Homotopy classification of gerbes

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Abstract

Gerbes are locally connected presheaves of groupoids on a small Grothendieck site $\mathcal{C}$. Gerbes are classified up to local weak equivalence by path components of a cocycle category taking values in the diagram $\mathbf{Grp}(\mathcal{C})$ of 2-groupoids consisting of all sheaves of groups, their isomorphisms and homotopies. If $\mathcal{F}$ is a full subpresheaf of $\mathbf{Grp}(\mathcal{C})$ then the set $[*, B\mathcal{F}]$ of morphisms in the homotopy category of simplicial presheaves classifies gerbes locally equivalent to objects of $\mathcal{F}$ up to weak equivalence. If $\mathrm{St}(\pi\mathcal{F})$ is the stack completion of the fundamental groupoid $\pi\mathcal{F}$ of $\mathcal{F}$, if L is a global section of $\mathrm{St}(\pi\mathcal{F})$, and if F_L is the homotopy fibre over L of the canonical map $B\mathcal{F} \to B\,\mathrm{St}(\pi\mathcal{F})$, then $[*, F_L]$ is in bijective correspondence with Giraud's non-abelian cohomology object $H^2(\mathcal{C}, L)$ of equivalence classes of gerbes with band L.

Introduction

Suppose that $\mathcal{M}$ is a closed model category, and that X and Y are objects of $\mathcal{M}$. A cocycle from X to Y is a picture

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

of morphisms in $\mathcal{M}$ such that f is a weak equivalence. A morphism of cocycles $(f, g) \to (f', g')$ is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

and these cocycles and their morphisms together form the category $H(X, Y)$ of cocycles from X to Y. The assignment $(f, g) \mapsto gf^{-1}$ defines a function

$$\phi : \pi_0 H(X, Y) \to [X, Y]$$

from the path components of cocycle category $H(X, Y)$ to the set of morphisms $[X, Y]$ from X to Y in the homotopy category $\mathrm{Ho}(\mathcal{M})$. Then it is a basic result of [9] that this function ϕ is a bijection if the model category $\mathcal{M}$ is right proper and if its class of weak equivalences is closed under finite products.

The right properness condition is a serious restriction, but right proper model structures are fairly common in nature, and include the standard model structures for spaces, simplicial sets, and spectra, as well as more exotic structures such as simplicial presheaves, simplicial sheaves and presheaves of spectra on small Grothendieck sites.

The cocycle approach to constructing morphisms in the homotopy category is proving to be very useful, particularly in connection with simplicial sheaves and presheaves. Applications have so far appeared in new, short and conceptual arguments for the homotopy classification of sheaf cohomology theories, both abelian and non-abelian [9]. Cocycle categories are involved in the explicit construction of the stack completion functor which is given in [8]. They have been used to show [7], in a variety of settings, that morphisms $[*, BI]$ in the homotopy category can be identified with path components of a suitably defined category of I-torsors for all small category objects I.

The present paper uses cocycles in presheaves of 2-groupoids, here called 2-cocycles, to give a homotopy classification of gerbes.

A gerbe is typically defined in the literature [2, p.129] to be a stack G which is locally path connected. Stacks themselves have no conceptual mystery: they are fibrant objects in model structures for sheaves of groupoids [11] or more generally presheaves of groupoids [4], and one now can identify a stack with the homotopy type that it represents in presheaves of groupoids. The model structure for presheaves (or sheaves) of groupoids, over any small Grothendieck site $\mathcal{C}$, is easy to describe: a map $f : G \to H$ of presheaves of groupoids is a local weak equivalence (respectively global fibration) if the induced map $BG \to BH$ of classifying objects is a local weak equivalence (respectively global fibration) of simplicial presheaves. Local path connectedness is an invariant of homotopy type in this sense, and we shall take the point of view that a gerbe is a presheaf of groupoids G such that the classifying simplicial presheaf BG is locally path connected. The local path connnectedness condition can be expressed this way: given objects $x, y \in \mathrm{Ob}(G)(U)$ in a section $G(U)$, there is a covering family $\phi : V \to U$ such that there is a morphism $\phi^*(x) \to \phi^*(y)$ in $G(V)$ for any ϕ in the cover.

Every gerbe G is locally equivalent to any of its sheaves of automorphism groups. The category $H(*, \mathbf{Grp}(\mathcal{C}))$ of 2-cocycles taking values in the diagram of all sheaves of groups, their isomorphisms and homotopies, is the vehicle by which we classify gerbes up to local weak equivalence. Theorem 20 says that the path components of this cocycle category are in one to one correspondence with the path components of the category $\mathbf{Gerbe}(\mathcal{C})$ of gerbes and their local weak equivalences, or that there is a bijection

$$\pi_0(\mathbf{Gerbe}(\mathcal{C})) \cong \pi_0 H(*, \mathbf{Grp}(\mathcal{C})).$$

One has to interpret a statement like this correctly, because the categories involved are not small. The path component functor π_0 means the class of equivalence classes of objects, where two objects are equivalent if and only if there is a finite string of arrows connecting them in the ambient category, and we show that there are functions

$$\Phi : \pi_0(\mathbf{Gerbe}(\mathcal{C})) \leftrightarrows \pi_0 H(*, \mathbf{Grp}(\mathcal{C})) : \Psi$$

which are inverse to each other. Here, Φ is induced by a canonical cocycle construction which is introduced in Example 12, and Ψ is defined by a generalized Grothendieck construction, which is the subject of much of Section 2.

The 2-groupoid diagram $\mathbf{Grp}(\mathcal{C})$ has subobjects which are honest presheaves of 2-groupoids. Examples include the sheaf of 2-groupoids G_* associated to a sheaf of groups of G on $\mathcal{C}$: it has one object, the sheaf of 1-cells is the sheaf of automorphisms of G, and its sheaf of 2-cells is the sheaf of homotopies (or conjugations) of automorphisms. More generally, any presheaf of sheaves of groups in $\mathbf{Grp}(\mathcal{C})$ determines a full subobject $\mathcal{F} \subset \mathbf{Grp}(\mathcal{C})$ which is a presheaf of 2-groupoids, and one can discuss the homotopy type of $\mathcal{F}$ and its classifying object $B\mathcal{F}$ in simplicial presheaves. It is shown in Theorem 23 that there is a one to one correspondences

$$\pi_0 H(*, \mathcal{F}) \cong \pi_0(\mathcal{F} - \mathbf{Gerbe})$$

between the set of path components of 2-cocycles taking values in the presheaf of 2-groupoids $\mathcal{F}$ and path components of the category $\mathcal{F} - \mathbf{Gerbe}$ of gerbes locally equivalent to sheaves of groups appearing in $\mathcal{F}$. By the result relating path components of cocycle categories to morphisms in the homotopy category displayed above, both of these objects are then in bijective correspondence with the set $[*, B\mathcal{F}]$ of morphisms in the homotopy category of simplicial presheaves — this statement appears formally as Corollary 24. The bijection of path components in the statement of Theorem 23 is a restriction of the bijection of Theorem 20.

In the special case where $\mathcal{F} = G_*$ for some sheaf of groups G, Theorem 23 says that gerbes locally equivalent to G are classified up to weak equivalence by morphisms $[*, BG_*]$ in the homotopy category of simplicial presheaves. This result was originally proved, in a very different form, by Breen [1].

Finally, the presheaf of 2-groupoids has a fundamental groupoid $\pi\mathcal{F}$ and a canonical morphism $\mathcal{F} \to \pi\mathcal{F}$. In the case where $\mathcal{F} = G_*$, the fundamental groupoid πG_* is the sheaf of outer automorphisms of G. The fundamental groupoid $\pi\mathcal{F}$ has a functorial stack completion $\pi\mathcal{F} \to \mathrm{St}(\pi\mathcal{F})$, and $\mathrm{St}(\pi\mathcal{F})$ is the stack of bands (liens) for $\mathcal{F}$. Suppose that the band L is a fixed choice of global section of $\mathrm{St}(\pi\mathcal{F})$, and consider the homotopy fibre F_L of the composite

$$B\mathcal{F} \to B\pi\mathcal{F} \to B\,\mathrm{St}(\pi\mathcal{F}).$$

Theorem 27 (see also Corollary 28) identifies the set of morphisms $[*, BF_L]$ in the homotopy category with path components in a suitably defined category of L-gerbes. In other words, Giraud's non-abelian invariant $H^2(\mathcal{C}, L)$ is isomorphic to $[*, BF_L]$. Once again, the real thrust of the proof is to identify the set of path components of the cocycle category $H(*, F_L)$ with path components in L-gerbes, and then use the general result about cocycles to conclude that $[*, F_L] \cong \pi_0 H(*, F_L)$.

Theorem 20, Theorem 23 and Theorm 27 are the main results of this paper. The demonstration of these results appear in Sections 3 and 4, but depend on some generalities about groupoids enriched in simplicial sets and presheaves of 2-groupoids which appear in Section 1, as well as the discussion of the generalized Grothendieck construction of Section 2.

Contents

1 Simplicial groupoids

There are various equivalent ways to define a groupoid enriched in simplicial sets. We shall initially take the point of view that such an object H is a simplicial groupoid such that the simplicial set $\mathrm{Ob}(H)$ of objects is simplicially discrete, or just a set. The morphisms $\mathrm{Mor}(H)$ is a simplicial sets and the source, target $s,t : \mathrm{Mor}(H) \to \mathrm{Ob}(H)$ and identity maps $e : \mathrm{Ob}(H) \to \mathrm{Mor}(H)$ are all simplicial set maps. The notation H_n will refer to the associated groupoid in simplicial degree n.

For objects x, y of H the simplicial set of morphisms $H(x,y)$ is defined by the pullback diagram

$$\begin{tikzcd} H(x,y) \arrow[r] \arrow[d] & \mathrm{Mor}(H) \arrow[d, "(s,t)"] \\ * \arrow[r, "(x,y)"'] & \mathrm{Ob}(H) \times \mathrm{Ob}(H) \end{tikzcd}$$

where $* = \Delta^0$ defines the one-point (terminal) simplicial set.

A 2-groupoid is a groupoid enriched in groupoids. Equivalently, a simplicial groupoid G is a groupoid enriched in simplicial sets such that the simplicial set $\mathrm{Mor}(G)$ is the nerve of a groupoid.

We shall routinely write BH for both the bisimplicial classifying space $n \mapsto BH_n$ associated to H and its associated diagonal simplicial set dBH. The vertical simplicial presheaf BH_n in horizontal degree n is the iterated fibre product

$$\mathrm{Mor}(H) \times_{t,s} \mathrm{Mor}(H) \times_{t,s} \cdots \times_{t,s} \mathrm{Mor}(H),$$

This simplicial set models composable strings of morphisms of length n, and is the inverse limit for a diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{Mor}(H) \arrow[dr, "t"] & & \mathrm{Mor}(H) \arrow[dl, "s"'] \arrow[dr, "t"] & & \mathrm{Mor}(H) \arrow[dl, "s"'] & \cdots \\
& \mathrm{Ob}(H) & & \mathrm{Ob}(H) & &
\end{tikzcd}$$

involving n copies of the morphism object $\mathrm{Mor}(H)$.

The vertices of BH are the objects of H, and two vertices of BH are in the same path component if and only if they are in the same path component of the space BH_1, or in the same path component of the groupoid H_1 in simplicial degree 1. It is well known and easily seen that the degeneracy morphism $H_0 \to H_n$ induces a bijection $\pi_0 BH_0 \cong \pi_0 BH_n$ for all $n \geq 0$. It follows that there are natural isomorphisms

$$\pi_0 BH \cong \pi_0 BH_1 \cong \pi_0 BH_n \tag{1}$$

for all $n \geq 0$. We shall say that H is *connected* if BH is a path-connected simplicial set. More generally, $\pi_0 H$ will often be written to denote $\pi_0 BH$, and will be called the set of *path components* of H.

There is a simplicial groupoid $H^{\mathbf{1}}$ whose objects in simplicial degree n are morphisms $h : x \to y$ of H_n and whose morphisms $h \to h'$ are the commutative squares

$$\begin{tikzcd}
x \arrow[r, "h"] \arrow[d, "\alpha"'] & y \arrow[d, "\beta"] \\
x' \arrow[r, "h'"'] & y'
\end{tikzcd}$$

in H_n. There is a simplicial groupoid functor $(s,t) : H^{\mathbf{1}} \to H \times H$ which is defined in degree n by sending the square diagram above to the pair of morphisms $(x \xrightarrow{\alpha} x', y \xrightarrow{\beta} y')$. The two projections $s, t : H^{\mathbf{1}} \to H$ are weak equivalences, because they are weak equivalences in each simplicial degree.

Lemma 1. *Suppose that H is a groupoid enriched in simplicial sets. Then the pullback diagram*

$$\begin{tikzcd}
\mathrm{Mor}(H) \arrow[r] \arrow[d] & BH^{\mathbf{1}} \arrow[d] \\
\mathrm{Ob}(H) \times \mathrm{Ob}(H) \arrow[r] & BH \times BH
\end{tikzcd}$$

is homotopy cartesian.

Proof. Suppose first that H is an ordinary groupoid. There is a functor $\hom : H \times H \to \mathbf{Set}$ defined by $(x,y) \mapsto H(x,y)$ and which sends the pair of morphisms $(\alpha : x \to x', \beta : y \to y')$ to the function

$$H(x,y) \to H(x',y')$$

defined by sending $f : x \to y$ to the composite

$$x' \xrightarrow{\alpha^{-1}} x \xrightarrow{f} y \xrightarrow{\beta} y'.$$

Observe that there is an isomorphism

$$\underrightarrow{\mathrm{holim}}_{H \times H} \hom \cong B(H^{\mathbf{1}}).$$

Suppose now that H is a groupoid enriched in simplicial sets. Applying the construction of the previous paragraph in all simplicial degrees gives a simplicial functor $\hom : H \times H \to s\mathbf{Set}$ and an isomorphism

$$\underrightarrow{\mathrm{holim}}_{H \times H} \hom \cong B(H^{\mathbf{1}}).$$

There is a homotopy cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_G X \arrow[d] \\ \mathrm{Ob}(G) \arrow[r] & BG \end{tikzcd}$$

for all simplicial set-valued diagrams X defined on groupoids G enriched in simplicial sets [7, Lemma 2], [14], and this specializes to give the homotopy cartesian diagram required by the statement of the Lemma. $\square$

Lemma 2. *If $f : G \to H$ is a morphism of groupoids enriched in simplicial sets and if $X : H \to s\mathbf{Set}$ is a simplicial diagram defined on H, then the induced diagram*

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_G X \cdot f \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_H X \arrow[d] \\ BG \arrow[r] & BH \end{tikzcd}$$

is homotopy cartesian.

Proof. Suppose given a factorization

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_H X \arrow[r, "j"] \arrow[dr] & Z \arrow[d, "p"] \\ & BH \end{tikzcd}$$

of the canonical map $\underrightarrow{\mathrm{holim}}_H X \to BH$ such that j is a weak equivalence and p is a fibration. The squares in the picture

$$\begin{tikzcd} X \cdot f \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}_H X \arrow[d] \\ \mathrm{Ob}(G) \arrow[r] & \mathrm{Ob}(H) \arrow[r] & BH \end{tikzcd}$$

are homotopy cartesian, so that the induced map

$$\begin{tikzcd} X \cdot f \arrow[rr] \arrow[dr] & & \mathrm{Ob}(G) \times_{BH} Z \arrow[dl] \\ & \mathrm{Ob}(G) & \end{tikzcd}$$

is a weak equivalence of objects over $\mathrm{Ob}(G)$. It follows that the induced map

$$\begin{tikzcd} \underrightarrow{\mathrm{holim}}_G X \cdot f \arrow[rr] \arrow[dr] & & BG \times_{BH} Z \arrow[dl] \\ & BG & \end{tikzcd}$$

induces a weak equivalence on all homotopy fibres. This map is also a homotopy colimit of a comparison of diagrams made up of the homotopy fibres of the respective maps, and is therefore a weak equivalence. $\square$

Corollary 3. *Suppose that H is a groupoid enriched in simplicial sets. Then the square*

$$\begin{tikzcd} H(x,y) \arrow[r] \arrow[d] & B(H/y) \arrow[d] \\ * \arrow[r, "x"'] & BH \end{tikzcd}$$

is homotopy cartesian.

Proof. In the diagram of pullback squares

$$\begin{tikzcd} H(x,y) \arrow[r] \arrow[d] & B(H/y) \arrow[r] \arrow[d] & B(H^{\mathbf{1}}) \arrow[d] \\ * \arrow[r, "x"'] & BH \arrow[r, "(1,y)"'] & BH \times BH \end{tikzcd}$$

the square on the right is homotopy cartesian, on account of Lemma 2 applied to the composite functor

$$H \xrightarrow{(1,y)} H \times H \xrightarrow{hom} s\mathbf{Set}.$$

Lemma 1 implies that the composite square is homotopy cartesian, and the desired result follows. $\square$

Let πH denote the groupoid of path components of a groupoid H enriched in simplicial sets. The object πH will typically be called the *path component* groupoid of H. It has the same objects as H, and the set of morphisms from x to y is the set $\pi H(x,y)$ of path components of the simplicial set $H(x,y)$. There is a canonical map $\eta : H \to \pi H$ which is the identity on objects and is the canonical map $\mathrm{Mor}(H) \to \pi_0 \mathrm{Mor}(H)$ on morphisms. The morphism η is one of the canonical maps for an adjunction: the functor $H \mapsto \pi H$ is left adjoint to the inclusion of groupoids in simplicial groupoids.

Corollary 4. *The induced map $\eta : BH \to B\pi H$ induces an isomorphism on path components and all fundamental groups, so that πH is naturally weakly equivalent to the fundamental groupoid of BH.*

Proof. The morphisms $H(x,x) \to \pi H(x,x)$ induce isomorphisms in path components, and so the map $BH \to B\pi H$ induces isomorphisms in path components of all loop spaces, by Corollary 3. It follows that all homomorphisms $\pi_1(BH, x) \to \pi_1(B\pi H, x)$ are isomorphisms. The claim that $\pi_0 BH \to \pi_0 B\pi H$ is a bijection follows from (1) and the observation that the function $\pi_0 BH_0 \to \pi_0 B\pi H$ is a bijection. $\square$

Suppose now that $\mathcal{C}$ is a small Grothendieck site.

If G is a presheaf of groupoids on $\mathcal{C}$ and x, y are objects of $G(U)$, there is a presheaf $G(x,y)$ of homomorphisms from x to y on $\mathcal{C}/U$. Write $G_x = G(x,x)$ for the presheaf of automorphisms of x in G, and let $\tilde{G}_x$ denote the associated sheaf of automorphisms on $\mathcal{C}/U$.

Say that a presheaf of groupoids G is a *Čech object* if the canonical map $G \to \pi_0 G$ is a local weak equivalence, where $\pi_0 G = \pi_0 BG$ is the presheaf of path components of G.

In particular, an ordinary groupoid H is a Čech groupoid if the groupoid moprhism $H \to \pi_0 H$ is a weak equivalence. Equivalently, H is a Čech groupoid if and only if there is at most one morphism between any two objects of H.

Example 5. The Čech groupoid $C(p)$ for a function $p : X \to Y$ has the objects $\mathrm{Ob}(C(p)) = X$, and there is a morphism $x \to y$ in $C(p)$ if and only if $p(x) = p(y)$ in Y. There is a canonical bijection $\pi_0 C(p) \cong p(X)$.

This construction is natural, and therefore applies to morphisms $p : X \to Y$ of presheaves on a site. If p is an epimorphism of sheaves, the simplicial presheaf map $BC(p) \to Y$ is the Čech resolution of Y corresponding to the epimorphism p, and is a local weak equivalence. This construction specializes to the standard Čech resolution when applied to an epimorphism $p : \bigsqcup_\alpha U_\alpha \to Y$ arising from a covering.

Lemma 6. *Suppose that H is a presheaf of 2-groupoids, and let $\eta : H \to \pi H$ be the canonical map to the presheaf of path component groupoids. Then η is a local weak equivalence if and only if all presheaves of groupoids $H(x,y)$ are Čech objects.*

Proof. If the map $\eta : H \to \pi H$ is a local weak equivalence, then the map

$$\mathrm{Mor}(H) \to \pi_0 \mathrm{Mor}(H)$$

is a local weak equivalence over $\mathrm{Ob}(H) \times \mathrm{Ob}(H)$, by Lemma 1. The object $\mathrm{Ob}(H) \times \mathrm{Ob}(H)$ is a constant simplicial presheaf, so pullback along any map

$Z \to \mathrm{Ob}(H) \times \mathrm{Ob}(H)$ preserves local weak equivalences over $\mathrm{Ob}(H) \times \mathrm{Ob}(H)$ [7]. In particular, for all choices $x, y \in \mathrm{Ob}(U)(U)$, the induced map $H(x,y) \to \pi_0 H(x,y)$ is a local weak equivalence of presheaves of groupoids over $\mathcal{C}/U$.

In general, one can show that a map

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr] & & W \arrow[dl] \\ & A & \end{tikzcd}$$

of simplicial presheaves fibred over a presheaf A is a local weak equivalence if and only if it induces weak equivalences $Z_x \to W_x$ of simplicial presheaves on $\mathcal{C}/U$ for all $x \in A(U), U \in \mathcal{C}$. One implication involved in this statement we already know about, from [7]. For the other, if we know that all induced maps on fibres are local weak equivalences, we can replace f by a sectionwise Kan fibration, and check local lifting with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, which must take place in individual fibres.

Thus, if all morphism groupoids $H(x,y)$ are Čech objects, then the map $\mathrm{Mor}(H) \to \pi_0 \mathrm{Mor}(H)$ is a local weak equivalence of simplicial presheaves over the presheaf $\mathrm{Ob}(H) \times \mathrm{Ob}(H)$. It follows that all maps

$$\begin{aligned} &\mathrm{Mor}(H) \times_{t,s} \mathrm{Mor}(H) \times_{t,s} \cdots \times_{t,s} \mathrm{Mor}(H) \\ &\qquad \to \pi_0 \mathrm{Mor}(H) \times_{t,s} \pi_0 \mathrm{Mor}(H) \times_{t,s} \cdots \times_{t,s} \pi_0 \mathrm{Mor}(H) \end{aligned}$$

of iterated fibre products over $\mathrm{Ob}(H)$ are local weak equivalences. These are the comparison maps of vertical simplicial presheaves making up the comparison $BH \to B\pi H$ of bisimplicial presheaves, and one concludes that this map is a local weak equivalence. $\square$

Lemma 7. *A presheaf of groupoids H is a Čech object if and only if for every two morphisms $f, g : x \to y$ in $H(U)$ there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^* f = \phi^* g$ for all $\phi : V \to U$ in R.*

Proof. Suppose that $H \to \tilde{H}$ is the canonical map taking values in the associated sheaf of groupoids $\tilde{H}$. Then H is a Čech object if and only if all sheaves $\tilde{H}(x,x)$ of automorphisms of $\tilde{H}$ are trivial in the sense that the canonical sheaf map $\tilde{H}(x,x) \to *$ are isomorphisms. This is equivalent to the assertion that all presheaf maps $H(x,x) \to *$ are local monomorphisms.

Thus, suppose that H is a Čech object, and suppose given $f, g : x \to y$ in $H(U)$ the composite $g^{-1}f \in H(x,x)(U)$, and there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(g^{-1}f) = 1_{\phi^* x}$ for all $\phi : V \to U$ in R. But then $\phi^*(g) = \phi^*(f)$ for all $\phi \in R$.

The converse is clear: the local coincidence of all $f, g : x \to y$ means that all presheaf maps $H(x,y) \to *$ are local monomorphisms, and so all sheaf maps $\tilde{H}(x,x) \to *$ are isomorphisms. $\square$

Lemma 8. *Suppose that A is a presheaf of 2-groupoids, and that $\pi_0 A$ is its presheaf of path components. Then the canonical map $A \to \pi_0 A$ is a local weak equivalence if and only if all presheaves of groupoids $A(x,y)$ and the path component groupoid πA are Čech objects.*

Proof. Suppose that $A \to \pi_0 A$ is a local weak equivalence. Then all sheaves of homotopy groups for BA are trivial, and so Lemma 1 implies that all maps $A(x,y) \to *$ are local weak equivalences. In particular, all $A(x,y)$ are Čech objects. But then $\eta : A \to \pi A$ is a local weak equivalence by Lemma 6, and so the induced map $\pi A \to \pi_0(\pi A)$ is a weak equivalence, so that the presheaf of groupoids πA is a Čech object.

Suppose conversely that all $A(x,y)$ and πA are Čech objects. Then Lemma 6 implies that $A \to \pi A$ is a local weak equivalence, and then the map

$$\pi A \to \pi_0(\pi A) \cong \pi_0 A$$

is a local weak equivalence. It follows that the map $A \to \pi_0 A$ is a composite of two local weak equivalences. □

2 The Grothendieck construction

Let **cat** denote the 2-category whose 0-cells are the small categories, whose 1-cells are the functors between small categories, and whose 2-cells are the homotopies of functors.

Suppose given a 2-category morphism $F : A \to \mathbf{cat}$, such that A is a small category enriched in groupoids. This morphism has an associated "Grothendieck construction", which is a category $E_A F$ that is constructed as follows.

Consider the collection of pairs (x,i) where i is an object or 0-cell of A and $x \in F(i)$. Look at all pairs

$$(f,\alpha) : (x,i) \to (y,j)$$

where $\alpha : i \to j$ is a 1-cell of A and $f : \alpha_*(x) \to y$ is a morphism of $F(j)$. Say that two such pairs

$$(f,\alpha), (f',\alpha') : (x,i) \to (y,j)$$

are equivalent if there is a 2-cell $h : \alpha \to \alpha'$ in A such that the diagram

$$\begin{tikzcd} \alpha_*(x) \arrow[dr, "f"] \arrow[dd, "F(h)"'] & \\ & y \\ \alpha'_*(x) \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes, where $F(h)$ is the homotopy associated to the 2-cell h by F. This is an equivalence relation, since the homotopies h are isomorphisms in the groupoids $A(x,y)$. Write $[(f,\alpha)]$ for the equivalence class containing the pair (f,α).

Suppose given strings

$$(x,i) \xrightarrow{(f,\alpha)} (y,j) \xrightarrow{(g,\beta)} (z,k)$$

and

$$(x,i) \xrightarrow{(f',\alpha')} (y,j) \xrightarrow{(g',\beta')} (z,k)$$

and suppose that $(f,\alpha) \overset{h_1}{\simeq} (f',\alpha')$ and $(g,\beta) \overset{h_2}{\simeq} (g',\beta')$ via the displayed homotopies. Then there is a commutative diagram

$$\begin{tikzcd}
\beta_*\alpha_*(x) \arrow[r, "\beta_*(f)"] \arrow[d, "F(h_2)"'] & \beta_*(y) \arrow[r, "g"] \arrow[d, "F(h_2)"'] & z \\
\beta'_*\alpha_*(x) \arrow[r, "\beta'_*(f)"] \arrow[d, "\beta'_*(F(h_1))"'] & \beta'_*(y) \arrow[ur, "g'"'] & \\
\beta'_*\alpha'_*(x) \arrow[ur, "\beta'_*(f')"'] & &
\end{tikzcd}$$

The composite homotopy $\beta'_*(F(h_1))(F(h_2)\alpha_*)$ is the image of the composite 2-cell $h_2 * h_1$ under the morphism F. It follows that the assignment

$$[(g,\beta)] \cdot [(f,\alpha)] = [(g\beta_*(f), \beta\alpha)]$$

gives a well defined law of composition. This composition law is associative, and has 2-sided identities. Write E_AF for the corresponding category.

Remark 9. This category E_AF should seem familiar. Suppose that I is a small category, and let x,y be objects of I. There is a small category $I_s(x,y)$ whose objects are the functors $\theta : \mathbf{n} \to I$ (strings of length n) such that $\theta(0) = x$ and $\theta(n) = y$. A morphism $\theta \to \gamma$ of $I_s(x,y)$ is a commutative diagram of functors

$$\begin{tikzcd}
\mathbf{n} \arrow[dr, "\theta"] \arrow[dd, "\alpha"'] & \\
& I \\
\mathbf{m} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

such that the ordinal number map α is end-point preserving in the sense that $\alpha(0) = 0$ and $\alpha(n) = m$. Concatenation of strings defines a composition law $I_s(x,y) \times I_s(y,z) \to I_s(x,z)$, and so there is a 2-category I_s with the same objects as I and a canonical weak equivalence $I_s \to I$ (see also [3, IX.3.2]). Write GI_s for the category enriched in groupoids, having the same 0-cells as I, and such that the groupoid $GI_s(x,y)$ is the free groupoid on the category $I_s(x,y)$. The groupoid $GI_s(x,y)$ is a Čech groupoid with path components given by the set $I(x,y)$ of morphisms from x to y in I.

A pseudo-functor F defined on I and taking values in small categories can be identified with a 2-category morphism $F : GI_s \to \mathbf{cat}$ [3, IX.3.3], and one can show that the Grothendieck construction $E_F GI_s$ as defined above is isomorphic to the standard Grothendieck construction for the pseudo-functor F.

The Grothendieck construction $E_A F$ given here for 2-catgory morphisms defined on categories A enriched in groupoids generalizes the usual construction for pseudo-functors, but there appears to be no corresponding construction for lax functors.

We shall henceforth specialize to 2-category morphisms $F : A \to \mathbf{cat}$ which are defined on small 2-groupoids A. In this case, there is a canonical functor $p : E_A F \to \pi A$, which is defined by the assignment $(x, i) \mapsto i$.

Lemma 10. *Suppose that $F : A \to \mathbf{cat}$ is a 2-category morphism, where A is a 2-groupoid. Suppose that $[(f, \alpha)] : (x, i) \to (y, j)$ is a morphism of $E_A F$ such that $f : \alpha_*(x) \to y$ is an invertible morphism of $F(j)$. Then $[(f, \alpha)]$ is invertible in $E_F A$.*

Proof. The inverse of $[(f, \alpha)]$ is represented by $[(\alpha_*^{-1}(f^{-1}), \alpha^{-1})]$. □

Corollary 11. *If the 2-category morphism $F : A \to \mathbf{cat}$ takes values in groupoids, then $E_A F$ is a groupoid.*

A diagram

$$B \xleftarrow[\simeq]{\alpha} A \xrightarrow{F} \mathbf{cat}$$

such that $\alpha : A \to B$ is a weak equivalence of 2-groupoids is a 2-*cocycle* taking values in small categories. A morphism of 2-cocycles is a commutative diagram of functors

$$\begin{tikzcd}
 & A \arrow[dl, "\alpha"', "\simeq"] \arrow[dd] \arrow[dr, "F"] & \\
B & & \mathbf{cat} \\
 & A' \arrow[ul, "\alpha'", "\simeq"'] \arrow[ur, "F'"'] &
\end{tikzcd}$$

and the corresponding 2-cocycle category is denoted by $H(B, \mathbf{cat})$. There are analogous definitions for 2-cocycles and 2-cocycle categories taking values in of groups and small groupoids. These 2-cocycle categories are typically not small.

In particular, write **grp** for the 2-groupoid whose objects are the groups, whose 1-cells are the isomorphisms of groups $G \to H$, and whose 2-cells are the homotopies of isomorphisms, and suppose now that there is a 2-cocycle

$$\pi A \xleftarrow[\simeq]{\eta} A \xrightarrow{K} \mathbf{grp}$$

taking values in groups. Then the associated Grothendieck construction $E_A K$ can be identified with a category having as objects all $i \in \mathrm{Ob}(A)$ and with morphisms consisting of equivalence classes of pairs $(f, \alpha) : i \to j$, where $\alpha : i \to j$ is a 1-cell of A and $f \in K(j)$. In this case, there is a relation $(f, \alpha) \sim (g, \beta)$ if the diagram

$$\begin{tikzcd}
* \arrow[dd, "h_*"'] \arrow[dr, "f"] & \\
 & * \\
* \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes in the group $K(j)$, where conjugation by h_* defines the image of the unique 2-cell $\alpha \to \beta$.

The category $E_A K$ is a groupoid by Lemma 10.

Example 12. Suppose that G is a groupoid. The *resolution 2-groupoid* $R(G)$ has the same objects and 1-cells as G, and has a unique 2-cell $f \to g$ between any two morphisms $f, g : x \to y$ of G. The path component groupoid $\pi R(G)$ of $R(G)$ is a Čech groupoid, and the natural maps

$$BR(G) \to B\pi R(G) \to \pi_0 B(\pi R(G))$$

weak equivalences. There are natural bijections

$$\pi_0 G = \pi_0 BG \cong \pi_0 B(\pi R(G)).$$

There is a canonical morphism $F(G) : R(G) \to \mathbf{grp}$ which takes the object $x \in G(U)$ to the group $G_x = G(x,x)$ on $\mathcal{C}/U$, takes a 1-cell $f : x \to y$ to the isomorphism $G_x \to G_y$ which is defined by conjugation by f, and takes the 2-cell $f \to g$ to the homotopy defined by conjugation by the element $gf^{-1} \in G_y$. It follows that G determines a canonical 2-cocycle

$$\pi_0 G \xleftarrow{\simeq} R(G) \xrightarrow{F(G)} \mathbf{grp}.$$

Lemma 13. *There is a natural isomorphism of groupoids $\psi : E_{R(G)}F(G) \xrightarrow{\cong} G$ which is defined fibrewise over $\pi R(G)$ in the sense that there is a commutative diagram*

$$\begin{tikzcd}
E_{R(G)}F(G) \arrow[rr, "\psi", "\cong"'] \arrow[dr] & & G \arrow[dl] \\
& \pi R(G) &
\end{tikzcd}$$

Proof. The functor ψ is the identity on objects. It is defined on morphisms by sending the pair (f, α) to the composite $f \cdot \alpha$ in G. If $(f,\alpha) \sim (g, \beta)$ and $\alpha \to \beta$ is the unique 2-cell in $R(G)$, then the diagram

$$\begin{tikzcd}
& j \arrow[dd, "\beta\alpha^{-1}"] \arrow[dr, "f"] & \\
i \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & j \\
& j \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes in G, so that $f \cdot \alpha = g \cdot \beta$ and the assignment $[(f,\alpha)] \mapsto f \cdot \alpha$ is well defined. The assignment is functorial, because the diagram

$$\begin{tikzcd}
& & k \arrow[dr, "\beta_*(f)"] & & \\
& j \arrow[ur, "\beta"] \arrow[dr, "f"] & & k \arrow[dr, "g"] & \\
i \arrow[ur, "\alpha"] & & j \arrow[ur, "\beta"] & & k
\end{tikzcd}$$

commutes in G.

The functor ψ plainly induces surjective functions

$$\psi : \hom_{E_{R(G)}F(G)}(i,j) \to \hom_G(i,j)$$

Finally, if the diagram

$$\begin{tikzcd}[row sep=small] & j \arrow[dr, "f"] & \\ i \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & j \\ & j \arrow[ur, "g"'] & \end{tikzcd}$$

commutes in the groupoid G then $g \cdot (\beta\alpha^{-1}) = f$, so that $(f,\alpha) \sim (g,\beta)$, and ψ is injective on morphisms. $\square$

3 Cocycle classification of gerbes

A *gerbe* G is a locally connected presheaf of groupoids. A *morphism of gerbes* is a local weak equivalence $G \to H$ of presheaves of groupoids. We shall write $\mathbf{Gerbe}(\mathcal{C})$ for the category of gerbes and morphisms of gerbes.

Remark 14. If G is a gerbe and x is a global section of $\mathrm{Ob}(G)$, then the inclusion map $G_x \to G$ is a local weak equivalence. It follows that every gerbe H is locally equivalent to a presheaf of groups, in the sense that there is a covering $U \to *$ by objects $U \in \mathcal{C}$ and section $x_U \in \mathrm{Ob}(H)(U)$ such that the morphisms $H_{x_U} \to H|_U$ are local weak equivalences over $\mathcal{C}/U$ for all U in the covering.

Remark 15. Suppose that E is a presheaf, and identify E with a presheaf of discrete groupoids. An E*-gerbe* is a morphism $G \to E$ of presheaves of groupoids such that the associated presheaf map $\pi_0 G \to E$ induces an isomorphism $\tilde{\pi}_0 G \cong \tilde{E}$ of associated sheaves. A morphism of E-gerbes is a commutative diagram

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr] & & H \arrow[dl] \\ & E & \end{tikzcd}$$

such that the morphism $f : G \to H$ is a local weak equivalence of presheaves of groupoids. Write $\mathbf{Gerbe}_E(\mathcal{C})$ for the corresponding category. Categories of E-gerbes do appear in applications — see [12, p.22].

There is an equivalence of categories

$$\mathbf{Gerbe}_E(\mathcal{C}) \simeq \mathbf{Gerbe}(\mathcal{C}/E)$$

between the category of E-gerbes on $\mathcal{C}$ and the category of gerbes for the fibred site $\mathcal{C}/E$. Equivalences of this sort are discussed at length in [6]. Classification results for E-gerbes can therefore be deduced from classification results for gerbes on the site $\mathcal{C}/E$.

We shall write $\mathbf{Grp}(\mathcal{C})$ for the following monster: it is a contravariant diagram defined on $\mathcal{C}$ and taking values in 2-groupoids, such that the 0-cells of $\mathbf{Grp}(\mathcal{C})(U)$ are the sheaves of groups on $\mathcal{C}/U$, the 1-cells are the isomorphisms of sheaves of groups on $\mathcal{C}/U$, and the 2-cells are the (global) homotopies of sheaf isomorphisms. $\mathbf{Grp}(\mathcal{C})$ is not a presheaf of groupoids, because it does not take values in small groupoids.

If G is a gerbe, then the corresponding resolution 2-groupoid $R(G)$ (Example 12) is weakly equivalent to a point in the sense that the map $R(G) \to *$ is a local weak equivalence of presheaves of 2-groupoids. There is a canonical morphism $F(G) : R(G) \to \mathbf{Grp}(\mathcal{C})$ for which the 0-cell $x \in R(G)(U)$ is mapped to the sheaf of groups $\tilde{G}_x$, the 1-cell $\alpha : x \to y$ is mapped to the sheaf isomorphism $c_\alpha : \tilde{G}_x \to \tilde{G}_y$ on $\mathcal{C}/U$ which is defined by conjugation by the global section α, and each 2-cell $h : \alpha \to \beta$ of 1-cells $x \to y$ maps to conjugation by the image of $h \in \tilde{G}_y(U)$ in global sections of $\tilde{G}_y$. In this way, each gerbe G has a canonically associated 2-cocycle

$$* \xleftarrow{\simeq} R(G) \xrightarrow{F(G)} \mathbf{Grp}(\mathcal{C}).$$

Write $H(*, \mathbf{Grp}(\mathcal{C}))$ for the category of 2-cocycles taking values in the 2-groupoid object $\mathbf{Grp}(\mathcal{C})$.

The assignment of the cocycle $F(G) : R(G) \to \mathbf{Grp}(\mathcal{C})$ to the gerbe G is not functorial. It is true, however, that a map $G \to H$ of gerbes induces a 2-groupoid morphisms $f_* : R(G) \to R(H)$. The sheaf isomorphisms $f_x : \tilde{G}_x \xrightarrow{\cong} \tilde{H}_{f(x)}$ induced by the local weak equivalence f determine a homotopy

$$R(G) \times \underline{\mathbf{1}} \to \mathbf{Grp}(\mathcal{C})$$

from $F(G)$ to $F(H) \cdot f_*$. It follows that $F(G)$ and $F(H)$ represent the same element of $\pi_0 H(*, \mathbf{Grp}(\mathcal{C}))$, and so the assignment $G \mapsto [F(G)]$ induces a function

$$\Phi : \pi_0 \mathbf{Gerbe}(\mathcal{C}) \to \pi_0 H(*, \mathbf{Grp}(\mathcal{C})).$$

Suppose that

$$* \xleftarrow{\simeq} A \xrightarrow{K} \mathbf{Grp}(\mathcal{C})$$

is a 2-cocycle with coefficients in $\mathbf{Grp}(\mathcal{C})$. Then K consists of 2-groupoid morphisms $K(U) : A(U) \to \mathbf{Grp}(\mathcal{C})(U)$, and hence induces composite morphisms

$$A(U) \xrightarrow{K(U)} \mathbf{Grp}(\mathcal{C})(U) \xrightarrow{ev_U} \mathbf{grp}.$$

Here, $ev_U : \mathbf{Grp}(\mathcal{C})(U) \to \mathbf{grp}$ is the 2-groupoid morphism which is defined by U-sections.

Write $E_A K(U)$ for the Grothendieck construction corresponding to the composite $ev_U K(U)$. Then the assignment $U \mapsto E_A K(U)$ defines a presheaf of groupoids $E_A K$. From Section 2, we see that there is a canonical morphism $p : E_A K \to \pi A$ of presheaves of groupoids; it is defined in sections to be the identity on objects, and it sends a class $[(f, \alpha)]$ to the class $[\alpha]$.

Lemma 16. *Suppose that $K : A \to \mathbf{Grp}(\mathcal{C})$ is a 2-cocycle over the terminal object $*$ taking values in sheaves of groups. Then the presheaf of groupoids $E_A K$ is a gerbe.*

Proof. The map $\pi_0 E_A K \to \pi_0(\pi A)$ is an isomorphism of presheaves, since each 2-functor $K(U)$ takes values in groups. $\square$

Take $i \in A(U)$ and let $K(i)$ be the corresponding sheaf of groups on $\mathcal{C}/U$. The functor $\phi : K(i) \to p/i$ of presheaves of groupoids on $\mathcal{C}/U$ is defined in sections corresponding to an object $\psi : V \to U$ of $\mathcal{C}/U$ by sending the group element $f \in K(i)(V) = K(\psi^*(i))(V)$ to the class $[(f, 1_i)]$.

Lemma 17. *Suppose that $K : A \to \mathbf{Grp}(\mathcal{C})$ is a 2-cocycle over $*$ taking values in sheaves of groups, and choose $i \in A(U)$. Then the homomorphism $\gamma_i : K(i) \to \hom(i,i) \subset E_A K$ defined by $f \mapsto [(f, 1_i)]$ induces an isomorphism of sheaves of groups on $\mathcal{C}/U$.*

Proof. Suppose that $[(g, \alpha)]$ is an element of $\hom(i,i)$. By Lemma 7 there is a covering sieve $R \subset \hom(\ ,U)$ such that there is a 2-cell $h_\phi : \phi^*(\alpha) \to 1_{\alpha^*(i)}$ for all $\phi \in R$. It follows that, locally, $[(g, \alpha)]$ is in the image of γ_i.

Take group elements $f, g \in K(i)(U)$ and suppose that $\gamma_i(f) = \gamma_i(g)$. Then there is a 2-cell $h : 1_i \to 1_i$ in $A(U)$ such that the diagram

$$\begin{tikzcd} * \arrow[dd, "h_*"'] \arrow[dr, "f"] & \\ & * \\ * \arrow[ur, "g"'] & \end{tikzcd}$$

commutes in $K(i)(U)$. The presheaf of groupoids $A(i,i)$ is a Čech object by Lemma 8 so that there is a covering $\phi : V \to U$ such that $\phi^*(h) = 1$ for all members ϕ of the cover. But then $\phi^*(h_*) = 1$ for all ϕ, and so $h_* = 1$ since $K(i)$ is a sheaf of groups. $\square$

Corollary 18. *Suppose that $K : A \to \mathbf{Grp}(\mathcal{C})$ is a 2-cocycle over $*$ taking values in sheaves of groups, and choose $i \in A(U)$. Then the corresponding map $\gamma_i : K(i) \to \pi/i$ is a local equivalence of presheaves of groupoids on $\mathcal{C}/U$.*

Proof. The map $\gamma_i : K(i) \to \pi/i$ takes the group element f to the automorphism $[(f, 1_i)]$ of the object $[1_i] : \pi(i) \to i$. The induced map $K(i) \to \hom([1_i],[1_i])$ is a surjection of presheaves of groups. The composite

$$K(i) \to \hom([1_i],[1_i]) \to \hom(i,i)$$

is locally monic by Lemma 17.

The map $\gamma_i : K(i) \to \pi/i$ is a sectionwise surjection on path components. $\square$

Corollary 19. *Suppose that the diagram*

$$\begin{tikzcd} & A \arrow[dl, "\simeq"'] \arrow[dr, "K"] \arrow[dd, "\theta"] & \\ * & & \mathbf{Grp}(\mathcal{C}) \\ & B \arrow[ul, "\simeq"] \arrow[ur, "G"'] & \end{tikzcd}$$

is a morphisms of 2*-cocycles. Then the induced map* $\theta : E_A K \to E_B G$ *is a local weak equivalence of presheaves of groupoids.*

Proof. The diagram

$$\begin{tikzcd} K(i) \arrow[r, "\gamma_i"] \arrow[d, "="'] & \hom(i,i) \arrow[d, "\theta_*"] \\ G(\theta(i)) \arrow[r, "\gamma_{\theta(i)}"'] & \hom(\theta(i), \theta(i)) \end{tikzcd}$$

commutes, so that θ induces an isomorphism on all sheaves of fundamental groups by Lemma 17. The map θ induces an isomorphism on sheaves of path components by Lemma 16. □

It follows that the assignment $K \mapsto E_A K$ defines a functor $H(*, \mathbf{Grp}(\mathcal{C})) \to \mathbf{Gerbe}(\mathcal{C})$ and hence a function

$$\Psi : \pi_0 H(*, \mathbf{Grp}(\mathcal{C})) \to \pi_0(\mathbf{Gerbe}(\mathcal{C})).$$

Theorem 20. *The functions* Φ *and* Ψ *are inverse to each other, and define a bijection*

$$\pi_0(\mathbf{Gerbe}(\mathcal{C})) \cong \pi_0 H(*, \mathbf{Grp}(\mathcal{C})).$$

Proof. The relation $\Psi\Phi = 1$ is a consequence of Lemma 13.

Suppose that $K : A \to \mathbf{Grp}(\mathcal{C})$ is a 2-cocycle over $*$. There is a 2-groupoid morphism $\omega : A \to R(E_A K)$ which is the identity on objects, sends the 1-cell $\alpha : i \to j$ to the 1-cell $[(e, \alpha)]$, and sends the 2-cell $h : \alpha \to \beta$ to the 2-cell

$$[(h_*, 1)] : [(e, \alpha)] \to [(e, \beta)].$$

The presheaf of groupoids $\pi R(E_A K)$ has the same objects as A; it is a Čech object (by Lemma 8), in which there is a morphism $i \to j$ in $R(E_A K)$ if and only if there is a 1-cell $i \to j$ in A. It follows that the morphism ω induces an isomorphism on presheaves of path components. It also follows that the composite

$$A \xrightarrow{\omega} R(E_A K) \xrightarrow{F(E_A K)} \mathbf{Grp}(\mathcal{C})$$

defines a group-valued 2-cocycle on $\pi_0 A$. This composite sends the object $i \in A$ to the presheaf of groups $E_A K(i,i)$, sends a 1-cell $\alpha : i \to j$ to the homomorphism $c_\alpha : E_A K(i,i) \to E_A K(j,j)$ which is defined by conjugation with $[(e, \alpha)]$, and sends a 2-cell $h : \alpha \to \beta$ to the homotopy defined by conjugation with the element $[(h_*, 1)]$.

The assignments $f \mapsto [(f, 1)]$ define homomorphisms

$$\gamma_i : K(i) \to E_A K(i, i).$$

which induce isomorphisms of associated sheaves, by Lemma 17. The morphisms γ_i further determine a homotopy

$$\gamma : A \times \underline{\mathbf{1}} \to \mathbf{Grp}(\mathcal{C}).$$

from the cocycle K to the cocycle $F(E_A K)\omega$. It follows that there is a path

$$F(E_A K) \sim F(E_A K)\omega \sim \gamma \sim K$$

in the cocycle category, and so $\Psi\Phi = 1$ as required. $\square$

4 Homotopy classification of gerbes

Suppose that G is a presheaf of groupoids on $\mathcal{C}$, with automorphism sheaves $\tilde{G}_x$, $x \in G(U)$. The presheaf of 2-groupoids G_* has the same objects as G; the 1-cells $x \to y$ of $G_*(U)$ are the sheaf isomorphisms $\tilde{G}_x \to \tilde{G}_y$, and the 2-cells of $G_*(U)$ are the homotopies of isomorphisms. There is a 2-functor $\nu_G : G_* \to \mathbf{Grp}(\mathcal{C})$ which is defined by sending x to $\tilde{G}_x$, and is the identity on sheaf isomorphisms and homotopies. The canonical 2-cocycle $F(G) : R(G) \to \mathbf{Grp}(\mathcal{C})$ factors uniquely through a cocycle $F(G)_* : R(G) \to G_*$ in the category of presheaves of 2-groupoids.

Suppose that $\mathcal{F} \subset \mathbf{Grp}(\mathcal{C})$ is a subobject of $\mathbf{Grp}(\mathcal{C})$ such that

1) the imbedding is full: all simplicial presheaf maps

$$\mathcal{F}(H, K) \to \mathbf{Grp}(\mathcal{C})(H, K)$$

are isomorphisms,

2) $\mathcal{F}$ is a presheaf of groupoids, so that all classes $\mathrm{Ob}(\mathcal{F})(U)$ are sets,

We shall say that a subobject $\mathcal{F}$ of the diagram of 2-groupoids $\mathbf{Grp}(\mathcal{C})$ which satisfies these conditions is a *full subpresheaf* of $\mathbf{Grp}(\mathcal{C})$.

The image of the presheaf of 2-groupoids G_* in $\mathbf{Grp}(\mathcal{C})$ which arises from a presheaf of groupoids G is an example of such an object $\mathcal{F}$.

Lemma 21. *Suppose that $\mathcal{F} \subset \mathcal{F}'$ are full subpresheaves of $\mathbf{Grp}(\mathcal{C})$. Suppose further that every automorphism group $\mathcal{F}'_x$ of $\mathcal{F}'$ is locally isomorphic to automorphism groups of $\mathcal{F}$. Then the inclusion $\mathcal{F} \subset \mathcal{F}'$ is a local weak equivalence of presheaves of 2-groupoids.*

Proof. Write $\alpha : \mathcal{F} \subset \mathcal{F}'$ for the inclusion morphism. Then α is full, and therefore induces a presheaf monomorphism $\pi_0 G_* \to \pi_0 \mathcal{F}$. Every sheaf of groups $\mathcal{F}'_x \in \mathcal{F}'(U)$ is locally isomorphic to objects in the image of α, by definition, so that $\pi_0 \mathcal{F} \to \pi_0 \mathcal{F}'$ is a local epimorphism.

The assertion that α induces an isomorphism in all possible sheaves of higher homotopy groups is a consequence of the fullness and Lemma 1. □

Say that two gerbes G and H are *locally equivalent* if there is a covering family $U \to *$, $U \in \mathrm{Ob}(\mathcal{C})$, such that the restricted gerbes $G|_U$ and $H|_U$ are locally weakly equivalent on $\mathcal{C}/U$ for each object U in the covering of the terminal object $*$. If there is a local weak equivalence $G \to H$ then G and H are locally equivalent in the sense just described, but the converse is not true.

Example 22. Suppose that the presheaf of 2-groupoids $\mathcal{F}$ is a full subpresheaf of $\mathbf{Grp}(\mathcal{C})$, and that there is a 2-cocycle

$$* \xleftarrow{\simeq} A \xrightarrow{F} \mathcal{F} \subset \mathbf{Grp}(\mathcal{C})$$

over the terminal sheaf $*$.

There is a covering family $U \to *$, $U \in \mathcal{C}$, such that $A(U) \neq \emptyset$. In effect, $\mathrm{Ob}(A) \to *$ is a local epimorphism, so there is a covering $U \to *$ such that there are liftings

$$\begin{tikzcd} & \mathrm{Ob}(A) \arrow[d] \\ U \arrow[ur, "x_U"] \arrow[r] & * \end{tikzcd}$$

where x_U represents an object of $A(U)$. The presheaf of groupoids $E_A F$ is locally connected by Lemma 16, and the maps

$$F(x_U) \to \hom_{E_A F}(x_U, x_U)$$

induce isomorphisms of associated sheaves of groups on $\mathcal{C}/U$ by Lemma 17. It follows that the automorphism groups of the Grothendieck construction $E_A F$ are locally equivalent to objects of $\mathcal{F}$.

Write $\mathcal{F} - \mathbf{Gerbe}$ for the full subcategory of the category of gerbes whose automorphism groups are locally equivalent to sheaves of groups in $\mathcal{F}$. The assignment $F \mapsto E_A F$ for a cocycle $F : A \to \mathcal{F}$ takes values in $\mathcal{F}$-gerbes, so that there is a commutative diagram

$$\begin{tikzcd} \pi_0 H(*, \mathcal{F}) \arrow[r] \arrow[d] & \pi_0 H(*, \mathbf{Grp}(\mathcal{C})) \arrow[d, "\cong"] \\ \pi_0(\mathcal{F} - \mathbf{Gerbe}) \arrow[r] & \pi_0(\mathbf{Gerbe}) \end{tikzcd}$$

Note that if $f : G \to H$ is a local weak equivalence of gerbes, then G is an $\mathcal{F}$-gerbe if and only if H is an $\mathcal{F}$-gerbe, and it follows that the induced map

$$\pi_0(\mathcal{F} - \mathbf{Gerbe}) \to \pi_0(\mathbf{Gerbe})$$

is an injection.

Theorem 23. *Suppose that $\mathcal{F}$ is a full subpresheaf of the* $\mathbf{Grp}(\mathcal{C})$*. Then the Grothendieck construction defines a function*

$$\pi_0 H(*, \mathcal{F}) \to \pi_0(\mathcal{F} - \mathbf{Gerbe})$$

which is a bijection.

Proof. Suppose given cocycles $F : A \to \mathcal{F}$ and $G : B \to \mathcal{F}$ such that F and G are in the same path component as cocycles taking values in $\mathbf{Grp}(\mathcal{C})$. Then there is a string of maps of cocycles

$$F = F_0 \leftrightarrow F_1 \leftrightarrow \cdots \leftrightarrow F_n = G \tag{2}$$

where $F_i : A_i \to \mathbf{Grp}(\mathcal{C})$ are cocycles in $\mathbf{Grp}(\mathcal{C})$.

Suppose that $F : A \to \mathbf{Grp}(\mathcal{C})$ is a cocycle taking values in sheaves of groups locally isomorphic to objects of $\mathcal{F}$ and that

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "F"'] & & A' \arrow[dl, "F'"] \\ & \mathbf{Grp}(\mathcal{C}) & \end{tikzcd}$$

is a morphism of $H(*, \mathbf{Grp}(\mathcal{C}))$. Take $x \in A'(U)$. Then there is a covering family $\phi : V \to U$ with 1-cells $\phi^*(x) \to \alpha(y_V)$ in $A'(V)$ for all ϕ. It follows that the group $F'(x)$ is locally isomorphic to groups of the form $F(y_V)$, and all of these are locally isomorphic to objects of $\mathcal{F}$. Thus, the cocycle F' takes values in sheaves of groups locally isomorphic to objects of $\mathcal{F}$

It follows that all cocycles F_i in the list (2) take values in groups locally isomorphic to objects of $\mathcal{F}$. Write $\mathcal{F}'$ for the presheaf of 2-groupoids which is the full subobject of $\mathbf{Grp}(\mathcal{C})$ on the sheaves of groups appearing in the sets $\mathcal{F}(U)$ and all $F_i(\mathrm{Ob}(A_i))(U)$. Then $\mathcal{F} \subset \mathcal{F}'$, and Lemma 21 implies that this map of presheaves of 2-groupoids is a weak equivalence. The string of cocycles F_i in (2) all take values in $\mathcal{F}'$ by construction, and the map

$$\pi_0 H(*, \mathcal{F}) \to \pi_0 H(*, \mathcal{F}')$$

is a bijection. It follows that the original cocycles F and G are in the same path component of $H(*, \mathcal{F})$. The function

$$\pi_0 H(*, \mathcal{F}) \to \pi_0 H(*, \mathbf{Grp}(\mathcal{C}))$$

is therefore a monomorphism, as is the function

$$\pi_0 H(*, \mathcal{F}) \to \pi_0(\mathcal{F} - \mathbf{Gerbe}).$$

Suppose that the every automorphism presheaf of the gerbe H is locally equivalent to an object of $\mathcal{F}$. Then all automorphism sheaves of H are locally isomorphic to automorphism sheaves of G. Choose a full subpresheaf $\mathcal{F}' \subset \mathbf{Grp}(\mathcal{C})$ whose 0-cells are sheaves of groups locally equivalent to objects of $\mathcal{F}$ and which contains both $\mathcal{F}$ and H_*. Then the canonical cocycle

$$F(H) : R(H) \to \mathbf{Grp}(\mathcal{C})$$

takes values in $\mathcal{F}'$. The map $\mathcal{F} \to \mathcal{F}'$ is a local weak equivalence, and so $F(H)$ can be represented by a cocycle taking values in $\mathcal{F}$. It follows that the function $\pi_0 H(*, \mathcal{F}) \to \pi_0(\mathcal{F} - \mathbf{Gerbe})$ is surjective. □

Corollary 24. *Suppose that $\mathcal{F}$ is a full subpresheaf of 2-groupoids in $\mathbf{Grp}(\mathcal{C})$. Then there is a bijection*

$$[*, dB\mathcal{F}] \cong \pi_0(\mathcal{F} - \mathbf{Gerbe}).$$

Suppose that G is a gerbe, and write $G - \mathbf{Gerbe}$ for the category of gerbes which are locally equivalent to G. This category coincides with the category $G_* - \mathbf{Gerbe}$ arising from the full subpresheaf of 2-groupoids G_*, and so we have the following:

Corollary 25. *Suppose that G is a gerbe on a site $\mathcal{C}$ with associated 2-groupoid object G_* of isomorphisms and homotopies of automorphism sheaves of G. Then there is a bijection*

$$[*, dBG_*] \cong \pi_0(G - \mathbf{Gerbe}).$$

A special case of Corollary 25, corresponding to the case of a sheaf of groups G, was proved by Breen in [1].

Remark 26. Recall that gerbes on $\mathcal{C}$ can be identified with gerbes on the fibred site $\mathcal{C}/E$ up to natural equivalence. Given an gerbe G, write G_E for the corresponding gerbe on $\mathcal{C}/E$. Then Corollary 25 gives a homotopy classification

$$[*, dBG_{E*}] \cong \pi_0(G_E - \mathbf{Gerbe}).$$

for gerbes, up to local equivalence defined on the site $\mathcal{C}/E$.

Suppose that $\mathcal{F}'$ is a full subpresheaf of $\mathbf{Grp}(\mathcal{C})$, and write $\mathbf{Gerbe}(\mathcal{F}')$ for the full subcategory of gerbes G such that $G_* \subset \mathcal{F}'$ — say that the objects of $\mathbf{Gerbe}(\mathcal{F}')$ are the gerbes in $\mathcal{F}'$. The category $\mathcal{F} - \mathbf{Gerbe}$ of gerbes with automorphism sheaves locally isomorphic to objects of $\mathcal{F}$ is a filtered colimit of subcategories $\mathbf{Gerbe}(\mathcal{F}')$, indexed over all inclusions $\mathcal{F} \subset \mathcal{F}'$ of full subpresheaves of $\mathbf{Grp}(\mathcal{C})$ such that every object of $\mathcal{F}'$ is locally isomorphic to objects of $\mathcal{F}$. It follows that there is an isomorphism

$$\pi_0(\mathcal{F} - \mathbf{Gerbe}) \cong \varinjlim_{\substack{\mathcal{F} \subset \mathcal{F}' \\ \simeq}} \pi_0(\mathbf{Gerbe}(\mathcal{F}')).$$

Write $\mathrm{St}(\mathcal{F})$ and $\mathrm{St}(\pi\mathcal{F})$ for the stack completions (really, fibrant models) for the presheaf of 2-groupoids $\mathcal{F}$ and its path component object $\pi\mathcal{F}$. The path component object is a groupoid of outer automorphisms and its stack completion $\mathrm{St}(\pi\mathcal{F})$ is the stack of *bands* (*liens*) for $\mathcal{F}$. These stack completion constructions are functorial, since the underlying model structures are cofibrantly generated [13].

A (global) *band* L is a global section of the presheaf of groupoids $\mathrm{St}(\pi\mathcal{F})$, or equivalently [8] a torsor for the presheaf of outer automorphism groupoids $\pi\mathcal{F}$.

Write $p_{\mathcal{F}}$ for the composite

$$\mathcal{F} \to \pi\mathcal{F} \to \mathrm{St}(\pi\mathcal{F}).$$

The homotopy fibre over a global band L of the induced map $B\mathcal{F} \to B\,\mathrm{St}(\pi\mathcal{F})$ is the classifying object $B(p_{\mathcal{F}}/L)$ of the simplicial groupoid $p_{\mathcal{F}}/L$ [7].

The objects of 2-cocycle category $H(*, B(p_{\mathcal{F}}/L))$ can be identified with the collection of pairs (ν, ϕ) consisting of a 2-cocycle

$$* \xleftarrow{\simeq} A \xrightarrow{\phi} \mathcal{F}$$

and a natural (iso)morphism $\nu : \phi_* \to L$ in $\mathrm{St}(\pi\mathcal{F})$, where $\phi_* : \pi(A) \to \mathrm{St}(\pi\mathcal{F})$ is the unique induced morphism in the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & \mathcal{F} \arrow[d] \\ \pi(A) \arrow[r, "\phi_*"'] & \mathrm{St}(\pi\mathcal{F}) \end{tikzcd} \tag{3}$$

and L has been identified with the composite

$$\pi(A) \to * \xrightarrow{L} \mathrm{St}(\pi\mathcal{F}).$$

The morphisms $f : (\phi, \nu) \to (\phi', \nu')$ are cocycle morphisms

$$\begin{tikzcd} A \arrow[dr, "\phi"] \arrow[dd, "f"'] & \\ & \mathcal{F} \\ A' \arrow[ur, "\phi'"'] & \end{tikzcd}$$

such that $\nu' \cdot f_* = \nu : \phi_* \to L$.

Suppose that G is a gerbe in $\mathcal{F}$. Consider the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & R(G) \arrow[r, "F(G)"] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathrm{St}(\mathcal{F}) \arrow[d] \\ & \pi R(G) \arrow[r, "F(G)_*"'] & \pi\mathcal{F} \arrow[r, "\eta"'] & \mathrm{St}(\pi\mathcal{F}) \end{tikzcd} \tag{4}$$

and write $\omega_G = \eta F(G)_* : \pi R(G) \to \mathrm{St}(\pi\mathcal{F})$.

An L-gerbe (G,ν_G) in $\mathcal{F}$ is a gerbe G in $\mathcal{F}$ together with a natural isomorphism $\nu_G : \omega_G \to L$, where L has been identified with the composite

$$G \to * \xrightarrow{L} \mathrm{St}(\pi\mathcal{F}).$$

Observe that there is a canonical natural isomorphism

$$h_{f*} : F(G)_* \xrightarrow{\cong} F(H)_* f_*$$

for any morphism $f : G \to H$ of gerbes. A morphism $f : (G,\nu_G) \to (H,\nu_H)$ of L-gerbes is a morphism $f : G \to H$ of gerbes such that the diagram of natural isomorphisms

$$\begin{tikzcd}
\omega_G \arrow[r, "\eta(h_{f*})"] \arrow[d, "\nu_G"'] & \omega_H f_* \arrow[d, "\nu_H f_*"] \\
L \arrow[r, "1"'] & L
\end{tikzcd} \tag{5}$$

commutes. The natural isomomorphisms h_{f*} arising from gerbe morphisms f are coherent; this gives the law of composition for a category of L-gerbes in $\mathcal{F}$, which will be denoted by $\mathbf{Gerbe}(\mathcal{F})/L$.

Every L-gerbe (G,ν_G) in $\mathcal{F}$ determines an object $(F(G),\nu_G)$ in the cocycle category $H(*, B(p_{\mathcal{F}}/L))$.

Suppose that $f : (G,\nu_G) \to (H,\nu_H)$ is a morphism of L-gerbes in $\mathcal{F}$. Then the homotopy of cocycles $h_f : R(G) \times \underline{\mathbf{1}} \to \mathcal{F}$ from $F(G)$ to $F(H)f_*$ determines a diagram of path component groupoid morphisms

$$\begin{tikzcd}
\pi R(G) \arrow[d] \arrow[drr, "F(G)_*"] & & & \\
\pi R(G) \times \underline{\mathbf{1}} \arrow[rr, "h_{f*}"] & & \pi\mathcal{F} \arrow[r, "\eta"] & \mathrm{St}(\pi\mathcal{F}) \\
\pi R(G) \arrow[u] \arrow[r, "f_*"'] & \pi R(H) \arrow[ur, "F(H)_*"'] & &
\end{tikzcd}$$

The natural isomorphism $\nu_G : \eta F(G)_* \to L$ extends uniquely to a natural isomorphism $\nu_h : \eta h_{f*} \to L$, and ν_h restricts to $\nu_H f_* : \eta F(H)_* \to L$ on $\pi R(G) \times \{1\}$ on account of the commutativity of the diagram (5).

It follows that every morphism $f : (G,\nu_G) \to (H,\nu_H)$ of L-gerbes determines a path between the associated objects $(F(G),\nu_G)$, $(F(H),\nu_H)$ in the cocycle category, and that there is a function

$$\Phi_{\mathcal{F}} : \pi_0(\mathbf{Gerbe}(\mathcal{F})/L) \to \pi_0 H(*, B(p_{\mathcal{F}}/L))$$

which is defined by $\Phi([(G,\nu_G)]) = [(F(G),\nu_G)]$.

Suppose that the 2-cocycle

$$\begin{tikzcd}
* & A \arrow[l, "\simeq"'] \arrow[r, "\phi"] \arrow[d] & \mathcal{F} \arrow[d] \\
& \pi(A) \arrow[r, "\phi_*"'] & \mathrm{St}(\pi\mathcal{F})
\end{tikzcd}$$

and the natural isomorphism $\nu : \phi_* \to L$ in $\mathrm{St}(\pi\mathcal{F})$ define an object (ν, ϕ) of the cocycle category $H(*, B(p_{\mathcal{F}}/L))$. Then the associated presheaf of groupoids $E_A\phi$ is a gerbe which has automorphism sheaves locally isomorphic to objects of $\mathcal{F}$, and then from the proof of Theorem 20 we know that there is a homotopy

$$\begin{tikzcd}
A \arrow[d] \arrow[drr, "\phi"] & & \\
A \times \underline{\mathbf{1}} \arrow[rr, "\gamma"] & & \mathcal{F}' \\
A \arrow[u] \arrow[r, "\omega"'] & R(E_A\phi) \arrow[ur, "F(E_A\phi)"'] &
\end{tikzcd}$$

where $\mathcal{F}'$ is a full subpresheaf of $\mathbf{Grp}(\mathcal{C})$ containing $\mathcal{F}$ such that the map $\mathcal{F} \subset \mathcal{F}'$ is a local weak equivalence. We also know that the induced map $\omega_* : \pi A \to \pi R(E_A\phi)$ is an isomorphism. It follows that the induced natural isomorphism (or homotopy)

$$\gamma_* : \phi_* \xrightarrow{\cong} F(E_A\phi)_*\omega_*$$

of functors $\pi A \to \mathrm{St}(\mathcal{F}')$ induces a unique natural isomorphism

$$\eta F(E_A\phi)_* \xrightarrow{\tilde{\nu}} L$$

which restricts to the isomorphism $\nu : \phi_* \to L$ along the homotopy

$$\pi(A) \times \underline{\mathbf{1}} \xrightarrow{\gamma_*} \pi\mathcal{F}' \to \mathrm{St}(\pi\mathcal{F}').$$

In other words, $(E_A\phi, \tilde{\nu})$ is an L-gerbe in $\mathcal{F}'$.

Suppose that $f : (\phi, \nu) \to (\phi', \nu')$ is a morphism of the 2-cocycle category $H(*, B(p_{\mathcal{F}}/L))$. Then there is a full subpresheaf $\mathcal{F}'' \subset \mathbf{Grp}(\mathcal{C})$ containing $\mathcal{F}$, such that $\mathcal{F} \subset \mathcal{F}''$ is a weak equivalence and such that the associated gerbes $E_A\phi$ and $E_{A'}\phi'$ are gerbes in $\mathcal{F}''$. There is a diagram of homotopies

$$\begin{tikzcd}
\phi_*(i) \arrow[r, "="] \arrow[d, "\gamma_i"'] & \phi'_*(f(i)) \arrow[d, "\gamma_{f(i)}"] \\
\eta\,\mathrm{Aut}(i) \arrow[r, "f_*"'] & \eta\,\mathrm{Aut}(f(i))
\end{tikzcd}$$

where $\mathrm{Aut}(i)$ is the sheaf of automorphisms of i in $E_A\phi$ and $\mathrm{Aut}(f(i))$ is the sheaf of automorphisms of $f(i)$ in $E_{A'}\phi'$. Then the morphisms $\nu_i : \phi(i) \to L$ and $\nu'_{f(i)} : \phi'(f(i)) \to L$ coincide on $\phi(i) = \phi'(f(i))$ since f is a morphism of the cocycle category $H(*, B(p_{\mathcal{F}}/L))$. Furthermore, the vertical isomorphisms uniquely determine the natural isomorphisms $\tilde{\nu} : \eta\,\mathrm{Aut}(i) \to L$ and $\tilde{\nu}' f_* : \eta\,\mathrm{Aut}(f(i)) \to L$, respectively. It follows that the map $f_* : E_A\phi \to E_{A'}\phi'$ defines a morphism of L-gerbes in $\mathcal{F}''$. We therefore have a well defined function

$$\Psi : \pi_0 H(*, B(p_{\mathcal{F}}/L)) \to \varinjlim_{\substack{\mathcal{F}\subset\mathcal{F}' \\ \simeq}} \pi_0(\mathbf{Gerbe}(\mathcal{F}')/L).$$

Theorem 27. *Suppose that L is a band. Then the function*

is a bijection.

Proof. Suppose that $\mathcal{F} \subset \mathcal{F}''$ is a weak equivalence of full subpresheaves of $\mathbf{Grp}(\mathcal{C})$. Then the diagram

$$\begin{tikzcd}
\pi_0(\mathbf{Gerbe}(\mathcal{F})/L) \arrow[r, "\Phi_{\mathcal{F}}"] \arrow[d] & \pi_0 H(*, B(p_{\mathcal{F}}/L)) \arrow[d, "\cong"] \\
\pi_0(\mathbf{Gerbe}(\mathcal{F}')/L) \arrow[r, "\Phi_{\mathcal{F}'}"'] & \pi_0 H(*, B(p_{\mathcal{F}'}/L))
\end{tikzcd}$$

commutes, where the indicated vertical map is a bijection since the comparison map $B(p_{\mathcal{F}}/L)) \to B(p_{\mathcal{F}'}/L)$ is a local weak equivalence. It follows that the maps $\Phi_{\mathcal{F}'}$ together induce a function

$$\Phi : \varinjlim_{\substack{\mathcal{F}\subset\mathcal{F}' \\ \simeq}} \pi_0(\mathbf{Gerbe}(\mathcal{F}')/L) \to \pi_0 H(*, B(p_{\mathcal{F}}/L)).$$

The function Φ is the inverse of Ψ. $\square$

Corollary 28. *Suppose that L is a band. Then there are bijections*

$$[*, B(p_{\mathcal{F}}/L)] \cong \pi_0 H(*, B(p_{\mathcal{F}}/L)) \cong \varinjlim_{\substack{\mathcal{F}\subset\mathcal{F}' \\ \simeq}} \pi_0(\mathbf{Gerbe}(\mathcal{F}')/L).$$

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