

M-STRUCTURES DETERMINE INTEGRAL HOMOTOPY TYPE

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ABSTRACT. This paper proves that the functor $\mathcal{C}(\ast)$ that sends pointed, simply-connected CW-complexes to their chain-complexes equipped with diagonals and iterated higher diagonals, determines their integral homotopy type — even inducing an equivalence of categories between the category of CW-complexes up to homotopy equivalence and a certain category of chain-complexes equipped with higher diagonals. Consequently, $\mathcal{C}(\ast)$ is an algebraic model for integral homotopy types similar to Quillen’s model of rational homotopy types. For finite CW complexes, our model is finitely generated.

Our result implies that the geometrically induced diagonal map with all “higher diagonal” maps (like those used to define Steenrod operations) collectively determine integral homotopy type.

1. INTRODUCTION

This paper forms a sequel to [11]. That paper developed the theory of m-coalgebras and defined a functor $\mathcal{C}(\ast)$ that associated canonical m-coalgebras to semi-simplicial complexes.

Our main result is:

Corollary 3.6 on page 25: *The functor (defined in 4.2 on page 30 of [11])*

$$\mathcal{C}(\ast): \underline{\text{Homotop}}_0 \rightarrow \hat{\mathfrak{M}}$$

(see 2.15 on page 13 for the definition of $\hat{\mathfrak{M}}$) defines an equivalence of categories, where $\underline{\text{Homotop}}_0$ is the category of pointed, simply-connected CW-complexes and continuous maps, in which homotopy equivalences have been inverted (i.e., it is the category of fractions by homotopy equivalences).

This, of course, implies the claim made in the title — that m-structures determine integral homotopy type.

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From the beginning, it has been a central goal of homotopy theory (and algebraic topology in general) to develop tractable models for spaces and mappings. The early models were combinatorial, including simplicial or semi-simplicial complexes, chain-complexes, DGA algebras and coalgebras and so on. These models tended to fall into two classes:

- powerful, but computationally intractable (i.e., minimal models, chain-complexes of free rather than abelian groups, etc.)
- weak (chain-complex) but well-behaved.

The first major breakthrough came with the work of Quillen in [9], in which he simplified the problem by focusing on rational homotopy types. Rationalizing eliminates much of homotopy theory’s complexity by killing off cohomology operations, like Steenrod operations. Quillen was able to create a complete and faithful model of rational homotopy theory — co-commutative DGA-coalgebras over \mathbb{Q} .

This paper is the outcome of a research program of several years duration. One of the main goals of this program was to understand the coproduct or cup-product structure of the total space of a fibration. In order to accomplish this, it was necessary to compute a topological coproduct on the cobar construction and on the canonical acyclic twisted tensor with fiber a cobar construction.

Although the cobar construction is defined for DGA-coalgebras, computing a “geometric” coproduct on the cobar construction requires more than the mere coproduct. I quickly realized that various cohomology operations entered into the cobar construction’s coproduct. It was necessary to equip the chain complex of a space with diagonals and higher diagonals defined on the *chain level* (rather than on cohomology with coefficients in a finite field).

These higher coproducts satisfy a complex web of relationships I call *coherence conditions*. In [11], I developed an algebraic device called an *m-coalgebra* over a formal coalgebra to encapsulate these relations. A referee of [11] pointed out that formal coalgebras had been defined and studied before under the name *operad*.

This research had a gratifying outcome: A coherent *m-coalgebra*’s cobar construction not only has a computable coproduct; it comes equipped with a well-defined and geometrically valid *m-coalgebra structure* (although the coherence condition must be weakened slightly). It, consequently, becomes possible to *iterate* the cobar construction. A side-effect was an explicit procedure for computing geometric *m-coalgebra* structures on the total space of a fibration (represented by a twisted tensor product).

This suggested to me a *possibility* of characterizing integral homotopy theory: If one can compute coproducts (and higher coproducts) on fibrations, one can in principal compute fibrations over fibrations, and so on. This suggested the possibility of purely algebraic computations of *Postnikov towers* — possibly along the lines of Sullivan in [13].

The present paper is the result.

In 1985, Smirnov proved a result similar to ours in [10] — showing that a functor whose value is a certain comodule over a certain operad determines the integral homotopy type of a space. The operad and comodule in question were uncountably generated in all dimensions and in the simplest case.

In contrast, our functor is finitely generated in all dimensions for finite simplicial complexes. Although it is considerably more complex than the co-commutative coalgebras Quillen derived, it is highly unlikely one can get away with something much simpler: *all* of our functor appears nontrivially in even the *coproduct* of a cobar construction.

At this point, I feel it is appropriate to compare and contrast my results with work of Michael Mandell. In [7], he proved

Main Theorem. *The singular cochain functor with coefficients in $\bar{\mathbb{Z}}_p$ induces a contravariant equivalence from the homotopy category of connected nilpotent p -complete spaces of finite p -type to a full subcategory of the homotopy category of $E_\infty \bar{\mathbb{Z}}_p$ -algebras.*

Here, p denotes a prime and $\bar{\mathbb{Z}}_p$ the algebraic closure of the finite field of p elements. E_∞ -algebras are defined in [4] — they are modules over a suitable operad.

At first glance, it would appear that his results are a kind of dual to mine: He characterizes nilpotent p -complete spaces in terms of $E_\infty \bar{\mathbb{Z}}_p$ -algebras. This is not the case, however. A complete characterization of nilpotent p -complete spaces does not lead to one of integral homotopy types: One must somehow know that p -local homotopy equivalences patch together. Consequently, his results do not imply mine.

The converse statement is also true: My results do not imply his.

My results in [11] imply that all the primes “mix” when one studies algebraic properties of homotopy theory (for instance the p -local structure of the cobar construction of a space depend on the q -local structure of the space for all primes $q \geq p$). This is intuitively clear when considers the composite $(1 \otimes \Delta) \circ \Delta$ (iterated coproducts) and notes that \mathbb{Z}_2 acting on both copies of Δ give rise to elements of the symmetric group on 3 elements.

Consequently, a characterization of integral homotopy does not lead to a p -local homotopy theory: In killing off all primes other than p , one also kills off crucial information needed to compute the cobar construction of a space.

In [7], Dr. Mandell proved that one *must* pass to the algebraic closure of \mathbb{Z}_p to get a characterization of p -complete homotopy theory. I conjecture that, in passing to the algebraic closure, one kills off additional data within the homotopy type — namely the data that depends on larger primes. Consequently, one restores algebraic consistence to the theory, regaining the ability to characterize local homotopy types.

I am indebted to Jim Stasheff for his encouragement and to Michael Mandell for pointing out errors and inconsistencies in an earlier version of this paper.

2. DEFINITIONS AND PRELIMINARIES

We recall a few relevant facts from [11].

Definition 2.1. If $f: C_1 \rightarrow D_1$, $g: C_2 \rightarrow D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$.

Remarks. 2.1.1. This convention simplifies many of the common expressions that occur in homological algebra — in particular it eliminates complicated signs that occur in these expressions. For instance the differential, ∂_{\otimes} , of the tensor product $C \otimes D$ is just $\partial_C \otimes 1 + 1 \otimes \partial_D$.

2.1.2. Throughout this entire paper we will follow the convention that group-elements act on the left. Multiplication of elements of symmetric groups will be carried out accordingly — i.e. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} =$ result of applying $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ first and then $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. The product is thus $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$.

2.1.3. Let f_i, g_i be maps. It isn't hard to verify that the Koszul convention implies that $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)$.

2.1.4. We will also follow the convention that, if f is a map between chain-complexes, $\partial f = \partial \circ f - (-1)^{\deg(f)} f \circ \partial$. The compositions of a map with boundary operations will be denoted by $\partial \circ f$ and $f \circ \partial$ — see [1]. This convention clearly implies that $\partial(f \circ g) = (\partial f) \circ g + (-1)^{\deg(f)} f \circ (\partial g)$. We will call any map f with $\partial f = 0$ a chain-map. We will also follow the convention that if C is a chain-complex and

$\uparrow: C \rightarrow \Sigma C$ and $\downarrow: C \rightarrow \Sigma^{-1}C$ are, respectively, the suspension and desuspension maps, then \uparrow and \downarrow are both chain-maps. This implies that the boundary of ΣC is $-\uparrow \circ \partial C \circ \downarrow$ and the boundary of $\Sigma^{-1}C$ is $-\downarrow \circ \partial C \circ \uparrow$.

2.1.5. We will use the symbol T to denote the transposition operator for tensor products of chain-complexes $T: C \otimes D \rightarrow D \otimes C$, where $T(c \otimes d) = (-1)^{\dim(c) \cdot \dim(d)} d \otimes c$, and $c \in C$, $d \in D$.

Definition 2.2. Let $\{U_n\}$ denote a sequence of differential graded \mathbb{Z} -chain-complexes with preferred \mathbb{Z} -bases, $\{b_\alpha\}$, with n running from 1 to ∞ . This sequence will be said to constitute an *operad* with U_n being the component of rank n if:

given \mathbb{Z} -basis elements, S_1 and S_2 , the following (possibly distinct) composites are defined: $\{S_1 \circ_k S_2\}$, where $1 \leq k \leq \text{rank}(S_2)$ and all are defined to have rank equal to $\text{rank}(S_1) + \text{rank}(S_2) - 1$ and degree equal to $\dim(S_1) + \dim(S_2)$. These composition operators are subject to the following identities:

1. $(S_1 \circ_i S_2) \circ_j S_3 = S_1 \circ_{i+j-1} S_2 \circ_j S_3$;
2. if $j < i$ then $S_1 \circ_{i+\text{rank}(S_2)-1} S_2 \circ_j S_3 = S_2 \circ_j S_1 \circ_i S_3$

The differential $\partial: U \rightarrow U$:

1. preserves rank;
2. imposes the following additional condition on composition operations $\partial(S_1 \circ_i S_2) = \partial S_1 \circ_i S_2 + (-1)^{\dim(S_1)} S_1 \circ_i \partial S_2$.

Remarks. 2.2.1. Multiple compositions are assumed to be right-associative unless otherwise stated — i.e. $S_1 \circ_i S_2 \circ_j S_3 = S_1 \circ_i (S_2 \circ_j S_3)$.

2.2.2. An operad will be called *unitary* if it contains an identity element with respect to the composition-operations $\{\circ_i\}$. This will clearly have to be an element of rank 1 and degree 0.

2.2.3. Our definition of an operad in the category of DGA algebras is slightly different from the standard one given in [4]. It is a simple exercise to see that the two definitions are equivalent: The fundamental degree- n operation of an operad, Z , (in the standard definition) is a $n + 1$ -linear map

$$Z_{i_1} \otimes Z_{i_2} \otimes \cdots \otimes Z_{i_n} \otimes Z_n \rightarrow Z_{i_1+\dots+i_n}$$

which is simply an n -fold iteration of our “higher” compositions:

$$z_1 \circ_1 z_2 \circ_2 \cdots z_n \circ_n b$$

where $b \in Z_n$ and $z_j \in Z_{i_j}$.

Our notation lends itself to the kinds of computations we want to do.

Definition 2.3. Let A and B be operads. A morphism $f: A \rightarrow B$ is a morphism of the underlying chain-complexes, that preserves the composition operations.

Now we give a few examples of operads:

Definition 2.4. The *trivial* operad, denoted I , is defined to have one basis element $\{b_i\}$ for all integers $i \geq 0$. Here the rank of b_i is i and the degree is 0 and these elements satisfy the composition-law: $b_i \circ_\alpha b_j = b_{i+j-1}$, regardless of the value of α , which can run from 1 to j . The differential of this formal coalgebra is identically zero.

Remark. 2.4.1 This is clearly a unitary operad — the identity element is b_1 .

Definition 2.5. Let C_1 and C_2 be operads. Then $C_1 \otimes C_2$ is defined to have:

1. component of rank $i = (C_1)_i \otimes (C_2)_i$, where $(C_1)_i$ and $(C_2)_i$ are, respectively, the components of rank i of C_1 and C_2 ;
2. composition operations defined via $(a \otimes b) \circ_i (c \otimes d) = (-1)^{\dim(b)\dim(c)}(a \circ_i c \otimes b \circ_i d)$, for $a, c \in C_1, b, d \in C_2$.

Definition 2.6. Let C be a DGA-module with augmentation $\epsilon: C \rightarrow \mathbb{Z}$, and with the property that $C_0 = \mathbb{Z}$. Then the *endomorphism operad* of C , denoted $\mathbf{P}(C)$ is defined to be the operad with:

1. component of rank $i = \text{Hom}_{\mathbb{Z}}(C, C^i)$, with the differential induced by that of C and C^i . The dimension of an element of $\text{Hom}_{\mathbb{Z}}(C, C^i)$ (for some i) is defined to be its degree as a map.
2. The \mathbb{Z} -summand is generated by one element, e , of rank 0.

Let $s_1 \in \text{Hom}_{\mathbb{Z}}(C, C^i)$ and $s_2 \in \text{Hom}_{\mathbb{Z}}(C, C^j)$ be elements of rank i and j , respectively, where $i, j \geq 1$. Then the composition $s_1 \circ_k s_2$, where $1 \leq k \leq j$, is defined by: $s_1 \circ_k s_2 = \underbrace{1 \otimes \cdots \otimes s_1 \otimes \cdots \otimes 1}_{k^{\text{th}} \text{ position}} \circ s_2: C \rightarrow$

C^{i+j-1} . The composition $e \circ_k s_2$ is defined in a similar way, by identifying e with the augmentation map of C — it follows that $e \circ_k s_2 \in \text{Hom}_{\mathbb{Z}}(C, C^{j-1})$, as one might expect.

The canonical subcomplex $\text{Hom}_{\mathbb{Z}}(C, C^i)$ of elements of rank i , is equipped with a natural S_i -action — it is defined by permutation of the factors of the target, C^i .

Remarks. 2.6.1. This is a unitary operad — its identity element is the identity map $\text{id} \in \text{Hom}_{\mathbb{Z}}(C, C)$.

2.6.2. In general, operads model structures like the iterated coproducts that occur in the endomorphism operad. We will use operads as an convenient algebraic framework for defining other constructs that have topological applications.

Proposition 2.7. *Let C be a DGA-module. Co-associative coalgebra structures on C can be identified with morphisms $f: I \rightarrow \mathbf{P}(C)$, the the trivial operad to the endomorphism of C .*

We now define a very important operad — the *symmetric construct*. It models the formal behavior of $\{\mathrm{Hom}_{\mathbb{Z}}(C, C^n)\}$ in which each C^n is equipped with an action of S_n that permutes the factors of C .

The symmetric construct will be denoted \mathfrak{S} . Its components are $\{\mathbf{R}(S_n)\}_{n \in \mathbb{Z}^+}$, where:

1. S_n denotes the symmetric group on n objects;
2. $\mathbf{R}(S_n)$ denotes the bar-resolution of \mathbb{Z} over $\mathbb{Z}S_n$;

Here we follow the convention that $\mathbf{R}(S_0) = \mathbf{R}(S_1) = \mathbb{Z}$, concentrated in dimension 0. Pure elements of \mathfrak{S} are canonical basis elements of $\mathbf{R}(S_n)$ for all values of n , or the generator 1 of the \mathbb{Z} -summand (by canonical basis elements, we mean elements of the form $[g_1 | \dots | g_k] \in \mathbf{R}(S_n)$).

See § 2 of [11] for a detailed description of the composition operations of \mathfrak{S} .

We are now in a position to define *m-structures*

Definition 2.8. Let C be a chain-complex with $H_0(C) = \mathbb{Z}$. Then:

1. An *m-structure* on C is defined to be a sequence of chain maps $\mathfrak{f}[C]_n: C \rightarrow \mathrm{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^n)$, where $\mathfrak{R} = \{\mathfrak{R}[C]_n\}$ is some \mathfrak{f} -resolution, and n is an integer that satisfies $0 \leq n < \infty$. We assume that:
 - (a) the composite $e_1 \circ f_1: C \rightarrow C^1$, is the identity map of C ;
 - (b) and the composite $e_0 \circ f_0: C \rightarrow C^0 = \mathbb{Z}$ coincides with the augmentation of C ;
 - (c) For any $c \in C$, at most a finite number of the $\{\mathfrak{f}[C]_n(c)\}$ are nonzero. Here C^n is equipped with the S_n -action that permutes the factors.
 - (d) The adjoint will be denoted $\widetilde{\mathfrak{f}}[C]_n: \mathfrak{R}[C]_n \otimes C \rightarrow C^n$, and is defined by $\widetilde{\mathfrak{f}}[C]_n(r \otimes c) = (-1)^{\dim(r) \cdot \dim(c)} \mathfrak{f}[C]_n(c)(r)$, where $r \in \mathfrak{R}[C]_n$ and $c \in C$. With this definition in mind, we require $\widetilde{\mathfrak{f}}[C]_n(\mathfrak{R}[C]_n \otimes C(k)) \subseteq C(k)^n$, where $C(k)$ is the k -skeleton of C .

2. An m-structure will be called *weakly-coherent* if the *adjoint maps* fit into commutative diagrams:

$$\begin{array}{ccc}
\mathfrak{R}[C]_n \otimes \mathfrak{R}[C]_m \otimes C & \xrightarrow{1_{\mathfrak{R}[C]_n \otimes \mathfrak{R}[C]_m \otimes C}} & \mathfrak{R}[C]_n \otimes \mathfrak{R}[C]_m \otimes C \\
\downarrow 1 \otimes \widetilde{f}[C]_m & & \downarrow \circ_i \\
\mathfrak{R}[C]_n \otimes C^m & & \mathfrak{R}[C]_{n+m-1} \otimes C \\
\downarrow V_{i-1} & & \downarrow \widetilde{f}[C]_{n+m-1} \\
C^{i-1} \otimes \mathfrak{R}[C]_n \otimes C \otimes C^{m-i} & \xrightarrow{1 \otimes \cdots \otimes \widetilde{f}[C]_n \otimes \cdots \otimes 1} & C^{n+m-1}
\end{array}$$

for all $n, m \geq 1$ and $1 \leq i \leq m$. Here $V: \mathfrak{R}[C]_n \otimes C^m \rightarrow C^{i-1} \otimes \mathfrak{R}[C]_n \otimes C \otimes C^{m-i}$ is the map that shuffles the factor $\mathfrak{R}[C]_n$ to the right of $i - 1$ factors of C .

3. An m-structure $\{f[C]_n: C \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^n)\}$, will be called *strongly coherent* (or just *coherent*) if it is weakly coherent, and $\mathfrak{R}[C] = \mathfrak{S}$.

A chain-complex, C , equipped with an m-structure will be called an *m-coalgebra*. The maps $f[C]_n: C \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^n)$, where n is an integer such that $0 \leq n < \infty$, will be called the *structure maps* of C .

Remarks. 2.8.1. If C is an incoherent m-coalgebra we may, without loss of generality, assume that $\mathfrak{R}[C] = \mathfrak{S}$, since the contracting homotopy, Φ , that is packaged with $\mathfrak{R}[C]$, allows us to construct a unique sequence of chain-map $\mathfrak{S}_n = R(S_n) \rightarrow \mathfrak{R}[C]_n$, for n an integer such that $0 \leq n < \infty$. We then compose the structure maps of the original m-coalgebra with the induced natural transformation $\text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C], *) \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{S}, *)$, to get the structure maps of the modified m-coalgebra.

2.8.2. An m-coalgebra can be given the following interpretation: The adjoint isomorphism allows us to regard the structure maps as a family of S_n -equivariant chain-maps $\widetilde{f}[C]_n: R(S_n) \otimes C \rightarrow C^n$. The map $\widetilde{f}[C]_2: R(S_2) \otimes C \rightarrow C^2$, restricted to $[\] \otimes C$, defines a kind of coproduct on \widetilde{C} , called the *underlying coproduct* of the m-coalgebra. Define $\mathfrak{D}_a = f[C]_i(a \otimes *): C \rightarrow C^i$. These maps will be called the *higher-coproducts* associated with the m-coalgebra. The map $\mathfrak{D}_{[(1,2)]}: C \rightarrow C^2$ defines a chain-homotopy between $\Delta = \mathfrak{D}_{[\]}$ and $T \circ \Delta$, where T is the transposition map defined in 2.1.5 on page 5.

2.8.3. The basic definitions can be stated in terms of *operads* in the category of graded differential modules. Operads were originally defined in terms of topological spaces by May in [8] and this concept was extended to DG-modules by Smirnov in [10]. Essentially:

1. the operad \mathfrak{S} , constitutes an operad, and
2. a coherent m-coalgebra is a *comodule over* this operad, in the sense of § 3 of [10].

2.8.4. My original definition of an m-coalgebra regarded a coherent m-structure as a morphism of operads $\mathfrak{S} \rightarrow \mathbf{P}(C)$, and a weakly coherent m-structure as a morphism $\mathfrak{R}[C] \rightarrow \mathbf{P}(C)$. Although this definition has the advantage of being much more elegant than the one given above it doesn't lend itself to effective computation unless C is finitely generated as a \mathbb{Z} -module — this means:

1. $C_i \neq 0$ for at most a finite number of values of i ;
2. each of these nonzero C_i is, itself, finitely generated as a \mathbb{Z} -module.

2.8.5. The definition of weak coherence of an m-structure can be re-stated in terms of the maps $\{f[C]_n\}$ themselves, rather than their *adjoints* $\{\widetilde{f[C]_n}\}$. An m-structure is weakly coherent if and only if the diagram in figure 2.2.2 on page 22 of [11] commutes for all integers n such that $0 \leq n < \infty$. In this diagram, the map V'_i represents the composite

$$\begin{aligned}
 (2.1) \quad & \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^{i-1} \otimes \text{Hom}_{\mathbb{Z}S_m}(\mathfrak{R}[C]_m, C^m) \otimes C^{n-i}) \\
 & \xrightarrow{i_1} \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^{i-1} \otimes \text{Hom}_{\mathbb{Z}S_m}(\mathfrak{R}[C]_m, C^m) \otimes C^{n-i}) \\
 & \xrightarrow{\text{Hom}_{\mathbb{Z}S_n}(1, i_2)} \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^{i-1} \otimes \text{Hom}_{\mathbb{Z}S_m}(\mathfrak{R}[C]_m, C^m) \otimes C^{n-i}) \\
 & \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathfrak{R}[C]_n \otimes \mathfrak{R}[C]_m, C^{n+m-1})
 \end{aligned}$$

where i_1 and i_2 are inclusion mappings of the $\text{Hom}_{\mathbb{Z}S_n}$ -functors in the respective $\text{Hom}_{\mathbb{Z}}$ -groups. We are also including $\text{Hom}_{\mathbb{Z}S_i}(*, *)$ in

$\text{Hom}_{\mathbb{Z}}(*, *)$, by simply forgetting that the elements are $\mathbb{Z}S_i$ linear.

$$\begin{array}{ccc}
C & \xrightarrow{f[C]_n} & \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^n) \\
\downarrow f_{n+m-1} & & \downarrow \text{Hom}_{\mathbb{Z}}(1, 1 \otimes \cdots \otimes f_m \otimes \cdots \otimes 1) \\
& & G \\
& & \downarrow V'_i \\
\text{Hom}_{\mathbb{Z}S_{n+m-1}}(\mathfrak{R}[C]_{n+m-1}, C^{n+m-1}) & \xrightarrow{\text{Hom}_{\mathbb{Z}}(\circ_i, 1)} & \text{Hom}_{\mathbb{Z}}(\mathfrak{R}[C]_n \otimes \mathfrak{R}[C]_m, C^{n+m-1})
\end{array}$$

where $G = \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C]_n, C^{i-1} \otimes \text{Hom}_{\mathbb{Z}S_m}(\mathfrak{R}[C]_m, C^m) \otimes C^{n-i})$. This diagram means that the composition-operations in the coordinate coalgebra correspond to actual compositions of the adjoint maps.

Coherence of an m-structure implies a number of identities involving compositions of higher coproducts. For instance, $\mathfrak{D}_{[(1,2)]} \otimes 1 \circ \mathfrak{D}_{[(1,2)]} = \mathfrak{D}_{[(1,2)] \otimes \mathfrak{r}_{2,1}[(1,2)]} = \mathfrak{D}_{[(1,3,2)] \otimes [(1,2)]} = \mathfrak{D}_{[(1,3,2)|(1,2)] - [(1,2)|(1,2,3)]} = \mathfrak{D}_{[(1,3,2)|(1,2)]} - \mathfrak{D}_{[(1,2)|(1,2,3)]}$. In fact, we can translate *any* formula involving compositions of higher-coproducts into one without compositions involving elements of the $\{\mathfrak{R}(S_n)\}$.

Proposition 2.9. *Let $\mathfrak{R}_1 = \{\mathfrak{R}_{1,n}\}$ and $\mathfrak{R}_2 = \{\mathfrak{R}_{2,n}\}$ be f-resolutions, and let C_1 and C_2 be chain-complexes. Then there exists a natural transformation of functors $\mathfrak{E}_n: \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}_{1,n}, C_1^n) \otimes \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}_{2,n}, C_2^n) \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}_{1,n} \otimes \mathfrak{R}_{2,n}, (C_1 \otimes C_2)^n)$, for all n .*

Remark. 2.9.1 If $u \in \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}_{1,n}, C_1^n)$, $v \in \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}_{2,n}, C_2^n)$, then \mathfrak{E}_n sends $u \otimes v$ to $(c_1 \otimes c_2 \rightarrow V_n((u \otimes v)(c_1 \otimes c_2)))$, where $c_1 \in C_1$, $c_2 \in C_2$ and $V_n: C_1^n \otimes C_2^n \rightarrow (C_1 \otimes C_2)^n$ is the map that shuffles the factors of together.

Now we recall how morphisms of m-coalgebras were defined in [11]:

Definition 2.10. Let C_1 and C_2 be m-coalgebras with sets of structure maps $\{f[C_i]_n: C_i \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{R}[C_i]_n, C_i^n)\}$, $i = 1, 2$, and all $0 \leq n < \infty$. A *strict morphism* $\{g, h\}: C_1 \rightarrow C_2$ consists of:

1. a chain-map from $g: C_1 \rightarrow C_2$;
2. a morphism of f-resolutions, $h: \mathfrak{R}[C_2] \rightarrow \mathfrak{R}[C_1]$ such that the diagram

$$\begin{array}{ccc}
 C_1 & \xrightarrow{f[C_1]_n} & \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{A}[C_1]_n, C_1^n) \\
 \downarrow g & & \downarrow \text{Hom}_{\mathbb{Z}S_n}(h, g^n) \\
 C_2 & \xrightarrow{f[C_2]_n} & \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{A}[C_2]_n, C_2^n)
 \end{array}$$

commutes for all n .

Definition 2.11. A *contraction* of chain-complexes

$$(f', p, \varphi): C \rightarrow D$$

is a pair of maps $f': C \rightarrow D$, $f: D \rightarrow C$ and a chain-homotopy $\varphi: C \rightarrow C$ such that:

1. $f' \circ f = 1_D$
2. $f \circ f' - 1_C = \partial\varphi$.
3. $\varphi^2=0$, $\varphi \circ f = 0$, and $f' \circ \varphi = 0$

The map f' is called the *projection* of the contraction and f is called its *injection* — see [2].

Remark. 2.11.1 In his thesis ([6]), Martin Majewsky called contractions *Eilenberg-Zilber* maps.

Definition 2.12. Let C and D be weakly-coherent m-coalgebras. A contraction

$$(f', f, \varphi): C \rightarrow D$$

with the injection, f , a strict morphism of m-coalgebras, will be called an *elementary equivalence* from $C \rightarrow D$. We will use the notation

$$C \overset{f}{\underset{f'}{\rightleftarrows}} D$$

to denote an elementary equivalence.

Remark. 2.12.1 It is well-known (for instance, see the discussion of Schanuel's Lemma in [5]) that any chain-homotopy equivalence of two chain-complexes can be decomposed into two iterated contractions.

This implies that contractions are of limited interest when one is studying *chain-complexes*. This is no longer true when the chain-complexes have additional structure — that of an m-coalgebra, for instance. In this case, the injection of a contraction induces a condition on m-structures somewhat similar equivalence of quadratic forms.

Definition 2.13. The category of weakly-coherent m-coalgebras, denoted \mathfrak{M} , is defined to be the localization of \mathfrak{M}_0 by the set of strict morphisms whose associated chain-maps of underlying chain-complexes are injections of contractions of chain-complexes.

Remarks. 2.13.1. The objects of this category are weakly-coherent m-coalgebras as before, but a morphism from A to B (say) is a formal composite:

$$A \xrightarrow{m_1} \cdots \quad A_i \xleftarrow[\underbrace{s'_i}]{s_i} \cdots \xrightarrow[\underbrace{s'_j}]{s_j} A_j \cdots B$$

where the $\{m_j\}$ are strict morphisms and the $\{s_k\}$ are elementary equivalences defined in 2.12 on the page before — which may go to the left or right. We have weakened the definition of morphism considerably in going from \mathfrak{M}_0 to \mathfrak{M} . Since projections of contractions are chain-maps, we can still regard a morphism as having an *underlying chain map* of chain-complexes.

We will also identify morphisms with the same underlying chain map.

A morphism will be an *equivalence* if all of its constituents are elementary equivalences or their formal inverses.

2.13.2. The definition is essentially set up so that the maps in the Eilenberg-Zilber theorem on page 31 of [11] are morphisms. Neither map is a strict morphism, but they both turn out to be equivalences.

2.13.3. Morphisms preserve m-structures up to a chain-homotopy.

Definition 2.14. Let $C = (C, \{f[C]_n : C \rightarrow \text{Hom}_{\mathbb{Z}S_n}(\mathfrak{A}[C]_n, C^n)\})$ be a weakly-coherent m-coalgebra. Then C will be called *strictly cellular* if there exist strict morphisms of formal coalgebras

$$g_k : \mathfrak{A}[C] \rightarrow \mathfrak{S}$$

supporting strict isomorphisms of m-coalgebras

$$f_k : S_{k, n_k} = \mathfrak{C} \left(\bigvee_{i=1}^{n_k} S^{k-1} \right) \rightarrow C(k-1)$$

such that

$$C(k) = \mathfrak{C} \left(\bigvee_{i=1}^{n_k} D^k \right) \bigcup_{f_k} C(k-1)$$

for all $k \geq 0$. Here, $C(k)$ denotes the k -skeleton of C , S_{k, n_k} is the canonical coherent m-coalgebra of the singular complex of a wedge

of spheres (see 4.2 on page 30 of [11]), and the D^k are disks whose boundaries are the S^{k-1} .

We will call a weakly coherent m-coalgebra *cellular* if it is equivalent (in \mathfrak{M}) to a strictly cellular m-coalgebra.

Remarks. 2.14.1. If X is a CW complex, $\mathfrak{C}(X) = \mathfrak{C}(\dot{\Delta}(X))$, where $\dot{\Delta}(\ast)$ is the singular semisimplicial complex functor.

2.14.2. Note that cellularity requires the m-structure of an m-coalgebra to be an iterated extension of m-structures of spheres.

2.14.3. Clearly, the canonical m-coalgebra of any CW-complex is cellular. The converse also turns out to be true — see 3.5 on page 25.

It is not hard to find non-cellular m-coalgebras: Consider the m-coalgebra, B , concentrated in dimensions 0 and 3 (say), where underlying chain groups are equal to \mathbb{Z} . Equip this with a trivial coproduct and higher coproducts (subject to the defining conditions in 3.3 on page 19 of [11]). Let $\{e_i\}$ be the generator $R(S_2)$ with boundary $\partial e_i = (1 + (-1)^i t)e_{i-1}$, where $t \in \mathbb{Z}_2$ is the generator. We define a map

$$\Delta: R(S_2) \otimes B \rightarrow B \otimes B$$

where

1. $B_0 = \mathbb{Z}$,
2. $B_3 = \mathbb{Z}$, generated by x ,
3. The “higher coproducts” are defined by

$$\Delta(e_i \otimes x) = \begin{cases} 1 \otimes x + x \otimes 1 & \text{if } i = 0 \\ 0 & \text{if } i = 2 \\ x \otimes x & \text{if } i = 3 \end{cases}$$

(the last condition is required by 3.3 on page 19 of [11] and implies that the Steenrod operation Sq^0 is the identity). Here, we assume that $t \in \mathbb{Z}_2$ acts trivially on B and multiplies $B_3 \otimes B_3 = \mathbb{Z}$ by -1 .

This is (trivially) coherent – indeed, it is the m-coalgebra induced on the *homology* of the 3-sphere. It *cannot possibly* be cellular because the Hopf invariant of any map from it to a 2-sphere is identically 0.

Definition 2.15. Define $\hat{\mathfrak{M}}$ be the full subcategory of cellular objects of \mathfrak{M} .

We conclude this section with two algebraic results used in the next section:

Proof. We will actually construct the more complicated diagram:

$$(2.4) \quad \begin{array}{ccccccc} A & \cdots & U_i & \xrightarrow[s'_i]{s_i} & \cdots & U_j & \xrightarrow[s'_j]{s_j} & \cdots & U_k & \xrightarrow[s'_k]{s_k} & B \\ \downarrow a & & \downarrow p_i & & & \downarrow p_j & & & \downarrow p_k & & \downarrow b \\ C & \cdots & Z_i & \xrightarrow[t'_i]{t_i} & \cdots & Z_j & \xrightarrow[t'_j]{t_j} & \cdots & Z_k & \xrightarrow[t'_k]{t_k} & C \\ \parallel & & \downarrow v_i & & & \downarrow v_j & & & \downarrow v_k & & \parallel \\ C & \cdots & C & \xrightarrow{=} & \cdots & C & \xrightarrow{=} & \cdots & C & \xrightarrow{=} & C \end{array}$$

We construct the lower rows by scanning the upper, from left to right, and:

1. Whenever we encounter a subdiagram of the form

$$\begin{array}{ccc} U_i & \xrightarrow[s'_i]{s_i} & U_{i+1} \\ \downarrow p_i & & \vdots \\ Z_i & \cdots \cdots \cdots & ? \\ \downarrow v_i & & \downarrow v'_i \\ C & & C \end{array}$$

We replace the “?” with the push-out $Z_{i+1} = Z_i \oplus U_{i+1}/U_i$ (embedded via (s_i, p_i)) — and the appropriate maps. This results in the subdiagram

$$\begin{array}{ccc} U_i & \xrightarrow[s'_i]{s_i} & U_{i+1} \\ \downarrow p_i & & \downarrow p_{i+1} \\ Z_i & \xrightarrow[t'_i]{t_i} & Z_{i+1} \\ \downarrow v_i & & \downarrow v'_{i+1} \\ C & \xrightarrow{=} & C \end{array}$$

where

- (a) p_{i+1} and t_i are defined by the canonical property of a push-out and are strict morphisms of m-coalgebras (see 2.10 on page 10).
- (b) $t'_i = (1, p_i \circ s'_i): Z_i \oplus U_{i+1}/U_i \rightarrow Z_i$. This map is surjective since s'_i is, and we have made explicit use of the fact that s'_i is a left-inverse of s_i .

We define a contracting homotopy $\varphi_{Z_{i+1}} = (0, \varphi_{U_{i+1}}): Z_{i+1} \rightarrow Z_{i+1}$, where $\varphi_{U_{i+1}}$ is the contracting homotopy of the upper

row (which exists because it is an elementary equivalence — see 2.12 on page 11). This makes the lower row an elementary equivalence.

- (c) $v_{i+1} = (v_i, 0): H \rightarrow Z_{i+1} = Z_i \oplus U_{i+1}/U_i$ and $v'_{i+1} = v'_i \circ t'_i$
 2. Whenever we encounter a subdiagram of the form

$$\begin{array}{ccc}
 U_i & \xrightarrow{s_i} & U_{i+1} \\
 p_i \downarrow & \swarrow s'_i & \downarrow \dots \\
 Z_i & \xleftarrow{\dots} & ? \\
 v_i \downarrow & & \downarrow v'_i \\
 C & & C
 \end{array}$$

we simply pull back Z_i to form the diagram

$$\begin{array}{ccc}
 U_i & \xleftarrow{s_i} & U_{i+1} \\
 p_i \downarrow & & \downarrow p_i \circ s_i \\
 Z_i & \xleftarrow{1} & Z_i \\
 v_i \downarrow & & \downarrow v_{i+1} \\
 C & \xlongequal{\quad} & C
 \end{array}$$

where $v_{i+1} = v_i$.

This procedure works until we come to the end (i.e., the right end of diagram 2.4 on the page before).

(2.5)

$$\begin{array}{c}
 B \\
 \bar{b} \downarrow \\
 Z_t \\
 v_t \downarrow \\
 C
 \end{array}$$

where \bar{b} is induced by b — its target is the embedded copy of C .

The commutativity of diagram 2.2 on page 14 implies that we can splice an extra column onto diagram 2.5 on the preceding page to get

$$(2.6) \quad \begin{array}{ccc} B & \xlongequal{\quad} & B \\ \bar{b} \downarrow & & \downarrow b \\ Z_t & \xrightarrow{v_t} & C \\ \left. \begin{array}{c} v_t \\ \downarrow \\ v'_t \end{array} \right\} & \xrightarrow{v'_t} & \parallel \\ C & \xlongequal{\quad} & C \end{array}$$

□

Corollary 2.17. *Suppose we have a commutative diagram of weakly-coherent m -coalgebras:*

$$(2.7) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & & \downarrow b \\ C & \xrightarrow{\cong} & C \end{array}$$

where the top row is an equivalence, and the downward-maps are strict morphisms.

Then there exists an equivalence of weakly-coherent m -coalgebras

$$\hat{f}: A \otimes_{\alpha \circ a} \mathcal{F}C \rightarrow B \otimes_{\alpha \circ b} \mathcal{F}C$$

where $\mathcal{F}(\ast)$ denotes the cobar construction, $\alpha: C \rightarrow \mathcal{F}C$ is the canonical twisting cochain, and the twisted tensor products are equipped with the canonical weakly-coherent m -structures described in Proposition 1.19 on page 84 of [11].

In addition, the following diagram commutes:

$$(2.8) \quad \begin{array}{ccc} A \otimes_{\alpha \circ a} \mathcal{F}C & \xrightarrow{f \otimes 1} & B \otimes_{\alpha \circ b} \mathcal{F}C \\ 1 \otimes \epsilon \downarrow & & \downarrow 1 \otimes \epsilon \\ A & \xrightarrow{f} & B \end{array}$$

Remark. 2.17.1 We will use this and the results of [11] to show that the equivalence $\mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$ implies the existence of an equivalence between the next stages of Postnikov towers of X_1 and X_2 .

Proof. This follows by taking diagram 2.3 on page 14 and putting a third row of cobar constructions and twisting cochains

$$(2.9) \quad \begin{array}{ccccccc} A & \cdots & U_i & \xrightarrow[\underbrace{\quad}_{s'_i}]{s_i} & \cdots & U_j & \xrightarrow[\underbrace{\quad}_{s'_j}]{s_j} & \cdots & U_k & \xrightarrow[\underbrace{\quad}_{s'_k}]{s_k} & B \\ \downarrow a & & \downarrow p_i & & & \downarrow p_j & & & \downarrow p_k & & \downarrow b \\ C & \cdots & Z_i & \xrightarrow[\underbrace{\quad}_{t'_i}]{t_i} & \cdots & Z_j & \xrightarrow[\underbrace{\quad}_{t'_j}]{t_j} & \cdots & Z_k & \xrightarrow[\underbrace{\quad}_{t'_k}]{t_k} & C \\ \downarrow \alpha & & \downarrow \alpha_i & & & \downarrow \alpha_j & & & \downarrow \alpha_k & & \downarrow \alpha \\ \mathcal{F}C & \cdots & \mathcal{F}Z_i & \xrightarrow[\underbrace{\quad}_{\mathcal{F}(t'_i)}]{\mathcal{F}(t_i)} & \cdots & \mathcal{F}Z_j & \xrightarrow[\underbrace{\quad}_{\mathcal{F}(t'_j)}]{\mathcal{F}(t_j)} & \cdots & \mathcal{F}Z_k & \xrightarrow[\underbrace{\quad}_{\mathcal{F}(t'_k)}]{\mathcal{F}(t_k)} & \mathcal{F}C \end{array}$$

where $\alpha_i: Z_i \rightarrow \mathcal{F}Z_i$ are the canonical twisting cochains.

The elementary equivalences on the bottom row are the result of applying Proposition 2.32 on page 58 of [11]. \square

3. TOPOLOGICAL REALIZATION OF MORPHISMS

In this section, we will prove the main results involving the topological realization of m -coalgebras and morphisms. We begin with a proof that *equivalences* topologically realizable m -coalgebras are topologically realizable.

Theorem 3.1. *Let X_1 and X_2 be pointed, simply-connected, locally-finite, simisimplicial sets, with associated canonical m -coalgebras, $\mathcal{C}(X_i)$, $i = 1, 2$.*

In addition, suppose there exists an equivalence of m -coalgebras

$$f: \mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$$

as defined in [11] or in 2.13 on page 12 and the surrounding discussion.

Then there exist refinements (simplicial subdivisions) X'_i , $i = 1, 2$, of X_i , respectively and a simplicial map

$$\hat{f}: X'_1 \rightarrow X'_2$$

such that

$$f' = \mathcal{C}(\hat{f}): \mathcal{C}(X'_1) \rightarrow \mathcal{C}(X'_2)$$

Consequently, any m -coalgebra equivalence is topologically realizable up to a chain-homotopy.

Remarks. 3.1.1. We work in the simplicial category because the functors $\mathcal{C}(\ast)$ were originally defined over it.

It is well-known that the category of locally-finite simplicial sets coincides with the category of CW complexes. We could also have worked with the functors $\mathcal{C}(\ast)$, computed from singular complexes.

3.1.2. The refinement is a barycentric subdivision whose degree is finite within a neighborhood of each vertex of the X_i , if they are finite dimensional. If the X_i are finite, we can bound this degree by a finite number.

In any case, however, there are canonical equivalences

$$\mathcal{C}(X_i) \cong \mathcal{C}(X'_i)$$

for $i = 1, 2$.

Proof. The hypothesis implies that the chain-complexes are chain-homotopy equivalent, hence that the X_i , $i = 1, 2$, have the same homology. This implies that the lowest-dimensional nonvanishing homology groups — say M in dimension k — are isomorphic. We get a diagram

$$(3.1) \quad \begin{array}{ccc} \mathcal{C}(X_1) & \xrightarrow{f} & \mathcal{C}(X_2) \\ \mathcal{C}(c_1) \downarrow & & \downarrow \mathcal{C}(c_2) \\ \mathcal{C}(K(M, k)) & \xrightarrow{=} & \mathcal{C}(K(M, k)) \end{array}$$

Here, the maps are defined as follows:

1. The maps $\{\mathcal{C}(c_i)\}$, $i = 1, 2$, are induced by geometric classifying maps;
2. f is the composite of rightward arrows in the equivalence between the $\mathcal{C}(X_i)$, $i = 1, 2$:

$$(3.2) \quad \mathcal{C}(X_1) \cdots U_i \xrightarrow[\underbrace{\quad}_{s'_i}]{s_i} \cdots \xrightarrow[\underbrace{\quad}_{s'_j}]{s_j} U_j \cdots U_k \xrightarrow[\underbrace{\quad}_{s'_k}]{s_k} \mathcal{C}(X_2)$$

where the $\{U_\alpha\}$ are all weakly-coherent m-coalgebras and the $\{s_\ast\}$ all define elementary equivalences (see 2.12 on page 11).

Claim: If we forget simplicial structures (i.e., regard the simplicial sets in 3.1 as CW-complexes), we may assume that diagram 3.1 commutes exactly. to be precise:

1. The cellular chain complexes of the X_i are naturally isomorphic to the underlying chain-complexes of the $\mathcal{C}(X_i)$.
2. We construct the map c_1 by finding a topological realization of the composite $\mathcal{C}(c_2) \circ f$. That this *can* be done follows by elementary obstruction theory and the fact that all the spaces in question are simply-connected — see [14], for instance. We replace the

simplicial map, c_1 , by a cellular map, c'_1 , homotopic to it. The result is a map of pairs

$$((X_1)_k, (X_1)_{k-1}) \rightarrow (X_2)_k, (X_2)_{k-1})$$

(where $(X_1)_k$ denotes the k -skeleton) for all $k \geq 0$, such that the induced map of cellular chain modules

$$\pi_k((X_1)_k, (X_1)_{k-1}) = \mathcal{C}(X_1)_k \rightarrow \pi_k((X_2)_k, (X_2)_{k-1}) = \mathcal{C}(X_2)_k$$

exactly coincides with f (regarded only as a map of chain complexes).

3. Now, we refine the simplicial sets until we can replace c'_1 by a simplicial approximation. The image of each simplex of X_1 lies in a finite subcomplex of $K(M, 1)$ and X_2 , so we can simplicially approximate the restriction of c'_1 to this simplex. Consequently, a *finite* (but, possibly, unbounded) number of subdivisions of each simplex suffices.

In the following discussion, we will assume that this subdivision and simplicial approximation has been carried out — and we will suppress the extra notation (i.e., the prime) for the subdivided complexes and induced maps.

All of the maps in 3.1 on the page before are strict morphisms of m -coalgebras (see 2.10 on page 10), except for the map f : The vertical maps and the lower horizontal map are strict because they were induced by *geometric* maps.

Corollary 2.17 on page 17 implies that there exists an equivalence

$$\hat{f}: \mathcal{C}(X_1) \otimes_{\alpha \circ \mathcal{C}(g_1)} \mathcal{FC}(K(M, k)) \rightarrow \mathcal{C}(X_2) \otimes_{\alpha \circ \mathcal{C}(g_1)} \mathcal{FC}(K(M, k))$$

such that the following diagram commutes:

(3.3)

$$\begin{array}{ccc} \mathcal{C}(X_1) \otimes_{\alpha \circ \mathcal{C}(g_1)} \mathcal{FC}(K(M, k)) & \xrightarrow{\hat{f}} & \mathcal{C}(X_2) \otimes_{\alpha \circ \mathcal{C}(g_1)} \mathcal{FC}(K(M, k)) \\ \downarrow 1 \otimes \epsilon & & \downarrow 1 \otimes \epsilon \\ \mathcal{C}(X_1) & \xrightarrow{f} & \mathcal{C}(X_2) \end{array}$$

Lemma 3.1 of page 93 and Corollary 3.5 on page 96 of [11] imply the existence of equivalences (of weakly-coherent m -coalgebras)

$$\mathcal{C}(X_i \times_{\hat{\alpha} \circ g_i} \Omega K(M, k)) \rightarrow \mathcal{C}(X_i) \otimes_{\alpha \circ \mathcal{C}(g_i)} \mathcal{FC}(K(M, k))$$

for $i = 1, 2$

We conclude that there is an equivalence

$$\hat{F}: \mathcal{C}(X_1 \times_{\hat{\alpha} \circ g_1} \Omega K(M, k)) \rightarrow \mathcal{C}(X_2 \times_{\hat{\alpha} \circ g_2} \Omega K(M, k))$$

where $\Omega(*)$ denotes the loop space functor and $\hat{\alpha}: K(M, k) \rightarrow \Omega K(M, k)$ is the canonical twisting function (defining a fibration as twisted Cartesian product — see [3]).

In addition, the commutativity of 3.3 on the facing page implies that

$$f^*(\mu_2) = \mu_1 \in H^{k+1}(X_1, M)$$

where μ_1 and μ_2 are the k -invariants of the fibrations $X_1 \times_{\hat{\alpha} \circ g_1} \Omega K(M, k)$ and $X_2 \times_{\hat{\alpha} \circ g_2} \Omega K(M, k)$, respectively.

Since the $X_i \times_{\hat{\alpha} \circ g_i} \Omega K(M, k)$ are homotopy fibers of the g_i maps for $i = 1, 2$, respectively, we conclude that the second stage of the Postnikov towers of X_1 and X_2 are equivalent.

A straightforward induction implies that all *finite stages* of the Postnikov tower of X_1 are equivalent to corresponding finite stages of the Postnikov tower of X_2 . It follows that all finite-dimensional obstructions to realizing the underlying chain-map of f by a geometric map of CW-complexes vanish.

It is necessary to make one last remark regarding our simplicial approximations to maps in diagrams like 3.1 on page 19 that arise during inductive steps. Clearly, after any *finite* number of inductive steps, we are still dealing with *finite* subdivisions of the simplicial sets from the hypothesis. If the original spaces were finite dimensional, we only need a finite number of inductive steps.

The conclusion follows. \square

Next, we prove a similar result for well-behaved morphisms that aren't *a priori* equivalences. We are heading toward a proof that arbitrary morphisms are topologically realizable.

Proposition 3.2. *Let X_1 and X_2 be pointed, simply-connected simplicial complexes, with associated canonical m -coalgebras, $\mathcal{C}(X_i)$, $i = 1, 2$.*

In addition, suppose there exists a strict morphism of weakly coherent m -coalgebras that induces homology isomorphisms in all dimensions

$$f: \mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$$

as defined in [11] or in 2.13 on page 12 and the surrounding discussion.

Then there exists a map of CW-complexes (i.e., we forget the semi-simplicial structure of the spaces and regard them as CW-complexes — or pass to suitable simplicial refinements, as in 3.1 on page 18):

$$\hat{f}: X_1 \rightarrow X_2$$

such that

$$f = \mathcal{C}(\hat{f})$$

Consequently, f is an equivalence.

Remarks. 3.2.1. This is interesting because strict morphisms don't generally define m -coalgebra equivalences — even when they are homology equivalences. The topological realizability of the m -coalgebras in question is crucial here.

3.2.2. We could actually have stated that the map f is a composite $e_1 \circ f' \circ e_2$, where e_1 and e_2 are equivalences of m -coalgebras and f' is a strict morphism inducing homology isomorphisms.

Proof. We follow an argument exactly like that used in 3.2 on the preceding page above. In each inductive step we have a morphism of the form $e_1 \circ f_i \circ e_2$, where e_1 and e_2 are equivalences of m -coalgebras and f_i is a strict morphism inducing homology isomorphisms. the only thing we must do differently, here, is to invoke the Serre Spectral Sequence of a fibration to verify that the f_{i+1} will be a homology equivalence, given that f_i is. \square

Corollary 3.3. *Suppose C_1 and C_2 are weakly coherent m -coalgebras that are topologically realizable — i.e., they are equivalent in \mathfrak{M} (see 2.13 on page 12) to $\mathcal{C}(X_i)$, respectively, for two pointed, simply-connected semi-simplicial complexes, X_i , $i = 1, 2$.*

Then a morphism

$$f: C_1 \rightarrow C_2$$

is an equivalence if and only if it induces isomorphisms in homology.

Theorem 3.4. *Let X_1 and X_2 be pointed, simply-connected, locally-finite, simisimplicial sets complexes, with associated canonical m -coalgebras, $\mathcal{C}(X_i)$, $i = 1, 2$.*

In addition, suppose there exists a morphism of m -coalgebras (see 2.10 on page 10):

$$f: \mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$$

Then there exists a map of CW-complexes (i.e, we forget the semi-simplicial structure of the spaces and regard them as CW-complexes — or form simplicial refinements, as in 3.1 on page 18):

$$\hat{f}: X_1 \rightarrow X_2$$

such that

$$f = \mathcal{C}(\hat{f})$$

Consequently, any morphism of m -coalgebras is topologically realizable up to a chain-homotopy.

Proof. We prove this result by an inductive argument somewhat different from that used in theorem 3.1 on page 18.

We build a sequence of fibrations

$$\begin{array}{c} F_i \\ \downarrow p_i \\ X_2 \end{array}$$

over X_2 in such a way that

1. the morphism $f: \mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$ lifts to $\mathcal{C}(F_i)$ — i.e., we have commutative diagrams

$$\begin{array}{ccc} & & \mathcal{C}(F_i) \\ & \nearrow f_i & \downarrow \mathcal{C}(p_i) \\ \mathcal{C}(X_1) & \xrightarrow{f} & \mathcal{C}(X_2) \end{array}$$

For all $i > 0$, F_i will be a fibration over F_{i-1} with fiber a suitable Eilenberg-MacLane space.

2. The map f_i is i -connected in homology.

If the morphism f were geometric, we would be building its *Postnikov tower*.

Assuming that this inductive procedure can be carried out, we note that it forms a convergent sequence of fibrations (see [12], chapter 8, § 3). This implies that we may pass to the inverse limit and get a commutative diagram

$$\begin{array}{ccc} & & \mathcal{C}(F_\infty) \\ & \nearrow f_\infty & \downarrow \mathcal{C}(p_\infty) \\ \mathcal{C}(X_1) & \xrightarrow{f} & \mathcal{C}(X_2) \end{array}$$

where \tilde{f}_∞ is a morphism of weakly-coherent m -coalgebras that is a *homology equivalence*. Now 3.3 on the preceding page implies that f_∞ is an equivalence of m -coalgebras, and 3.1 on page 18 implies that it is topologically realizable.

It follows that we get a (geometric) map

$$\bar{f}_\infty: X_1 \rightarrow F_\infty$$

and the composite of this with the projection $p_\infty: F_\infty \rightarrow X_2$ is a topological realization of the original map $f: \mathcal{C}(X_1) \rightarrow \mathcal{C}(X_2)$.

It only remains to verify the inductive step:

Suppose we are in the k^{th} iteration of this inductive procedure. Then the mapping cone, $\mathcal{A}(f)$ is acyclic below dimension k . Suppose that $H_k(\mathcal{A}(f_k)) = M$. Then we get a long exact sequence in cohomology:

$$(3.4) \quad \dots \rightarrow H^{k+1}(X_1, M) \rightarrow H^k(\mathcal{A}(f_k), M) = \text{Hom}_{\mathbb{Z}}(M, M) \\ \rightarrow H^k(F_k, M) \rightarrow H^k(X_1, M) \rightarrow 0$$

Let $\mu \in H^k(F_k, M)$ be the image of $1_M \in H^k(\mathcal{A}(f_k), M) = \text{Hom}_{\mathbb{Z}}(M, M)$ and consider the map

$$h_\mu: X_2 \rightarrow K(M, k)$$

classified by μ . We pull back the contractible fibration

$$K(M, k) \times_{\bar{\alpha}} \Omega K(M, k)$$

over h_μ to get a fibration

$$F_{k+1} = F_k \times_{\bar{\alpha} \circ h_\mu} \Omega K(M, k)$$

where, as before, $\Omega(\ast)$ represents the loop space.

Claim: The morphism f_k lifts to a morphism $f_{k+1}: \mathcal{C}(X_1) \rightarrow \mathcal{C}(F_{k+1})$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}(X_1) & \xrightarrow{f_{k+1}} & \mathcal{C}(F_{k+1}) \\ & \searrow f_k & \downarrow p \\ & & \mathcal{C}(F_k) \end{array}$$

where $p'_{k+1}: F_{k+1} \rightarrow F_k$ is that fibration's projection map.

Proof of Claim: We begin by using Lemma 3.1 of page 93 and Corollary 3.5 on page 96 of [11] to conclude the existence of a commutative diagram:

$$(3.5) \quad \begin{array}{ccc} \mathcal{C}(F_k \times_{\bar{\alpha} \circ h_\mu} \Omega K(M, k)) & \xrightarrow{e} & \mathcal{C}(F_k) \otimes_{\bar{\alpha} \circ h_\mu} \mathcal{FC}(K(M, k)) \\ & \searrow p'_{k+1} & \downarrow 1 \otimes \epsilon \\ & & \mathcal{C}(F_k) \end{array}$$

where e is an m-coalgebra equivalence.

If we pull back this twisted tensor product over the map f_k , we get a trivial twisted tensor product (i.e., an untwisted tensor product), because the image of $f^*(\mu) = 0 \in H^k(X_1, M)$, by the exactness of 3.4.

Theorem 1.20 on page 85 of [11] implies the existence of a morphism

$$(3.6) \quad \begin{aligned} \mathcal{C}(X_1) &\rightarrow \mathcal{C}(X_1) \otimes 1 \subset \mathcal{C}(X_1) \otimes \mathcal{FC}(K(M, k)) \\ &\rightarrow \mathcal{C}(F_k) \otimes_{\bar{\alpha} \circ h_\mu} \mathcal{FC}(K(M, k)) \end{aligned}$$

The composition of this map with e in 3.5 on the facing page is the required map

$$\mathcal{C}(X_1) \rightarrow \mathcal{C}(F_{k+1})$$

To see that $H_k(\mathcal{A}(f_{k+1})) = 0$, note that:

1. $\mu \in H^k(F_k, M) = H^k(\mathcal{C}(F_k), M)$ is the pullback of the class in $H^k(\mathcal{A}(f_k), M)$ inducing a homology isomorphism

$$\mu: H_k(\mathcal{A}(f_k)) \rightarrow H_k(K(M, k))$$

(by abuse of notation, we identify μ with a cochain) or

$$\mu: H_k(\mathcal{A}(f_k)) \rightarrow H_k(\mathcal{C}(K(M, k)))$$

2. in the stable range, $\mathcal{C}(F_k) \otimes_{\bar{\alpha} \circ h_\mu} \mathcal{FC}(K(M, k))$ is nothing but the *algebraic mapping cone* of the chain-map, μ , above. But the algebraic mapping cone of μ clearly has vanishing homology in dimension k since μ induces homology isomorphisms.

□

Corollary 3.5. *A weakly-coherent m -coalgebra is topologically realizable if and only if it is cellular (see 2.14 on page 12).*

Proof. Clearly, topologically realizable m -coalgebras are cellular.

Theorem 3.4 on page 22 implies the converse, because all of the attaching morphisms in 2.14 on page 12 are topologically realizable. □

Corollary 3.6. *The functor*

$$\mathcal{C}(*): \underline{\text{Homotop}}_0 \rightarrow \hat{\mathfrak{M}}$$

(see 2.15 on page 13 for the definition of $\hat{\mathfrak{M}}$) defines an equivalence of categories, where $\underline{\text{Homotop}}_0$ is the category of pointed, simply-connected CW-complexes and continuous maps, in which homotopy equivalences have been inverted (i.e., it is the category of fractions by homotopy equivalences).

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