CLASSIFYING SPACES AND A SUBGROUP OF THE EXCEPTIONAL LIE GROUP G_2

BY
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Abstract. We consider a problem on the conditions of a compact Lie group that its loop space of the p–completed classifying space be a p–compact group, as well as some related problems. A previously obtained necessary condition is shown to be not sufficient. Our counterexample is given by a quotient group of a subgroup of the exceptional Lie group G_2 at $p = 3$. The K-theory of the space is isomorphic to $K(BG_2; \mathbb{Z}_3^\wedge)$, though its loop space is not a 3-compact group.

The notion of a p–compact group X, [4], is a good generalization of a compact Lie group G at the prime p. The structure of the classifying space BX is similar to that of $(BG)_p^\wedge$. Here we say that a space is a *p–compact classifying space* if its loop space is a p–compact group. It is well-known that $(BG)_p^\wedge$ is p–compact if $\pi_0(G)$ is a p–group. In [14] the author has tried to find the conditions on G that $(BG)_p^\wedge$ be a p–compact classifying space, and obtained some results mostly for a special case. Theorem 2 of [14] implies that the loop space $\Omega(BG)_p^\wedge$ is a p–compact toral group if and only if the compact Lie group G is p–nilpotent, [8]. Thus the connected component of G is necessarily a torus. For the general case, necessary conditions are stated in [14, Proposition 3.1]. Our work in this paper has been motivated by showing that the converse is false, even though the rational cohomology of $(BG)_p^\wedge$ is expressed as an invariant ring of pseudoreflections. We will use a subgroup of the exceptional Lie group G_2 to find a counterexample and to see simplicity of the p–completed classifying space of a non-connected compact Lie group.

Suppose that G is simple and simply–connected, and that the order of its Weyl group is divisible by a prime p. According to the results of [11] and [12], a map $f : BG \longrightarrow BX$ is essential if and only if $Ker\ f$ is included in the center of G, except that G is the exceptional Lie group G_2 at $p = 3$. The Lie group G_2 contains $SU(3)$. Assume $H = SU(3) \rtimes \mathbb{Z}/2$ is the

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subgroup of G_2 discussed in [11, p220, proof of Theorem 2]. The center of $SU(3)$, isomorphic to $\mathbb{Z}/3$, is a normal subgroup of H. Let Γ_2 denote the quotient group $H/(\mathbb{Z}/3)$. Then we see $(BH)^\wedge_3 \simeq (BG_2)^\wedge_3$ and the quotient map induces a map $f : (BH)^\wedge_3 \longrightarrow (B\Gamma_2)^\wedge_3$ with $f|_{B\mathbb{Z}/3} = 0$. Note that the center of G_2 is trivial. According to [14, Proposition 3.1], if G is a compact Lie group and $(BG)^\wedge_p$ is p–compact, then $\pi_0 G$ is p–nilpotent and $\pi_1((BG)^\wedge_p)$ is isomorphic to a p–Sylow subgroup of $\pi_0 G$. These two conditions are satisfied if p does not divide the order of $\pi_0(G)$. Consequently the 3–completed classifying space $(B\Gamma_2)^\wedge_3$, rationally equivalent to $(BG_2)^\wedge_3$, satisfies the necessary conditions.

Theorem 1. *Let $G = \Gamma_2$, the quotient group of a subgroup $SU(3) \rtimes \mathbb{Z}/2$ of the exceptional Lie group G_2. For $p = 3$, the following hold:*

1. *$\pi_0 G$ is p–nilpotent and $\pi_1((BG)^\wedge_p)$ is isomorphic to a p–Sylow subgroup of $\pi_0 G$.*
2. *$(BG)^\wedge_p$ is rationally equivalent to $(BG_2)^\wedge_p$.*
3. *$(BG)^\wedge_p$ is not a p-compact classifying space.*

Next we discuss a minimality of this example. As usual, for a compact Lie group G, the connected component with the identity is denoted by G_0, and $\pi_0 G = G/G_0$. When $rank(G_0) = 1$, the following result implies the converse of [14, Proposition 3.1] is true.

Theorem 2. *Suppose G is a compact Lie group with $rank(G_0) = 1$. If the group $\pi_0 G$ is p–nilpotent, then $(BG)^\wedge_p$ is a p–compact classifying space.*

We recall a few more examples. Suppose a compact connected Lie group G is simple, and NT denotes the normalizer of a maximal torus T of G. In [14] it is determined exactly when $(BNT)^\wedge_p$ is p–compact. The Weyl group being p–nilpotent is sufficient except the following cases. At $p = 2$, it is shown that $(BNT)^\wedge_2$ is not 2–compact if the Weyl group is one of the following four cases: $W(A_2)$, $W(B_3) = W(C_3)$, $W(D_3)$, $W(G_2)$, though $\pi_0(NT)$ is 2–nilpotent in each case.

We turn back to the subgroup $H = SU(3) \rtimes \mathbb{Z}/2$ of G_2 and its quotient group $\Gamma_2 = H/(\mathbb{Z}/3)$. Suppose T_1 is a maximal torus of $H_0 = SU(3)$ and T_2 is the image of T_1 under the quotient homomorphism $H \longrightarrow \Gamma_2$. Considering the normalizers, we obtain the following commutative diagram of groups:

$$\begin{tikzcd}
\mathbb{Z}/3 \arrow[r, equal] \arrow[d] & \mathbb{Z}/3 \arrow[r] \arrow[d] & * \arrow[d] \\
T_1 \arrow[r] \arrow[d] & N_H(T_1) \arrow[r] \arrow[d] & W_1 \arrow[d, equal] \\
T_2 \arrow[r] & N_{\Gamma_2}(T_2) \arrow[r] & W_2
\end{tikzcd}$$

Here $W_1 = N_H(T_1)/T_1$ and $W_2 = N_{\Gamma_2}(T_2)/T_2$, and both groups are isomorphic to the Weyl group of G_2. The map $T_1 \longrightarrow T_2$ is the admissible map, [1], for $SU(3) \longrightarrow PU(3)$. We notice that the W_2–action on T_2 is the dual representation of the W_1–action on T_1. The main result of [13] states that, in general, if the dual representation of $W(SU(n))$ is denoted by $W(SU(n))^*$, then $K(BPU(n); \mathbb{Z}_p^\wedge) = K(BT^{n-1}; \mathbb{Z}_p^\wedge)^{W(SU(n))^*}$ as λ–rings for any p, and the integral representation of $W(SU(n))$ is not isomorphic to $W(SU(n))^*$.

Since the exceptional Lie group G_2 is 3–torsion free, the mod 3 cohomology $H^*(BG_2, \mathbb{F}_3)$ is isomorphic to the ring of invariants $H^*(BT^2; \mathbb{F}_3)^{W(G_2)}$. The Weyl group $W(G_2)$ is the dihedral group of order 12 presented as $D_{12} =< r, s \mid r^6 = s^2 = 1, srs = r^5 >$. The matrix (integral) representation can be taken as follows:

$$r = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

Consequently, if we write $H^*(BT^2; \mathbb{F}_3) = \mathbb{F}_3[t_1, t_2]$ with $deg(t_i) = 2$, then the ring of invariants is the following polynomial ring:

$$H^*(BT^2; \mathbb{F}_3)^{W(G_2)} = \mathbb{F}_3[x_4, x_{12}]$$

where $x_4 = (t_1 - t_2)^2$ and $x_{12} = t_1^2 t_2^2 (t_1 + t_2)^2$.

The group G_2 contains $SU(3)$, and $W(SU(3))$ can be a subgroup of $W(G_2)$ such that $W(SU(3)) = \{1, r^2, r^4, s, sr^2, sr^4\}$. Consider the following invariant rings:

$$H^*(BT^2; \mathbb{F}_3)^{W(SU(3))} = \mathbb{F}_3[y_4, y_6]$$

where $y_4 = (t_1 - t_2)^2$ and $y_6 = t_1 t_2 (t_1 + t_2)$, and for the dual representation we have

$$H^*(BT^2; \mathbb{F}_3)^{W(SU(3))^*} = \mathbb{F}_3[z_2, z_{12}]$$

where $z_2 = t_1 + t_2$ and $z_{12} = t_1^2 t_2^2 (t_1 - t_2)^2$. Recall [20, §3] and [21, Ch10] that an unstable algebra is said to be *realizable* if it is isomorphic to the mod p cohomology of a space over the Steenrod algebra. Obviously the unstable

algebra $H^*(BT^2;\mathbb{F}_3)^{W(SU(3))}$ is realizable, however, $H^*(BT^2;\mathbb{F}_3)^{W(SU(3))^*}$ is not realizable, [13]. As the following result indicates, the case of G_2 is different. In fact, we see $H^*(BT^2;\mathbb{F}_3)^{W(G_2)} = H^*(BT^2;\mathbb{F}_3)^{W(G_2)^*}$ as unstable algebras.

Theorem 3. *Let Γ_2 be the compact Lie group as in Theorem 1. Then the following hold:*

(1) *The 3–adic K-theory $K(B\Gamma_2;\mathbb{Z}_3^\wedge)$ is isomorphic to $K(BG_2;\mathbb{Z}_3^\wedge)$ as a λ–ring.*
(2) *Let Γ be a compact Lie group such that $\Gamma_0 = PU(3)$ and the order of $\pi_0(\Gamma)$ is not divisible by 3. Then any map from $(B\Gamma)_3^\wedge$ to $(BG_2)_3^\wedge$ is null homotopic. In particular $[(B\Gamma_2)_3^\wedge,(BG_2)_3^\wedge]=0$.*

We recall that if a connected compact Lie group G is simple, the following results hold:

(1) For any prime p, the space $(BG)_p^\wedge$ has no nontrivial retracts. ([9])
(2) Assume $|W(G)| \equiv 0 \bmod p$. If a self-map $(BG)_p^\wedge \longrightarrow (BG)_p^\wedge$ is not null homotopic, it is a homotopy equivalence. ([18])
(3) Assume $|W(G)| \equiv 0 \bmod p$. For a compact Lie group K, if a map $f : (BG)_p^\wedge \longrightarrow (BK)_p^\wedge$ is trivial in mod p cohomology, then f is null homotopic. ([11])

Replacing G by Γ_2 at $p=3$, we will see that (3) still holds. On the other hand it is not known if (1) and (2) hold, though on the level of K-theory they do.

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1. The p–completion of BG and p–compact groups.

We will prove Theorem 1 and Theorem 2 in this section. Our proof of Theorem 1 uses a result of [15]. This result implies that there is no fibration of the type $B\mathbb{Z}/3 \longrightarrow (BG_2)_3^\wedge \longrightarrow Y$, despite the existence of a map $f : (BG_2)_3^\wedge \longrightarrow Y$ with $f|_{B\mathbb{Z}/3} = 0$. The proof of Theorem 2 uses a result of [7], which determines topology and algebra of the non-modular unstable polynomial algebras at any odd prime.

Lemma 1. *Suppose G is a compact Lie group such that a prime p does not divide the order of $\pi_0(G)$. Then $(BG)_p^\wedge$ is 1–connected.*

Proof. Recall that there exists a Postnikov fibration $F \longrightarrow (BG)^{\wedge}_p \xrightarrow{q} B\pi$ where $\pi = \pi_1((BG)^{\wedge}_p)$ and $q_* : \pi_1((BG)^{\wedge}_p) \longrightarrow \pi_1(B\pi)$ is isomorphic. Consider the following diagram:

$$\begin{tikzcd}
BG_0 \arrow[d] & F \arrow[d, "i"] \\
BG \arrow[r] \arrow[d] & (BG)^{\wedge}_p \arrow[d, "q"] \\
B\pi_0(G) \arrow[r] & B\pi
\end{tikzcd}$$

The map $B\pi_0(G) \longrightarrow B\pi$ is induced from $BG \longrightarrow (BG)^{\wedge}_p$, since the composite $BG_0 \longrightarrow B\pi$ is null homotopic. We note that p does not divide $|\pi_0(G)|$, and that π is a finite p–group, [3]. Consequently the composite $BG \longrightarrow B\pi$ is null homotopic. So there is a map $r : BG \longrightarrow F$ such that the p–completion $(i \cdot r)^{\wedge}_p$ is homotopy equivalent to the identity map of $(BG)^{\wedge}_p$. Thus $\pi_1(F) = 0$ implies $\pi_1((BG)^{\wedge}_p) = 0$. This completes the proof. □

Lemma 2. *Let* Γ_2 *be the compact Lie group as in Theorem 1. Then we have* $\pi_1((B\Gamma_2)^{\wedge}_3) = 0$ *and* $\pi_2((B\Gamma_2)^{\wedge}_3) = \mathbb{Z}/3$.

Proof. Without 3–completion, we have $\pi_1(B\Gamma_2) = \mathbb{Z}/2$ and $\pi_2(B\Gamma_2) = \mathbb{Z}/3$ from the exact sequences of homotopy groups for the following commutative diagram of groups:

$$\begin{tikzcd}
\mathbb{Z}/3 \arrow[r, equal] \arrow[d] & \mathbb{Z}/3 \arrow[r] \arrow[d] & * \arrow[d] \\
SU(3) \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[d, equal] \\
PU(3) \arrow[r] & \Gamma_2 \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

Thus Lemma 1 shows $\pi_1((B\Gamma_2)^{\wedge}_3) = 0$.

Next consider the Postnikov system $\{X_n\}$ of the space $B\Gamma_2$ so that

$$\pi_i(X_n) = \begin{cases} 0 & \text{if } i > n \\ \pi_i(B\Gamma_2) & \text{if } i \leq n \end{cases}$$

and we have the fibrations $K(\pi_n(B\Gamma_2), n) \longrightarrow X_n \longrightarrow X_{n-1}$. We note that $X_1 = K(\mathbb{Z}/2, 1)$ and $X_2 = K(\mathbb{Z}/3, 2) \times X_1$. For $n \geq 2$, we see $\pi_2((X_n)^{\wedge}_3) = \mathbb{Z}/3$. Tower lemma [3] implies, therefore, that $\pi_2((B\Gamma_2)^{\wedge}_3) = \varprojlim_n \pi_2((X_n)^{\wedge}_3) = \mathbb{Z}/3$. □

Proof of Theorem 1. From the definition of Γ_2 together with Lemma 2, it is easy to see that the first two conditions are satisfied. Consequently it suffices to show that the space $(B\Gamma_2)^\wedge_3$ is not 3-compact.

If $X = \Omega(B\Gamma_2)^\wedge_3$ is a 3-compact group, Lemma 2 implies that there is a Postnikov fibration $B\mathbb{Z}/3 \longrightarrow B\widetilde{X} \longrightarrow BX$ where $\widetilde{X}$ is a 1–connected 3-compact group, [19]. Notice that the projective group $PSU(3)$ is a subgroup of Γ_2, and that the map $(BH)^\wedge_3 \longrightarrow (BG_2)^\wedge_3$ is homotopy equivalent.

Consider the following diagram:

$$\begin{tikzcd}
B\mathbb{Z}/3 \arrow[r] & B\widetilde{X} \arrow[r] & BX \\
(BG_2)^\wedge_3 & (BH)^\wedge_3 \arrow[l, "\sim"'] \arrow[u] \arrow[r] & (B\Gamma_2)^\wedge_3 \arrow[u, equal] \\
 & BSU(3) \arrow[u] \arrow[r] & BPSU(3) \arrow[u]
\end{tikzcd}$$

where the map $(BH)^\wedge_3 \longrightarrow B\widetilde{X}$ is the lift of $(BH)^\wedge_3 \longrightarrow BX$, since $(BH)^\wedge_3$ is 2–connected.

First, suppose that the 3–completion of the composite $BSU(3) \longrightarrow B\widetilde{X}$ is a monomorphism of 3-compact groups. Since the composition of monomorphisms is also a monomorphism, the map $(BG_2)^\wedge_3 \longrightarrow B\widetilde{X}$ is a monomorphism. According to a result of Møller, [18, Lemma 2.5], this map is an isomorphism. This means that there is a fibration $B\mathbb{Z}/3 \longrightarrow (BG_2)^\wedge_3 \longrightarrow BX$. The result [15, Theorem 4] shows that this is a contradiction, since the center of the exceptional Lie group G_2 is trivial.

Next, suppose that the 3–completion of the composite $BSU(3) \longrightarrow B\widetilde{X}$ is not a monomorphism of 3–compact groups. If we consider the upper right of the previous diagram on the level of the maximal tori of 3–compact groups, we obtain the following diagram:

$$\begin{tikzcd}
B\mathbb{Z}/3 \arrow[r] & (BT^2)^\wedge_3 \arrow[r, "B\phi"] & (BT^2)^\wedge_3 \\
B\gamma \arrow[u] \arrow[r] & (BT^2)^\wedge_3 \arrow[u, "B\psi"'] \arrow[r] & (BT^2)^\wedge_3 \arrow[u, equal] \\
B\alpha \arrow[u] \arrow[r, equal] & B\alpha \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd}$$

where ϕ and ψ are the admissible map for $B\widetilde{X} \longrightarrow BX$ and $(BH)^\wedge_3 \longrightarrow B\widetilde{X}$ respectively, and α and γ are suitable finite 3–groups. Since $B\psi$ is not a monomorphism, the group α is non–trivial, [4, Proposition 9.11]. Consequently the kernel of the map $BSU(3) \longrightarrow BX$ contains $\mathbb{Z}/3$ as a

proper subgroup. This is a contradiction, since a map $f : BSU(3) \longrightarrow BX$ is essential if and only if $Ker\ f$ is included in the center of $SU(3)$. Consequently $\Omega(B\Gamma_2)^\wedge_3$ can not be a 3–compact group. $\square$

Proof of Theorem 2. First assume $p \not| |\pi_0 G|$. Then $H^*(BG;\mathbb{F}_p)$ is isomorphic to $H^*(BG_0;\mathbb{F}_p)^{\pi_0 G}$. Since $rank(G_0) = 1$, the connected compact Lie group is one of S^1, S^3, $SO(3)$. Recall that, if $H^*(BS^1;\mathbb{F}_p) = \mathbb{F}_p[t]$ with $deg(t) = 2$ and $H^*(B(\mathbb{Z}/2)^2;\mathbb{F}_2) = \mathbb{F}_2[x,y]$ with $deg(x) = deg(y) = 1$, then for odd p, we see $H^*(BS^3;\mathbb{F}_p) = H^*(BSO(3);\mathbb{F}_p) = \mathbb{F}_p[t]^{\mathbb{Z}/2} = \mathbb{F}_p[t^2]$, and for $p = 2$, $H^*(BS^3;\mathbb{F}_2) = \mathbb{F}_2[t^2]$ and $H^*(BSO(3);\mathbb{F}_2) = \mathbb{F}_2[x,y]^{GL(2,\mathbb{F}_2)} = \mathbb{F}_2[c_2,c_3]$. The mod p cohomology of each of classifying spaces of these Lie groups is a polynomial algebra, and $H^*(BG;\mathbb{F}_p)$ is also a polynomial algebra. If p is odd, a result of Dwyer–Miller–Wilkerson [7, Theorem 1.2] shows $(BG)^\wedge_p$ is a p–compact classifying space. For the case $p = 2$, the unstable algebra $H^*(BG;\mathbb{F}_2)$ must be one of the following:

$$H^*(BS^1;\mathbb{F}_2), H^*(BS^3;\mathbb{F}_2), H^*(BSO(3);\mathbb{F}_2)$$

It is known [6] that, in these cases, the mod 2 cohomology determines the homotopy type of each of the classifying spaces. Thus $(BG)^\wedge_2$ is homotopy equivalent to one of $(BS^1)^\wedge_2$, $(BS^3)^\wedge_2$, $(BSO(3))^\wedge_2$.

Next consider the general case. Since the group $\pi_0 G$ is p–nilpotent, if $\gamma = \pi_0 G$, then $\gamma = \nu \rtimes \gamma_p$ where ν is the subgroup generated by all elements of order prime to p, and where γ_p is the p–Sylow subgroup. We obtain the following commutative diagram of groups:

$$\begin{tikzcd}
G_0 \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & \nu \arrow[d] \\
G_0 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \pi_0 G \arrow[d] \\
* \arrow[r] & \gamma_p \arrow[r, equal] & \gamma_p
\end{tikzcd}$$

where H is the subgroup of G induced by the inclusion $\nu \longrightarrow \pi_0 G$. Since p does not divides $|\nu|$, the previous argument implies $(BH)^\wedge_p$ is a p–compact classifying space. Notice that the fibration $BH \longrightarrow BG \longrightarrow B\gamma_p$ is preserved by the p–completion, since γ_p is a finite p–group. Consequently $(BG)^\wedge_p$ is a p–compact classifying space. $\square$

2. Invariant rings and some properties of $B\Gamma_2$ and BG_2 at $p = 3$.

We will prove Theorem 3 in this section. Suppose G is a compact connected Lie group. The Weyl group $W(G)$ acts on the maximal torus T^n, and the integral representation $W(G) \longrightarrow GL(n, \mathbb{Z})$ is obtained. If such Lie groups are locally isomorphic, the representations of the Weyl groups are equivalent over $\mathbb{Q}$. They need not be, however, equivalent as $\mathbb{Z}$–representations. For instance, as mentioned before [13], the integral representation of $W(PU(n))$ is not equivalent to $W(SU(n))$, since the $\mathbb{Q}$–equivalence is induced by admissible map as follows:

$$\phi = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}$$

with $\det \phi \neq \pm 1$. Turning to the rings of invariant, we note that an isomorphism of cohomology rings need not imply the equivalence of the two representations. Such an example will be given by $W(G_2)$ and $W(G_2)^*$.

Lemma 3. *There is $\psi \in GL(2, \mathbb{Z})$ such that $\psi W(G_2) \psi^{-1} = W(G_2)^*$.*

Proof. As mentioned before, the representation of the Weyl group $W(G_2)$ is generated by $r = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$. Let $\psi = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. A calculation shows

$$\psi r \psi^{-1} = {}^t r \quad \text{and} \quad \psi s \psi^{-1} = {}^t(sr)$$

where ${}^t r = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and ${}^t(sr) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. This completes the proof. □

Remark. From Lemma 3 we see that, over the Steenrod algebra,

$$H^*(BT^2; \mathbb{F}_3)^{W(G_2)} \cong H^*(BT^2; \mathbb{F}_3)^{W(G_2)^*}$$

We notice, however, that the mod 3 reductions of the integral representations of $W(G_2)$ and $W(G_2)^*$ are not equivalent. In fact, the composition series for the $W(G_2)$–module V is

$$0 \longrightarrow \mathbb{F}_3 < t_1 - t_2 > \longrightarrow V \longrightarrow \mathbb{F}_3 < [t_1 + t_2] > \longrightarrow 0\ ,$$

while the composition series for the $W(G_2)^*$–module U is

$$0 \longrightarrow \mathbb{F}_3 < t_1 + t_2 > \longrightarrow U \longrightarrow \mathbb{F}_3 < [t_1 - t_2] > \longrightarrow 0\ .$$

The actions on $\mathbb{F}_3 < t_1 + t_2 >$ and $\mathbb{F}_3 < [t_1 + t_2] >$ are trivial, and those on $\mathbb{F}_3 < t_1 - t_2 >$ and $\mathbb{F}_3 < [t_1 - t_2] >$ are non-trivial.

As in [13], the $\mathbb{Q}$–representations of $W(G_2)$ and $W(G_2)^*$ are equivalent. The equivalence is given by the admissible map $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Consequently, under the projection $\mathbb{Z} \longrightarrow \mathbb{F}_p$ for $p \neq 3$, the mod p reductions of the integral representations are equivalent. For $p = 2$, the kernel of each of the reduction is $\mathbb{Z}/2$, and the image is $GL(2, \mathbb{F}_2)$. Otherwise the kernel is trivial.

Proof of Theorem 3. (1) Recall [2] that if G is a compact connected Lie group and T is its maximal torus, we have $K(BG; \mathbb{Z}_p^\wedge) \cong K(BT; \mathbb{Z}_p^\wedge)^{W(G)}$ for any prime p. Hence $K(BG_2; \mathbb{Z}_3^\wedge) \cong K(BT^2; \mathbb{Z}_3^\wedge)^{W(G_2)}$. Next, for the group extension $PU(3) \longrightarrow \Gamma_2 \longrightarrow \mathbb{Z}/2$, applying the spectral sequence $H^*(B\mathbb{Z}/2; K(BPU(3); \mathbb{Z}_3^\wedge)) \Rightarrow K(B\Gamma_2; \mathbb{Z}_3^\wedge)$ we see the following:

$$K(B\Gamma_2; \mathbb{Z}_3^\wedge) \cong K(BPU(3); \mathbb{Z}_3^\wedge)^{\mathbb{Z}/2} \cong K(BT^2; \mathbb{Z}_3^\wedge)^{W(G_2)^*}$$

Lemma 3 shows $K(BT^2; \mathbb{Z}_3^\wedge)^{W(G_2)} \cong K(BT^2; \mathbb{Z}_3^\wedge)^{W(G_2)^*}$. In fact, the map ψ is the admissible map between the two λ–rings, [23].

(2) A map $\alpha : (B\Gamma)_3^\wedge \longrightarrow (BG_2)_3^\wedge$ is null homotopic if and only if so is the restriction of α on $BPU(3)$, since $PU(3)$ contains all 3–toral subgroups of Γ. Let $f = \alpha|_{BPU(3)}$. It suffices to show that the induced homomorphism of mod 3 cohomology

$$f^* : H^*(BG_2; \mathbb{F}_3) \longrightarrow H^*(BPU(3); \mathbb{F}_3)$$

is trivial. We note that

$$H^*(BG_2; \mathbb{F}_3) = \mathbb{F}_3[x_4, x_{12}]$$

and that

$$H^*(BPU(3); \mathbb{F}_3) = \mathbb{F}_3[y_2, y_8, y_{12}] \otimes \Lambda(y_3, y_7)/J$$

where J is the ideal generated by y_2y_3, y_2y_7, and $y_3y_7 + y_2y_8$, [16, Theorem 3.10] . For the map f, there is $\phi : BT^2 \longrightarrow (BT^2)_3^\wedge$ which makes the following diagram commutative:

$$\begin{tikzcd} BPU(3) \arrow[r, "f"] & (BG_2)_3^\wedge \\ BT^2 \arrow[u] \arrow[r, "\phi"] & (BT^2)_3^\wedge \arrow[u, "Bi"'] \end{tikzcd}$$

If f was not null homotopic, then $Ker\ f$ should be trivial and hence the map ϕ would be mod 3 homotopy equivalence. Consequently we obtain the following commutative diagram:

$$\begin{array}{ccc} H^*(BG_2;\mathbb{F}_3) & \xrightarrow{f^*} & H^*(BPU(3);\mathbb{F}_3) \\ {\scriptstyle (Bi)^*}\downarrow & & \downarrow \\ H^*(BT^2;\mathbb{F}_3) & \xrightarrow{\cong} & H^*(BT^2;\mathbb{F}_3) \end{array}$$

Since $(Bi)^*$ is a monomorphism, so is f^*. Thus we have $f^*(x_4) = \pm y_2^2$ and $f^*(x_{12}) = \pm y_{12} + ay_2^6$ for some $a \in \mathbb{F}_3$. Thus the image of f^* would be included in the algebra generated by y_2 and y_{12}. According to [16, Theorem 3.20] , however, we see that $\mathcal{P}^1(y_{12}) = y_8^2 + y_{12}y_2^2$. This means that $Im\ f^*$ would not be closed under the action of a Steenrod operation. This contradiction implies the desired result. $\square$

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