VECTOR BUNDLES OVER CLASSIFYING SPACES OF COMPACT LIE GROUPS

by Stefan Jackowski & Bob Oliver

The completion theorem of Atiyah and Segal [AS] says that the complex K-theory group $K(BG)$ of the classifying space of any compact Lie group G is isomorphic to $\mathrm{R}(G)\widehat{\ }$: the representation ring completed with respect to its augmentation ideal. However, the group $K(BG) = [BG, \mathbb{Z} \times BU]$ does not directly contain information about vector bundles over the infinite dimensional complex BG itself. The purpose of this paper is to compare the Grothendieck group of vector bundles over BG, which we denote $\mathbb{K}(BG)$, with both $K(BG)$ and $R(G)$. The main result is an algebraic description of $\mathbb{K}(BG)$ in terms of the representation rings of certain subgroups of G. As one consequence, we show that of the natural maps

$$\begin{array}{ccc} \mathrm{R}(G) & \xrightarrow{\ \lambda_G\ } & \mathrm{R}(G)\widehat{\ } \\ {\scriptstyle \bar\alpha_G}\downarrow & \searrow^{\alpha_G} & \downarrow{\scriptstyle \cong} \\ \mathbb{K}(BG) & \xrightarrow{\ \beta_G\ } & K(BG), \end{array}$$

β_G is always injective and $\bar\alpha_G$ is surjective whenever G is finite or all elements of $\pi_0(G)$ have prime power order (Corollary 2.11), but $\bar\alpha_G$ is not surjective in general (Theorem 4.7). Also, for arbitrary G, any vector bundle can be embedded as a summand of a bundle associated to a G-representation (Corollary 2.11); while if $\bar\alpha_G$ is onto, then any bundle is the difference of two bundles associated to G-representations.

More precisely, we show that for any compact Lie group G,

$$\mathbb{K}(BG) \cong \mathrm{R}_{\mathcal{P}}(G) \stackrel{\text{def}}{=} \varprojlim_{P} \mathrm{R}(P),$$

where the limit is taken over all p-toral subgroups of G (i.e., subgroups which are extensions of tori by groups of prime power order); and with respect to inclusion and conjugation of subgroups. When G is itself a p-toral group, this says that $\mathbb{K}(BG) \cong \mathrm{R}(G)$, and follows from results of Dwyer & Zabrodsky [DZ] (when G is a p-group) and Notbohm [Nb] (when G is p-toral).

The "limit" groups $\mathrm{R}_{\mathcal{P}}(G)$ are descibed more precisely in Section 2 below (Definition 2.2). The kernel of the map $\mathrm{R}(G) \to \mathrm{R}_{\mathcal{P}}(G)$ is easily determined: it is the

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set of elements whose characters vanish on all components of G of prime power order in $\pi_0(G)$. The cokernel of this map is determined in Section 4 (Theorem 4.7). In particular, we will see that $\mathrm{R}_{\mathcal{P}}(G) \cong \mathrm{R}(G)$ whenever $\pi_0(G)$ has prime power order.

Our results can be summarized by saying that the Grothendieck group of the monoid of vector bundles over BG is very well behaved, and is quite close to the representation ring of G. For a p-toral group G, this monoid itself is understood: it is isomorphic to the monoid of representations of G [DZ] [Nb]. In contrast, for more general groups, one can construct examples using the methods in [JMO, Section 7] to show that the monoid $\coprod_{n=0}^{\infty}[BG, BU(n)]$ can be extremely complicated. If G is cyclic of order not a prime power, or if $G = SO(3)$, this monoid does inject into $\mathbb{K}(BG)$ (i.e., stably isomorphic vector bundles over BG are isomorphic), but bundles do not have unique decompositions as sums of indecomposable bundles. For larger groups G, one can find uncountable families of pairwise nonisomorphic vector bundles over BG all of which are isomorphic after stabilizing (by arbitrary bundles). One can also construct a pair of stably isomorphic vector bundles such that one is indecomposable and the other decomposable. On the other hand, we do show (Proposition 2.17 below) that two vector bundles over BG are isomorphic if they are isomorphic after stabilizing by trivial bundles.

Although it is not stated there explicitly, Adams in his paper [Ad2] was clearly motivated in part by this question of determining the subgroup generated by elements of $K(BG)$ which can be represented by vector bundles over BG (i.e., the image of $\mathbb{K}(BG)$ in $K(BG)$). He considered there the subgroup $FF(BG) \subseteq K(BG)$ generated by the "formally finite dimensional" elements in $K(BG)$; i.e., those elements x such that $\lambda^k(x) = 0$ for k sufficiently large (where $\lambda^k(-)$ denotes the k-th exterior power). Clearly, $FF(BG)$ contains all elements represented by vector bundles over BG. Adams studied the composite of the inclusions

$$\alpha_G(\mathrm{R}(G)) \subseteq \beta_G(\mathbb{K}(BG)) \subseteq FF(BG),$$

and showed in particular that $FF(BG) = \alpha_G(\mathrm{R}(G))$ (and hence that $\alpha_G(\mathrm{R}(G)) = \beta_G(\mathbb{K}(BG))$) whenever G is finite or $\pi_0(G)$ has prime power order. Hence, since β_G is always injective (Corollary 2.10 below), $\bar{\alpha}_G : \mathrm{R}(G) \to \mathbb{K}(BG)$ is surjective in these cases. For an arbitrary compact Lie group G, our results in this paper, together with those of Adams, imply that β_G defines an isomorphism

$$\beta_G : \mathbb{K}(BG) \xrightarrow{\cong} FF(BG).$$

In particular, the subgroup generated by formally finite dimensional elements of $K(BG)$ coincides with the subgroup generated by vector bundles over BG.

For an arbitrary space X, $K(X)$ is the group of connected components of the topological group $\mathrm{map}(X, \mathbb{Z}\times BU)$. Since $\coprod_{n=0}^{\infty} BU(n)$ is a topological monoid and commutative up to homotopy, the space of maps from X into it (the "topological monoid of vector bundles" over X) is also a homotopy commutative topological monoid. Thus, we can take its topological group completion

$$\mathfrak{K}(X) = \Omega B\,\mathrm{map}\Big(X, \coprod_{n=0}^{\infty} BU(n)\Big)$$

(where B is the classifying space functor applied to the monoid). When X is a finite complex, this has the homotopy type of the mapping space $\mathrm{map}(X, \mathbb{Z} \times BU)$. The group completion theorem (cf. [MS]) implies that $\mathbb{K}(X) \cong \pi_0(\mathfrak{K}(X))$ for any X.

The Atiyah-Segal completion theorem [AS] also describes the higher homotopy groups $\pi_*\bigl(\mathrm{map}(X, \mathbb{Z}\times BU)\bigr)$. Here, in Proposition 2.15, we show that when G is finite, the connected components of $\mathfrak{K}(BG)$ have the same homotopy type as the components of $\mathrm{map}(BG, \mathbb{Z}\times BU)$. In contrast, when $\dim(G) > 0$, we will see (Proposition 2.16) that the components of $\mathfrak{K}(BG)$ are quite different from those of $\mathrm{map}(BG, \mathbb{Z}\times BU)$; and in particular that their odd dimensional homotopy groups are nonvanishing.

The above discussion has focused on the case of complex bundles, but most of the results also hold for real bundles. In particular, for any G, $\mathbb{KO}(BG)$ (defined analogously) maps isomorphically onto the subgroup $FFO(BG) \subseteq KO(BG)$ generated by formally finite dimensional elements. And $\alpha_G : \mathrm{RO}(G) \to \mathbb{KO}(BG)$ is surjective whenever G is finite or $\pi_0(G)$ has prime power order.

The main ingredients in our computation of $\mathbb{K}(BG)$ are the theorems of Dwyer-Zabrodsky and Notbohm on p-toral groups (Theorem 1.1 below), a decomposition of BG at any prime p as a homotopy direct limit of classifying spaces of p-toral subgroups of G [JMO, Theorem 2.1]; together with the vanishing of certain higher derived functors of inverse limits. The vanishing of these higher limits depends in turn (somewhat surprisingly) on the equivariant Bott periodicity theorem.

Section 1 contains some introductory material and examples involving the functor $\mathbb{K}(-)$. The main results about $\mathbb{K}(BG)$ and $\mathbb{KO}(BG)$, and about $\mathfrak{K}(BG)$ and $\mathfrak{KO}(BG)$, are all shown in Section 2. In Section 3, we prove using Smith theory a vanishing theorem for higher inverse limits used in Section 2. And the representation groups $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$ are studied in Section 4.

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Notation: For any homomorphism $\rho : G \to U(n)$, we let V_ρ denote the corresponding G-representation on $\mathbb{C}^n$. Conversely, if V is any unitary representation of G, we let $\rho_V : G \to U(n)$ ($n = \dim(V)$) denote the corresponding homomorphism (of course, only well defined up to conjugacy). Also, when convenient for simplicity of notation, we write $\mathrm{d}_V = \dim(V)$.

It will be convenient to state some of the results simultaneously for real and complex vector bundles, or for orthogonal and unitary groups. In such situations, we set $F = \mathbb{C}$ or $\mathbb{R}$, and write $U(n, F)$ for $\mathrm{U}(n)$ or $\mathrm{O}(n)$, respectively. Also, $KF(-) = K(-)$ or $KO(-)$, $\mathbb{K}F(-) = \mathbb{K}(-)$ or $\mathbb{KO}(-)$, and $\mathrm{R}^F(-) = \mathrm{R}(-)$ or $\mathrm{RO}(-)$.

1. An introduction to $\mathbb{K}(X)$

For any space X, we let $\mathbb{V}\mathrm{ect}_n^{\mathbb{C}}(X) = [X, BU(n)]$ denote the set of isomorphism classes of n-dimensional complex bundles over X, and $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(X) = \coprod_{n=0}^{\infty} \mathbb{V}\mathrm{ect}_n^{\mathbb{C}}(X)$ the monoid of all bundles. Then $\mathbb{K}(X) = \mathcal{G}\mathfrak{r}(\mathbb{V}\mathrm{ect}^{\mathbb{C}}(X))$, the Grothendieck group of

$\mathbb{V}\mathrm{ect}^{\mathbb{C}}(X)$. When X is a finite complex, then of course $\mathbb{K}(X) \cong K(X)$ by definition. In general, $\mathbb{K}(-)$ is a functor from spaces to the category of λ-rings, and there is a natural homomorphism $\beta : \mathbb{K}(-) \to K(-)$. In this section, we give examples which point out how many of the nice properties of ordinary topological K-theory fail for $\mathbb{K}(X)$ when X is an infinite dimensional complex.

We start with a theorem of Dwyer-Zabrodsky and Notbohm, which implies a description of $\mathbb{K}(BP)$ for any p-toral group P. As noted in the introduction, this result is our starting point for determining $\mathbb{K}(BG)$ for arbitrary G; and it will also be useful for constructing some of the examples in this section.

For any group G, we let $\mathrm{Rep}_n^{\mathbb{C}}(G)$ and $\mathrm{Rep}_n^{\mathbb{R}}(G)$ denote the sets of n-dimensional unitary or orthogonal representations of G, and $\mathrm{Rep}^{\mathbb{C}}(G)$ and $\mathrm{Rep}^{\mathbb{R}}(G)$ the monoids of all unitary or orthogonal representations. In the statement of the following theorem, when V is a unitary or orthogonal G-representation, we write $\mathrm{Aut}_G(V)$ to denote the group of G-equivariant unitary or orthogonal automorphisms of V (and $\mathrm{Aut}(V)$ to denote the group of all unitary or orthogonal automorphisms).

Theorem 1.1. [Dwyer-Zabrodsky and Notbohm] *Set $F = \mathbb{C}$ or $\mathbb{R}$. For any prime p, any p-toral group P, and any $n > 0$, the natural map*

$$\mathrm{Rep}_n^F(P) \xrightarrow{\cong} \mathbb{V}\mathrm{ect}_n^F\, BP = [BP, BU(n, F)]$$

which sends V to $B\rho_V$ is a bijection. For any (unitary or orthogonal) P-representation V, the homomorphism $\mathrm{Aut}_P(V)\times P \to \mathrm{Aut}(V)$ induces an $\mathbb{F}_p$-homology equivalence

$$\tau_V : B\,\mathrm{Aut}_P(V) \longrightarrow \mathrm{map}(BP, BU(\mathrm{d}_V, F))_{B\rho_V}.$$

And if $F = \mathbb{C}$ or $p = 2$ or $\dim(V)$ is odd, this in turn extends to a homotopy equivalence

$$\widehat{\tau}_V : B\,\mathrm{Aut}_P(V)^\wedge_p \xrightarrow{\simeq} \mathrm{map}(BP, BU(\mathrm{d}_V, F)^\wedge_p)_{B\rho_V}.$$

Proof. For any compact Lie group G',

$$[BP, BG'] \cong \mathrm{Hom}(P, G')/\,\mathrm{Inn}(G');$$

and for any $\rho \in \mathrm{Hom}(P, G')$ the adjoint map

$$BC_{G'}(\mathrm{Im}(\rho)) \longrightarrow \mathrm{map}(BP, BG')_{B\rho} \tag{1}$$

is an $\mathbb{F}_p$-homology equivalence. This was shown by Dwyer & Zabrodsky [DZ] when P is a finite p-group, and by Notbohm [Nb] when P is p-toral. Also, if $\pi_0(G')$ is a p-group, then (1) extends to a homotopy equivalence

$$BC_{G'}(\mathrm{Im}(\rho))^\wedge_p \longrightarrow \mathrm{map}(BP, (BG')^\wedge_p)_{B\rho}$$

(cf. [JMO, Theorem 3.2(iii)]). The above formulation is now just the special case where $G' = \mathrm{U}(n)$, or where $G' = \mathrm{O}(n)$ and $p = 2$, or where $G' = \mathrm{O}(2k+1) \cong \mathrm{SO}(2k+1) \times \{\pm 1\}$. $\square$

We next note that $\mathbb{K}(-)$ is not an exact functor. Let $\widetilde{\mathbb{K}}(-)$ denote the reduced functor $\widetilde{\mathbb{K}}(X) = \mathbb{K}(X)/\mathbb{K}(\mathrm{pt}) \cong \mathrm{Ker}[\mathbb{K}(X) \xrightarrow{\dim} \mathbb{Z}]$.

Example 1.2. *Fix a prime p, and let $C_p \subseteq S^1$ denote the subgroup of order p. Then the induced sequence*

$$\widetilde{\mathbb{K}}(BS^1/BC_p) \xrightarrow{\text{proj}} \mathbb{K}(BS^1) \xrightarrow{\text{restr}} \mathbb{K}(BC_p)$$

is not exact.

Proof. For each $n \in \mathbb{Z}$, let $\xi_n{\downarrow}BS^1$ denote the line bundle with Chern class n times some fixed generator of $H^2(BS^1)$. By Theorem 1.1, $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(BC_p) \cong \mathrm{Rep}^{\mathbb{C}}(C_p)$ and $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(BS^1) \cong \mathrm{Rep}^{\mathbb{C}}(S^1)$. In particular, a bundle over BS^1 whose restriction to BC_p is trivial contains no summand ξ_n for $p \nmid n$. So $\xi_1|BC_p \cong \xi_{p+1}|BC_p$, but $[\xi_1] - [\xi_{p+1}] \in \mathbb{K}(BS^1)$ is not in the image of $\mathbb{K}(BS^1/BC_p)$. $\square$

The following lemma will be useful for computing $\mathbb{K}(-)$ for certain spaces with retractions. For later use in Section 2, we state it here in the equivariant form, and for real and complex bundles both. If X is any G-space, then we let $\mathbb{K}_G(X)$ and $\mathbb{KO}_G(X)$ denote the Grothendieck groups of complex and real G-vector bundles, respectively, over X.

Lemma 1.3. *Set $F = \mathbb{C}$ or $\mathbb{R}$. Fix a pair of G-complexes (X, A), together with a G-retraction $r : X \to A$. For $n \geq 1$, let $* \in S^n$ be a base point, and consider the relative suspension*

$$\Sigma^n_A(X) = (S^n \times X) \cup_{*\times r} A \cong (D^n \times X) \cup_{S^{n-1}\times r} A.$$

Let $A \subseteq \Sigma^n_A(X)$ be the obvious subspace, and write $\bar{r} : \Sigma^n_A(X) \to A$ to denote the retraction induced by r. Let $\mathbb{V}\mathrm{ect}^F_G(A)$ denote the monoid of G-F-vector bundles over A, also regarded as a category in the obvious way. For any G-F-vector bundle $\eta{\downarrow}A$, let $\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X), [\eta{\downarrow}A]\big)$ denote the set of isomorphism classes of G-F-vector bundles ξ over $\Sigma^n_A(X)$ with given isomorphism $\xi|A \cong \eta$ (a group via the suspension structure on $\Sigma^n_A(X)$). Then

$$\mathbb{K}^F_G(\Sigma^n_A(X))/\bar{r}^*\mathbb{K}^F_G(A) \cong \varinjlim_{\eta\in\mathbb{V}\mathrm{ect}^F_G(A)} \mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X), [\eta{\downarrow}A]\big).$$

Proof. Fix any vector bundle $\xi{\downarrow}\Sigma^n_A(X)$, and set $\eta = \xi|A$. Let $\bar{r}^*\eta{\downarrow}\Sigma^n_A(X)$ denote the pullback of η: the identity element in the group $\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X), [\eta{\downarrow}A]\big)$. Let ξ' denote the inverse of ξ in this group. Then, in $\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X), [\eta\oplus\eta{\downarrow}A]\big)$,

$$\begin{aligned}[\xi\oplus\xi'] &= [(\xi{\cdot}\bar{r}^*\eta)\oplus(\bar{r}^*\eta{\cdot}\xi')] = [(\xi\oplus\bar{r}^*\eta){\cdot}(\bar{r}^*\eta\oplus\xi')] \\ &= [(\xi\oplus\bar{r}^*\eta){\cdot}(\xi'\oplus\bar{r}^*\eta)] = [\bar{r}^*(\eta\oplus\eta)];\end{aligned}$$

where the third equality holds since the monoids $\coprod_{n=0}^\infty BU(n)$ and $\coprod_{n=0}^\infty BO(n)$ are homotopy commutative (via conjugation by an appropriate matrix $\left(\begin{smallmatrix} 0 & \pm I_m \\ \pm I_n & 0\end{smallmatrix}\right)$)

in $\mathrm{U}(n+m)$ or $\mathrm{SO}(n+m)$). In particular, every G-F-vector bundle over $\Sigma^n_A(X)$ is a summand of a vector bundle pulled back from A.

For each bundle $\eta{\downarrow}A$, let $\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X),\langle\eta{\downarrow}A\rangle\big)$ denote the set of isomorphism classes of G-F-vector bundles ξ over $\Sigma^n_A(X)$ such that $\xi|A\cong\eta$ (but without a fixed choice of isomorphism). This is a pointed set with basepoint $\bar{r}^*\eta$ (but not a group). Then

$$\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X)\big)\cong\coprod_{\eta\in\mathbb{V}\mathrm{ect}^F_G(A)}\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X),\langle\eta{\downarrow}A\rangle\big). \tag{1}$$

Also, in the limit,

$$\varinjlim_{\eta\in\mathbb{V}\mathrm{ect}^F_G(A)}\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X),\langle\eta{\downarrow}A\rangle\big)\cong\varinjlim_{\eta\in\mathbb{V}\mathrm{ect}^F_G(A)}\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X),[\eta{\downarrow}A]\big): \tag{2}$$

since every bundle is a summand of a pullback. Hence, there is a homomorphism

$$\mathbb{K}^F_G(\Sigma^n_A(X))\longrightarrow\mathbb{K}^F_G(A)\times\varinjlim_{\eta\in\mathbb{V}\mathrm{ect}^F_G(A)}\mathbb{V}\mathrm{ect}^F_G\big(\Sigma^n_A(X),[\eta{\downarrow}A]\big)$$

which sends $[\xi]$ to $([\xi|A],\varinjlim[\xi])$. And this is an isomorphism by (1) and (2), and the universality properties of Grothendieck groups. $\square$

The following example illustrates the failure of two formulations of Bott periodicity for $\mathbb{K}(-)$. At the same time, this gives a second example of the failure of exactness for the functor.

Example 1.4. *Fix a prime p and a finite p-group $P\neq1$. Set $r=\mathrm{rk}(\mathrm{R}(P))$; i.e., the number of conjugacy classes of elements in P. Then*

$$\mathbb{K}(BP)\cong\mathrm{R}(P)\cong\mathbb{Z}^r,$$

$$\widetilde{\mathbb{K}}(\Sigma^2(BP_+))=\widetilde{\mathbb{K}}\big((S^2\times BP)/BP\big)\cong\mathbb{Z},$$

and

$$\mathbb{K}(S^2\times BP)/\mathbb{K}(BP)\cong\mathbb{Z}\times(\hat{\mathbb{Z}}_p)^{r-1}.$$

In particular, these groups are pairwise nonisomorphic.

Proof. The formula for $\mathbb{K}(BP)$ follows from Theorem 1.1. By Lemma 1.3,

$$\begin{aligned}\widetilde{\mathbb{K}}(\Sigma^2(BP_+))&\cong\varinjlim_n\mathbb{V}\mathrm{ect}^{\mathbb{C}}(\Sigma^2(BP_+),[\mathbb{C}^n{\downarrow}*])\\&\cong\varinjlim_n\pi_2\big(\mathrm{map}(BP,BU(n))_{\mathrm{null}}\big).\end{aligned}$$

Also, $\mathrm{map}(BP,BU(n))_{\mathrm{null}}\simeq BU(n)$ for all n: since

$$\mathrm{map}_*(BP,BU(n))_{\mathrm{null}}\simeq\mathrm{map}_*(BP,BU(n)^\wedge_p)_{\mathrm{null}}$$

(spaces of pointed maps) is contractible by Theorem 1.1. Hence,

$$\widetilde{\mathbb{K}}(\Sigma^2(BP_+)) \cong \pi_2(BU) \cong \mathbb{Z}.$$

By Lemma 1.3 again (applied this time with $X = A = BP$),

$$\begin{aligned}
\mathbb{K}(S^2 \times BP)/\mathbb{K}(BP) &\cong \varinjlim_{\xi\in\mathbb{V}\mathrm{ect}^{\mathbb{C}}(P)} \mathbb{V}\mathrm{ect}^{\mathbb{C}}(S^2 \times BP, [\xi{\downarrow}BP]) \\
&\cong \varinjlim_{V\in\mathrm{Rep}^{\mathbb{C}}(P)} \pi_2\big(\mathrm{map}(BP, BU(\mathrm{d}_V))_{B\rho_V}\big) \\
&\cong \pi_2(\mathfrak{K}(BP)_0) && \text{(Prop. 2.14)} \\
&\cong \mathbb{Z} \times (\hat{\mathbb{Z}}_p)^{r-1}. \qquad \square && \text{(Prop. 2.16)}
\end{aligned}$$

By Example 1.4, if P is a nontrivial p-group, then $\mathbb{K}(S^2{\times}BP)/\mathbb{K}(BP) \cong \mathbb{Z} \times (\hat{\mathbb{Z}}_p)^{r-1}$ (with $r > 1$), while $\widetilde{\mathbb{K}}((S^2{\times}BP)/BP) \cong \mathbb{Z}$. In particular, excision fails for $\mathbb{K}(-)$, even in cases where the subspace is a retract. Note also that for such P, $\mathbb{K}(S^2{\times}BP)$ is not isomorphic to $\mathbb{K}(S^2) \otimes \mathbb{K}(BP)$.

The Atiyah-Segal theorem not only describes $K(BG)$ as the completion of the representation ring, but in fact for any finite G-complex X it says that $K(EG{\times}_G X) \cong K_G(X)\widehat{\ }$ (the completion with respect to the augmentation ideal of $\mathrm{R}(G)$). This, together with our description of $\mathbb{K}(BG)$ as the inverse limit of representation rings of p-toral subgroups of G, makes it natural to ask whether for any finite G-complex X, $\mathbb{K}(EG{\times}_G X)$ is isomorphic to the inverse limit over p-toral subgroups $P \subseteq G$ (for all primes p) of the groups $K_G(G/P{\times}X) \cong K_P(X)$. In fact, this relation fails even in the simplest case where $G \neq 1$ is a p-group (and so the inverse limit collapses). For any such G, let $X = S^2$ with the trivial G-action, and set $r = \mathrm{rk}(\mathrm{R}(G))$. Then $\mathbb{K}(BG \times S^2) \cong \mathbb{Z}^{r+1} \times (\hat{\mathbb{Z}}_p)^{r-1}$ by Lemma 1.4 (where $r = \mathrm{rk}(\mathrm{R}(G))$); while by equivariant Bott periodicity [At, Theorem 4.3],

$$K_G(S^2) \cong K_G(\mathrm{pt}) \oplus K_G^{-2}(\mathrm{pt}) \cong \mathrm{R}(G) \oplus \mathrm{R}(G) \cong \mathbb{Z}^{2r}.$$

Now define, for any space X, $\mathfrak{Vect}^{\mathbb{C}}_n(X) = \mathrm{map}(X, BU(n))$ (the "space" of complex n-dimensional bundles over X), and

$$\mathfrak{Vect}^{\mathbb{C}}(X) = \coprod_{n=0}^{\infty} \mathfrak{Vect}^{\mathbb{C}}_n(X) = \mathrm{map}\Big(X, \coprod_{n=0}^{\infty} BU(n)\Big).$$

Since $\coprod_{n=0}^{\infty} BU(n)$ is a topological monoid and commutative up to homotopy, $\mathfrak{Vect}^{\mathbb{C}}(X)$ is also a homotopy commutative topological monoid. It thus has a classifying space $B\mathfrak{Vect}^{\mathbb{C}}(X)$, and we define the "$K$-theory space" of X

$$\mathfrak{K}(X) \overset{\mathrm{def}}{=} \Omega B\big(\mathfrak{Vect}^{\mathbb{C}}(X)\big) :$$

to be the topological group completion of $\mathfrak{Vect}^{\mathbb{C}}(X)$ (cf. [MS]). The topological monoid

$$\mathfrak{Vect}^{\mathbb{R}}(X) = \mathrm{map}\Big(X, \coprod_{n=0}^{\infty} BO(n)\Big)$$

and its topological group completion $\mathfrak{KO}(X)$ are defined analogously.

Lemma 1.5. *For any complex X,*

$$\mathbb{K}(X) \cong \pi_0(\mathfrak{K}(X)) \quad \textit{and} \quad \mathbb{KO}(X) \cong \pi_0(\mathfrak{KO}(X)). \tag{1}$$

If X is a finite complex, then

$$\mathfrak{K}(X) \simeq \mathrm{map}(X, \mathbb{Z}\times BU) \quad \textit{and} \quad \mathfrak{KO}(X) \simeq \mathrm{map}(X, \mathbb{Z}\times BO). \tag{2}$$

Proof. The group completion theorem (cf. [MS]) says in particular that if M is a homotopy commutative topological monoid and if $\mathcal{G}\mathrm{r}(M) = \Omega BM$ denotes its topological group completion, then $\pi_0(\mathcal{G}\mathrm{r}(M)) \cong \mathcal{G}\mathrm{r}(\pi_0(M))$, and there is a homology equivalence

$$\underset{x\in\pi_0(M)}{\underrightarrow{\mathrm{hocolim}}}(M_x) \longrightarrow \mathcal{G}\mathrm{r}(M)_0.$$

Statement (1) of the lemma follows immediately. If X is a finite complex, then

$$\mathrm{map}(X, BU) \simeq \underset{n}{\underrightarrow{\mathrm{hocolim}}}\big(\mathrm{map}(X, BU(n))\big)$$

(and similarly for maps to $BO(n)$). And since the components of $\mathrm{map}(X, BU)$ have commutative fundamental group (BU being a monoid), statement (2) follows by the second part of the group completion theorem (and the fact that trivial vector bundles are cofinal in $\mathbb{V}\mathrm{ect}(X)$). □

2. $\mathbb{K}(BG)$ for a compact Lie group G

We have already seen (Theorem 1.1) that $\mathbb{K}(BP) \cong \mathrm{R}(P)$ for any p-toral group P. We are now ready to generalize this to a computation of $\mathbb{K}(BG)$ for an arbitrary compact Lie group G. This will be based on a homotopy decomposition at each prime p of BG as a direct limit of classifying spaces of p-toral subgroups of G; together with the vanishing of the relevant higher inverse limits. The arithmetic pullback square is then used to combine the p-adic calculations to get a global result. At the end of the section, a similar strategy is used to study the group completed mapping spaces $\mathfrak{K}(BG)$.

Throughout this section, G will be a fixed compact Lie group, $T = T_G \subseteq G$ is a maximal torus of G, and $W_G = N(T)/T$ is the Weyl group. We also fix Sylow p-subgroups $N_p(T)/T \subseteq N(T)/T$ for each prime $p\big||W_G|$. When G is finite ($T = 1$), we also write $\mathrm{Syl}_p(G) = N_p(T)$ for the Sylow p-subgroup.

These subgroups $N_p(T)$ are *maximal* p-toral subgroups of G, in the sense that any other p-toral subgroup of G is conjugate to a subgroup of $N_p(T)$ (cf. [JMO, Lemma A.1]). The main idea in this section is to describe $\mathbb{K}(BG)$ and $\mathfrak{K}(BG)$, by combining Theorem 1.1 (the description of $\mathfrak{Vect}_n^{\mathbb{C}}(BP)$ and $\mathfrak{Vect}^{\mathbb{C}}(BP)$ for a p-toral group P), with the decomposition in [JMO] of BG as a homotopy direct limit of classifying spaces of p-toral subgroups of G.

For any prime p, we define the category $\mathcal{R}_p(G)$ to be the category whose objects are the orbits G/P such that (1) $P \subseteq G$ is p-toral, (2) $N(P)/P$ is finite, and (3) there is no normal p-subgroup $1 \neq Q \lhd N(P)/P$. A morphisms in $\mathcal{R}_p(G)$ is any G-map between objects.

Theorem 2.1. *For any prime p, there is an $\mathbb{F}_p$-homology isomorphism*

$$\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\,(EG/P) \longrightarrow BG.$$

In particular, for any p-complete space X,

$$\mathrm{map}(BG,X) \simeq \mathrm{map}\Big(\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\,(EG/P), X\Big).$$

Furthermore, the category $\mathcal{R}_p(G)$ has the following properties.

(a) $\mathcal{R}_p(G)$ is a finite category, in that it has finitely many isomorphism classes of objects and finite morphism sets.

(b) For any given G and p, there is an integer $d = d(G,p) \geq 0$, with the following property: for any functor $F : \mathcal{R}_p(G) \to \hat{\mathbb{Z}}_p\text{-}\mathtt{mod}$, $\underleftarrow{\lim}^*{}_{\mathcal{R}_p(G)}(F) = 0$ for all $* > d$. If G is connected and has Weyl group W, then we can take $d = 0$ if $p\nmid|W|$; or $d = 1$ if $p^2\nmid|W|$.

Proof. The first statement is shown in [JMO, Section 1], and the fact that $\mathcal{R}_p(G)$ is a finite category in [JMO, Proposition 1.6].

In part (b), the existence of an upper bound d is shown in [JMO2, Proposition 4.11]. The last statement follows from [JMO3, Corollary 1.11 and Proposition 1.13]. □

We have already given, in the introduction, one definition of certain groups $\mathrm{R}_{\mathcal{P}}(G)$. The following definition of these groups (and of the corresponding groups $\mathrm{RO}_{\mathcal{P}}(G)$ in the orthogonal case) will be easier to use in practice.

Definition 2.2. *(a) For any $H \subseteq G$, a (unitary or orthogonal) H-representation V will be called G-*invariant *if $\chi_V(g) = \chi_V(g')$ for any pair of elements $g, g' \in H$ which are conjugate in G.*

(b) Define, for $F = \mathbb{C}$ or $\mathbb{R}$,

$$\mathrm{R}^F(H)^{G\text{-inv}} = \{v \in \mathrm{R}^F(H) \,|\, \chi_v(g) = \chi_v(g') \text{ if } g, g' \in H \text{ are conjugate in } G\};$$

(c) Define, for $F = \mathbb{C}$ or $\mathbb{R}$,

$$\mathrm{R}^F_{\mathcal{P}}(G) = \Big\{(v_p) \in \prod_{p\,|\,|W|} \mathrm{R}^F(N_p(T))^{G\text{-inv}} \,\Big|\, v_p|T \cong v_{p'}|T \;\forall p,p'\Big\}$$

Recall (cf. [Br, Theorem 0.4.2]) that any finite dimensional H-representation is contained in the restriction to H of some finite dimensional G-representation. Using this fact, one sees easily that $\mathrm{R}(H)^{G\text{-inv}} \subseteq \mathrm{R}(H)$ and $\mathrm{RO}(H)^{G\text{-inv}} \subseteq \mathrm{RO}(H)$ are the subgroups generated by all classes of G-invariant H-representations.

The next proposition characterizes $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$ as inverse limits of representation rings of p-toral subgroups; thus showing that the above definition of $\mathrm{R}_{\mathcal{P}}(G)$ is equivalent to the one given in the introduction.

Lemma 2.3. *Let $\mathcal{O}_p(G)$ (for any prime p) and $\mathcal{O}_{\mathcal{P}}(G)$ denote the full subcategories of the orbit category of G, where the objects of $\mathcal{O}_p(G)$ are the orbits G/P where P is p-toral, and where $\mathrm{Ob}(\mathcal{O}_{\mathcal{P}}(G)) = \cup_{p||W_G|} \mathrm{Ob}(\mathcal{O}_p(G))$. Then for $F = \mathbb{C}$ or $\mathbb{R}$,*

$$\mathrm{R}^F(N_p(T))^{G\text{-inv}} \cong \varprojlim_{G/P \in \mathcal{O}_p(G)} \mathrm{R}^F(P) \ (\forall p \big| |W_G|) \quad \textit{and} \quad \mathrm{R}^F_{\mathcal{P}}(G) \cong \varprojlim_{G/P \in \mathcal{O}_{\mathcal{P}}(G)} \mathrm{R}^F(P).$$

Proof. Since every p-toral subgroup of G is conjugate to a subgroup of $N_p(T)$, $\varprojlim_{G/P \in \mathcal{O}_p(G)} \mathrm{R}(P)$ is the group of elements $v \in \mathrm{R}(N_p(T))$ with the following property: for any p-toral subgroup $P \subseteq N_p(T)$ and any $g \in G$ such that $gPg^{-1} \subseteq N_p(T)$, the restrictions of v to P and to gPg^{-1} induce the same (virtual) representation of P. In particular, v lies in the inverse limit if its character is constant on G-conjugacy classes in $N_p(T)$, i.e., if $v \in \mathrm{R}(N_p(T))^{G\text{-inv}}$. Conversely, if v lies in the inverse limit, then consideration of the case where P is finite and cyclic shows that the character of v must be constant on G-conjugacy classes of elements of p-power order. And since such elements are dense in $N_p(T)$, we see that $v \in \mathrm{R}(N_p(T))^{G\text{-inv}}$.

The argument in the orthogonal case is identical. And the descriptions of $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{RO}_{\mathcal{P}}(G)$ as inverse limits now follow directly from their definitions. □

The next proposition describes how G-invariant representations of $N_p(T)$ arise: as algebraic invariants of maps $BG \to BU(n)$.

Proposition 2.4. *For any map $f : BG \to BU(n)$, there is for each prime p a unique G-invariant representation V_p (depending only on the homotopy class of f) such that $f|BN_p(T) \simeq B\rho_{V_p}$. Similarly, for any map $f : BG \to BO(n)$, there is for each prime p a unique G-invariant orthogonal representation V_p such that $f|BN_p(T) \simeq B\rho_{V_p}$. In either case, the V_p all restrict to the same representation of T.*

Proof. We prove this in the unitary case; the proof of the orthogonal case is similar. By Theorem 1.1, for any p-toral subgroup $P \subseteq G$, $[BP, BU(n)] \cong \mathrm{Rep}_n(P)$. Thus for any $f : BG \to BU(n)$, restriction to p-toral subgroups induces a map

$$[BG, BU(n)] \longrightarrow \varprojlim_{G/P \in \mathcal{O}_p(G)} [BP, BU(n)] \longrightarrow \varprojlim_{G/P \in \mathcal{O}_p(G)} \mathrm{R}(P) \cong \mathrm{R}(N_p(T))^{G\text{-inv}}.$$

In particular, $f|BN_p(T) \simeq B\rho_{V_p}$ for some $N_p(T)$-representation V_p which is G-invariant. And since $[BT, BU(n)] \cong \mathrm{Rep}_n(T)$, the restrictions $V_p|T$ are all isomorphic as T-representations, independently of p. □

Recall that for any space X, $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(X)$ and $\mathbb{V}\mathrm{ect}^{\mathbb{R}}(X)$ denote the monoids of complex and real vector bundles over X, and $\mathbb{K}(X)$ and $\mathbb{KO}(X)$ denote their Grothendieck groups. For each prime p, restriction defines a homomorphism

$$\begin{aligned}\psi_p : \mathbb{K}(BG) \longrightarrow \mathbb{K}(BN_p(T)) &= \mathcal{G}\mathsf{r}\big(\mathbb{V}\mathrm{ect}^{\mathbb{C}}(BN_p(T))\big) \\ &\cong \mathcal{G}\mathsf{r}\big(\mathrm{Rep}^{\mathbb{C}}(N_p(T))\big) \qquad \text{(Theorem 1.1)} \\ &= \mathrm{R}(N_p(T)).\end{aligned}$$

By Proposition 2.4, for any $f : BG \to BU(n)$, $\psi_p(f) \in \mathrm{R}(N_p(T))^{G\text{-inv}}$, and the $\psi_p(f)$ agree after restriction to T. The ψ_p thus combine to define a homomorphism

$$\psi_G : \mathbb{K}(BG) \longrightarrow \mathrm{R}_{\mathcal{P}}(G).$$

A homomorphism

$$\psi_G : \mathbb{KO}(BG) \longrightarrow \mathrm{RO}_{\mathcal{P}}(G)$$

is defined in a similar fashion.

The main result of this paper can now be stated.

Theorem 2.5. *For any compact Lie group G, the homomorphisms*

$$\psi_G^{\mathbb{C}} : \mathbb{K}(BG) \xrightarrow{\ \cong\ } \mathrm{R}_{\mathcal{P}}(G) \qquad \text{and} \qquad \psi_G^{\mathbb{R}} : \mathbb{KO}(BG) \xrightarrow{\ \cong\ } \mathrm{RO}_{\mathcal{P}}(G)$$

are isomorphisms.

When G is p-toral for any prime p, $\mathrm{R}_{\mathcal{P}}(G) = \mathrm{R}(G)$, and Theorem 2.5 follows from Theorem 1.1. The general case will be reduced to this using the decomposition of BG shown in Theorem 2.1. The next two lemmas are needed to deal with the higher limits which come up during this reduction process.

In the following lemma, we want to take limits over all (finite dimensional) G-representations of the homotopy groups of certain mapping spaces. The simplest way to make this precise is to restrict attention to any cofinal sequence of G-representations, and take the limit over the representations in that sequence. This is clearly independent of the choices of basepoints and stabilizing maps (a priori defined only up to homotopy). It is also independent of the choice of cofinal sequence: since given any two cofinal sequences $\{V_i\}$ and $\{W_i\}$, the functor applied to each sequence can be applied to the functor mapped into the $\{V_i \oplus W_i\}$.

Set $F = \mathbb{C}$ or $\mathbb{R}$. For each $i > 0$ and each finite G-complex X, we define a natural homomorphism

$$\Phi_i^G(X) : KF_G^{-i}(X) \longrightarrow \varinjlim_{V \in \mathrm{Rep}^F(G)} \Big(\pi_i\big(\mathrm{map}(EG \times_G X, BU(\mathrm{d}_V, F))_{B\rho_V}\big)\Big)$$

as follows. By Lemma 1.3,

$$KF_G^{-i}(X) = \widetilde{KF}_G(\Sigma^i(X_+)) \cong \varinjlim_{V \in \mathrm{Rep}^F(G)} \Big(\mathbb{V}\mathrm{ect}^{G,F}\big(S^i \wedge X_+, [V{\downarrow}*]\big)\Big),$$

where, $\mathbb{V}\mathrm{ect}_n^{G,F}(\cdots)$ denotes the set isomorphism classes of n-dimensional orthogonal G-vector bundles over $S^i \wedge X_+$ whose fiber over the basepoint has a given isomorphism with V. We define $\Phi_i^G(X)$ to be the composite of this isomorphism with the following homomorphism, induced by the Borel construction on G-vector bundles:

$$\begin{aligned}
&\varinjlim_{V\in\mathrm{Rep}^F(G)} \Big(\mathbb{V}\mathrm{ect}^{G,F}\big(S^i\wedge X_+,[V{\downarrow}*]\big)\Big) \\
&\longrightarrow \varinjlim_{V\in\mathrm{Rep}^F(G)} \Big(\mathbb{V}\mathrm{ect}^{F}\big(EG\times_G (S^i\wedge X_+),[EG{\times_G}V{\downarrow}EG{\times_G}*]\big)\Big) \\
&\cong \varinjlim_{V\in\mathrm{Rep}^F(G)} \Big(\pi_i\big(\mathrm{map}(EG\times_G X, BU(\mathrm{d}_V,F))_{B\rho_V}\big)\Big).
\end{aligned}$$

Here, in the last formula, the subscript $B\rho_V$ designates the connected component of the composite

$$EG{\times_G}X \xrightarrow{\text{proj.}} BG \xrightarrow{B\rho_V} BU(\mathrm{d}_V,F)^\wedge_p.$$

Lemma 2.6. *Set $F=\mathbb{C}$ or $\mathbb{R}$. Then for any $i>0$, any prime p and any p-toral subgroup $P\subseteq G$, $\Phi_i^G(G/P)$ induces an isomorphism*

$$\hat{\mathbb{Z}}_p\otimes KF_G^{-i}(G/P)\xrightarrow{\cong}\varinjlim_{V\in\mathrm{Rep}^F(G)}\Big(\pi_i\big(\mathrm{map}(EG\times_G G/P, BU(\mathrm{d}_V,F)^\wedge_p)_{B\rho_V}\big)\Big). \quad (1)$$

Proof. We show this in the orthogonal case. The unitary case is similar but slightly simpler. For each $V\in\mathrm{Rep}^{\mathbb{R}}(G)$,

$$\mathbb{V}\mathrm{ect}^{G,\mathbb{R}}\big(S^i\wedge(G/P_+),[V{\downarrow}*]\big)\cong\mathbb{V}\mathrm{ect}^{P,\mathbb{R}}\big(S^i,[(V|P){\downarrow}*]\big)\cong\pi_i\big(B\,\mathrm{Aut}_P(V)\big).$$

Also,

$$\begin{aligned}
\mathbb{V}\mathrm{ect}^{\mathbb{R}}\big(EG\times_G(S^i\wedge(G/P_+)),&[EG{\times_G}V{\downarrow}EG{\times_G}*]\big)\\
&\cong\pi_i\big(\mathrm{map}(BP,BO(\mathrm{d}_V))_{B\rho_V}\big).
\end{aligned}$$

So $\Phi_i^G(G/P)$ is isomorphic to the map

$$\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)}\Big(\pi_i(B\,\mathrm{Aut}_P(V))\Big)\longrightarrow\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)}\Big(\pi_i\big(\mathrm{map}(BP,BO(\mathrm{d}_V))_{B\rho_V}\big)\Big) \quad (2)$$

induced by the group homomorphism

$$\mathrm{Aut}_P(V)\times P\xrightarrow{(\mathrm{incl},\rho_V)}\mathrm{Aut}(V)=\mathrm{O}(\mathrm{d}_V).$$

Since odd dimensional representations are cofinal in $\mathrm{Rep}^{\mathbb{R}}(G)$, Theorem 1.1 now applies to show that (2) extends to an isomorphism

$$\hat{\mathbb{Z}}_p\otimes\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)}\Big(\pi_i(B\,\mathrm{Aut}_P(V))\Big)\longrightarrow\varinjlim_{V\in\mathrm{Rep}^{\mathbb{R}}(G)}\Big(\pi_i\big(\mathrm{map}(BP,BO(\mathrm{d}_V)^\wedge_p)_{B\rho_V}\big)\Big);$$

and hence that the homomorphism (1) induced by $\Phi_i^G(G/P)$ is an isomorphism. $\square$

We now combine Lemma 2.6 with results from Section 3 to give:

Corollary 2.7. *Set $F = \mathbb{C}$ or $\mathbb{R}$. For each $i > 0$, let $\Pi_i^F : \mathcal{R}_p(G) \to \hat{\mathbb{Z}}_p$-mod be the functor defined by setting*

$$\Pi_i^F(G/P) = \varinjlim_{V \in \mathrm{Rep}^F(G)} \pi_i(\mathrm{map}(EG/P, BU(\mathrm{d}_V, F)\hat{_p})_{B\rho_V}).$$

Then

$$\Pi_i^F \cong \hat{\mathbb{Z}}_p \otimes KF_G^{-i}(-) \tag{1}$$

as functors on $\mathcal{R}_p(G)$, and

$$\varprojlim_{\mathcal{R}_p(G)}{}^j \, \Pi_i^F = 0 \tag{2}$$

for all $i, j > 0$.

Proof. The isomorphism (1), as an isomorphism of functors on $\mathcal{R}_p(G)$, is immediate from Lemma 2.6. And hence by Corollary 3.7 below,

$$\varprojlim_{\mathcal{R}_p(G)}{}^j (\Pi_i^{\mathbb{R}}) \cong \varprojlim_{\mathcal{R}_p(G)}{}^j \big(KF_G^{-i}(-)\big) = 0$$

for all $i, j > 0$. □

In general, of course, direct and inverse limits cannot be switched. The following lemma describes one case where this can be done.

Lemma 2.8. *Fix a finite category $\mathcal{C}$ and a directed category $\mathcal{D}$, and let $F : \mathcal{C}\times\mathcal{D} \to$ Ab be any functor to abelian groups. Then for any $i \geq 0$,*

$$\varprojlim_{\mathcal{C}}{}^i \Big(\varinjlim_{\mathcal{D}} (F) \Big) \cong \varinjlim_{\mathcal{D}} \Big(\varprojlim_{\mathcal{C}}{}^i (F) \Big).$$

Proof. The term finite category was used earlier to mean a category with finitely many isomorphism classes of objects and finite morphism sets. But we can of course assume here in the proof that $\mathcal{C}$ actually has only finitely many objects.

For any functor $F' : \mathcal{C} \to$ Ab, the higher limits $\varprojlim_{\mathcal{C}}{}^* F'$ of F' are the homology groups of a cochain complex

$$0 \to \prod_{c \in \mathcal{C}} F'(c) \longrightarrow \prod_{c_0 \to c_1} F'(c_1) \longrightarrow \prod_{c_0 \to c_1 \to c_2} F'(c_2) \longrightarrow \dots$$

(cf. [BK, XI.6.2] or [O3, Lemma 2]). Since $\mathcal{C}$ is a finite category, all of the products are finite, and hence commute with direct limits over the directed category $\mathcal{D}$. And since direct limits over $\mathcal{D}$ commute with taking homology, we now see that they commute with inverse limits over $\mathcal{C}$. □

Using Lemma 2.8, one can show that under certain conditions, homotopy direct limits and homotopy inverse limits commute for functors $\mathcal{C}\times\mathcal{D} \to$ Top. An example of this is shown in the proof of Proposition 2.14 below, but the conditions for a general result seem too complicated to be worth stating here.

We are now ready to prove the main theorem. At the same time, we will prove the following result:

Proposition 2.9. *For any (unitary or orthogonal) G-representation V, let $V_G{\downarrow}BG$ denote the induced vector bundle $V_G = EG \times_G V$. Then the following hold:*

(a) Any element $x \in \mathbb{K}(BG)$ ($\mathbb{KO}(BG)$) has the form $x = [\xi] - [V_G]$, for some complex (real) bundle $\xi{\downarrow}BG$ and some unitary (orthogonal) G-representation V.

(b) If two complex (real) vector bundles ξ_1, ξ_2 over BG are stably isomorphic, then $\xi_1 \oplus V_G \cong \xi_2 \oplus V_G$ for some unitary (orthogonal) G-representation V.

In particular, this says that for any G, it suffices to invert representations when constructing the group completions of $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(BG)$ and $\mathbb{V}\mathrm{ect}^{\mathbb{R}}(BG)$. This will be seen in Proposition 2.14 below to also hold on the space level: $\mathfrak{K}(BG)$ is obtained from $\mathfrak{Vect}^{\mathbb{C}}(BG)$ by inverting G-representations.

Proof of Theorem 2.5 and Proposition 2.9. By Proposition 2.4,

$$\psi_G : \mathbb{K}(BG) \longrightarrow \mathrm{R}_{\mathcal{P}}(G) \qquad \text{and} \qquad \psi_G : \mathbb{KO}(BG) \longrightarrow \mathrm{RO}_{\mathcal{P}}(G)$$

are well defined homomorphisms. We must show that they are isomorphisms.

The following result will allow us to compare homotopy classes of maps $BG \to BU(n)$ with homotopy classes of maps to the p-adic completions $BU(n)^\wedge_p$. For any compact connected Lie group G' (for example, $G' = U(n)$ or $SO(n)$), and any homomorphism $\phi : T \to G'$, let $[BG, BG']_\phi$ and $[BG, BG'^\wedge_p]_\phi$ denote the sets of homotopy classes of maps whose restriction to BT is homotopic to $B\phi$. Then

$$[BG, BG']_\phi \cong \prod_{p|n} [BG, BG'^\wedge_p]_\phi. \tag{1}$$

This follows from Sullivan's arithmetic pullback square for BG'; see [JMO3, Proposition 1.2] or [JMO2, Proposition 3.4] for more detail.

We first consider the unitary case. For any G-invariant representation V, define functors

$$\Pi_j^V : \mathcal{R}_p(G) \longrightarrow \hat{\mathbb{Z}}_p\text{-}\mathsf{mod}, \qquad\qquad (\text{for all } j \geq 1)$$

by setting

$$\Pi_j^V(G/P) = \pi_j\big(\mathrm{map}(BP, BU(\mathrm{d}_V)^\wedge_p)_{B(\rho_V|P)}\big) \cong \pi_j\left(B\,\mathrm{Aut}_P(V)^\wedge_p\right).$$

(Recall that we write $\mathrm{d}_V = \dim(V)$ for short.) For any V_0, the direct systems $\Pi_i^{V_0 \oplus V}$ and Π_i^V are equivalent as functors for $V \in \mathrm{Rep}(G)$: since V_0 is a summand of the restriction to $N_p(T)$ of some G-representation (cf. [Br, Theorem 0.4.2]). Also, $\mathcal{R}_p(G)$ is a finite category by Theorem 2.1(a). Hence by Lemma 2.8 and Corollary 2.7, for all $i, j > 0$,

$$\begin{aligned}
\varinjlim_{V \in \mathrm{Rep}(G)} \Big(\varprojlim_{\mathcal{R}_p(G)}{}^{j}\, \Pi_i^{V_0 \oplus V} \Big) &\cong \varprojlim_{\mathcal{R}_p(G)}{}^{j} \Big(\varinjlim_{V \in \mathrm{Rep}(G)} \Pi_i^{V_0 \oplus V} \Big) \\
&\cong \varprojlim_{\mathcal{R}_p(G)}{}^{j} \Big(\varinjlim_{V \in \mathrm{Rep}(G)} \Pi_i^{V} \Big) = 0.
\end{aligned} \tag{2}$$

Fix any element $v = \big(v_p\big)_{p\big||W_G|} \in \mathrm{R}_{\mathcal{P}}(G)$. For any given $p\big||W_G|$, write $v_p = [V'] - [V'']$, where V' and V'' are $N_p(T)$-representations. Since V'' is contained in the restriction of some G-representation ([Br, Theorem 0.4.2] again), we can assume after stabilizing that V'' is the restriction of a G-representation, and hence that V' is G-invariant. The successive obstructions to the existence of a map

$$f_p \in [BG, BU(n)\hat{_p}] \cong \Big[\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} \big(EG/P\big), BU(n)\hat{_p}\Big]$$

such that $f_p|BN_p(T) \simeq B\rho_{V'}$ lie in the groups $\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}^{i+1} \Pi_i^{V'}$ for all $i > 0$ (cf. [Wo]). Hence, by (2), each obstruction in turn vanishes after adding a sufficiently large G-representation to V' (and V''). In addition, by Theorem 2.1(b), all of the higher limits vanish in degrees above some fixed $d(G,p)$ (which is independent of V'); and so there are only finitely many obstructions to constructing f_p.

After carrying out this procedure for each prime $p\big||W_G|$, we can arrange, by stabilizing further, that the same G-representation is subtracted for each p. In other words, we end up with

(a) a G-representation V,

(b) G-invariant $N_p(T)$-representations V_p (for each p), and

(c) maps $f_p : BG \to BU(m)\hat{_p}$;

such that for each $p\big||W_G|$,

(d) $\dim(V_p) = m$ and $v_p = [V_p] - [V|N_p(T)]$, and

(e) $f_p|BN_p(T) \simeq B\rho_{V_p}$.

By definition of $\mathrm{R}_{\mathcal{P}}(G)$, the v_p all restrict to the same element $[V_T] - [V|T] \in R(T)$, and $f_p|BT \simeq B\rho_{V_T}$ for all p. So by (1) there is a map $f : BG \to BU(m)$ whose completion at each prime p is homotopic to f_p. Since $[BN_p(T), BU(-)]$ injects into $[BN_p(T), BU(-)\hat{_p}]$ (by (1) again), this implies that $f|BN_p(T) \simeq B\rho_{V_p}$ for each p; and hence that

$$v = \psi_G\Big([f] - [B\rho_V]\Big).$$

This proves that ψ_G is surjective. In terms of vector bundles, it says that $v = \psi_G([\xi] - [V_G])$ (where $(\xi{\downarrow}BG)$ is the pullback of the universal vector bundle over $BU(m)$). So Proposition 2.9(a) will follow once we have shown ψ_G to be injective.

To show that ψ_G is injective, and simultaneously prove Proposition 2.9(b), we will show that for any n and any two maps $f, g : BG \to BU(n)$ such that $\psi_G(f) = \psi_G(g)$, there is some G-representation V for which $f \oplus B\rho_V \simeq g \oplus B\rho_V$. Since $\psi_G(f) = \psi_G(g)$, there is for each p a G-invariant representation V_p of $N_p(T)$ such that $f|BN_p(T) \simeq B\rho_{V_p} \simeq g|BN_p(T)$. By (1) again, it will suffice to find V such that $f \oplus B\rho_V \simeq g \oplus B\rho_V$ after completion at any prime $p\big||W_G|$. The obstructions to the maps

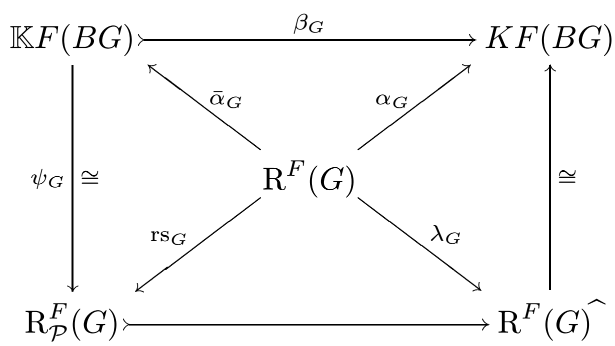
$$(f \oplus B\rho_V)\hat{_p},\ (g \oplus B\rho_V)\hat{_p} \in [BG, BU(n+\mathrm{d}_V)\hat{_p}] \cong \Big[\underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} \big(EG/P\big), BU(n+\mathrm{d}_V)\hat{_p}\Big]$$

being homotopic lie in the groups $\varprojlim_{\mathcal{R}_p(G)}^i \Pi_i^{V_p \oplus V}$ for $i \geq 1$ (cf. [Wo]). By (2) again, each obstruction vanishes for V large enough; and by Theorem 2.1(b) there are only finitely many nonvanishing obstruction groups.

When proving that ψ_G sends $\mathbb{KO}(BG)$ isomorphically to $\mathrm{RO}_{\mathcal{P}}(G)$, the only difference is that one has to restrict attention to odd dimensional representations: since $BO(n) \simeq BSO(n) \times B\mathbb{Z}/2$ for odd n (while $BO(n)$ is not nilpotent for n even). □

We now list some corollaries to Theorem 2.5. The first one follows immediately from it together with the Atiyah-Segal completion theorem [AS]. We want to compare the isomorphism $\mathbb{K}(BG) \cong \mathrm{R}_{\mathcal{P}}(G)$ just shown with the isomorphism $K(BG) \cong \mathrm{R}(G)\widehat{\ }$ of [AS].

Corollary 2.10. *Set $F = \mathbb{C}$ or $\mathbb{R}$. For any compact Lie group G, consider the commutative diagram*



where $\bar{\alpha}_G$ sends a representation V to its associated vector bundle $(V_G{\downarrow}BG)$ ($V_G = EG \times_G V$), β_G is the natural homomorphism, and λ_G is the completion homomorphism. In both the unitary and orthogonal cases, β_G is always injective, and $\mathrm{Ker}(\bar{\alpha}_G) \subseteq \mathrm{R}^F(G)$ *is the subgroup of elements whose characters vanish on all elements in components of prime power order in $\pi_0(G)$. Also, $\bar{\alpha}_G$ is surjective if G is finite or if $\pi_0(G)$ has prime power order.*

Proof. The diagrams commute by construction, ψ_G is an isomorphism by Theorem 2.5, and $KF(BG) \cong \mathrm{R}^F(G)\widehat{\ }$ by [AS, Theorems 2.1 & 7.1]. The injectivity of β_G follows from the fact [Seg, Proposition 3.10] that $\mathrm{R}(P)$ injects into $\mathrm{R}(P)\widehat{\ }$ (and hence that $\mathrm{RO}(P)$ injects into $\mathrm{RO}(P)\widehat{\ }$), for any P such that $\pi_0(P)$ has prime power order (in particular, for any p-toral group P).

Since ψ_G is an isomorphism, $\mathrm{Ker}(\bar{\alpha}_G) = \mathrm{Ker}\big[\mathrm{rs}_G : \mathrm{R}^F(G) \to \mathrm{R}^F_{\mathcal{P}}(G)\big]$, and this by definition is the set of elements whose characters vanish on all elements which lie in any p-toral subgroup of G for any prime p. And by Proposition 4.6(a) below, for any prime p, any element in a connected component of p-power order in $\pi_0(G)$ is contained in some p-toral subgroup of G. Alternatively, $\mathrm{Ker}(\bar{\alpha}_G) = \mathrm{Ker}(\alpha_G)$, and this was calculated (in the unitary case, at least) by Segal [Seg, Proposition 3.10].

Adams, in [Ad2], showed that $\alpha_G(\mathrm{R}(G)) = \beta_G(\mathbb{K}(BG))$ whenever G is finite or $\pi_0(G)$ has prime power order; and the surjectivity of $\bar{\alpha}_G$ in these cases then follows from the injectivity of β_G. Another proof of the surjectivity of $\bar{\alpha}_G$ (or equivalently, of rs_G) in these and other cases is given in Corollary 4.8 below. In Proposition 4.9, $\bar{\alpha}_G$ is shown to be surjective in the orthogonal case whenever G is finite or $\pi_0(G)$ has prime power order. $\square$

In general, if $\dim(G) > 0$ and $\pi_0(G)$ does not have prime power order, then the maps $\bar{\alpha}_G$ need not be surjective. The precise cokernel of $\bar{\alpha}_G$ (in the unitary case) will be computed in Theorem 4.8.

The following is just a reinterpretation of the above results in geometric terms.

Corollary 2.11. *For any group G, any (real or complex) vector bundle over BG is a summand of the bundle $(V_G{\downarrow}BG)$ ($V_G = V{\times_G}EG$) associated to some G-representation V. If G is finite or if $\pi_0(G)$ has prime power order, then for any (real or complex) vector bundle $\xi{\downarrow}BG$, there exist G-representations V, V' such that $\xi{\oplus}V'_G \cong V_G$.*

Proof. Let $\xi{\downarrow}BG$ be any bundle. By Proposition 2.9(a), we can write $-[\xi] = [\eta]-[V_G]$ in $\mathbb{K}F(BG)$, for some bundle $\eta{\downarrow}BG$ and some G-representation V. Then $\xi \oplus \eta$ is stably isomorphic to V_G; and by Proposition 2.9(b) $\xi \oplus \eta \oplus V'_G \cong (V \oplus V')_G$ for some V'.

The second statement follows immediately from the surjectivity of $\bar{\alpha}_G : \mathrm{R}^F(G) \twoheadrightarrow \mathbb{K}F(BG)$ in the given cases. $\square$

If G is finite, then the relation between $K(BG)$ and $\mathbb{K}(BG)$ can be made more explicit. For each $p\big||G|$, let r_p denote the number of conjugacy classes of elements $g{\in}G{\smallsetminus}1$ of p-power order. Then by Corollary 2.10,

$$\mathbb{K}(BG) \cong \mathrm{R}(G)/\operatorname{Ker}(\bar{\alpha}_G) \cong \mathbb{Z}\times \prod_{p\big||G|} \mathbb{Z}^{r_p}.$$

In contrast,

$$K(BG) \cong \mathrm{R}(G)\widehat{\ } \cong \mathbb{Z}\times \prod_{p\big||G|} (\hat{\mathbb{Z}}_p)^{r_p} :$$

this follows from [Ja, proof of Theorem 2.2] (although not stated explicitly there).

We adopt the notation of Adams in [Ad2], and let $FF(X) \subseteq K(X)$ (for any space X) denote the subgroup of "formally finite" elements; i.e., the subgroup generated by those elements $x \in K(X)$ such that $\lambda^n(x) = 0$ for n sufficiently large. Similarly, $FFO(X) \subseteq KO(X)$ will denote the subgroup of formally finite elements in the real K-theory of X.

Corollary 2.12. *For any G,*

$$\beta_G(\mathbb{K}(BG)) = FF(BG) \qquad \text{and} \qquad \beta_G(\mathbb{KO}(BG)) = FFO(BG).$$

Proof. For each prime p, let $G_p \subseteq G$ be a subgroup of finite index such that $\pi_0(G_p)$ is a Sylow p-subgroup of $\pi_0(G)$. By [Ad2, Theorem 1.11], $FF(BG)$ is the set of those elements of $K(BG)$ whose restriction to $K(BG_p)$ (for any prime p) lies in the image of $\mathrm{R}(G_p)$. By Theorem 2.5 (and the definition of $\mathrm{R}_{\mathcal{P}}(G)$), $\beta_G(\mathbb{K}(BG))$ is the set of elements of $K(BG)$ whose restriction to $K(BN_p(T))$ (for any prime p) lies in the image of $\mathrm{R}(N_p(T))$. Thus, $\beta_G(\mathbb{K}(BG)) \supseteq FF(BG)$; and the opposite inclusion follows immediately from the definitions.

To prove the corresponding result in the orthogonal case, the first step is to show that $KO(BG)$ is torsion free. This follows exactly the same lines as the proof in [Ad2, Lemma 1.12] that $K(BG)$ is torsion free. Hence, since the composite

$$KO(BG) \xrightarrow{\mathbb{C}\otimes_{\mathbb{R}}} K(BG) \xrightarrow{\text{forget}} KO(BG)$$

is multiplication by 2, we can regard $KO(BG)$ as a subgroup of $K(BG)$. It is then clear that $FFO(BG) = FF(BG) \cap KO(BG)$. So given any element $x \in FFO(BG)$, we must show that $x|BP \in \mathrm{Im}(\mathrm{RO}(P))$ for any p-toral subgroup $P \subseteq G$; and we know that $x|BP \in \mathrm{Im}(\mathrm{R}(P)) \cap KO(BP)$. Also, any representation of a p-toral subgroup P whose restriction to all finite p-subgroups of P comes from real representations is itself a real representation. Thus, it remains only to show, for any finite p-group P, that the square

$$\begin{array}{ccc}
\mathrm{RO}(P) & \longrightarrow & KO(BP) \cong \mathrm{RO}(P)^{\wedge} \\
\downarrow & & \downarrow \\
\mathrm{R}(P) & \longrightarrow & K(BP) \cong \mathrm{R}(P)^{\wedge}
\end{array}$$

is a pullback square. And this follows since the only torsion in $\mathrm{R}(P)/\mathrm{RO}(P)$ is p-torsion (and only when $p = 2$); and since the I-adic completions are the same in this case as the p-adic completions of the augmentation ideals (cf. [AT, Proposition III.1.1]). $\square$

We now turn to the problem of describing the individual components of the group-like topological monoids $\mathfrak{K}(BG)$ and $\mathfrak{KO}(BG)$. This will be done by comparing them with the direct limits of the mapping spaces $\mathrm{map}(BG, BU(\mathrm{d}_V))_{B\rho_V}$ or $\mathrm{map}(BG, BO(\mathrm{d}_V))_{B\rho_V}$, where V runs over all finite dimensional unitary or orthogonal G-representations. We will also have to consider the spaces of maps to localizations and completions of the $BU(n)$ and $BO(n)$, and so the following notation will be useful.

Definition 2.13. *Set $F = \mathbb{C}$ or $\mathbb{R}$. Fix a G-space X. For any $n \geq 0$ and any $V \in \mathrm{Rep}_n^F(G)$, set*

$$\mathfrak{Vect}^F(EG \times_G X)_V = \mathrm{map}(EG \times_G X, BU(n, F))_{B\rho_V}$$

(the connected component of the composite $EG\times_G X \xrightarrow{(X\to *)} BG \xrightarrow{B\rho_V} BU(n, F)$),

$$\mathfrak{Vect}^F(EG \times_G X; \mathbb{Q})_V = \mathrm{map}(EG \times_G X, BU(n, F)_{\mathbb{Q}})_{B\rho_V},$$

$$\mathfrak{Vect}^F(EG\times_G X;\hat{\mathbb{Z}}_p)_V = \operatorname{map}(EG\times_G X, BU(n,F)^{\wedge}_p)_{B\rho_V} \qquad (p \text{ any prime}),$$

$$\begin{aligned}\mathfrak{Vect}^F(EG\times_G X;\widehat{\mathbb{Z}})_V &= \operatorname{map}(EG\times_G X, BU(n,F)\widehat{\ })_{B\rho_V} \\ &\cong \prod_p \mathfrak{Vect}^F(EG\times_G X;\hat{\mathbb{Z}}_p)_V\end{aligned}$$

(where $BU(n,F)\widehat{\ } = \prod_p BU(n,F)^{\wedge}_p$), and

$$\mathfrak{Vect}^F(EG\times_G X;\widehat{\mathbb{Q}})_V = \operatorname{map}\big(EG\times_G X, (BU(n,F)\widehat{\ })_{\mathbb{Q}}\big)_{B\rho_V}.$$

Finally, define

$$\mathfrak{Vect}^F(EG\times_G X;-)_\infty = \underset{V\in \operatorname{Rep}^F(G)}{\underrightarrow{\operatorname{hocolim}}} \big(\mathfrak{Vect}^F(EG\times_G X;-)_V\big).$$

The relationship between these spaces, and $\mathfrak{K}(BG)$ and $\mathfrak{KO}(BG)$, is described in the next proposition.

Proposition 2.14. *Set $w=|W_G|$, and let $F=\mathbb{C}$ or $\mathbb{R}$. There is a homotopy pullback square*

$$\begin{array}{ccc} \mathfrak{K}_F(BG)_0 \simeq \mathfrak{Vect}^F(BG)_\infty & \longrightarrow & \mathfrak{Vect}^F(BG;\widehat{\mathbb{Z}})_\infty \\ \downarrow & & \downarrow \\ \mathfrak{Vect}^F(BG;\mathbb{Q})_\infty & \longrightarrow & \mathfrak{Vect}^F(BG;\widehat{\mathbb{Q}})_\infty. \end{array} \tag{1}$$

Furthermore,

$$\mathfrak{Vect}^F(BG;\hat{\mathbb{Z}}_p)_\infty \simeq \underset{\mathcal{R}_p(G)}{\underleftarrow{\operatorname{holim}}}\, \mathfrak{Vect}^F(EG\times_G -;\hat{\mathbb{Z}}_p)_\infty \tag{2}$$

for each prime p. And for each $i>0$,

$$\pi_i\big(\mathfrak{Vect}^F(BG;\hat{\mathbb{Z}}_p)_\infty\big) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}\ \pi_i\big(\mathfrak{Vect}^F(EG\times_G -;\hat{\mathbb{Z}}_p)_\infty\big) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}\ \big(\hat{\mathbb{Z}}_p\otimes KF_G^{-i}(-)\big) \tag{3}$$

and

$$\pi_i\big(\mathfrak{Vect}^F(BG;\widehat{\mathbb{Z}})_\infty\big) \cong \prod_{p|w}\pi_i\big(\mathfrak{Vect}^F(BG;\hat{\mathbb{Z}}_p)_\infty\big) \times \bigg(\Big(\prod_{p\nmid w}\hat{\mathbb{Z}}_p\Big)\otimes KF_G^{-i}(G/T)^{W_G}\bigg). \tag{4}$$

Proof. We prove this in the orthogonal case. The unitary case is similar, but slightly simpler since $BU(n)$ is simply connected for each n.

If n is odd, then $BO(n) \simeq BSO(n) \times B\{\pm I\}$. Hence, the arithmetic pullback square for the simply connected space $BSO(n)$ (cf. [BK, VI.8.1]) induces a homotopy pullback square

$$\begin{array}{ccc} \mathrm{map}(BG, BO(n)) & \longrightarrow & \mathrm{map}(BG, BO(n)\widehat{\ }) \\ \downarrow & & \downarrow \\ \mathrm{map}(BG, BO(n)_{\mathbb{Q}}) & \longrightarrow & \mathrm{map}(BG, (BO(n)\widehat{\ })_{\mathbb{Q}}) \end{array} \tag{5}$$

of mapping spaces. Also, the top row in (5) induces an injection on $\pi_0(-)$ (cf. [JMO, Theorem 3.1]). Hence for any n-dimensional orthogonal G-representation V, (5) restricts to a homotopy pullback square involving the components of $B\rho_V$. Upon taking the homotopy direct limit over all (odd dimensional) representations, and using the exactness of direct limits and the 5-lemma, we see that the square in (1) is a homotopy pullback square.

By Theorem 2.1 (the approximation at p of BG as a homotopy direct limit),

$$\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_V \simeq \underset{\mathcal{R}_p(G)}{\underleftarrow{\mathrm{holim}}} \big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_V\big)$$

for each V. By Lemma 2.8 and Corollary 2.7, for each $i, j > 0$,

$$\begin{aligned} &\underset{V \in \mathrm{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \Big(\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}{}^j \, \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_V\big)\Big) \\ &\qquad \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}{}^j \, \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_\infty\big) = 0. \end{aligned} \tag{6}$$

By Theorem 2.1(b), there is an integer $d > 0$ such that

$$\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}{}^j \, \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_V\big) = 0$$

for all V and all $j > d$. The obstruction theory of Wojtkowiak [Wo], or the spectral sequence of Bousfield & Kan [BK, §XI.7 & §XII.4], now applies to prove that

$$\begin{aligned} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_\infty\big) &\cong \underset{V \in \mathrm{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_V\big) \\ &\cong \underset{V \in \mathrm{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \Big(\underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}} \, \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_V\big)\Big) \\ &\cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}} \Big(\underset{V \in \mathrm{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_V\big)\Big) \\ &\cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}} \, \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_\infty\big). \end{aligned}$$

This is the first isomorphism in (3), and the second follows from Corollary 2.7.

By (6), again, the space $\underleftarrow{\text{holim}}_{\mathcal{R}_p(G)} \mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_\infty$ is connected, and

$$\pi_i\Big(\underset{\mathcal{R}_p(G)}{\underleftarrow{\text{holim}}}\, \mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_\infty\Big) \cong \underset{\mathcal{R}_p(G)}{\underleftarrow{\lim}}\; \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(EG \times_G -; \hat{\mathbb{Z}}_p)_\infty\big). \qquad (7)$$

Together with (3), this proves point (2).

We next check formula (4). For each $p\nmid w{=}|W_G|$, $\mathcal{R}_p(G)$ contains the orbit G/T as a representative of its unique isomorphism class of object. Hence (3) takes the form

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_\infty\big) \cong \hat{\mathbb{Z}}_p \otimes \big(KO_G^{-i}(G/T)^{W_G}\big).$$

Also, for each V (and $p \nmid w$),

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_V\big) \cong \pi_i\big(\text{map}(BT, BO(\text{d}_V)\hat{{}_p})_{B\rho_V}\big)^{W_G} \cong \Big[\hat{\mathbb{Z}}_p \otimes \pi_i\big(B\,\text{Aut}_T(V)\big)\Big]^{W_G}$$

is finitely generated; and so

$$\underset{V \in \text{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \Big(\prod_{p\nmid w} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_V\big)\Big) \cong \Big(\prod_{p\nmid w} \hat{\mathbb{Z}}_p\Big) \otimes \Big(K_G^{-i}(G/T)^{W_G}\Big).$$

Since direct limits commute with *finite* products, it now follows that

$$\begin{aligned}\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \widehat{\mathbb{Z}})_\infty\big) &\cong \underset{V \in \text{Rep}^{\mathbb{R}}(G)}{\underrightarrow{\lim}} \Big(\prod_{p} \pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_V\big)\Big) \\ &\cong \prod_{p|w}\Big(\pi_i\big(\mathfrak{Vect}^{\mathbb{R}}(BG; \hat{\mathbb{Z}}_p)_\infty\big)\Big) \times \Big(\Big(\prod_{p\nmid w} \hat{\mathbb{Z}}_p\Big) \otimes K_G^{-i}(G/T)^{W_G}\Big).\end{aligned}$$

By [MS], the natural map

$$\mathfrak{Vect}^{\mathbb{R}}(BG)_\infty \longrightarrow [\Omega B\mathfrak{Vect}^{\mathbb{R}}(BG)]_0 = \mathfrak{KO}(BG)_0 \qquad (8)$$

is an equivalence of homology with any twisted coefficients. In particular, it induces a surjection on $\pi_1(-)$, whose kernel is the commutator subgroup of $\pi_1(\mathfrak{Vect}^{\mathbb{R}}(BG)_\infty)$ and is perfect. But the pullback square (1), together with (3) and (4) above (and the fact that $BO(n)_{\mathbb{Q}}$ is a product of Eilenberg-Maclane spaces for n odd), show that $\pi_1(\mathfrak{Vect}^{\mathbb{R}}(BG)_\infty)$ is solvable. Hence $\pi_1(\mathfrak{Vect}^{\mathbb{R}}(BG)_\infty)$ is abelian, and (8) is a homotopy equivalence. □

Proposition 2.14 will first be applied to study the components of $\mathfrak{K}(BG)$ and $\mathfrak{KO}(BG)$ when G is finite, and afterwards to study $\mathfrak{K}(BG)$ (the unitary case only) for arbitrary G.

Proposition 2.15. *If G is finite, then each connected component of $\mathfrak{K}(BG)$ is homotopy equivalent to* $\mathrm{map}(BG, BU)_0$*, and each connected component of $\mathfrak{KO}(BG)$ is homotopy equivalent to* $\mathrm{map}(BG, BO)_0$.

Proof. We prove this in the unitary case; the orthogonal case is identical. Let $\kappa : \mathfrak{K}(BG)_0 \to \mathrm{map}(BG, BU)_0$ be the map induced by the group completion of the inclusion map $\coprod_{n=0}^{\infty} \mathrm{map}(BG, BU(n)) \to \mathrm{map}(BG, \mathbb{Z} \times BU)$. We will show that κ is a homotopy equivalence by showing that each of the other terms in the homotopy pullback square (1) of Proposition 2.14 maps via a homotopy equivalence to the corresponding term in the pullback square

$$\begin{array}{ccc} \mathrm{map}(BG, BU)_0 & \longrightarrow & \mathrm{map}(BG, BU\widehat{\ })_0 \\ \downarrow & & \downarrow \\ \mathrm{map}(BG, BU_{\mathbb{Q}})_0 & \longrightarrow & \mathrm{map}(BG, (BU\widehat{\ })_{\mathbb{Q}})_0. \end{array}$$

This is clear for the $\mathbb{Q}$-local terms: since

$$\begin{aligned} \mathfrak{Vect}^{\mathbb{C}}(BG; \mathbb{Q})_\infty &= \underset{V}{\underrightarrow{\mathrm{hocolim}}}\Big(\mathrm{map}\big(BG, BU(\mathrm{d}_V)_{\mathbb{Q}}\big)\Big) \simeq \underset{V}{\underrightarrow{\mathrm{hocolim}}}\big(BU(\mathrm{d}_V)_{\mathbb{Q}}\big) \\ &\simeq BU_{\mathbb{Q}} \simeq \mathrm{map}(BG, BU_{\mathbb{Q}})_0. \end{aligned}$$

It remains to show that $\widehat{\kappa} : \mathfrak{Vect}^{\mathbb{C}}(BG; \widehat{\mathbb{Z}})_\infty \to \mathrm{map}(BG, BU\widehat{\ })_0$ is a homotopy equivalence. Since G is finite, $K_G^{-i}(G/T)$ is finitely generated for all i, and hence Proposition 2.14(4) implies that $\mathfrak{Vect}^{\mathbb{C}}(BG; \widehat{\mathbb{Z}})_\infty \simeq \prod_p \mathfrak{Vect}^{\mathbb{C}}(BG; \hat{\mathbb{Z}}_p)_\infty$. So it will suffice to show that

$$\kappa_{\hat{p}} : \mathfrak{Vect}^{\mathbb{C}}(BG; \hat{\mathbb{Z}}_p)_\infty \to \mathrm{map}(BG, BU_{\hat{p}})_0 \tag{1}$$

is a homotopy equivalence for each p. Also,

$$\mathfrak{Vect}^{\mathbb{C}}(BG; \hat{\mathbb{Z}}_p)_\infty \simeq \underset{G/P \in \mathcal{R}_p(G)}{\underleftarrow{\mathrm{holim}}} \mathfrak{Vect}^{\mathbb{C}}(EG/P; \hat{\mathbb{Z}}_p)_\infty$$

by Proposition 2.14, and

$$\mathrm{map}(BG, BU_{\hat{p}}) \simeq \underset{G/P \in \mathcal{R}_p(G)}{\underleftarrow{\mathrm{holim}}} \mathrm{map}(EG/P, BU_{\hat{p}})$$

by Theorem 2.1. So we are reduced to showing that (1) is an equivalence when G is a finite p-group.

By the Atiyah-Hirzebruch spectral sequence for K-theory (and the fact that $K^{-i}(BG^{(m)})$ is finitely generated for each i and each finite skeleton $BG^{(m)}$),

$$\pi_i\big(\mathrm{map}(BG, BU_{\hat{p}})_0\big) \cong \big(K^{-i}(BG)\big)_{\hat{p}};$$

i.e., the p-adic completion. Also, by Proposition 2.14,

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{C}}(BG;\hat{\mathbb{Z}}_p)_\infty\big) \cong \hat{\mathbb{Z}}_p \otimes K_G^{-i}(\mathrm{pt}).$$

And by the Atiyah-Segal theorem [AS], $K^{-i}(BG) \cong \widehat{K_G^{-i}(\mathrm{pt})}$: the completion with respect to the augmentation ideal $\mathrm{IR}(G) \subseteq \mathbb{R}(G)$.

It remains only to check that $\hat{\mathbb{Z}}_p \otimes \mathrm{R}(G) = \mathrm{R}(G)^{\wedge}_p$ is complete with respect to its augmentation ideal; or equivalently that $\mathrm{IR}(G)^m \subseteq p\,\mathrm{IR}(G)$ for some m. And this is shown in [AT, Proposition III.1.1]. (Note that the same proof as that in [AT] also applies to show the corresponding result for the real representation ring.) □

We now turn to the case $\dim(G) > 0$, and consider only maps to $BU(n)$. As will be seen, the components of $\mathfrak{K}(BG)$ are in this case very different from the components of $\mathrm{map}(BG, BU)$.

Proposition 2.16. *The even dimension homotopy of $\mathfrak{K}(BG)$ is described by the following pullback square (for any $i > 0$):*

$$\begin{array}{ccc}
\pi_{2i}\big(\mathfrak{K}(BG)\big) & \longrightarrow & \prod_{p\,\big|\,|W_G|} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(N_p(T))^{G\text{-inv}} \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{R}(T)^{W_G} & \longrightarrow & \prod_{p\,\big|\,|W_G|} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(T)^{W_G}.
\end{array} \tag{1}$$

Also, if $\dim(G) > 0$, then for all $i > 0$, $\pi_{2i-1}\big(\mathfrak{K}(BG)\big)$ is an infinite dimensional rational vector space.

Proof. The proof will be carried out in three steps. The homotopy groups of $\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty$ and of $\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty$ will be computed in Steps 1 and 2, respectively. The homotopy groups of $\mathfrak{K}(BG)$ will then be computed in Step 3, based on the homotopy pullback square

$$\begin{array}{ccc}
\mathfrak{K}(BG) & \longrightarrow & \mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty \\
\Big\downarrow & & \Big\downarrow \\
\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty & \longrightarrow & \mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{Q})_\infty
\end{array} \tag{2}$$

of Proposition 2.14.

Step 1 We first consider the homotopy groups of $\mathfrak{Vect}(BT;\mathbb{Q})_V$, and in particular their limit over $V \in \mathrm{Rep}(T)$. For any $\varphi \in \pi_{2i}(\mathfrak{Vect}(BT;\mathbb{Q})_V)$, let $f : S^{2i} \times BT \to BU(\mathrm{d}_V)$ denote the adjoint map to φ, and define

$$\widetilde{\delta}^i_V(\varphi) = C(f)/[S^{2i}] \in H^*(BT;\mathbb{Q})^{\le 2(\mathrm{d}_V - i)}.$$

Here, $C(f)$ is the total Chern class, $[S^{2i}]\in H_{2i}(S^{2i})$ is the orientation class, and

$$/ : H^n(X\times Y)\otimes H_q(X)\longrightarrow H^{n-q}(Y)$$

denotes the slant product. Since the map

$$\prod c_i : BU(n)_{\mathbb{Q}}\longrightarrow \prod_{i=1}^{n} K(\mathbb{Q},2i)$$

is a homotopy equivalence for any n, $\widetilde{\delta}^i_V$ defines an isomorphism

$$\widetilde{\delta}^i_V : \pi_{2i}\left(\mathfrak{Vect}(BT;\mathbb{Q})_V\right)\xrightarrow{\ \cong\ } H^*(BT;\mathbb{Q})^{\leq 2(\mathrm{d}_V-i)}.$$

If W is any other T-representation, then for $\varphi\in\pi_{2i}(\mathfrak{Vect}(BT;\mathbb{Q})_V)$ as above,

$$\widetilde{\delta}^i_{V\oplus W}\circ\pi_{2i}(-\oplus B\rho_W)(\varphi)=\big(C(f)\cdot C(W)\big)/[S^{2i}]=C(W)\cdot\widetilde{\delta}^i_V(\varphi).$$

So the $\widetilde{\delta}^i_V$ can be combined to give an isomorphism

$$\begin{aligned}\delta_i(T):\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}(BT;\mathbb{Q})_\infty\big)\xrightarrow{\ \cong\ }\Delta(T)&\overset{\mathrm{def}}{=}\bigcup_{V\in\mathrm{Rep}(T)}\frac{1}{C(V)}H^*(BT;\mathbb{Q})^{\leq 2(\mathrm{d}_V-i)}\\&\cong\bigcup_{V\in\mathrm{Rep}(T)}\frac{1}{C(V)}H^*(BT;\mathbb{Q});\end{aligned}$$

where $\delta_i(T)$ is the union of the maps $\frac{1}{C(V)}\cdot\delta^i_V$. The last isomorphism follows from the fact that the Chern class of any representation with trivial action is 1. There is an analogous isomorphism

$$\widehat{\delta}_i(T):\pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}(BT;\widehat{\mathbb{Q}})_\infty\big)\xrightarrow{\ \cong\ }\widehat{\mathbb{Q}}\otimes\Delta(T).$$

Now consider the map

$$\tau_V : B\,\mathrm{Aut}_T(V)\longrightarrow\mathfrak{Vect}(BT;\mathbb{Q})_V=\mathrm{map}(BT,BU(\mathrm{d}_V)_{\mathbb{Q}})_{B\rho_V}$$

which is the adjoint to the map induced by multiplication. The centralizer is a product of unitary groups, one for each distinct irreducible representation occurring as a summand of V. If an irreducible representation W occurs with multiplicity k, then the centralizer contains a corresponding factor $U(k)$, and τ_V restricts to a map

$$\tau^W_V : BU(k)\longrightarrow\mathfrak{Vect}(BT;\mathbb{Q})_V.$$

For any $1 \leq i \leq k$, we let $\theta_{i,W}(V){\in}\pi_{2i}(\mathfrak{Vect}(BT;\mathbb{Q})_V)$ denote the image of the generator $\eta^{\wedge i}{\in}\pi_{2i}(BU(k))$. The elements $\theta_{i,W}(-)$ are consistent with respect to stabilization, and hence define an element

$$\theta_{i,W} \in \pi_{2i}\big(\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty\big).$$

To calculate $\delta_i(T)(\theta_{i,W})$, for any irreducible T-representation W, consider the maps

$$S^{2i}{\times}BT \xrightarrow{\eta^{\wedge i}\times B\rho_W} BU(n){\times}BS^1 \xrightarrow{\mu_n} BU(n)$$

for any $n \geq i$, where μ_n is the product of the identity map with the inclusion of the center. Let $t \in H^2(BS^1)$ be a generator. If we identify $H^*(BU(n))$ as the ring of Σ_n-invariants in $H^*(BT^n) \cong \mathbb{Z}[x_1,\ldots,x_n]$ (and extend μ_n^* accordingly), then $\mu_n^*(x_i) = x_i + t$, and hence

$$C(\mu_n) = \mu_n^*\Big(\prod_{j=1}^n(1+x_j)\Big) = \prod_{j=1}^n[(1+t)+x_j] = \sum_{j=0}^n(1+t)^{n-j}c_j.$$

Hence

$$\begin{aligned}\widetilde{\delta}^i_{W^n}(\theta_{i,W}) &= \Big[(\eta^{\wedge i}{\times}B\rho_W)^*\Big(\sum_{j=0}^n(1+t)^{n-j}c_j\Big)\Big]/[S^{2i}] \\ &= \sum_{j=0}^n\langle c_j(\eta^{\wedge i}),[S^{2i}]\rangle{\cdot}(1+c_1(W))^{n-j} \\ &= \langle c_i(\eta^{\wedge i}),[S^{2i}]\rangle{\cdot}C(W)^{n-i}.\end{aligned}$$

It now follows that

$$\delta_i(T)(\theta_{i,W}) = \langle c_i(\eta^{\wedge i}),[S^{2i}]\rangle{\cdot}C(W)^{-i}.$$

Write $\eta = [H]-1$, where H is a line bundle over S^2, and $t \overset{\text{def}}{=} c_2(H)$ generates $H^2(S^2)$. Then $\text{ch}(\eta) = \text{ch}(H) - 1 = (1+t) - 1 = t$; and hence $\text{ch}(\eta^{\wedge i}) = t^{\otimes i} \in H^{2i}(S^{2i})$. This implies that $c_i(\eta^{\wedge i}) \neq 0$; and in fact an easy computation with Newton polynomials now shows that $\langle c_i(\eta^{\wedge i}),[S^{2i}]\rangle = (-1)^{i-1}(i-1)!$. See [Kar, Proposition 2.1], or [Hu, Corollary 18.9.8], for more details.

Now define

$$\gamma_i : \mathbb{Q}{\otimes}\,\text{R}(T) \longrightarrow \Delta(T)$$

by setting $\gamma_i([W]) = \delta_i(T)(\theta_{i,W}) = \langle c_i(\eta^{\wedge i}),[S^{2i}]\rangle{\cdot}C(W)^{-i}$ for any irreducible representation W. Since $c_i(\eta^{\wedge i}) \neq 0$, γ_i is always injective, and it is an isomorphism if and only if $T = 1$.

Step 2 Write $w = |W_G|$ for convenience. By Proposition 2.14,

$$\pi_i\big(\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty\big) \simeq \prod_{p|w} \pi_i\big(\mathfrak{Vect}^{\mathbb{C}}(BG;\hat{\mathbb{Z}}_p)_\infty\big) \times \bigg(\Big(\prod_{p\nmid w} \hat{\mathbb{Z}}_p\Big) \otimes K_G^{-i}(G/T)^{W_G}\bigg);$$

where for each prime p,

$$\begin{aligned}\pi_i\big(\mathfrak{Vect}^{\mathbb{C}}(BG;\hat{\mathbb{Z}}_p)_\infty\big) &\cong \varprojlim_{G/P\in\mathcal{R}_p(G)} \pi_i\big(\mathfrak{Vect}^{\mathbb{C}}(EG/P;\hat{\mathbb{Z}}_p)_\infty\big) \\ &\cong \varprojlim_{\mathcal{R}_p(G)} \big(\hat{\mathbb{Z}}_p \otimes K_G^{-i}(-)\big) \cong \begin{cases} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(N_p(T))^{G\text{-inv}} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.}\end{cases}\end{aligned}$$

In other words,

$$\pi_i(\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty) \cong \begin{cases} \Big[\prod_{p|w} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(N_p(T))^{G\text{-inv}}\Big] \oplus \Big[\Big(\prod_{p\nmid w} \hat{\mathbb{Z}}_p\Big) \otimes \mathrm{R}(T)^W\Big] & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.}\end{cases}$$

Step 3 Since the odd dimensional homotopy groups of the spaces

$$\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty, \quad \mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty, \quad \text{and} \quad \mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Q}})_\infty$$

all vanish, the homotopy pullback square (2) above induces for each $i > 0$ an exact sequence

$$\begin{aligned} 0 \to \pi_{2i}(\mathfrak{K}(BG)) &\longrightarrow \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty) \oplus \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Z}})_\infty) \\ &\longrightarrow \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Q}})_\infty) \longrightarrow \pi_{2i-1}(\mathfrak{K}(BG)) \to 0.\end{aligned}$$

Furthermore,

$$\pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BG;\mathbb{Q})_\infty) \cong \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BN(T);\mathbb{Q})_\infty) \cong \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}}(BT;\mathbb{Q})_\infty)^W$$

(and similarly for $\mathfrak{Vect}^{\mathbb{C}}(BG;\widehat{\mathbb{Q}})_\infty$) since the inclusion $BN(T) \hookrightarrow BG$ is a $\mathbb{Q}$-equivalence. So after substituting the groups computed in Steps 1 and 2, the localization exact sequence takes the form

$$\begin{aligned} 0 \to \pi_{2i}(\mathfrak{K}(BG)) \longrightarrow \Delta(T)^W \oplus \Big[\prod_{p|w} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(N_p(T))^{G\text{-inv}}\Big] \oplus \Big[\Big(\prod_{p\nmid w} \hat{\mathbb{Z}}_p\Big) \otimes \mathrm{R}(T)^W\Big] \\ \longrightarrow \widehat{\mathbb{Q}} \otimes \Delta(T)^W \longrightarrow \pi_{2i-1}(\mathfrak{K}(BG)) \to 0. \qquad (3)\end{aligned}$$

Here, the map to $\widehat{\mathbb{Q}} \otimes \Delta(T)^W$ is induced by the homomorphism $\gamma_i : \mathbb{Q} \otimes \mathrm{R}(T) \to \Delta(T)$ defined at the end of Step 1.

The short exact sequence $0 \to \mathbb{Z} \to \widehat{\mathbb{Z}} \oplus \mathbb{Q} \to \widehat{\mathbb{Q}} \to 0$ is still exact after tensoring with the free $\mathbb{Z}$-module $\mathrm{R}(T)^W$, and hence induces an exact sequence

$$0 \to \mathrm{R}(T)^W \xrightarrow{(\mathrm{incl},\gamma_i)} \bigl(\widehat{\mathbb{Z}} \otimes \mathrm{R}(T)^W\bigr) \oplus \Delta(T)^W \xrightarrow{\gamma_i - \mathrm{incl}} \widehat{\mathbb{Q}} \otimes_{\mathbb{Q}} \Delta(T)^W$$
$$\longrightarrow (\widehat{\mathbb{Q}}/\mathbb{Q}) \otimes_{\mathbb{Q}} \bigl(\Delta(T)/\gamma_i(\mathbb{Q} \otimes \mathrm{R}(T))\bigr)^W \to 0.$$

Upon comparing this with sequence (3), we see that

$$\pi_{2i-1}(\mathfrak{K}(BG)) \cong (\widehat{\mathbb{Q}}/\mathbb{Q}) \otimes_{\mathbb{Q}} \bigl(\Delta(T)/\gamma_i(\mathbb{Q} \otimes \mathrm{R}(T))\bigr)^W; \tag{4}$$

and that the sequence

$$0 \to \pi_{2i}(\mathfrak{K}(BG)) \longrightarrow \Bigl(\prod_{p|w} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(N_p(T))^{G\text{-inv}}\Bigr) \oplus \mathrm{R}(T)^W$$
$$\longrightarrow \prod_{p|w} \hat{\mathbb{Z}}_p \otimes \mathrm{R}(T)^W \to 0$$

is exact. In particular, $\pi_{2i}(\mathfrak{K}(BG))$ sits in the pullback square (1), and $\pi_{2i-1}(\mathfrak{K}(BG))$ is an infinite dimensional $\mathbb{Q}$-vector space if $\dim(G) > 0$. $\square$

For example, when $G = T$ is a positive dimensional torus, then for $i > 0$, $\pi_{2i}(\mathfrak{K}(BT)_0) \cong \mathrm{R}(T)$, and $\pi_{2i-1}(\mathfrak{K}(BT)_0)$ is a nonzero $\mathbb{Q}$-vector space. In contrast, $\pi_{2i}(\mathrm{map}(BG, BU)_0) \cong \mathrm{R}(T)\widehat{\ }$, a power series ring; and $\pi_{2i-1}(\mathrm{map}(BG, BU)_0) = 0$.

It follows from the Bott periodicity theorem for K-theory that the map

$$\pi_i\bigl(\mathrm{map}(BG, BU)\bigr) \cong K^{-i}(BG) \longrightarrow K^{-i-2}(BG) \cong \pi_{i+2}\bigl(\mathrm{map}(BG, BU)\bigr),$$

induced by multiplication by a generator of $K^{-2}(\mathrm{pt}) = \widetilde{K}(S^2)$, is an isomorphism for all $i > 0$. In contrast, the proof of Proposition 2.16 shows that the corresponding map

$$\pi_i(\mathfrak{K}(BG)) \to \pi_{i+2}(\mathfrak{K}(BG))$$

is *not* an isomorphism when $i > 0$ is odd. To see this, note that this "periodicity" map commutes with the maps $\delta_i(T) : \pi_{2i}(\mathfrak{Vect}^{\mathbb{C}} BT; \mathbb{Q}) \to \Delta(T)$ and $\delta_{i+1}(T)$ in Step 1 of the proof of Proposition 2.16. But the image of $\gamma_i : \mathbb{Q} \otimes \mathrm{R}(T) \to \Delta(T)$ depends on i, and so the formula (4) in the proof shows that the periodicity map is not an isomorphism.

It is not hard to construct examples to show that the monoid $\mathbb{V}\mathrm{ect}^{\mathbb{C}}(BG)$ is very far from having any general cancellation property. To finish this section, we note that in contrast, trivial summands of complex vector bundles over BG can always be cancelled. Let ϵ denote the trivial 1-dimensional representation of G. Recall that for any G-representation V, V_G denotes the associated bundle over BG.

Proposition 2.17. *Two vector bundles $\xi_1, \xi_2 \downarrow BG$ are isomorphic if $\xi_1 \oplus \epsilon \cong \xi_2 \oplus \epsilon$. For any vector bundle $\xi \downarrow BG$, ξ splits as a sum $\xi \cong \xi' \oplus \epsilon$ if for each prime $p \big| |W_G|$, $\xi | BN_p(T) \cong V_{N_p(T)}$ for some $N_p(T)$-representation V which contains a trivial summand.*

Proof. For any prime p and any G-invariant representation V of $N_p(T)$, consider the functors

$$\Pi_i^V, \Pi_i^{V\oplus\epsilon} : \mathcal{R}_p(G) \longrightarrow \hat{\mathbb{Z}}_p\text{-}\mathtt{mod};$$

where

$$\Pi_i^V(G/P) = \pi_i\big(\mathrm{map}(BP, BU(\mathrm{d}_V)\hat{_p})_{B\rho_V|P}\big) \cong \pi_i(B\,\mathrm{Aut}_P(V))\hat{_p}$$

and similarly for $\Pi_i^{V\oplus\epsilon}$. The automorphism groups $\mathrm{Aut}_P(V)$ and $\mathrm{Aut}_P(V\oplus\epsilon)$ are both products of unitary groups, and differ only in the factor corresponding to the trivial summand: a factor which for each orbit G/P is fixed by the action of $N(P)/P$. Hence

$$\Pi_1^V = \Pi_1^{V\oplus\epsilon} = 0; \tag{1}$$

and by [JMO, Lemma 5.4 & Proposition 5.5]:

$$\varprojlim{}^j\big(\mathrm{Ker}[\Pi_i^V \to \Pi_i^{V\oplus\epsilon}]\big) = 0 = \varprojlim{}^j\big(\mathrm{Coker}[\Pi_i^V \to \Pi_i^{V\oplus\epsilon}]\big)$$

for all $j > 0$. It follows that

$$\varprojlim{}^j\Pi_i^V \cong \varprojlim{}^j\Pi_i^{V\oplus\epsilon} \ \ (\text{all } j \geq 2) \qquad \text{and} \qquad \varprojlim{}^1\Pi_i^V \twoheadrightarrow \varprojlim{}^1\Pi_i^{V\oplus\epsilon}. \tag{2}$$

Assume that $f, g : BG \to BU(n)$ are such that $f \oplus B\epsilon \simeq g \oplus B\epsilon$, and fix a homotopy $\bar{F} : BG \times I \to BU(n+1)$. We will construct a homotopy $F : BG \times I \to BU(n)$ such that $F \oplus B\epsilon \simeq \bar{F}$: homotopic relative to the edges. By Theorem 1.1, for each prime p, $f|BP \simeq g|BP$ for each p-toral subgroup $P \subseteq G$. Hence, since

$$[BG, BU(n)\hat{_p}] \cong \Big[\underset{G/P \in \mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}\big(EG/P\big), BU(n)\hat{_p}\Big]$$

by Theorem 2.1 (and similarly for $[BG, BU(n+1)\hat{_p}]$), the obstructions to constructing some homotopy F' between $f\hat{_p}$ and $g\hat{_p}$ lie in $\varprojlim_{\mathcal{R}_p(G)}{}^i\, \Pi_i^V$ for $i \geq 1$ [Wo]; and the obstructions to $F' \oplus B\epsilon$ being homotopic (relative edges) to $\bar{F}$ lie in $\varprojlim_{\mathcal{R}_p(G)}{}^i\, \Pi_{i+1}^{V\oplus\epsilon}$. At each stage in constructing F, the obstruction to constructing some homotopy vanishes by (1) or (2) (and the existence of $\bar{F}$); and the homotopy can then be changed to a lifting of $\bar{F}$ since $\varprojlim{}^i\Pi_{i+1}^V$ surjects onto $\varprojlim{}^i\Pi_{i+1}^{V\oplus\epsilon}$ by (1) or (2) again.

The proof of the second statement (that a bundle over BG can be destabilized under certain assumptions) is similar. In this case, we need to know that $\varprojlim{}^j\Pi_i^V \cong \varprojlim{}^j\Pi_i^{V\oplus\epsilon}$ whenever $j = i+1$ or $j = i$, and this follows from (1) or (2) again. □

3. Acyclicity of Mackey functors

The proof of the main theorem relies on the vanishing of certain higher inverse limits (shown in Theorem 3.5). In [JM1] and [JMO, Theorem 1.7], it was noted that higher limits of functors defined on orbit categories can be interpreted as equivariant cohomology groups of certain spaces with group action. For this reason, we begin by considering results related to Smith theory and equivariant cohomology.

One source of difficulties when working with actions of positive dimensional groups is that (in contrast to the case of finite p-groups) not all subgroups of a p-toral group are p-toral. Smith theory (among other things) implies that if a compact Lie group G acts on a finite dimensional $\mathbb{F}_p$-acyclic space X with finitely many orbit types, then for any p-toral subgroup $P \subseteq G$, the fixed point set X^P is also $\mathbb{F}_p$-acyclic. The main technical result needed in this section (Proposition 3.3) is that under the additional assumption that all isotropy subgroups of the G-action are p-toral, then X^Q is $\mathbb{F}_p$-acyclic for any subgroup $Q \subseteq G$ which is *contained in* a p-toral subgroup of G. Simple examples of circle actions make it clear that this extra assumption about the isotropy subgroups is necessary.

For convenience, we define here a sub-p-toral subgroup of a group G to be a subgroup which is contained in a p-toral subgroup. If $Q \subseteq P \subseteq G$, and P is p-toral, then $Q \cap P_0$ is a normal abelian subgroup of p-power index in Q. By the "singular subgroup" Q_s of a sub-p-toral subgroup Q will be meant the (unique) minimal such subgroup. Thus, $Q_s \lhd Q$ is characterized by the properties that $[Q : Q_s]$ is a power of p, and that Q_s is the product of a torus with a finite abelian group of order prime to p. Note that Q_s is a characteristic subgroup of Q (invariant under any automorphism).

Lemma 3.1. *Fix a compact Lie group G and a prime p.*

(a) Assume that G acts smoothly on a compact manifold M. Then for any sub-p-toral subgroup $Q \subseteq G$, $\chi(M^Q) \equiv \chi(M) \pmod p$.

(b) Let $Q \subsetneqq P \subseteq G$ be such that Q is sub-p-toral and P is p-toral. Then $N_P(Q)/Q$ is p-toral and nontrivial.

(c) Let $Q_1 \lhd Q_2$ be a pair of subgroups of the compact Lie group G, such that $[Q_2 : Q_1] = p^m$ for some m. Then Q_2 is sub-p-toral if and only if Q_1 is.

Proof. **(a)** Let $Q_s \lhd Q$ be the singular subgroup. By definition, there is a torus S in G such that $Q_s \subseteq S$. Then Q/Q_s is a finite p-group and S/Q_s is a torus. So for any compact manifold M with smooth G-action,

$$\begin{aligned}\chi(M^Q) &= \chi((M^{Q_s})^{Q/Q_s}) \equiv \chi(M^{Q_s}) \\ &= \chi((M^{Q_s})^{S/Q_s}) = \chi(M^S) = \chi(M) \pmod p.\end{aligned}$$

(b) We are given $Q \subsetneqq P \subseteq G$, where P is p-toral. Let $Q_s \lhd Q$ be the singular subgroup. Then $N_P(Q) \subseteq N_P(Q_s)$, since Q_s is characteristic in Q; and $N_P(Q_s)$ is p-toral since it contains the identity component of P. It follows that

$$N_P(Q)/Q \cong N_{N_P(Q_s)/Q_s}(Q/Q_s)/(Q/Q_s);$$

and this is p-toral since the normalizer of one p-toral subgroup in another is p-toral [JMO, Lemma A.3]. Finally, by part (a),

$$\chi(N_P(Q)/Q) = \chi((P/Q)^Q) \equiv \chi(P/Q) \equiv 0 \pmod p,$$

and so $N_P(Q)/Q \neq 1$.

(c) Let $N_p(T) \subseteq G$ be a maximal p-toral subgroup. Then a subgroup $Q \subseteq G$ is sub-p-toral if and only if $(G/N_p(T))^Q$ is non-empty. So by (a), Q is sub-p-toral if and only if

$$\chi((G/N_p(T))^Q) \equiv \chi(G/N_p(T)) \not\equiv 0 \pmod p.$$

But if $Q_1 \lhd Q_2$ and Q_2/Q_1 is a p-group, then

$$\chi((G/N_p(T))^{Q_1}) \equiv \chi((G/N_p(T))^{Q_2}) \pmod p;$$

and so Q_1 is sub-p-toral if and only if Q_2 is sub-p-toral. □

It will be convenient here to write $O_p(\Gamma)$, for a finite group Γ and a prime p, to denote the intersection of the Sylow p-subgroups of Γ. As in [JMO], for any compact Lie group G and any prime p, a subgroup $P \subseteq G$ will be called *p-stubborn* if P is p-toral, $N(P)/P$ is finite, and $O_p(N(P)/P) = 1$. In other words, P is a p-stubborn subgroup of G if and only if the orbit G/P lies in the category $\mathcal{R}_p(G)$.

Lemma 3.2. *Let G be any compact Lie group, and let $Q \subseteq G$ be a sub-p-toral subgroup. Assume that $N(Q)/Q$ is finite and $O_p(N(Q)/Q) = 1$. Then Q is p-toral.*

Proof. Let $Q_s \lhd Q$ be the singular subgroup. Set $H = \big(C_G(Q_s)\big)_0$ (the identity component of the centralizer), and $Q' = Q \cap H$.

Recall that Q_s is characterized by the property that $p \nmid [Q_s{:}Q_0]$, while Q/Q_s is a p-group. Since Q is sub-p-toral, this means that Q_s is contained in some torus $S \subseteq G$, and

$$Q_s \subseteq S \subseteq \big(C_G(Q_s)\big)_0 = H. \tag{1}$$

In particular, $Q' \supseteq Q_s$, and so Q/Q' is a p-group.

Consider the group

$$K = \{g \in H \mid [g, Q] \subseteq Q'\} \lhd N(Q).$$

Here, K/Q' is finite (since $|N(Q)/Q| < \infty$); and so $K/Q' = C_{H/Q'}(Q/Q')$ is a p-group by [JMO, Proposition A.4] (the group of components of the centralizer of a p-toral subgroup is a p-group). Thus, $K/Q' \subseteq O_p(N(Q)/Q') = Q/Q'$ by assumption. It follows that, $K = Q' = Q \cap H$; and so

$$\chi(H/Q') \equiv \chi\big((H/Q')^{Q/Q'}\big) = \chi(K/Q') = 1 \pmod p.$$

In particular, $\chi(H/Q') \neq 0$, and hence $\dim(H/Q') = 0$. By (1), this implies that $Q_s = S$ is a torus, and hence that Q is p-toral. □

We are now ready to look at fixed point sets of sub-p-toral subgroups.

Proposition 3.3. *Fix a compact Lie group G and a prime p. Let X be any finite dimensional $\mathbb{F}_p$-acyclic G complex with finitely many orbit types, such that all isotropy subgroups are p-toral. Then for any sub-p-toral subgroup $Q \subseteq G$, X^Q is $\mathbb{F}_p$-acyclic.*

Proof. Let $\mathcal{S}(G)$ be the compact space of all closed subgroups of G with the Hausdorff topology (cf. [tD, Proposition IV.3.2(i)]). Let $\mathcal{T} \subseteq \mathcal{S}(G)$ be the subset of all sub-p-toral subgroups $Q \subseteq G$ such that X^Q is not $\mathbb{F}_p$-acyclic.

Step 1 We first show that $\mathcal{T}$ contains a maximal element. Set

$$k = \max\bigl\{\dim(Q) \mid Q \in \mathcal{T}\bigr\}.$$

If $\mathcal{T}$ does not contain a maximal element, then there exists an infinite chain

$$Q_1 \subsetneqq Q_2 \subsetneqq Q_3 \subsetneqq \cdots$$

of k-dimensional sub-p-toral subgroups for which X^{Q_i} is not $\mathbb{F}_p$-acyclic. Let Q be the closure of the union of the Q_i. Then $\dim(Q) > k$, and we will get a contradiction upon showing that $Q \in \mathcal{T}$. Since X has only finitely many orbit types, $X^Q = X^{Q_i}$ for i sufficiently large [tD, Proposition IV.3.4], and hence X^Q is not $\mathbb{F}_p$-acyclic. So it remains only to show that Q is sub-p-toral.

Choose p-toral subgroups $P_i \supseteq Q_i$. Since the space $\mathcal{S}(G)$ of closed subgroups of G is compact, as noted above, we can assume (after restricting to a subsequence if necessary) that the P_i converge to a closed subgroup $P \supseteq Q$. Then $\pi_0(P)$ is a p-group, since $\pi_0(P_i)$ surjects onto it for i sufficiently large. Also, by a theorem of Jordan (cf. [tD, Proposition IV.6.4]), there exists some integer j such that each finite (hence each p-toral) subgroup of P contains a normal abelian subgroup of index $< j$. In particular, P contains a normal abelian subgroup of finite index, and hence has torus identity component. Thus, P is p-toral, and so Q is sub-p-toral.

Step 2 Now let $Q \in \mathcal{T}$ be a maximal element. In other words, Q is sub-p-toral in G and X^Q is not $\mathbb{F}_p$-acyclic, but $X^{Q'}$ is $\mathbb{F}_p$-acyclic for any sub-p-toral subgroup $Q' \supsetneqq Q$. Also, Q is not p-toral, since otherwise X^Q would be $\mathbb{F}_p$-acyclic by Smith theory (cf. [Br, Chapter III]). So by Lemma 3.2, either $\dim(N(Q)/Q) > 0$, or $O_p(N(Q)/Q) = 1$.

Consider the action of $N(Q)/Q$ on X^Q. For any $x \in X^Q$, the isotropy subgroup G_x is p-toral by assumption; and $G_x \supsetneqq Q$ since Q is not p-toral. Hence $(N(Q)/Q)_x = N_{G_x}(Q)/Q$ is p-toral and nontrivial by Lemma 3.1(b). For any subgroup $L/Q \subseteq N(Q)/Q$ of order p, L is sub-p-toral by Lemma 3.1(c), and hence $(X^Q)^{L/Q} = X^L$ is $\mathbb{F}_p$-acyclic by assumption. We thus have an action of $N(Q)/Q$ on X^Q with no free orbits, with finitely many orbit types (since the action of $N(Q)$ on any G-orbit has finitely many orbit types), where all isotropy subgroups are p-toral, and such that any order p subgroup has $\mathbb{F}_p$-acyclic fixed point set. So X^Q is $\mathbb{F}_p$-acyclic: by [JMO, Lemma 2.12] if $\dim(N(Q)/Q) > 0$ or by [JMO, Lemma 2.13] if $O_p(N(Q)/Q) \neq 1$. □

In order to translate this to a result about higher limits over orbit categories, we consider, for any full subcategory $\mathcal{C}$ of the orbit category $\mathcal{O}(G)$, the G-space

$$E\mathcal{C} \stackrel{\text{def}}{=} \underset{G/H \in \mathcal{C}}{\underrightarrow{\text{hocolim}}}(G/H).$$

By [JMO, Theorem 1.7], the higher inverse limits of a functor $F : \mathcal{C} \to \mathsf{Ab}$ can be interpreted as equivariant ordinary cohomology groups $H^*_G(E\mathcal{C}; F)$. See, e.g., [JMO, appendix] for more details on equivariant ordinary cohomology.

Proposition 3.4. *Fix a compact Lie group G and a prime p, and let $N_p(T)$ be a maximal p-toral subgroup of G. Then for any contravariant functor $F : \mathcal{O}(G) \to \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$,*

$$H^i_{N_p(T)}(E\mathcal{R}_p(G); F) = \begin{cases} F(G/N_p(T)) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases} \tag{1}$$

Proof. Let X be a finite dimensional $\mathbb{F}_p$-acyclic G-complex, all of whose isotropy subgroups are p-stubborn: as constructed in [JMO, Section 2]. By [JMO, Proposition 1.1], $E\mathcal{R}_p(G)$ is characterized by the properties that all isotropy subgroups are p-stubborn, and that $E\mathcal{R}_p(G)^P$ is contractible for each p-stubborn subgroup $P \subseteq G$. So there is a G-map $f : X \to E\mathcal{R}_p(G)$ which induces an $\mathbb{F}_p$-homology equivalence on the fixed point set of any isotropy subgroup. Then by [JMO, Lemma A.10], f^H is a G-$\mathbb{F}_p$-homology equivalence for all subgroups $H \subseteq G$. For each $Q \subseteq N_p(T)$, X^Q is $\mathbb{F}_p$-acyclic by Proposition 3.3, so $E\mathcal{R}_p(G)^Q$ is also $\mathbb{F}_p$-acyclic, and hence is $\mathbb{Z}_{(p)}$-acyclic since its homology is finitely generated in each dimension (cf. [JMO, Proposition 1.1]). Formula (1) now follows from [JMO, Lemma A.13]. □

We now consider the stable orbit category $\mathcal{O}^{\rm st}(G)$: the category whose objects are the orbits of G, and where

$$\mathrm{Mor}_{\mathcal{O}^{\rm st}(G)}(G/H, G/K) = \varinjlim_{V \in \mathrm{Rep}^{\mathbb{R}}(G)} \left[S^V \wedge (G/H_+), S^V \wedge (G/K_+)\right]^G_*.$$

Here, $[-,-]^G_*$ means pointed homotopy classes of pointed G-maps. Following the notation of Lewis, May, and McClure [LMM], the term "Mackey functor" will be used here to denote an additive contravariant functor defined on $\mathcal{O}^{\rm st}(G)$. We want to show that p-local Mackey functors (i.e., Mackey functors which take values in $\mathbb{Z}_{(p)}$-modules) are acyclic over $\mathcal{R}_p(G)$. This is really a problem only when $\dim(G) > 0$: for finite G it follows from results shown in [JM2].

Theorem 3.5. *Let $F : \mathcal{O}^{\rm st}(G)^{\rm op} \longrightarrow \mathbb{Z}_{(p)}\text{-}\mathsf{mod}$ be any p-local Mackey functor. Then*

$$\varprojlim_{\mathcal{R}_p(G)}{}^i (F) = 0 \qquad \textit{for all } i > 0.$$

Proof. Set $N = N_p(T)$, for short. By [JMO, Theorem 1.7],

$$\varprojlim_{\mathcal{R}_p(G)}{}^* (F) \cong H^*_G(E\mathcal{R}_p(G); F),$$

where $H^*_G(-; F)$ denotes ordinary equivariant cohomology with coefficients in F. Also,

$$H^i_G((G/N) \times E\mathcal{R}_p(G); F) \cong H^i_G(G \times_N E\mathcal{R}_p(G); F) \cong H^i_N(E\mathcal{R}_p(G); F) = 0$$

for $i>0$ by Proposition 3.4. So the proposition follows from [LMM], which says that the homomorphism

$$H^*_G(E\mathcal{R}_p(G);F) \longrightarrow H^*_G\big((G/N)\times E\mathcal{R}_p(G);F\big)$$

(induced by projection) is a split monomorphism.

We sketch here one simple way to see this. Choose an embedding $G/N \hookrightarrow V$, for some G-representation V, and let

$$f : S^V \longrightarrow S^V\wedge(G/N_+)$$

be the map given by the Pontrjagin-Thom construction. By [O1, Lemma], $f\wedge E\mathcal{R}_p(G)_+$ is homotopic to a map

$$g : S^V\wedge E\mathcal{R}_p(G)_+ \longrightarrow S^V\wedge(G/N\times E\mathcal{R}_p(G))_+$$

which preserves the skeleta of the spaces (under any G-CW-structure on the product space $G/N\times E\mathcal{R}_p(G)$). Thus, since F is defined on the stable category, any map $S^V(G/H_+)\to S^V(G/K_+)$ induces a homomorphism $F(G/K)\to F(G/H)$, and so g induces a chain homomorphism

$$g^* : C^*_G(G/N\times E\mathcal{R}_p(G);F) \longrightarrow C^*_G(E\mathcal{R}_p(G);F).$$

This in turn induces a homomorphism in cohomology, and the composite

$$H^*_G(E\mathcal{R}_p(G);F) \xrightarrow{(pr_2)^*} H^*_G((G/N)\times E\mathcal{R}_p(G);F) \xrightarrow{g^*} H^*_G(E\mathcal{R}_p(G);F) \qquad (1)$$

is induced by $\varphi\times$ Id, where $\varphi : S^V \to S^V$ is the composite

$$S^V \xrightarrow{\ f\ } S^V\wedge(G/N_+) \longrightarrow S^V.$$

We claim that for any p-toral subgroup $P\subseteq G$, $\varphi|P$ is invertible in the localized P-equivariant stable homotopy ring $\left(\omega_0^P\right)_{(p)}$. To see this, note that φ corresponds to the class of the orbit G/N under the isomorphism of $\omega_0^P \overset{\text{def}}{=} \varinjlim_V [S^V,S^V]^P_*$ with the Burnside ring $A(P)$ (see [tD, §II.8]). Here, $A(P)$ is the ring of equivalence classes of finite P-complexes X, where $[X]=[Y]$ if $\chi(X^H)=\chi(Y^H)$ for all $H\subseteq P$. By [tD, Theorem IV.4.2], any prime ideal of $A(P)$ has the form

$$q(Q,\mathfrak{p}) = \{[X]\,|\,\chi(X^Q)\in\mathfrak{p}\}$$

for some prime ideal $\mathfrak{p}\subseteq\mathbb{Z}$ and some $Q\subseteq P$. Also, by Lemma 3.1(a), $q(Q,p\mathbb{Z}) = q(1,p\mathbb{Z})$ for all $Q\subseteq G$. So $A(P)_{(p)}$ is a local ring with maximal ideal $q(1,p\mathbb{Z})$, and (since $p\nmid\chi(G/N)$) $[G/N]$ is invertible in $A(P)_{(p)}$.

Hence, for each p-toral subgroup $P \subseteq G$,

$$\varphi \wedge \mathrm{Id} : S^V \wedge (G/P_+) \longrightarrow S^V \wedge (G/P_+) \cong (S^V \times_P G)/(* \times_P G)$$

is an isomorphism in $\mathrm{Mor}_{\mathcal{O}^{\mathrm{st}}(G)}(G/P, G/P)_{(p)}$. In particular, this applies to each orbit in $E\mathcal{R}_p(G)$, and so the composite in (1) is an isomorphism. $\square$

Note that it does not necessarily follow, under the conditions in Theorem 3.5, that $\varprojlim^0_{\mathcal{R}_p(G)} F \cong F(G/G)$. The functor $K_G(-)$ provides a simple counterexample.

As a first easy corollary of Theorem 3.5, we note:

Corollary 3.6. *Fix a prime p, and let R be one of the rings $\hat{\mathbb{Z}}_p$, $\mathbb{Z}_{(p)}$, or $\mathbb{F}_p$. Then for any compact Lie group G,*

$$\varprojlim_{G/P \in \mathcal{R}_p(G)}{}^{i} \; H^*(EG/P; R) = 0$$

for $i > 0$.

Proof. By [LMM] (or [O1]), for any G-representation V, any G-map $f : S^V(X) \to S^V(Y)$ induces a homomorphism $f^* : H^*(Y/G; R) \to H^*(X/G; R)$ in a natural way. In particular, the functor $X \mapsto H^*(EG \times_G X; R)$ is defined on the stable category, and so Theorem 3.5 applies. $\square$

The vanishing result needed directly for the results in Section 2 is the following:

Corollary 3.7. *For any prime p and any compact Lie group G,*

$$\varprojlim_{\mathcal{R}_p(G)}{}^{j} \; \hat{\mathbb{Z}}_p \otimes K_G^{-i}(-) = \varprojlim_{\mathcal{R}_p(G)}{}^{j} \; \hat{\mathbb{Z}}_p \otimes KO_G^{-i}(-) = 0$$

for all $i, j > 0$.

Proof. This follows from Theorem 3.5, once one shows that the functors $K_G^*(-)$ and $KO_G^*(-)$ are defined on the stable category. And that follows from the Bott periodicity theorem for equivariant K-theory [At, Theorems 4.3 and 6.1]. $\square$

4. Representation theory

Recall the groups

$$\mathrm{R}_{\mathcal{P}}(G) = \Big\{ (v_p) \in \prod_{p \,\big|\, |W|} \mathrm{R}(N_p(T))^{G\text{-inv}} \,\Big|\, v_p|T \cong v_{p'}|T \;\forall p, p' \Big\}$$

and

$$\mathrm{RO}_{\mathcal{P}}(G) = \Big\{ (v_p) \in \prod_{p \,\big|\, |W|} \mathrm{RO}(N_p(T))^{G\text{-inv}} \,\Big|\, v_p|T \cong v_{p'}|T \;\forall p, p' \Big\}$$

defined in Section 2, and shown to be isomorphic to the Grothendieck groups $\mathbb{K}(BG)$ and $\mathbb{KO}(BG)$, respectively, of vector bundles over BG. In this section, we study more closely the natural "restriction" maps

$$\mathrm{rs}_G^{\mathrm{U}} : \mathrm{R}(G) \longrightarrow \mathrm{R}_{\mathcal{P}}(G) \cong \mathbb{K}(BG) \quad \text{and} \quad \mathrm{rs}_G^{\mathrm{O}} : \mathrm{RO}(G) \longrightarrow \mathrm{RO}_{\mathcal{P}}(G) \cong \mathbb{KO}(BG).$$

The homomorphisms rs_G are shown to split as a direct sums of homomorphisms between finitely generated groups, one for each G/G_0-orbit of irreducible G_0-representations, and the cokernel of each summand is computed (Theorem 4.7). In particular, this yields necessary and sufficient conditions for $\mathrm{rs}_G^{\mathrm{U}}$ to be onto (Theorem 4.7 and Corollary 4.8). The orthogonal case seems to be much more complicated; but we do at least show that $\mathrm{rs}_G^{\mathrm{O}}$ is onto whenever G is finite or $\pi_0(G)$ has prime power order (Proposition 4.9), and then give some examples which show that $\mathrm{rs}_G^{\mathrm{O}}$ can fail to be onto even when $\mathrm{rs}_G^{\mathrm{U}}$ is onto.

We first show that standard induction techniques can be used to study $\mathrm{R}_{\mathcal{P}}(G)$ and $\mathrm{rs}_G^{\mathrm{U}}$. When doing this, it is useful to define the "character" of an element of $\mathrm{R}_{\mathcal{P}}(G)$. For any compact Lie group G, let $G_{\mathcal{P}}$ denote the union of the connected components in G of prime power order in $\pi_0(G)$. Then every element of $G_{\mathcal{P}}$ is conjugate to an element of $N_p(T)$ for some prime p (see Proposition 4.6(a) below). Hence, for any $v = (v_p) \in \mathrm{R}_{\mathcal{P}}(G)$, the characters χ_{v_p} extend in a unique way to define a "character" $\chi_v : G_{\mathcal{P}} \to \mathbb{C}$ which is constant on G-conjugacy classes. We can thus identify $\mathrm{R}_{\mathcal{P}}(G)$ with the group of class functions $\chi \in \mathrm{Cl}(G_{\mathcal{P}})$ whose restriction to any p-toral subgroup, for any prime p, is a character.

As usual, a finite group Γ is p-elementary if it is a product of a p-group and a cyclic group, and is elementary if it is p-elementary for some prime p.

Proposition 4.1. *Let G be any compact Lie group, with identity component G_0.*

(a) For any subgroup $H \subseteq G$ of finite index, there is an induction homomorphism

$$\mathrm{Ind}_H^G : \mathrm{R}_{\mathcal{P}}(H) \longrightarrow \mathrm{R}_{\mathcal{P}}(G),$$

with the property that for any $v \in \mathrm{R}_{\mathcal{P}}(H)$ and any $g \in G_{\mathcal{P}}$,

$$\chi_{\mathrm{Ind}(v)}(g) = \big(\mathrm{Ind}_H^G(\chi_V)\big)(g) \overset{\text{def}}{=} \sum_{\substack{aH \in G/H \\ a^{-1}ga \in H}} \chi_v(a^{-1}ga). \tag{1}$$

(b) Let $\mathcal{E}(G)$ denote the set of subgroups $E \subseteq G$ of finite index such that E/G_0 is elementary. Then restriction induces an isomorphism

$$\mathrm{Coker}(rs_G^{\mathrm{U}}) \xrightarrow{\;\cong\;} \varprojlim_{E \in \mathcal{E}(G)} \mathrm{Coker}(rs_E^{\mathrm{U}})$$

where the limits are taken with respect to inclusion and conjugation in G.

Proof. We regard Ind_H^G as a homomorphism $\mathrm{Cl}(H) \to \mathrm{Cl}(G)$ or $\mathrm{Cl}_{\mathcal{P}}(H) \to \mathrm{Cl}_{\mathcal{P}}(G)$, defined via formula (1) (and only when $[G : H] < \infty$). Let $K \subseteq G$ be any other subgroup (not necessarily of finite index), let $g_1, \ldots, g_r$ be representatives for the double cosets in $K\backslash G/H$, and set $K_i = g_iHg_i^{-1} \cap K$ for each i. Let $c(g_i^{-1}) : K_i \to H$ be the conjugation homomorphism ($x \mapsto g_i^{-1}xg_i$). Then for any $f \in \mathrm{Cl}(H)$,

$$\big(\mathrm{Ind}_H^G(f)\big)\big|K = \sum_{i=1}^{r} \mathrm{Ind}_{K_i}^K\big(f \circ c(g_i^{-1})\big). \tag{2}$$

This is the standard "double coset formula", shown for representations of finite groups in [Ser, §7.3, Proposition 22]. To prove it in this situation, write each Kg_iH as the disjoint union of cosets $b_{ij}g_iH$, where $b_{ij} \in K$ and $1 \le j \le s_i$. Then $K = \coprod_{j=1}^{s_i} b_{ij}K_i$ for each i; and so for any $g \in K$,

$$\mathrm{Ind}_H^G(f)(g) = \sum_{i=1}^{r}\Bigg[\sum_{\substack{j=1\\ b_{ij}^{-1}gb_{ij}\in g_iHg_i^{-1}}}^{s_i} (f\circ c(g_i^{-1}))(b_{ij}^{-1}gb_{ij})\Bigg] = \sum_{i=1}^{r}\mathrm{Ind}_{K_i}^K\big(f\circ c(g_i^{-1})\big)(g).$$

(a) For any given $v \in \mathrm{R}_{\mathcal{P}}(H)$, let $\chi = \chi_v \in \mathrm{Cl}(H_{\mathcal{P}})$ be its character. If $P \subseteq G$ is any p-toral subgroup (for any prime p), and if $g_1, \ldots, g_r$ are double coset representatives for $P\backslash G/H$, then $P_i = g_iPg_i^{-1} \cap H$ is p-toral for each i (it has finite index in $g_iPg_i^{-1}$), so each $\chi|(g_i^{-1}P_ig_i)$ is a character, and $\mathrm{Ind}_H^G(\chi)|P$ is a character by (2). And this shows that $\mathrm{Ind}_H^G(\chi) \in \mathrm{Cl}(G_{\mathcal{P}})$ is the character of an element of $\mathrm{R}_{\mathcal{P}}(G)$.

(b) Let $\mathcal{F}(G)$ be the class of subgroups of G of finite index. The functor $H \mapsto \mathrm{R}(H/G_0)$ satisfies the double coset formula and Frobenius reciprocity relations for induction and restriction, and hence is a Green ring over $\mathcal{F}(G)$ in the sense of Dress [Dr]. Also, the double coset formula (2) for characters says that $H \mapsto \mathrm{R}_{\mathcal{P}}(H)$ and $H \mapsto \mathrm{Coker}(\mathrm{rs}_H^{\mathrm{U}})$ are both Mackey functors over $\mathcal{F}(G)$ (again in the sense of Dress); and both are modules over $\mathrm{R}(-/G_0)$ satisfying Frobenius reciprocity. Since $\mathrm{R}(G/G_0)$ is generated by induction from the $\mathrm{R}(E/G_0)$ for $E \in \mathcal{E}(G)$ [Ser, §10.5, Theorem 19], the "fundamental theorem" of Mackey functors and Green rings says that $F(G) \cong \varprojlim_{E\in\mathcal{E}(G)}(F(E))$ for any such module over $\mathrm{R}(-/G_0)$. This is shown in [Dr, Propositions 1.1' and 1.2], and a more direct proof is given in [O2, Theorem 11.1]. □

For any torus T, $T^* = \mathrm{Hom}(T, S^1)$ will denote the group of irreducible characters of T. This will also be regarded as a lattice in $L(T)^* = \mathrm{Hom}(L(T), \mathbb{R})$, where $L(T)$ denotes the Lie algebra of T. The following definitions establish some of the notation which will be used when dealing with irreducible characters and representations of groups with torus identity component.

Definition 4.2. *If G is a compact Lie group with identity component T, then the* support *of a G-representation V is the (G/T-invariant) subset* $\mathrm{Supp}(V) \subseteq T^*$ *of all characters of irreducible summands of $V|T$. More generally, for any $v \in \mathrm{R}(G)$,* $\mathrm{Supp}(v) \in T^*$ *is the union of the supports of the irreducible G-representations which occur in the decomposition of v. For any G/T-invariant subset $\Phi \subseteq T^*$,* $\mathrm{Irr}(G, \Phi)$

denotes the set of irreducible G-representations with support in Φ, and $\mathrm{R}(G,\Phi) \subseteq \mathrm{R}(G)$ *denotes the subgroup of elements with support in Φ. For $\phi \in T^*$, we write (ϕ) for the G/T-orbit of ϕ (and write* $\mathrm{Irr}(G,\phi)$*, etc., if ϕ is G/T-invariant). Finally, if V is any G-representation, then $V\langle\Phi\rangle$ and $V\langle\phi\rangle$ denote the largest summands of V with support in Φ or ϕ, respectively.*

The descriptions of $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ in Lemma 4.5 and Theorem 4.7 below will be given in terms of a certain function $\delta(G)$, defined for compact Lie groups whose identity component is a torus and central.

Definition 4.3. *Assume that G lies in a central extension $1 \to T \to G \to \Gamma \to 1$, where T is a torus and Γ is a finite group. Then we define*

$$\delta(G) = \mathrm{lcm}\bigl\{\delta(G,\phi) \bigm| \phi{\in}T^*\bigr\};$$

where for each $\phi{\in}T^*$,

$$\delta(G,\phi) = \gcd\bigl\{\dim(V) \bigm| V \in \mathrm{Irr}(G,\phi)\bigr\}.$$

The next lemma gives a partial description of this function independantly of representations; and also lists some of its more technical properties which will be needed in later proofs.

Lemma 4.4. *Assume that G lies in a central extension $1 \to T \to G \to \Gamma \to 1$, where T is a torus and Γ is finite. Set $e = \mathrm{expt}\bigl(T\cap[G,G]\bigr)$. For each prime $p\bigm||\Gamma|$, let G_p be a maximal p-toral subgroup of G: an extension of T by a Sylow p-subgroup of Γ. Then*

- *(a)* $\delta(G) = 1$ *if and only if* $e = 1$, *if and only if* $G \cong T\times\Gamma$
- *(b)* $e\bigm|\delta(G)$ *and* $\delta(G)^2\bigm||\Gamma|$
- *(c)* $\delta(G) = \prod_{p||\Gamma|}\delta(G_p)$, *and* $\delta(G,\phi) = \prod_{p||\Gamma|}\delta(G_p,\phi)$ *for all* $\phi{\in}T^*$
- *(d)* $\delta(G,\phi') = \delta(G,\phi)$ *for all* $\phi',\phi \in T^*$ *with* $\phi' \equiv \phi \pmod{e}$
- *(e)* $\delta(G,n\phi) = \delta(G,\phi)$ *for all* $\phi{\in}T^*$, *and all* $n\in\mathbb{Z}$ *with* $(n,e) = 1$.

Proof. Note first that for any $H \subseteq G$ of finite index, and any $\phi \in T^*$,

$$\delta(H,\phi)\bigm|\delta(G,\phi)\bigm|[G:H]{\cdot}\delta(H,\phi). \tag{1}$$

The first relation holds since each G-representation with support in ϕ can be regarded as an H-representation; and the second since $\mathrm{Ind}_H^G(V)$ has support in ϕ for any H-representation V with support in ϕ.

(b) Fix any $\phi \in T^*$, and choose $a \in T\cap[G,G]$ such that $\phi(a)$ generates $\phi(T\cap[G,G])$. Then for any G-representation V with support in ϕ, a acts on V via multiplication by $\phi(a)$; and since $a \in [G,G]$, $\phi(a){\cdot}\mathrm{Id}_V$ has determinant $\phi(a)^{\dim(V)} = 1$. Thus, $|\phi(a)|\bigm|\dim(V)$ for all such V, and so

$$|\phi(a)| = |\phi(T\cap[G,G])|\bigm|\delta(G,\phi). \tag{2}$$

In particular, $e = \mathrm{expt}(T \cap [G,G])$ divides $\delta(G)$.

Now fix any $\phi \in T^*$, and let V_ϕ be the corresponding 1-dimensional irreducible T-representation. Let $V_1, \ldots, V_k$ be the irreducible G-representations with support in ϕ. Then for each i, the multiplicity of V_i in $\mathrm{Ind}_T^G(V_\phi)$ is

$$\dim_{\mathbb{C}}\bigl(\mathrm{Hom}_G(\mathrm{Ind}_T^G(V_\phi), V_i)\bigr) = \dim_{\mathbb{C}}\bigl(\mathrm{Hom}_T(V_\phi, V_i)\bigr) = \dim_{\mathbb{C}} V_i.$$

Thus, $|\Gamma| = \dim(\mathrm{Ind}_T^G(V_\phi)) = \sum_{i=1}^k \dim(V_i)^2$. And so $\delta(G,\phi)$, the greatest common divisor of the $\dim(V_i)$, is such that $\delta(G,\phi)^2 \big| |\Gamma|$.

(a) We prove here the slightly more general equivalence that

$$\delta(G,\phi) = 1 \iff \phi(T \cap [G,G]) = 1 \iff G/\mathrm{Ker}(\phi) \cong T/\mathrm{Ker}(\phi) \times \Gamma. \qquad (3)$$

The third statement clearly implies the first, and the first implies the second by (2).

By the universal coefficient theorem, $H^2(\Gamma;T) \cong \mathrm{Hom}(H_2(\Gamma),T)$; and $T \cap [G,G]$ is the image of the homomorphism $\eta_G : H_2(\Gamma) \to T$ which corresponds to $[G]$ as an element if $H^2(\Gamma;T)$. So $G \cong T \times \Gamma$ if $T \cap [G,G] = 1$, and $G/\mathrm{Ker}(\phi) \cong T/\mathrm{Ker}(\phi) \times \Gamma$ if $\phi(T \cap [G,G]) = 1$.

(c) This formula follows immediately from (1), and the fact that $\delta(G_p,\phi) \big| |G_p/T|$ is a power of p for each p.

(d) If $\phi \equiv 0 \pmod{e}$, then $\phi(T \cap [G,G]) = 1$, and so $\delta(G,\phi) = 1$ by (3). If $\phi' \equiv \phi \not\equiv 0 \pmod{e}$, then the two composites

$$H_2(\Gamma) \xrightarrow{\ \eta_G\ } T \overset{\phi}{\underset{\phi'}{\rightrightarrows}} S^1$$

are equal. Hence $(G/\mathrm{Ker}(\phi), \phi) \cong (G/\mathrm{Ker}(\phi'), \phi')$ as pairs, and $\delta(G,\phi) = \delta(G,\phi')$.

(e) For any $n \in \mathbb{Z}$ and any G-representation V with support ϕ, $\psi^n(V)$ is a virtual representation with support $n\phi$: since $\chi_{\psi^n V}(gt) = \chi_V(g^n t^n) = \chi_{\psi^n V}(g) \cdot \phi(t)^n$ for any $g \in G$ and $t \in T$. Cf. [Ad1, Lemma 3.61] for details. Also, V and $\psi^n(V)$ have the same (virtual) dimension, and hence $\delta(G, n\phi) \big| \delta(G,\phi)$. So by (d), $\delta(G,n\phi) = \delta(G,\phi)$ if n is invertible mod e. $\square$

Whenever $G_0 = T$ is a torus, $\mathrm{R}(G)$ splits as the direct sum, taken over all G/T-orbits $(\phi) \subseteq T^*$, of the subgroups $\mathrm{R}(G,(\phi))$ of finite rank. In a similar fashion, $\mathrm{rs}_G^{\mathrm{U}}$ splits as the direct sum over all $(\phi) \subseteq T^*$ of homomorphisms

$$\mathrm{rs}_{G,(\phi)} : \mathrm{R}(G,(\phi)) \longrightarrow \mathrm{R}_{\mathcal{P}}(G,(\phi)).$$

We are now ready to describe the cokernel of each of these summands for such G. The key case to consider is that when $T = G_0$ is central and ϕ is faithful.

Lemma 4.5. *Assume that G lies in a central extension $1 \to T \to G \xrightarrow{\sigma} \Gamma \to 1$, where T is a torus of dimension 0 or 1, and where Γ is finite. Fix a faithful (injective) character $\phi \in T^*$. Let S be the set of all conjugacy classes of elements $g \in \Gamma$ such* *that no two elements in $\sigma^{-1}g$ are conjugate; and let $S_{\mathcal{P}} \subseteq S$ be the set of conjugacy classes of elements of prime power order. For each $g \in S_{\mathcal{P}}$, let $\eta(g)$ be the largest divisor of $\delta(C_G(g), \phi)$ which is prime to the order of g. Then*

$$\mathrm{R}(G,\phi) \cong \mathbb{Z}^{|S|}, \quad \mathrm{R}_{\mathcal{P}}(G,\phi) \cong \mathbb{Z}^{|S_{\mathcal{P}}|}, \quad \text{and} \quad \mathrm{Coker}(\mathrm{rs}_{G,\phi}) \cong \bigoplus_{1\neq g\in S_{\mathcal{P}}} \mathbb{Z}/\eta(g).$$

Proof. Note first that a character χ of G has support in ϕ if and only if it satisfies the relation $\chi(gt) = \chi(g)\phi(t)$ for all $g \in G$ and $t \in T$. In particular, since ϕ is injective, $\chi(g) = 0$ for any g which is conjugate to gt for some $1 \neq t \in T$. Thus, $\mathrm{Cl}(G,\phi)$ is a complex vector space of dimension $|S|$; and by the Peter-Weyl theorem (and the independence of irreducible characters) $\mathrm{R}(G,\phi)$ is a free abelian group of rank $|S|$. Also, $\mathrm{R}_{\mathcal{P}}(G,\phi)$ is torsion free (it is detected by characters defined on $G_{\mathcal{P}}$), and $\mathrm{Ker}(\mathrm{rs}_{G,\phi})$ is the set of elements of $\mathrm{R}(G,\phi)$ whose characters vanish on $G_{\mathcal{P}}$. So the image of $\mathrm{rs}_{G,\phi}$ is free of rank $|S_{\mathcal{P}}|$; and once we have shown that $\mathrm{rs}_{G,\phi}$ has finite cokernel it will follow that $\mathrm{R}_{\mathcal{P}}(G,\phi)$ is a free abelian group of the same rank.

The computation of the cokernel of $\mathrm{rs}_{G,\phi}$ will be carried out in two steps.

Step 1 Assume first that Γ is p-elementary for some prime p. Then we can write $G = C_n \times P$, where C_n is cyclic of order n prime to p, and where P is p-toral. In particular, $\mathrm{R}(G) \cong \mathrm{R}(C_n) \otimes \mathrm{R}(P)$ and $\mathrm{R}(G,\phi) \cong \mathrm{R}(C_n) \otimes \mathrm{R}(P,\phi)$. Let $\mathrm{IR}(-)$ denote the augmentation ideal of $\mathrm{R}(-)$, and similarly for $\mathrm{IR}_{\mathcal{P}}(-)$. Consider the following commutative diagram with split short exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IR}(C_n) \otimes \mathrm{R}(P,\phi) & \longrightarrow & \mathrm{R}(G,\phi) & \rightleftarrows & \mathrm{R}(P,\phi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{rs}_{C_n}\otimes\mathrm{augm.}} & & \downarrow{\scriptstyle \mathrm{rs}_{G,\phi}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \mathbb{Z} & \longrightarrow & \mathrm{R}_{\mathcal{P}}(G,\phi) & \rightleftarrows & \mathrm{R}(P,\phi) & \longrightarrow & 0.
\end{array}$$

Here, $\mathrm{IR}_{\mathcal{P}}(C_n)$ is the product of the $\mathrm{IR}(\mathrm{Syl}_q(C_n))$ for $q|n$, and any $v \in \mathrm{IR}(\mathrm{Syl}_q(C_n))$ lifts to an element of $\mathrm{IR}(C_n)$ whose character vanishes on other Sylow subgroups. Hence $\mathrm{IR}(C_n)$ surjects onto $\mathrm{IR}_{\mathcal{P}}(C_n)$, and so

$$\mathrm{Coker}(\mathrm{rs}_{G,\phi}) \cong \mathrm{Coker}(\mathrm{rs}_{C_n} \otimes \mathrm{augm.}) \cong \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \mathrm{Coker}\big[\mathrm{R}(P,\phi) \xrightarrow{\text{augm.}} \mathbb{Z}\big].$$

The cokernel of this augmentation map is by definition $\mathbb{Z}/\delta(P,\phi)$, and so

$$\mathrm{Coker}(\mathrm{rs}_{G,\phi}) \cong \mathrm{IR}_{\mathcal{P}}(C_n) \otimes \big(\mathbb{Z}/\delta(P,\phi)\big). \tag{1}$$

Step 2 Now assume that G is arbitrary. Let $\mathcal{E}(G)$ be the set of subgroups of G of finite index such that E/T is elementary, and (for each prime $p\big||\Gamma|$) let $\mathcal{E}_p(G)$ be the set of those $E \in \mathcal{E}(G)$ such that E/T is p-elementary. By Proposition 4.1, $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ is the inverse limit of the groups $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$, taken over all $E \in \mathcal{E}(G)$. By (1), $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$ is a finite p-group for all $E \in \mathcal{E}_p(G)$. Hence $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ is finite; and (for each p) $\mathrm{Coker}(\mathrm{rs}_{G,\phi})_{(p)}$ is the inverse limit of the $\mathrm{Coker}(\mathrm{rs}_{E,\phi})$ for $E \in \mathcal{E}_p(G)$.

Fix a prime $p\big||\Gamma|$; we want to determine the p-power torsion in $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$. If $K' \subseteq K$ are finite cyclic subgroups of order prime to p, then the composite

$$\mathrm{IR}(K')_{(p)} \xrightarrow{\mathrm{Ind}} \mathrm{IR}(K)_{(p)} \xrightarrow{\mathrm{Res}} \mathrm{IR}(K')_{(p)} \tag{2}$$

is multiplication by $[K{:}K']$, and hence an isomorphism. Thus, if K is cyclic of order prime to p, we can split

$$\mathrm{IR}_{\mathcal{P}}(K)_{(p)} = \bigoplus_{q\big||K|} \mathrm{IR}(\mathrm{Syl}_q(K)) \cong \bigoplus_{1\neq K'\subseteq K_{\mathcal{P}}} \widetilde{\mathrm{IR}}(K')_{(p)}$$

(i.e., taking the second sum over subgroups of prime power order). Here, $\widetilde{\mathrm{IR}}(K') \subseteq \mathrm{IR}(K')$ is the kernel of the map given by restriction to the subgroup of prime index, and is free with rank equal to the number of generators of K'.

For each $n\big||\Gamma|$ prime to p, let Cyc_n be the set of all cyclic subgroups $K \subseteq \Gamma$ of order n if n is a prime power, and set $\mathrm{Cyc}_n = \emptyset$ otherwise. By Lemma 4.4(c), for any maximal p-toral subgroup $P \subseteq H$, $\delta(P,\phi)$ is the largest power of p dividing $\delta(H,\phi)$. So with the help of (1) we now get

$$\begin{aligned} \mathrm{Coker}(\mathrm{rs}_{G,\phi})_{(p)} &\cong \varprojlim_{E\in\mathcal{E}_p(G)} \mathrm{Coker}(\mathrm{rs}_{E,\phi}) \\ &\cong \bigoplus_{p\nmid n\big||\Gamma|} \Big(\varprojlim_{K\in\mathrm{Cyc}_n} \big(\widetilde{\mathrm{IR}}(K)\otimes \mathbb{Z}/\delta(\sigma^{-1}(C_\Gamma(K)),\phi)\big)_{(p)}\Big). \end{aligned} \tag{3}$$

For each $n = q^k$ (where $q \neq p$ is prime), set

$$\mathrm{Cyc}'_n = \big\{K{=}\langle g\rangle{\in}\,\mathrm{Cyc}_n \,\big|\, \text{no two elts. in } \sigma^{-1}g \text{ conjugate in } G\big\}.$$

Fix some $K \in \mathrm{Cyc}_{q^k} \smallsetminus \mathrm{Cyc}'_{q^k}$, and let $K' \subseteq K$ be the subgroup of index q. Then there exists $x \in N_G(\sigma^{-1}K)$ such that for each $g \in \sigma^{-1}(K{\smallsetminus}K')$, $xgx^{-1} = gt$ for some $1 \neq t \in T$. The character of any element $v \in \widetilde{\mathrm{IR}}(K) \cong \widetilde{\mathrm{IR}}(\sigma^{-1}K,\phi)$ vanishes on $\sigma^{-1}K'$; and hence (since $\chi_v(gt) = \chi_v(g){\cdot}\phi(t)$) v is fixed by the action of x only if $v = 0$. Thus, x acts on $\widetilde{\mathrm{IR}}(K)$ with trivial fixed point set; and in particular such terms contribute nothing to the limit in (3).

Formula (3) thus reduces to a sum, over conjugacy class representatives for all $K \in \mathrm{Cyc}'_n$, of the groups

$$H^0\big(N_G(\sigma^{-1}K);\widetilde{\mathrm{IR}}_{\mathcal{P}}(K)\big) \otimes \big(\mathbb{Z}/\delta(C_G(\sigma^{-1}K),\phi)\big)_{(p)}.$$

The first factor here is free of rank equal to the number of Γ-conjugacy classes of generators of K. The formula for $\mathrm{Coker}(\mathrm{rs}_{G,\phi})$ now follows upon taking the product over all primes $p\big||\Gamma|$. $\square$

As an example, consider the group $G = C_n \times (S^1 \times_{C_2} Q(8))$, where n is odd, $Q(8)$ is a quaternion group of order 8, and the second product is taken while identifying the central elements of order 2 in S^1 and $Q(8)$. By Lemma 4.5, if $\phi \in T^*$ is a generator, then $\mathrm{rs}_{G,k\phi}$ is onto for k even, while $\mathrm{Coker}(\mathrm{rs}_{G,k\phi}) \cong \mathbb{Z}/2 \otimes \mathrm{IR}_{\mathcal{P}}(C_n) \neq 0$ if k is odd.

We now turn to the case of groups whose identity component is not a torus. Recall that a *weight* of a compact Lie group G is an irreducible representation (or irreducible character) of its maximal torus T. The set of weights of G can thus be identified with $T^* = \mathrm{Hom}(T, S^1) \subseteq L(T)^*$. If V is any representation of G, then by the "weights of V" is meant the set of characters of irreducible components of $V|T$.

Consider the partial ordering of the weights of G, where $\phi_1 \leq \phi_2$ if ϕ_1 is contained in the convex hull of the W_G-orbit of ϕ_2 (cf. [Ad1, Definition 6.23]). One of the basic theorems of representation theory says that if G is connected, then any irreducible G-representation V has a unique W_G-orbit of highest (maximal) weights, each of which occurs with multiplicity one. Furthermore, distinct irreducible representations have distinct orbits of higher weights. For more detail, see, e.g., [Ad1, Theorem 6.33] or [BtD, Section VI.2].

Now assume that G is not connected, and let G_0 denote its identity component. If V is an irreducible G-representation, and V_0 is any irreducible component of $V|G_0$, then V is an irreducible summand of $\mathrm{Ind}_{G_0}^{G}(V_0)$. Hence each irreducible summand of $V|G_0$ is obtained from V_0 by conjugation by some element of $\pi_0(G)$. Thus, there is still a uniquely defined W_G-orbit of highest weights for V. In this case, however, the highest weights can occur with multiplicity greater than one; and there usually are several irreducible G-representations with the same orbit of highest weights.

We also recall the role played by the Weyl chambers in representation theory. Let $R \subseteq L(T)^*$ denote the set of roots of G_0: these are nonzero elements which occur in pairs $\pm\theta$. Any choice of $x_0 \in L(T)$ such that $\theta(x_0) \neq 0$ for all $\theta \in R$ determines a choice of positive roots

$$R_+ = \{\theta \in R \,|\, \theta(x_0) > 0\}.$$

And this in turn determines a Weyl chamber

$$C = \{x \in L(T) \,|\, \theta(x) \geq 0 \ \forall \theta \in R_+\} \subseteq L(T)$$

and a dual Weyl chamber

$$C^* = \{x \in L(T)^* \,|\, \langle \theta, x \rangle \geq 0 \ \forall \theta \in R_+\} \subseteq L(T)^*.$$

Here, in the definition of C^*, $\langle -, - \rangle$ denotes any W_{G_0}-invariant inner product on $L(T)^*$; and (since such an inner product is uniquely defined up to scalar on each simple component of G_0) C^* is independent of the choice of inner product. Then W_{G_0} permutes the Weyl chambers simply and transitively (cf. [Ad1, Theorem 5.13]). Also, each Weyl chamber contains exactly one element in each W_{G_0} orbit in $L(T)$ (cf. [Ad1, Corollary 5.16]), and hence each dual Weyl chamber contains exactly one element in each W_{G_0} orbit in $L(T)^*$. In particular, the irreducible representations of G_0 are in one-to-one correspondence with the weights of T in any given dual Weyl chamber C^*.

Proposition 4.6. *Fix a compact Lie group G, a maximal torus $T \subseteq G$, and a Weyl chamber $C \subseteq L(T)$. Set $N = N_G(T,C)$: the subgroup of elements in $N(T)$ which leave C invariant under conjugation. Write $C^*_T = C^* \cap T^*$. Then the following hold.*

(a) $N \cap G_0 = T$, $N{\cdot}G_0 = G$, and hence $N/T \cong G/G_0$. Also, any element of G is conjugate to an element of N.

(b) A continuous class function $f : G \to \mathbb{C}$ is a character of G if and only if $f|N$ is a character of N.

(c) Let $\mathrm{Irr}(G)$ be the set of irreducible representations of G, and let $\mathrm{Irr}(N, C^*_T)$ be the set of irreducible representations of N with support in C^*_T (i.e., whose weights all lie in C^*). Then there is a bijection

$$\beta_G : \mathrm{Irr}(G) \xrightarrow{\;\cong\;} \mathrm{Irr}(N, C^*_T) \qquad \textit{defined by} \qquad \beta_G([V]) = \big[V\langle\, \mathrm{mx}_{C^*}(V)\rangle\big],$$

where $\mathrm{mx}_{C^*}(V) \subseteq C^*_T$ denotes the set of those maximal weights of the irreducible summands of $V|G_0$ which lie in C^*. In particular, $\beta_G(V)$ is always an irreducible summand of $V|N$ having multiplicity one.

Proof. **(a)** By [Bo9, §5.3, Theorem 1(b)], any automorphism of G_0 leaves invariant some maximal torus and some Weyl chamber in G_0. Hence, any element $g{\in}G$ is contained in $N(T', C')$ for some maximal torus T' and some Weyl chamber $C' \subseteq T'$. Also, T' and T are conjugate in G_0 (cf. [Ad1, Corollary 4.23]), and the Weyl group $N_{G_0}(T)/T$ permutes the Weyl chambers for T simply and transitively [Ad1, Lemma 5.13]. Hence there is $a \in G_0$ such that $T = aT'a^{-1}$ and $C = aC'a^{-1}$; and $aga^{-1} \in N = N(T,C)$. This also shows that $N{\cdot}G_0 = G$.

Since $N_{G_0}(T)/T$ permutes the Weyl chambers of T simply and transitively, each coset of $N_{G_0}(T)/T$ in $N(T)/T$ contains a unique element which leaves the given Weyl chamber C invariant. Thus, $N = N(T,C)$ has one connected component for each component of G. And this finishes the proof of part (a).

The following statement will be needed in the proof of point (b):

$$\forall \phi \in T^* \ \exists w \in W_G \text{ such that } w(\phi) \in C^*_T \text{ and } wN_\phi w^{-1} \subseteq N. \tag{1}$$

Here, $N_\phi \subseteq N$ denotes the subgroup of elements fixing ϕ. To show (1), fix $\phi \in T^*$, and choose any $\psi \in \mathrm{interior}(C^*)^N$ (N/T acts linearly on $L(T)^*$ and leaves the dual Weyl chamber C^* invariant). Then $\phi + \mathbb{R}\psi$ is not contained in the wall of any dual Weyl chamber (since ψ is not); and so there is a dual Weyl chamber C^*_1 such that $\phi + \epsilon\psi \in \mathrm{interior}(C^*_1)$ for small $\epsilon > 0$. Let $w \in W_G$ be any element such that $w(C^*_1) = C^*$ (W_{G_0} permutes the Weyl chambers transitively). Then $w\phi \in C^*$, since $\phi \in C^*_1$. And for any $a \in N_\phi$, $a(\psi) = \psi$ and $a(\phi) = \phi$ by assumption, so a leaves C^*_1 invariant, and hence waw^{-1} leaves $C^* = w(C^*_1)$ invariant. This shows that $wN_\phi w^{-1} \subseteq N$, and finishes the proof of (1).

(c) Fix an irreducible G_0-representation V_0, and let ϕ be the maximal weight of V_0 lying in C^*. Set $\Psi = W_G{\cdot}\phi$, the W_G-orbit of ϕ, and set $\Phi = \Psi \cap C^*$. For any $\phi' \in \Phi$,

$\phi' = g(\phi)$ for some $g \in W_G$; and if $g' \in gW_{G_0} \cap (N/T)$ then ϕ' and $g'(\phi)$ are two elements of C^* in the same W_{G_0}-orbit. So $g'(\phi) = \phi'$ by [Ad2, Corollary 5.16] again. This shows that Φ is the N/T-orbit of ϕ.

Let $\mathrm{Irr}(G, (V_0))$ denote the set of irreducible G-representations with support in (V_0); i.e., the set of those irreducible G-representations V such that all irreducible summands of $V|G_0$ lie in the G/G_0-orbit of V_0. For any irreducible G-representation V, $\mathrm{Hom}_{G_0}(V_0, V) \cong \mathrm{Hom}_G(\mathrm{Ind}_{G_0}^G(V_0), V)$ by Frobenius reciprocity; and thus $\mathrm{Irr}(G, (V_0))$ is the set of distinct irreducible summands of $\mathrm{Ind}_{G_0}^G(V_0)$. Similarly, if V_ϕ denotes the (1-dimensional) irreducible representation with weight (character) ϕ, then the set $\mathrm{Irr}(N, \Phi)$ of irreducible N-representations with support in Φ coincides with the set of distinct irreducible summands of $\mathrm{Ind}_T^N(V_\phi)$.

Since $\mathrm{Ind}_T^N(V_\phi) = \mathrm{Ind}_{G_0}^G(V_0)\langle\Phi\rangle$, any G-linear endomorphism of $\mathrm{Ind}_{G_0}^G(V_0)$ restricts to an N-linear endomorphism of $\mathrm{Ind}_T^N(V_\phi)$. We thus get a commutative diagram of restriction maps

$$\begin{array}{ccc} \mathrm{End}_G\big(\mathrm{Ind}_{G_0}^G(V_0)\big) & \xrightarrow{\text{restr.}} & \mathrm{End}_N\big(\mathrm{Ind}_T^N(V_\phi)\big) \\ \cong\Big\downarrow & & \cong\Big\downarrow \\ \mathrm{Hom}_{G_0}\big(V_0, \mathrm{Ind}_{G_0}^G(V_0)\big) & \xrightarrow[\cong]{\text{restr.}} & \mathrm{Hom}_T\big(V_\phi, \mathrm{Ind}_T^N(V_\phi)\big) \end{array} \tag{2}$$

where the vertical maps are isomorphisms by Frobenius reciprocity. The one-to-one correspondence between irreducible G_0-representations and highest weights contained in C^* shows that $\mathrm{Hom}_{G_0}(V_0, V') \cong \mathrm{Hom}_T(V_\phi, V'\langle\Phi\rangle)$ whenever V' is a G_0-representation with support in the G/G_0-orbit $(V_0) \subseteq \mathrm{Irr}(G_0)$, and in particular that the bottom map in (2) is an isomorphism.

Thus, the map between the endomorphism rings in (2) is an isomorphism. In particular, for any $[V] \in \mathrm{Irr}(G, (V_0))$, $\mathrm{End}_N(V\langle\Phi\rangle) \cong \mathrm{End}_G(V) \cong \mathbb{C}$, and so $V\langle\Phi\rangle$ is irreducible. If $[V] \neq [V'] \in \mathrm{Irr}(G, (V_0))$, the same argument shows that $V\langle\Phi\rangle \ncong V'\langle\Phi\rangle$. And finally, any irreducible N-representation with support in Φ is a summand of $\mathrm{Ind}_T^N(V_\phi) \cong \mathrm{Ind}_{G_0}^G(V_0)\langle\Phi\rangle$, and hence has the form $V\langle\Phi\rangle$ for some $V \in \mathrm{Irr}(G_0, (V_0))$.

This shows that $\beta_\Phi : \mathrm{Irr}(G, \Phi) \xrightarrow{\cong} \mathrm{Irr}(N, \Phi)$ is a well defined bijection. Since the restriction to G_0 of any irreducible G-representation is a sum of representations in just one G/G_0-orbit of irreducible G_0-representations, $\beta_G : \mathrm{Irr}(G) \to \mathrm{Irr}(N, C_T^*)$ is the disjoint union of the β_Φ taken over all N/T-orbits $\Phi \subseteq C_T^*$ and hence a bijection. This proves point (c). At the same time, this shows that the homomorphism

$$\bar{\beta}_G : \mathrm{R}(G) \to \mathrm{R}(N, C_T^*),$$

defined by sending $[V]$ to $[V\langle C_T^*\rangle]$, is an isomorphism (its matrix with respect to the basis of irreducible representations is triangular with 1's along the diagonal).

(b) Fix a continuous class function $f : G \to \mathbb{C}$ such that $f|N$ is a character of N. We must show that f is a character of G. Let $v_0 \in \mathrm{R}(N)$ be such that $\chi_{v_0} = f|N$, let χ be the character of $\bar{\beta}_G^{-1}(v_0\langle C_T^*\rangle) \in \mathrm{R}(G)$, and set $f' = f - \chi$. By construction, $f'|N$ is the character of an element $v \in \mathrm{R}(N)^{G\text{-inv}}$ such that $v\langle C_T^*\rangle = 0$. We will show that $v = 0$. It then follows that $f' = 0$ (since every element of G is conjugate to an element of N), and hence that $f = \chi$ is a character of G.

Choose N/T-orbit representatives $\phi_1, \ldots, \phi_k \in T^*$ for the support of v, and write $N_i = N_{\phi_i}$ (the subgroup of elements which fix ϕ_i). Then $v = \sum_{i=1}^k \mathrm{Ind}_{N_i}^N(v\langle\phi_i\rangle)$. For any $\phi \in T^*$, we apply (1) to choose $w \in W_G$ such that $w(\phi) \in C^*$ and $wN_\phi w^{-1} \subseteq N$. Then $v\langle w\phi\rangle = 0 \in \mathrm{R}(wN_\phi w^{-1})$, since $v\langle C_T^*\rangle = 0$; and so $v\langle\phi\rangle = 0 \in \mathrm{R}(N_\phi)$ since v is G-invariant. In particular, $v\langle\phi_i\rangle = 0$ for all i, and hence $v = 0$. □

We are now ready to describe $\mathrm{Coker}\big[\mathrm{rs}_G^{\mathrm{U}} : \mathrm{R}(G) \to \mathrm{R}_{\mathcal{P}}(G)\big]$ for arbitrary G.

Theorem 4.7. *Let G be any compact Lie group. Fix a maximal torus $T \subseteq G$ and a Weyl chamber $C \subseteq L(T)$, and set $N = N(T,C) \subseteq G$ and $C_T^* = T^* \cap C^*$.*

(a) Let $\mathcal{E}'(N)$ denote the set of subgroups $E \subseteq N$ of finite index such that E/T is elementary but not of prime power order. Then

$$\mathrm{expt}\big(\mathrm{Coker}(rs_G^{\mathrm{U}})\big) = \mathrm{lcm}\big\{\delta(E/[E,T]) \,\big|\, E \in \mathcal{E}'(N)\big\}. \tag{1}$$

In particular, $\mathrm{rs}_G^{\mathrm{U}}$ is surjective if and only if $\mathrm{rs}_N^{\mathrm{U}}$ is surjective, if and only if $T \cap [E,E] = [E,T]$ for all $E \in \mathcal{E}'(N)$, if and only if $E/[E,T] \cong E/T \times T/[E,T]$ for all $E \in \mathcal{E}'(N)$.

(b) $\mathrm{rs}_G^{\mathrm{U}}$ splits as a direct sum of homomorphisms

$$rs_{G,(V_0)} : \mathrm{R}(G,(V_0)) \longrightarrow \mathrm{R}_{\mathcal{P}}(G,(V_0)),$$

taken over all G/G_0-orbits $(V_0) \subseteq \mathrm{Irr}(G_0)$.

(c) Fix any $V_0 \in \mathrm{Irr}(G_0)$, and set ϕ be the maximal weight of V_0 in the dual Weyl chamber C^*. Let $N_\phi \subseteq N$ be the subgroup of elements which fix ϕ, and set $K_\phi = \mathrm{Ker}(\phi) \subseteq T$. Then the assignment $\big([V] \mapsto [V\langle\phi\rangle]\big)$ induces isomorphisms $\mathrm{R}(G,(V_0)) \cong \mathrm{R}(N_\phi/K_\phi,\phi)$, $\mathrm{R}_{\mathcal{P}}(G,(V_0)) \cong \mathrm{R}_{\mathcal{P}}(N_\phi/K_\phi,\phi)$, and

$$\mathrm{Coker}(rs_{G,(V_0)}) \xrightarrow{\cong} \mathrm{Coker}(rs_{N_\phi/K_\phi,\phi});$$

where $\mathrm{Coker}(rs_{N_\phi/K_\phi,\phi})$ is described by Lemma 4.5.

Proof. **(b)** If G/G_0 is a p-group for some prime p, then N is p-toral, and every element of G is conjugate to an element of N (Proposition 4.6(a)). So any G-invariant character of N extends to a unique class function of G, which is a character by Proposition 4.6(b). In particular, $\mathrm{rs}_G^{\mathrm{U}}$ is an isomorphism in this case.

If G is arbitrary, and if G_p (for each prime $p\big||G/G_0|$) denotes an extension of G_0 by a Sylow p-subgroup of G/G_0, it now follows that

$$\mathrm{R}_{\mathcal{P}}(G) \cong \Big\{(v_p) \in \prod_{p\big||G/G_0|} \mathrm{R}(G_p)^{G\text{-inv}} \,\Big|\, v_p|G_0 {=} v_{p'}|G_0 \;\forall p,p'\Big\}.$$

In other words, $\mathrm{R}_{\mathcal{P}}(G)$ is the inverse limit of the representation rings $\mathrm{R}(H)$, taken over all $H \subseteq G$ of finite index such that H/G_0 has prime power order. Since each $\mathrm{R}(G_p)^{G\text{-inv}}$ splits as a sum of finitely generated groups $\mathrm{R}(G_p,(V_0))^{G\text{-inv}}$, indexed by the G/G_0-orbits $(V_0) \in \mathrm{Irr}(G_0)$, we now see that $\mathrm{R}_{\mathcal{P}}(G)$ also splits as such a sum. And hence $\mathrm{rs}_G^{\mathrm{U}}$ also splits as a direct sum of homomorphisms $\mathrm{rs}_{G,(V_0)}$.

(c) Write $\Phi = (\phi)$ for short: the N/T-orbit of $\phi \in C_T^*$. By Proposition 4.6(c), the assignment $[V] \mapsto [V\langle\Phi\rangle]$ defines a bijection from $\mathrm{Irr}(G,(V_0))$ to $\mathrm{Irr}(N,\Phi)$, and hence an isomorphism $\mathrm{R}(G,(V_0)) \xrightarrow{\cong} \mathrm{R}(N,\Phi)$. Similarly, it induces isomorphisms $\mathrm{R}(H,(V_0)) \xrightarrow{\cong} \mathrm{R}(H \cap N,\Phi)$ for each $H \subseteq G$ of finite index, and upon taking the inverse limit over all such H for which H/G_0 has prime power order we get an isomorphism $\mathrm{R}_{\mathcal{P}}(G,(V_0)) \xrightarrow{\cong} \mathrm{R}_{\mathcal{P}}(N,\Phi)$. And this in turn induces an isomorphism between the cokernels of $\mathrm{rs}_{G,(V_0)}$ and $\mathrm{rs}_{N,\Phi}$.

The homomorphism $\mathrm{R}(N,\Phi) \to \mathrm{R}(N_\phi,\phi) \cong \mathrm{R}(N_\phi/K_\phi,\phi)$, defined by sending $[V]$ to $[V\langle\phi\rangle]$, is an isomorphism: its inverse is the induction map $[V] \mapsto [\mathrm{Ind}_{N_\phi}^N(V)]$. This same assignment also defines an isomorphism $\mathrm{R}_{\mathcal{P}}(N,\Phi) \xrightarrow{\cong} \mathrm{R}_{\mathcal{P}}(N_\phi/K_\phi,\phi)$ (whose inverse is again the induction map); and hence defines an isomorphism between the cokernels of $\mathrm{rs}_{N,\Phi}$ and $\mathrm{rs}_{N_\phi/K_\phi,\phi}$.

(a) It is clear from part (c) that the exponent of $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ divides the number given in (1). To show that these are equal, fix any prime p, and choose $E \subseteq N$ of finite index such that E/T is p-elementary but not a p-group. We must show that $\delta(E/[E,T]) \big| \operatorname{expt}\big(\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})\big)$. Choose any $\phi' \in (T/[E,T])^* \subseteq T^*$ such that $\delta(E/[E,T],\phi') = \delta(E/[E,T])$. Since N/T acts linearly on $L(T)^*$ and leaves C^* invariant, the fixed set $(C^*)^E$ is a cone shaped subspace of $(L(T)^*)^E$ with nonempty interior. Hence, we can choose $\phi \in C^* \cap (T/[E,T])^* = (C_T^*)^E$ such that $\phi \equiv \phi'$ modulo the exponent of $\frac{T \cap [E,E]}{[E,T]}$. If $q \neq p$ is any other prime dividing $|E/T|$, then

$$\delta(E/[E,T],q\phi) = \delta(E/[E,T],\phi) = \delta(E/[E,T],\phi') = \delta(E/[E,T])$$

by Lemma 4.4(d,e). And finally, if $gT \in E/T$ is the element of order q, then $gT \in S$ in the notation of Lemma 4.5: no two elements in $gT/\mathrm{Ker}(q\phi)$ are conjugate. Thus,

$$\delta(E/[E,T]) = \delta(E/[E,T],q\phi) \big| \operatorname{expt}\big(\mathrm{Coker}(\mathrm{rs}_{E,q\phi})\big) \big| \operatorname{expt}\big(\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})\big)$$

by Lemma 4.5; and this finishes the proof of formula (1). The necessary and sufficient conditions for $\mathrm{rs}_G^{\mathrm{U}}$ to be surjective now follow from Lemma 4.4(a). □

Since the general condition for $\mathrm{rs}_G^{\mathrm{U}}$ to be surjective is rather complicated, we now list some special cases which are simpler to formulate.

Corollary 4.8. *For any compact Lie group G, $\mathrm{Coker}(rs_G^{\mathrm{U}})$ has finite exponent, and*

$$\operatorname{expt}\big(\mathrm{Coker}(rs_G^{\mathrm{U}})\big)^2 \big| |\pi_0(G)|. \tag{1}$$

Furthermore, rs_G^{U} is surjective if G satisfies any of the following conditions:

(a) G is finite or connected.

(b) All elements of $\pi_0(G)$ have prime power order.

(c) $\pi_0(G)$ is a periodic group: all of its Sylow subgroups are cyclic or quaternion.

(d) $Z(G_0) = 1$.

(e) G is a semidirect product of the form $G = G_0 \rtimes \Gamma$, where $\Gamma \subseteq G$ normalizes some maximal torus T and leaves invariant some Weyl chamber in T.

Proof. Fix a maximal torus $T \subseteq G_0$, and a Weyl chamber C. Set $N = N(T, C)$. As in Theorem 4.7, let $\mathcal{E}'(N)$ be the set of subgroups $H \subseteq N$ of finite index such that H/T is elementary but not of prime power order.

By Lemma 4.4(b), $\delta(H/[T,H])^2 \big| |H/T| \big| |\pi_0(G)|$ for each $H \subseteq N$ of finite index. So (1) follows from Theorem 4.7(a).

(a) $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Lemma 4.5 if G is finite, and by (1) if G is connected.

(b) If all elements of $\pi_0(G) = \pi_0(N)$ have prime power order, then $\mathcal{E}'(N) = \emptyset$, and so $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Theorem 4.7(a).

(c) Note that $H_2(\Gamma) = 0$ for any finite periodic group Γ. Hence, if $\pi_0(G)$ is periodic, then for any $H \in \mathcal{E}'(N)$, $H/[H,T] \cong T/[H,T] \times H/T$. So $\mathrm{rs}_N^{\mathrm{U}}$ and $\mathrm{rs}_G^{\mathrm{U}}$ are onto by Theorem 4.7(a).

(e) The conditions on Γ imply that N is a semidirect product of T with Γ, and hence that $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Theorem 4.7(a).

(d) By [Bo9, §4.10, Corollaire], the surjection $\mathrm{Aut}(G_0) \twoheadrightarrow \mathrm{Out}(G_0)$ is split by outer automorphisms which fix T and C. Let $\Gamma \subseteq G$ be the subgroup of elements whose conjugation action lies in the image of any given splitting map. Then $G = G_0 \rtimes \Gamma$ (since $G_0 \cap \Gamma = Z(G_0) = 1$); and so $\mathrm{rs}_G^{\mathrm{U}}$ is onto by (e). □

We remark here that G being a semidirect product $G_0 \rtimes \Gamma$ does not in itself imply that $\mathrm{rs}_G^{\mathrm{U}}$ is onto. As an example, set

$$G = C_3 \times \big(\mathrm{SU}(2) \times_{C_2} Q(8)\big),$$

where C_3 is cyclic of order 3, $Q(8)$ is a quaternion group of order 8, and the product is taken by identifying the central subgroups of order 2 in $\mathrm{SU}(2)$ and $Q(8)$. Then Theorem 4.7(a) applies to show that $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{U}})$ has exponent 2. But $\mathrm{SU}(2) \times_{C_2} Q(8)$ is also a semidirect product of $\mathrm{SU}(2)$ with $C_2 \times C_2$: the splitting comes from the diagonal subgroup

$$(C_2)^3 \subseteq Q(8) \times_{C_2} Q(8) \subseteq \mathrm{SU}(2) \times_{C_2} Q(8).$$

So far, we have dealt only with the case of unitary representations. The corresponding problem for orthogonal representations seems to be much more complicated, and we deal here only with some simple cases.

As usual, we say that a G-representation V (over $\mathbb{C}$) has *real type* if it has the form $V = \mathbb{C} \otimes_{\mathbb{R}} V'$ for some $\mathbb{R}G$-representation V'; and that V has *quaternion type* if it is the restriction of an $\mathbb{H}G$-representation. If V is irreducible and its character is real-valued, then V has real or quaternion type, but not both [Ad1, Proposition 3.56]. By a real character will be meant the character of a virtual representation of real type (i.e., the difference of two representations of real type).

For any G, any maximal torus $T \subseteq G$, and any Weyl chamber $C \subseteq L(T)$, we let $N(T, \pm C)$ denote the subgroup of those elements in $N(T)$ whose conjugation action sends C to $\pm C$. The $N(T, \pm C)$ play a role in detecting real characters similar to the role of $N(T, C)$ in detecting (complex) characters.

Proposition 4.9. *(a) For any compact Lie group G, a class function $f : G \to \mathbb{C}$ is a real character if and only if $f\big|N(T, \pm C)$ is a real character.*

(b) If G is finite, or if $\pi_0(G)$ has p-power order for some prime p, then $\mathrm{rs}_G^{\mathrm{O}}$ is surjective.

Proof. **(a)** Fix a class function $f : G \to \mathbb{C}$ such that $f|N(T, \pm C)$ is a real character. Then f is a character by Proposition 4.6(b), and $f(G) \subseteq \mathbb{R}$ since any element of G is conjugate to an element of $N(T, C) \subseteq N(T, \pm C)$ (Proposition 4.6(a)). So in the decomposition of f as a combination of irreducible characters, all irreducible characters which are not real-valued occur in conjugate pairs. Hence f is a real character if (and only if) the multiplicity in f of each irreducible character of quaternion type is even.

We are thus reduced to considering the case where $f = \chi_V$, $V = \sum_{i=1}^{k} V_i$, and the V_i are distinct irreducible G-representations of quaternion type. Choose a W_G-orbit Ψ of maximal weights in one of the V_i — say V_1 — which does not occur in any of the others except possibly as maximal weights.

Write $N = N(T, C)$ and $N_{\pm} = N(T, \pm C)$, for short. Set $\Phi = \Psi \cap C^*$ and $\Phi_{\pm} = \Psi \cap (\pm C^*)$. By Proposition 4.6(c) (and the original assumption on Ψ), $V_1\langle\Phi\rangle$ is irreducible as an N-representation, and does not occur as a summand of $V_i|N$ for any $i \neq 1$. So the $N_{\pm}$-representation $V_1' \overset{\text{def}}{=} V_1\langle\Phi_{\pm}\rangle$ is irreducible — since

$$V_1'|N \cong V_1\langle\Phi\rangle \oplus V_1\langle\Phi_{\pm} \smallsetminus \Phi\rangle$$

— and V_1' does not occur as a summand of $V_i|N_{\pm}$ for any $i \neq 1$. Also, since V_1 is self-conjugate, the elements of Ψ, and hence of $\Phi_{\pm}$, occur in pairs $\pm\phi$. This shows that $V_1' = V_1\langle\Phi_{\pm}\rangle$ is invariant under the conjugate linear automorphism $j : V_1 \to V_1$, and hence that it also has quaternion type. Thus, $V|N_{\pm}$ contains with multiplicity one the irreducible summand V_1' of quaternion type, and this contradicts the assumption that $V|N_{\pm}$ is a representation of real type.

(b) The proof of this point will be split into three cases. In all of them, $\mathrm{rs}_G^{\mathrm{U}}$ is onto by Corollary 4.8(a,b). Hence, if we regard $\mathrm{RO}_{\mathcal{P}}(G)$ as a subgroup of $\mathrm{R}_{\mathcal{P}}(G)$, then any element of $\mathrm{RO}_{\mathcal{P}}(G)$ is represented by an element $x \in \mathrm{R}(G)$ whose character is real valued, and whose restriction to any p-toral subgroup of G (for any prime p) has real type. We must show that x itself can be chosen to have real type.

Case 1 If $\pi_0(G)$ is a 2-group, then $N(T,\pm C)$ is 2-toral. Hence by part (a), any G-invariant real character on $N_2(T)$ extends to a unique real character on G. So $\mathrm{rs}_G^{\mathrm{O}}$ is onto.

Case 2 Assume that p is an odd prime, and that $\pi_0(G)$ is a p-group. By Case 1, $\mathrm{rs}_{G_0}^{\mathrm{O}}$ is onto. So it will suffice to show that a real-valued character χ of G is a real character if its restriction to G_0 is a real character. By induction on $|\pi_0(G)|$, we can assume that $\chi|G'$ is a real character for some $G' \lhd G$ of index p.

After replacing χ by its sum with some real character of G, we can assume that $\chi = \chi_V$ for some (complex) G-representation $V = V_1 \oplus \cdots \oplus V_k$, where the V_i are distinct irreducible G-representations of quaternion type. We must show that $V = 0$. If not (if $k > 0$), then by assumption, the G-representation

$$\mathrm{Ind}_{G'}^{G}(V|G') \cong \mathbb{C}[G/G'] \otimes_{\mathbb{C}} V \cong \bigoplus_{\phi \in (G/G')^*} V_\phi \otimes_{\mathbb{C}} V$$

has real type. For each i, either $\mathbb{C}[G/G'] \otimes_{\mathbb{C}} V_i \cong (V_i)^p$ (if the character χ_{V_i} vanishes on $G \smallsetminus G'$); or $\mathbb{C}[G/G'] \otimes_{\mathbb{C}} V_i$ is a sum of p distinct irreducible representations of which V_i is the only one with real-valued character. Thus, each V_i occurs with odd multiplicity in $\mathrm{Ind}_{G'}^{G}(V|G')$, which contradicts the assumption that this representation has real type.

Case 3 Assume now that G is finite. Recall that G is called 2-$\mathbb{R}$-elementary if it contains a normal cyclic subgroup C_m of 2-power index such that any element of G either centralizes C_m, or acts on it via $(a \mapsto a^{-1})$. By [Ser, §12.6, Theorem 27], for any finite G, the real representation ring $\mathrm{RO}(G)$ is generated by induction from elementary and 2-$\mathbb{R}$-elementary subgroups of G. So by standard induction theory (as in the proof of Proposition 4.1), $\mathrm{Coker}(\mathrm{rs}_G^{\mathrm{O}})$ is detected by restriction to such subgroups. It thus suffices to prove that $\mathrm{rs}_G^{\mathrm{O}}$ is surjective when G is elementary or 2-$\mathbb{R}$-elementary.

Fix an element $\big(v_p\big)_{p \mid |G|} \in \mathrm{RO}_{\mathcal{P}}(G)$. In other words, $v_p \in \mathrm{RO}(\mathrm{Syl}_p(G))^{G\text{-inv}}$ for each p, and by subtracting a constant character we can assume that $\chi_{v_p}(1) = 0$ for each p. We must show that each v_p extends to an element $v_p' \in \mathrm{RO}(G)$ whose character vanishes on all Sylow q-subgroups for primes $q \neq p$. This is clear if $\mathrm{Syl}_p(G)$ has a normal complement, since in that case v_p' can be taken to be the composite of v_p with the surjection $G \twoheadrightarrow \mathrm{Syl}_p(G)$.

The only case left to consider is that where G is 2-$\mathbb{R}$-elementary and p is odd. In this situation, if $\mathrm{Syl}_p(G)$ is cyclic of order p^k, then there is a surjection $G \twoheadrightarrow D(2p^k)$, where $D(2p^k)$ is dihedral of order $2p^k$. One easily checks that any $v_p \in \mathrm{RO}(C_{p^k})$ such that $\chi_{v_p}(1) = 0$ extends to an element $v_p'' \in \mathrm{RO}(D(2p^k))$ such that $\chi_{v_p''}(g) = 0$ for all g of order 1 or 2. And hence if $v_p' \in \mathrm{RO}(G)$ is the composite of v_p'' with the surjection $G \twoheadrightarrow D(2p^k)$, then $v_p'|\,\mathrm{Syl}_p(G) = v_p$ and $\chi_{v_p'}$ vanishes on all elements of order prime to p. □

With a little more work, one can also show that if G is a central extension of a torus by a finite group, then $\mathrm{rs}_G^{\mathrm{O}}$ is onto if and only if $\mathrm{rs}_G^{\mathrm{U}}$ is onto. In contrast, the following example provides a simple way of constructing groups G for which $\mathrm{rs}_G^{\mathrm{O}}$ is not onto but $\mathrm{rs}_G^{\mathrm{U}}$ is onto.

Example 4.10. *Fix any pair (G',V'), where G' is a compact connected Lie group, and V' an irreducible G'-representation of real type having the additional property that some central element $z\in Z(G')$ of order 2 acts on V' by $(-\mathrm{Id})$. Choose any odd prime power $n>1$, and set $G=G'\times_{C_2}Q(4n)$: the central product of G' with the quaternion group of order $4n$, where z is identified with the central element of $Q(4n)$. Then $\mathrm{rs}_G^{\mathrm{O}}$ is not onto.*

Proof. Let W be any effective irreducible representation of $Q(4n)$, and set $V = V'\otimes_{\mathbb{C}}W$. Then V is an irreducible G-representation of quaternion type, but its restriction to any p-toral subgroup of G (for any prime p) has real type. In particular, $[V]$ represents an element of $\mathrm{RO}_{\mathcal{P}}(G)$; but since $\mathrm{rs}_G^{\mathrm{O}}$ and $\mathrm{rs}_G^{\mathrm{U}}$ are injective (all elements of $\pi_0(G)\cong D(2n)$ have prime power order), it does not lie in the image of $\mathrm{rs}_G^{\mathrm{O}}$. □

For example, we can take $G'=\mathrm{SO}(2m)$ for any $m\geq 2$, and let V' be the standard representation on $\mathbb{C}^{2m}$. Then for any odd prime power $n\geq 3$, $G=G'\times_{C_2}Q(4n)$ has the property that $\mathrm{rs}_G^{\mathrm{U}}$ sends $\mathrm{R}(G)$ isomorphically onto $\mathrm{R}_{\mathcal{P}}(G)$ (Corollary 4.8(b)), but $\mathrm{rs}_G^{\mathrm{O}}:\mathrm{RO}(G)\to\mathrm{RO}_{\mathcal{P}}(G)$ fails to be onto.

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