of dimension $2k$ with $2k > m$.*

Notice that $\mathbb{Z}^{2n}$ is the fundamental group of the torus $T^{2n}$. Since $\pi_2(T^{2n}) = 0$, $\mathbb{Z}^{2n}$ can be realized as the fundamental group of a symplectically aspherical manifold of dimension $2k$ with $2 \leq k \leq n$.

**2.5. Remark.** Because of Propositions 2.2 and 2.3, it makes sense to introduce the following invariant of symplectically aspherical groups. Namely, given a symplectically aspherical group $\pi$, we define $\nu(\pi)$ to be the largest $n$ such that $\pi$ can be realized as the fundamental group of a closed symplectically aspherical manifold $M^{2n}$. For example, $\nu(\mathbb{Z}^{2n}) = n$. Furthermore, if $\pi$ is the fundamental group of a closed orientable surface then $\nu(\pi) = 1$, and if $G$ is the direct product of $n$ such groups then $\nu(G) = n$.

## 3. Two classes of symplectically aspherical groups

Let $\mathcal{A}$ be the class of symplectically aspherical groups which can be realized as the fundamental groups of symplectically aspherical manifolds with trivial $\pi_2$, and let $\mathcal{B}$ be the class of symplectically aspherical groups which can be realized as the fundamental groups of symplectically aspherical manifolds with non-trivial $\pi_2$. In this section we want to investigate the relation between the classes $\mathcal{A}$ and $\mathcal{B}$. First, some trivial remarks.

1. If $\pi \in \mathcal{B}$ and $\tau$ is simplectically aspherical, then $\pi \times \tau \in \mathcal{B}$.
2. Let $G$ be the fundamental group of a closed orientable surface. Then $G \notin \mathcal{B}$ (by Corollary 2.2).

**3.1. Theorem.** *Let $(M^4, \omega)$ be a 4-dimensional closed symplectically aspherical manifold and let $\pi_1(M) = \pi$. If $\pi_2(M) = 0$ then $b_1(\pi) \geq b_3(\pi)$.*

*Proof.* Because of the Hopf Theorem 1.1, there is an epimorphism $H_3(M) \to H_3(\pi)$. So, $b_1(\pi) = b_1(M) = b_3(M) \geq b_3(\pi)$. □

**3.2. Corollary.** *If $\pi$ is a symplectically aspherical group and $b_1(\pi) < b_3(\pi)$, then $\pi \in \mathcal{B}$.*

*Proof.* Because of Corollary 2.2, $\pi$ can be realized as the fundamental group of a 4-dimensional symplectically aspherical manifold. Now the result follows from Theorem 3.1 □

**3.3. Corollary.** *Suppose that $\pi$ can be realized as the fundamental group of a symplectically aspherical manifold $M$ with $\pi_3(M) = 0$ and $b_1(M) < b_3(M)$. Then $\pi \in \mathcal{B}$.*

*Proof.* If $\pi_2(M) \neq 0$ then we are done. So, suppose that $\pi_2(M) = 0$. Since $\pi_3(M) = 0$, the Hopf exact sequence from Theorem 1.1 for $n = 3$ yields an isomorphism $H_3(M) \cong H_3(\pi)$. So,

$$b_1(\pi) = b_1(M) < b_3(M) = b_3(\pi)$$

and the result follows from 3.2. □

**3.4. Corollary.** *Let $\pi$ and $\tau$ be two symplectically aspherical groups.*

(i) *If $\max\{b_3(\pi), b_3(\tau)\} \geq 1$, then $\pi \times \tau \in \mathcal{B}$.*

(ii) *If $\max\{b_2(\pi), b_2(\tau)\} \geq 2$ and $\max\{b_1(\pi), b_1(\tau)\} \geq 1$, then $\pi \times \tau \in \mathcal{B}$.*

*Proof.* Notice that $b_2(G) > 0$ for every symplectically aspherical group $G$. Now, $b_1(\pi \times \tau) = b_1(\pi) + b_1(\tau)$, while (by the Künneth formula)

$$b_3(\pi \times \tau) = b_3(\tau) + b_1(\pi)b_2(\tau) + b_2(\pi)b_1(\tau) + b_3(\pi).$$

Now, each of the conditions in (i), (ii) implies that

$$b_3(\pi \times \tau) > b_1(\pi \times \tau),$$

and the result follows from 3.3. □

**3.5. Corollary.** *If $\pi$ is a symplectically aspherical group, then $\pi \times \mathbb{Z}^4 \in \mathcal{B}$.*

According to the results of Sections 2 and 3, it seems reasonable to introduce the classes $\mathcal{A}_{2n}$ an $\mathcal{B}_{2n}$ as follows. The group $\pi$ belongs to $\mathcal{A}_{2n}$ if $\pi$ can be realized as the fundamental group of a symplectically aspherical $2n$-dimensional manifold with $\pi_2(M) = 0$. Similarly, the group $\pi$ belongs to $\mathcal{B}_{2n}$ if $\pi$ can be realized as the fundamental group of a symplectically aspherical $2n$-dimensional manifold with $\pi_2(M) \neq 0$. Because of what is done above, we have the following Proposition.

**3.6. Proposition.**

$$\mathcal{A}_{2n+2} \subset \mathcal{A}_{2n} \subset \cdots \subset \mathcal{A}_6 \subset \mathcal{A}_4 \cup \mathcal{B}_4, \quad \mathcal{B}_{2n+2} \subset \mathcal{B}_{2n} \subset \cdots \subset \mathcal{B}_4.$$

*Proof.* Consider the $(n-2)$ equivalence $i : V^{2n-2} \to M^{2n}$ from the proof of Theorem 2.1. The homomorphism $i_* : \pi_2(V) \to \pi_2(M)$ is an isomorphism for $n \geq 44$ and an epimorphism for $n = 3$. Therefore the result holds. □

**3.7. Remarks and Questions.** 1. Notice that $\mathcal{B}_2 = \emptyset \neq \mathcal{A}_2$.

2. We have $\mathcal{A}_6 \cap \mathcal{B}_6 \neq \emptyset$ since $\mathbb{Z}^8 \in \mathcal{A}_6 \cap \mathcal{B}_6$. Indeed, $\mathbb{Z}^8 \in \mathcal{A}_8 \subset \mathcal{A}_6$. On the other hand, $\mathbb{Z}^6 = \pi_1(T^6)$ and therefore $\mathbb{Z}^6 \in \mathcal{B}_4$ by Theorem 3.1. Thus, $\mathbb{Z}^8 = \mathbb{Z}^6 \times \mathbb{Z}^2 \in \mathcal{B}_6$.

3. Similalry, $\mathcal{A}_{2n} \cap \mathcal{B}_{2n} \neq \emptyset$ for $n \geq 3$.

4. We don't know whether $\mathcal{A}_4 \cap \mathcal{B}_4 \neq \emptyset$. In particular, is it true that $\mathbb{Z}^4 \in \mathcal{B}_4$?

3. Is $\mathbb{Z}^{2n+1}$ symplectically aspherical if $n > 1$? ($\mathbb{Z}$ and $\mathbb{Z}^3$ are not by Proposition 2.3.) If the answer is negative, the proof should be delicate because the answer is positive at c-cymplectical leveL, see Proposition 3.8 below.

5. Generally, is it true that $\mathcal{B} \subset \mathcal{A}$?

**3.8. Proposition.** *For every $n > 3$ there exists a manifold $N^{2n}$ and a cohomology class $a \in H^2(N;\mathbb{R})$ such that $\pi_1(N) = \mathbb{Z}^{2n+1}$ and $a|_{\pi_2(N)} = 0$.*

*Proof.* Take the torus $T^{2n+2}$ and consider its hyperplane section as in Theorem 1.2. Then we get a 2n-dimensional symplectically aspherical manifold $(M,\omega)$ with the fundamental group $\mathbb{Z}^{2n+2}$. Then, by proposition 1.3, there exists a map $f : M \to T^{2n+2}$ which induces an isomorphism of fundamental groups and such that $f^*(u^n) = [\omega]^n \neq 0$ for some $u \in H^2(T^{2n+2};\mathbb{R})$. So, there are cohomology classes $x_i \in H^1(T^{2n+2};\mathbb{R}), i = 1, 2, \dots 2n$ such that $f^*(x_1 \cdots x_{2n}) \neq 0$. This implies, in turn, that there exists a map $g : M \to T^{2n}$ with $g^*[\omega_T]^n \neq 0$. Here $\omega_T$ is the symplectic form on $T^{2n}$. In particular, the degree of $g$ is non-zero.

Consider the induced homomorphism

$$g_* : \mathbb{Z}^{2n+2} = \pi_1(M) \to \pi_1(T^{2n}) = \mathbb{Z}^{2n}$$

and take any $a \in \operatorname{Ker} g_*$. Let $A$ be the subgroup generated by $a$. Then

$$\mathbb{Z}^{2n+2}/A \cong \mathbb{Z}^{2n+1} \oplus F$$

where $F$ is a finite abelian group.

Now, we represent $a$ by an embedded circle $S$ and perform the surgery of $g$ along $S$. Then we get a map $h : N \to T^{2n}$ which is bordant to $g$, and therefore $h$ has non-zero degree. So, $h^*[\omega]^n \neq 0$, and thus $N$ is c-symplectic. Furthermore, $\pi_1(N) = \mathbb{Z}^{2n+1} \oplus F$. Now, passing to a finite cover of of $N$, we obtain a c-symplectic manifold with the fundamental group $\mathbb{Z}^{2n+1}$. finally, the class $h^*[\omega]$ vanishes on the image of the Hurewicz map since $[\omega]$ does. □

## 4. Some results about realization

Now we describe some classes of symplectically aspherical groups. For this purpose, we recall several notions.

A *lattice* in a Lie group $G$ is a discrete subgroup $\pi \subset G$. A lattice $\pi$ in $G$ is called *uniform* if $G/\pi$ is compact.

**4.1. Definition.** A Lie group $G$ is called *completely solvable*, if any adjoint linear operator $ad\, V : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ of $G$ has only real eigenvalues.

It is well known that every completely solvable Lie group is solvable, [VGS] .

**4.2. Lemma.** *If $\pi$ is a uniform lattice in a simply-connected completely solvable Lie group $G$ of dimension $2n$ and $\pi$ is c-symplectic, then $\pi \in \mathcal{A}$. In particular, $\pi$ is symplectically aspherical.*

*Proof.* Consider the closed manifold $M := G/\pi$. Since $G$ is solvable, and, hence, diffeomorphic to euclidean space, we conclude that $M = K(\pi, 1)$ and so $H^*(\pi) \cong H^*(M)$. Now, since $\pi$ is c-symplectic, then $M$ is c-symplectic, so there exists a cohomology class $\alpha \in H^2(M, \mathbb{R})$ such that $\alpha^n \neq 0 \in H^{2n}(M; \mathbb{R})$. By the Hattori theorem [Ha], there is an isomorphism

$$H^*(M; \mathbb{R}) \cong H^*(\Lambda\mathfrak{g}^*, \delta),$$

where $(\Lambda\mathfrak{g}^*, \delta)$ denotes the standard Chevalley–Eilenberg complex for the Lie algebra $\mathfrak{g}$. Therefore $\alpha$ can be represented by a closed differential 2-form $\omega$ whose pullback $\widetilde{\omega}$ to $G$ is a left-invariant form. Furthermore, $\widetilde{\omega}$ is non-degenerate since it is left-invariant and $[\widetilde{\omega}]^n \neq 0$ on $H^{2n}(\Lambda\mathfrak{g}^*, \delta)$. Hence, $\omega$ is non-degenerate, and so $(M, \omega)$ is a symplectic manifold. □

Let $\pi$ be a polycylic group. Let $\alpha \in \mathrm{Aut}(\pi)$. There exists a subnormal series $\pi = \pi_n \supset \pi_{n-1} \supset \ldots \supset \pi_0$ such that $\alpha(\pi_i) \subset \pi_i$, [Gb]. (Here subnormality means that $\pi_i$ is normal in $\pi_{i+1}$ and $F_i = \pi_{i+1}/\pi_i$ are finitely generated abelian groups.) Hence $\alpha$ induces automorphisms $\alpha_i \in \mathrm{Aut}(F_i \otimes \mathbb{C}) = GL(k_i, \mathbb{C})$. One can easily check that the set of eigenvalues of all operators $\alpha_i$ does not depend on the choice of a subnormal series. We call the elements of this set eigenvalues of $\alpha$.

**4.3. Definition.** *A polycyclic group $\pi$ is called a group of type (R), if for all $\gamma \in \pi$ all eigenvalues of the inner automorphism* $\mathrm{Int}(\gamma)$ *are real and positive.*

**4.4. Theorem.** *A group $\pi$ is isomorphic to a uniform lattice in a completely solvable simply-connected Lie group if and only if $\pi$ is of type $(R)$.*

*Proof.* See Gorbatsevich [Gb]. □

**4.5. Corollary.** *Every c-symplectic group $\pi$ of type $(R)$ belongs to $\mathcal{A}$. Furthermore, $\pi \in \mathcal{B}$ if $b_1(\pi) < b_3(\pi)$.* □

Obviously, the class of completely solvable Lie groups contains all nilpotent Lie groups. Furthermore, it is well known that every finitely generated torsion free nilpotent group is of type $(R)$. Now we show that some of these groups really belong to $\mathcal{B}$.

**4.6. Corollary.** *The fundamental group of any 6-dimensional symplectic nilmanifold is a symplectically aspherical group of class $\mathcal{B}$.*

*Proof.* All 6-dimensional symplectic nilmanifolds are classified (see [Sa, IRTU]). In particular, the first and second Betti numbers of each of 34 such manifolds can be found in the corresponding tables in these papers. Note that since the Euler characteristic of any nilmanifold is zero, we get the following relation for the Betti numbers: $2 - 2b_1 + 2b_2 - b_3 = 0$. Hence $b_1 < b_3$ is the same as

$$2 + 2b_2 > 3b_1.$$

One can check that each of the symplectic nilmanifolds from the tables [Sa, IRTU] satisfies this inequality. □

Notice that one can also get groups of type $(R)$ which are solvable but non-nilpotent. For example, consider the following simply-connected completely solvable Lie group $G$ consisting of matrices

$$\begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & y_1 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & y_2 \\ 0 & 0 & e^t & 0 & 0 & z_1 \\ 0 & 0 & 0 & e^{-t} & 0 & z_2 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is shown in [FLS], that this group contains a uniform lattice $\pi$, and that the compact solvmanifold $M := G/\pi$ has $b_1(M) = 2 < b_3(M) = 4$. Thus, $\pi \in \mathcal{B}$.

**4.7. Example.** Tori, products of complex curves and hyperplane sections of these manifolds give us examples of symplectically aspherical algebraic (and therefore Kähler) manifolds. Here we show how to construct symplectically aspherical closed Kähler manifolds. Let $G$ be a semisimple simply-connected Lie group of non-compact type, and let $K$ be a maximal compact connected subgroup of $G$. If the homogeneous space $G/K$ is a symmetric Hermitian space then $G/K$ turns out to be a Kähler manifold with the invariant Kähler metric. All such pairs $(G, K)$ are listed in [He, Ch. IX]. Moreover, every such group $G$ contains a uniform lattice $\pi$, [VGS]. Thus, $M := \pi\backslash G/K$ is a symplectically aspherical Kähler manifold with the fundamental group $\pi$ (since $G/K$ is diffeomorphic to Euclidean space). In particular, $\pi \in \mathcal{A}$.

Now we give an example of $\pi$ as above with $\pi \in \mathcal{B}$. Let $\chi(\pi)$ denote the Euler characteristic of $\pi$. It was shown in [VGS, Theorem 7.9] that $\chi(\pi) \neq 0$ if and only if $\operatorname{rank}(G) = \operatorname{rank}(K)$, and in the latter case one has also the sign of $\chi(\pi)$ equal to $(-1)^n$, where $n = 1/2 \dim G/K$. Now, consider $G = Sp(2, \mathbb{R})$ and $K = U(2)$. Then $G/K$ is a 6-dimensional Hermitian symmetric space of non-compact type, and therefore $\chi(\pi) < 0$. Furthermore, $b_1(\pi) = 0$, see [VGS, Theorem 7.1]. On the other hand, $\chi(\pi) = 2 - 2b_1 + 2b_2 - b_3 = 2 + 2b_2 - b_3 < 0$, which implies $b_3(\pi) > 0$. Thus, $\pi \in \mathcal{B}$.

## 5. Nilpotent groups in $\mathcal{A}_4$

In this section we describe the nilpotent groups which can be realized as the fundamental groups of symplectic manifolds with $\pi_2(M) = 0$. Here we use some ideas from [R1].

Let $\pi$ be a finitely presented group, and let $X$ be a $CW$-space with $\pi_1(X) = \pi$ and finite 2-skeleton. Let $\widetilde{X}$ be the universal covering space of $X$, and let $H^1_c(\widetilde{X})$ be the 1-dimensional cohomology with compact supports of $\widetilde{X}$.

**5.1. Proposition–Definition.** *The group $H^1_c(\widetilde{X})$ depends on the group $\pi$ only. We denote it by $H^1_c(\pi)$ and call the 1-dimensional cohomology with compact supports of $\pi$.*

*Proof.* Consider two spaces $X_1$ and $X_2$ as the above described space $X$. First, assume that both $X_1$ and $X_2$ are $K(\pi,1)$'s. Consider homotopy equivalences $f : X_1 \to X_2$ and $g : X_2 \to X_1$ with $gf \simeq 1_{X_1}$ and $fg \simeq 1_{X_2}$. We can assume that $f(X_1^{(2)}) \subset X_2^{(2)}$ and $g(X_2^{(2)}) \subset X_1^{(2)}$. Moreover, the homotopies $H : gf \simeq 1$ and $H' : fg \simeq 1$ can be chosen so that $H(X_1^{(1)} \times I) \subset X_1^{(2)}$ and $H'(X_2^{(1)} \times I) \subset X_2^{(2)}$.

Passing to the universal coverings, we get the homotopy equivalences $\widetilde{f} : \widetilde{X}_1 \to \widetilde{X}_2$ and homotopies $\widetilde{H} : \widetilde{X}_1 \times I \to \widetilde{X}_1$ and $\widetilde{H}' : \widetilde{X}_2 \times I \to \widetilde{X}_2$. Clearly, the maps

$$\widetilde{f}|_{X_1^{(2)}} : X_1^{(2)} \to X_2, \quad \widetilde{g}|_{X_2^{(2)}} : X_1^{(2)} \to X_1$$

and the homotopies

$$\widetilde{H}|_{X_1^{(1)} \times I} : X_1^{(1)} \times I \to X_1, \quad \widetilde{H}'|_{X_2^{(1)} \times I} : X_2^{(1)} \times I \to X_2$$

are proper maps. Therefore $\widetilde{f}$ induces an isomorphism $H^1_c(\widetilde{X}_2) \to H^1_c(\widetilde{X}_1)$.

Now we consider an arbitrary space $X$ as above. We attach to $X$ cells of dimension $\geq 3$ and get an embedding $X \subset Y$ where $Y = K(\pi, 1)$.

Since $X^{(2)} = Y^{(2)}$, we conclude that $H^1_c(\widetilde{X}) = H^1_c(\widetilde{Y})$. This completes the proof. $\square$

**5.2. Remark.** Certainly, the group $H^1_c(\pi)$ admits a purely algebraic description in terms of the group $\pi$, cf [R1, N]. However, the description from 5.1 is enough for our goals.

**5.3. Theorem** (cf. [R1])**.** *Let $M^n$ be a closed manifold with $\pi_1(M) = \pi$ and $\pi_i(M) = 0$ for $2 \leq i \leq n-2$. Then $\pi_{n-1}(M) = H^1_c(\pi)$.*

*Proof.* Let $\widetilde{M}$ be the universal covering space for $M$. Because of the Poincaré duality we have

$$H^1_c(\pi) = H^1_c(\widetilde{M}) = H_{n-1}(\widetilde{M}).$$

But, by the Hurewicz Theorem, $H_{n-1}(\widetilde{M}) = \pi_{n-1}(\widetilde{M}) = \pi_{n-1}(M)$. $\square$

**5.4. Lemma.** *If $\pi$ is finitely generated nilpotent group with* $\operatorname{rank} \pi > 1$*, then $H^1_c(\pi) = 0$.*

*Proof.* First, we assume that $\pi$ is torsion free. We embed $\pi$ as a uniform lattice in a contractible nilpotent Lie group $G$ with $\dim G = \operatorname{rank} \pi = n$, [M]. Since $n > 1$, we conclude that $\pi_{n-1}(G) = 0$. Thus, since $\pi_1(G/\pi) = \pi$, we deduce from Theorem 5.3 that $H^1_c(\pi) = 0$.

Now, if $\pi$ is not torsion free then it contains a torsion free subgroup $\pi'$ of finite index, [Ku]. Then $K(\pi', 1)$ can be regarded as a finite covering over $K(\pi, 1)$. So, $K(\pi', 1)$ and $K(\pi, 1)$ have the same universal covering, and thus, $H^1_c(\pi) = H^1_c(\pi') = 0$. $\square$

**5.5. Corollary.** *Let $M$ be a closed $n$-dimensional manifold, $n > 1$ with $\pi_i(M) = 0$ for $i = 2, \dots, n-2$. If $\pi_1(M)$ is a nilpotent group $\pi$ with* $\operatorname{rank} \pi > 1$*, then $\pi$ is torsion free and* $\operatorname{rank} \pi = n$.

*Proof.* By Lemma 5.4, $H^1_c(\pi) = 0$. Therefore, by 5.3, $\pi_{n-1}(M) = 0$. Furthermore, $H_n(\widetilde{M}) = 0$ because $\pi$ is infinite. So, $\pi_n(\widetilde{M}) = 0$, i.e. $\widetilde{M}$ is contractible, i.e. $M = K(\pi, 1)$. So, $\pi$ is torsion free since $M$ is finite dimensional. Finally, $M$ is homotopy equivalent to a closed nilmanifold $G/\pi$ of dimension $n$, and therefore $\operatorname{rank} \pi = n$. $\square$

**5.6. Theorem.** *Let $M$ be a closed* 4*-dimensional symplectic manifold $M$ with $\pi_2(M) = 0$. If $\pi_1(M)$ is a nilpotent, then $\pi_1(M)$ is a torsion free nilpotent group of rank* 4*. Conversely, every finitely presented torsion free group can be realized as the fundamental group of closed* 4*-dimensional symplectic manifold with $\pi_2(M) = 0$.*

*Proof.* First, notice that $\operatorname{rank}\pi_1(M) > 1$. Indeed, if $\operatorname{rank}\pi_1(M) = 1$ then $\pi_1(M)$ contains $\mathbb{Z}$ as a subgroup of finite index. Considering the finite covering with respect to the inclusion $\mathbb{Z} \subset \pi_1(M)$, we get a 4-dimensional closed symplectic manifold with the fundamental group $\mathbb{Z}$. But this is impossible by Proposition 2.3.

Now, by Lemma 5.5, $\pi_1(M)$ must be a torsion free nilpotent group of the rank 4.

Finally, consider a torsion free finitely presented nilpotent group $\pi$, $\operatorname{rank}\pi = 4$. It is easy to see that $H^2(\pi;\mathbb{R}) \neq 0$. (You can use classification of such groups, [VGS], or notice that $b_1(\pi) > 1$ while $\chi(\pi) = 0$.) We embed $\pi$ as a uniform lattice in a 4-dimensional contractible nilpotent group $G$, [M]. Consider the closed oriented manifold $M := G/\pi, \dim M = 4$. Then $H^2(M;\mathbb{R}) = H^2(\pi;\mathbb{R}) \neq 0$. Take any $a \in H^2(M;\mathbb{R}), a \neq 0$. Then, by Poincaré duality, there exists $b \in H^2(M;\mathbb{R})$ with $ab \neq 0$. Since $ab = ba$, we have

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and so at least one of elements $a^2, b^2$ or $(a+b)^2$ must be non-zero. Thus, $M$ is a c-symplectic manifold. Now, asserting as in 4.2, we conclude that the nilmanifold $M$ is symplectic and symplectically aspherical. □

**5.7. Corollary.** *Let $\pi$ be a torsion free finitely generated c-symplectic nilpotent group. If $\operatorname{rank}\pi > 4$ then $\pi \in \mathcal{A}_6, \pi \in \mathcal{B}_4$ and $\pi \notin \mathcal{A}_4$.*

This Corollary strength Corollary 4.6.

*Proof.* Recall that $\pi$ is a uniform lattice in a certain simply connected group $G$, $\dim G = \operatorname{rank}\pi$. Asserting as in Lemma 4.2, we conclude that $\pi \in \mathcal{A}_{2n}$. Therefore $\pi \in \mathcal{A}_6$, see Proposition 3.6. Furthermore, $\pi \notin \mathcal{A}_4$ by Theorem 5.6. Thus, $\pi \in \mathcal{B}_4$ since, by proposition 3.6, $\mathcal{A}_6 \subset \mathcal{A}_4 \cup \mathcal{B}_4$. □

## 6. Gompf symplectic sum and symplectic asphericity

Here we mention briefly how to built symplectically aspherical manifold from other ones. Certainly, this yields to other examples of symplectically aspherical groups. We do not dwell these things here, but hope to do it somewhere later.

First, recall the construction of the connected sum of two manifolds along a submanifold, with the aim to emphasize the symplectic version of this construction, [G1] .

Let $M_1^n, M_2^n$ and $N^{n-2}$ be smooth closed oriented manifolds (not necessarily connected), of dimensions $n$ and $n-2$, respectively. Assume that we are giventwo embeddings $j_1 : N \to M_1$ and $j_2 : N \to M_2$, with the normal bundles $\nu_1$ and $\nu_2$, respectively, such that their Euler classes differ only by sign: $e(\nu_1) = -e(\nu_2)$. It turns out to be that there exists an orientation-reversing bundle isomorphism $\alpha : \nu_1 \to \nu_2$. Let $V_i$ denote a tubular neighborhood of $j_i(N)$, which we identify with the total space of $\nu_i$. Then $\alpha$ yields a diffeomorphism $\psi : V_1 \to V_2$, which maps $j_1(N)$ to $j_2(N)$. Then $\psi$ determines an orientation-preserving diffeomorphism

$$\varphi : (V_1 - j_1(N)) \to (V_2 - j_2(N)), \ \ \varphi = \theta \circ \psi,$$

where

$$\theta(p, v) = (p, \frac{v}{||v||^2})$$

is a diffeomorphism which turns each punctured normal fiber inside out.

**6.1. Definition.** Let $M_1 \cup_\psi M_2$ denote the smooth, closed oriented manifold obtained from the disjoint union $M_1 - (j_1(N)) \sqcup M_2 - (j_2(N))$ via gluing $V_1 - j_1(N)$ and $V_2 - j_2(N)$ by $\varphi$:

$$M_1 \cup_\psi M_2 = M - (j_1(N) \cup j_2(N))/ \simeq$$

where $a \simeq b$ if and only if $b = \varphi(a), \ a \in V_1 - j_1(N), b \in V_2 - j_2(N)$.

It was noted in [G1] that there exists a cobordism $X$ between $M_1 \sqcup M_2$ and $M_1 \cup_\psi M_2$. It will be important for us to notice that the cobordism $X$ is obtained from $(M_1 \sqcup M_2) \times I$ $(I = [0, 1])$ by identifying closed tubular neighborhoods of $j_1(N) \times 1$ and $j_2(N) \times 1$ by $\psi$ and rounding corners.

Now, we need the following observation. Every closed $k$-form $\omega_M$ on $M$ for which $j_1^*\omega_M = j_2^*\omega_M$ induces a cohomology class $[\Omega] \in H^k(X; \mathbb{R})$ and, hence, by restriction, a class $[\omega] \in H^k(M_1 \cup_\psi M_2; \mathbb{R})$. Note that $[\omega] = i^*[\Omega]$, where $i : M_1 \cup_\psi M_2 \to X$ is the canonical embedding.

In the sequel we will need the following result.

**6.2. Theorem.** *Let* $(M_1, M_2, N$ *and* $j_i : N \to M_i, i = 1, 2$ *be as in Definition* 6.1. *Suppose in addition that* $M_1, M_2$ *and* $N$ *are symplectic manifolds and both embeddings* $j_i : N \to M_i$ *are symplectic. Then, for any choice of (orientation reversing)* $\psi : V_1 \cong V_2$*, the manifold* $M_1 \cup_\psi M_2$ *admits a canonical symplectic structure* $\omega$*, which is induced by the symplectic form on* $M_1 \sqcup M_2$ *after a perturbation near* $j_2(N)$*. More precisely, there is a unique isotopy class of symplectic forms on*

*$M_1 \cup_\psi M_2$ (independent of fiber isotopies of $\psi$) that contains forms $\omega$ with the following characterization: the class $[\omega] \in H^2(MM_1 \cup_\psi M_2; \mathbb{R})$ is the restriction of the class $[\Omega] \in H^2(X; \mathbb{R})$ canonically induced on the cobordism $X$ by the symplectic on $M_1 \sqcup M_2$.*

*Proof.* See Gompf [G1]. □

Let $(X; A, B)$ be a $CW$-triad. We set $C = A \cap B$ and denote by $j_1 : A \to X$, $j_2 : B \to X$, $i_1 : C \to A$, $i_2 : C \to B$ the obvious inclusions.

**6.3. Proposition.** *Fix any $k$ and any coefficient group. If the homomorphism $i_1^* : H^k(A) \to H^k(C)$ is injective and the homomorphism $i_1^* : H^{k-1}(A) \to H^{k-1}(C)$ is surjective, then the $j_2^* : H^k(X) \to H^k(B)$ is injective and the homomorphism $j_2^* : H^{k-1}(X) \to H^{k-1}(B)$ is surjective.*

*Proof.* The exactness of the sequence

$$H^{k-1}(A) \xrightarrow{i_1^*} H^{k-1}(C) \to H^k(A, C) \xrightarrow{i_1^*} H^k(A) \to H^k(C)$$

implies that $H^k(A, C) = 0$. So, because of the excision property, $H^k(X, B) = H^k(A, C) = 0$. Now, the exactness of the sequence

$$H^{k-1}(X) \xrightarrow{j_2^*} H^{k-1}(B) \to H^k(X, B) \xrightarrow{i_2^*} H^k(X) \to H^k(B)$$

implies the required claim on $j_2^*$. □

We say that a 2-dimensional cohomology class $a$ is *decomposable* if it can be represented as $a = \sum_i a_i a_i'$ where $a_i$ and $a'i$ are 1-dimensional classes. Notice that a symplectic form $\omega$ on a symplectic manifold is aspherical if its cohomology class $[\omega]$ is decomposable.

**6.4. Theorem.** *Let $M_1, M_2, N$ and $j_i : N \to M_i, i = 1, 2$ be as in Theorem 6.2, and suppose that $j_1$ induces a surjection on the first cohomology group and an injection on the second cohomology group. Assume that $H^2(M_2; \mathbb{R})$ consists on decomposable elements. Then the symplectic manifold $(M_1 \cup_\psi M_2)$ is symplectically aspherical.*

*Proof.* Let $X$ be the cobordism described in after Definition 6.1. It suffices to prove that the cohomology class $[\Omega]$ of the form $\Omega$ on $X$ is decomposable. Notice that $X$ is homotopy equivalent to the space $Y = M_1 \cup_N M_2$. So, we have the triad $(Y; M_1, M_2)$ with $M_1 \cap M_2 = N$. Since smooth manifolds are triangulable, we can regard the above triad as a $CW$-triad. Let $j : M_2 \to Y$ be the inclusion. Because of the conditions of the Theorem,

$$j^*[\Omega] = \sum b_i b_i', \quad b_i, b_i' \in H^1(M_2).$$

By the Proposition 6.3, the map $j^* : H^1(Y) \to H^1(M_2)$ is an epimorphism. So, there are $a_i, a_i \in H^1(Y)$ with $j^*(a_i) = b_i, j^*(a_i) = b_i$. So, $j^*([\Omega] - \sum a_i a_i) = 0$. But, again by the Proposition 6.3, the map $j^* : H^2(Y) \to H^2(M_2)$ is a monomorphism, and thus $[\Omega] - \sum a_i a_i$. □

The Proposition below gives us one more source of symplectically aspherical manifolds.

**6.5. Proposition.** *Let $M_1, M_2, N$ and $j_i : N \to M_i, i = 1, 2$ be as in Theorem 6.2. Suppose in addition that $\pi_k(N) = 0 = \pi_k(M_i \setminus j_i(N)), i = 1, 2$ for $k > 1$ and the induced homomorphisms*

$$(j_i)_* : \pi_1(\partial N_i) \to \pi_1(M_i \setminus N_i), \quad i = 1, 2$$

*are monomorphisms Then $\pi_k(M_1 \cup_\psi M_2) = 0$ for $k > 1$. In particular the group*

$$\pi_1(M_1 \setminus N_1) *_{\pi_1(N_1 \setminus f_1(S))} \pi_1(M_2 \setminus N_2)$$

*is symplectically aspherical.*

*Proof.* See [Proposition 3.1][K]. □

One more sourse of symplectically aspherical groups comes from the observation of Gompf [G2, Lemma 1] who proved that a branched covering over a 4-dimensional symplectically aspherical manifold is symplectically aspherical.

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R. Ibáñez, Departamento de Matemáticas, Facultad de Ciencias, Universidad del Pais Vasco, Apdo. 644, 48080 Bilbao, Spain,

*E-mail address*: `mtpibtor@lg.ehu.es`

Yu. Rudyak, Department of Mathematics, Universoty of Florida, 358 Little Hall, Gainesville, FL 32601, USA

*E-mail address*: rudyak@math.ufl.edu

*E-mail address*: rudyak@mathi.uni-heidelberg.de

A. Tralle, Department of Mathematics, University of Warmia and Mazura, 10561 Olsztyn, Poland

*E-mail address*: tralle@matman.uwm.edu.pl