The Eilenberg-Moore spectral sequence for K-theory; applications to p-compact homogeneous spaces

A. Jeanneret & A. Osse*

Abstract

We construct the Eilenberg-Moore spectral sequence for some generalized cohomology theories, along the lines of Smith and Hodgkin. We prove its multiplicativity and give some sufficient conditions for its convergence to the desired target. As applications, we compute the K-theory of various spaces associated to p-compact groups.

AMS subject classification (1991): 55T20, 55N15, 55P35, 57T35.

Keywords: Spectral sequences, K-theory, completed tensor product, p-compact groups.

Introduction

The topology of homogeneous spaces constitutes an important aspect of Lie theory. The celebrated works of Borel, Bott, Baum, Smith and others completely describe their rational cohomology and provide important informations on their integral cohomology. For the K-theory of homogeneous spaces, the Eilenberg-Moore type spectral sequence constructed by Hodgkin has been the most successful tool.

Recently Dwyer and Wilkerson [7] have introduced a homotopical generalization of the Lie groups, namely the p-compact groups. In this homotopy Lie theory, the notions of Weyl group, maximal torus, homogeneous spaces are well defined, even if there is still no "differentiable structure" in sight.

The present work originates in understanding the topology of the p-compact homogeneous spaces of Dwyer and Wilkerson. An important invariant of these spaces is their p-adic cohomology. Its rational part can be

*Supported by grant No 20-46899.96 of the Swiss National Fund for Scientific Research

computed by combining a classical Eilenberg-Moore spectral sequence argument with the result of [7]. At the moment, the main indications about the torsion in p-adic cohomology are conjectural (see however the results of [16]). Because of the "lack of differentiability", the K-theoretical constructions of Hodgkin do not apply to the p-compact setting. To circumvent to this difficulty, we have to go back to the original definition of the spectral sequence, by basing all the constructions on the geometric cobar resolution. This has led us to rewrite, adapt and simplify many classical arguments. This approach turns out to be successful and allows us to deal with other generalized cohomology theories. To be more precise, let $\mathcal{E}^*(-)$ be a generalized multiplicative cohomology theory such that $\mathcal{E}^*(pt)$ is a graded field. Examples of such theories are given by singular cohomology with coefficient in a field, complex mod p K-theory and Morava K-theories. With our assumption, $\mathcal{E}^*(X)$ is a complete Hausdorff topological $\mathcal{E}^*(pt)$-algebra for any space X. Moreover, $\mathcal{E}^*(-)$ satisfies the Künneth isomorphism (with completed tensor products). These two properties are the crucial ingredients for the construction and the study of the Eilenberg-Moore spectral sequence for $\mathcal{E}^*(-)$. Our main result can be stated as follows:

Main Theorem *Let B be a connected space and $\mathcal{E}^*(-)$ as above. For any pull-back diagram*

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ f\ } & B \end{array}$$

there exists a strongly convergent spectral sequence $\{E_r^{*,*}(X,Y); d_r\}_{r\geq 2}$ satisfying the following properties:

1. *The spectral sequence is multiplicative and compatible with the stable operations of $\mathcal{E}^*(-)$.*

2. *$E_2^{i,*} \cong \mathcal{T}or^{-i}_{\mathcal{E}^*(B)}(\mathcal{E}^*(X), \mathcal{E}^*(Y))$ as algebras, where $\mathcal{T}or^{-i}_{\mathcal{E}^*(B)}(-,-)$ is the i-th derived functor of the completed tensor product.*

3. *If $p : X \longrightarrow B$ is a fibration and $\mathcal{E}^*(\Omega B)$ is an exterior algebra on odd degree generators, then the spectral sequence converges to $\mathcal{E}^*(X \times_B Y)$.*

The third point of the Theorem desserves the following comment. As easily seen, if $\mathcal{E}^*(\Omega B)$ is an exterior algebra, then $\mathcal{E}^*(\Omega B)$ is connected; hence B is 1-connected. It is well known that this weaker property of B suffices to ensure the convergence of the spectral sequence, when $\mathcal{E}^*(-) = H^*(-; K)$, the singular cohomology with coefficient in a field K (see [18])

With respect to our original aim, we apply the main theorem to compute the K-theory of various p-compact homogeneous spaces. These computations lead to a new proof of the main result in [12]. We refer to Section 5 for more precise statements.

The paper is organized as follows. The construction of the spectral sequence is given in Sections 1 and 2. The ideas are standard but our methods are slightly different from those in [18] and [11]. The multiplicative structure is discussed in Section 3. The reader will notice that this is the most delicate part of the paper; we assure her/him that we did our best to present the ideas as clearly as possible. The convergence questions are treated in Sections 2 and 4. Proposition 2.2 of Section 2 is already stated in Hodgkin's paper, but the proof presented here is new and hopefully simpler. Section 4 is crucial for the application to the p-compact groups. In contrast to Hodgkin, we are able here to get rid of the "differentiability hypothesis". In addition, the geometrical nature of our constructions readily implies that the spectral sequence is compatible with the stable operations of $\mathcal{E}^*(-)$. We mention that the results of Section 1 to Section 4 are stated for sectionned spaces over B and imply the main theorem, via the basepoint adjunction trick discussed in Section 1. The applications are discussed in Section 5. We end with an appendix on profinite rings and modules. The theme of the appendix is well known and has been considered by some topologists (see [3] and [23]). We have decided to discuss it in some details because of its central role in our arguments and also because we didn't find any reference suitable to our needs.

Aknowledgements. We would like to thanks U. Suter and U. Würgler for helpful discussions. We also thank J.M. Boardman for sending us a copy of [2].

1 The geometric cobar resolution

For the main construction of this section, we need some recollections about fiberwise topology. Until further notice, B is a fixed pointed and connected space; Top/B will denote the category of spaces over B with fibre maps as morphisms. The base point inclusion $\{pt\} \hookrightarrow B$, viewed as an element of Top/B, will simply be denoted pt. The categorical product in Top/B is the familiar fibred product.

In the sequel we will work in the pointed version of this category, namely the category $(Top/B)_*$ of sectionned spaces over B. The objects are triples

$B \xrightarrow{s} X \xrightarrow{p} B$ such that $p \circ s = id$ and morphisms are commutative diagrams

$$\begin{array}{ccccc} & & X_1 & & \\ & {\scriptstyle s_1}\nearrow & & \searrow{\scriptstyle p_1} & \\ B & & \downarrow{\scriptstyle \phi} & & B \\ & {\scriptstyle s_2}\searrow & & \nearrow{\scriptstyle p_2} & \\ & & X_2 & & \end{array}$$

When there is no danger of confusion, we simply write X (respectively $\Phi : X_1 \longrightarrow X_2$) instead of $B \xrightarrow{s} X \xrightarrow{p} B$ (respectively the diagram above). As the reader may have noticed, we recover the categories Top of topological spaces and Top_* of pointed topological spaces by setting $B = pt$. It is well known that all the usual homotopic theoretic constructions can be performed in the category $(Top/B)_*$. For instance, we have fiberwise homotopy, fiberwise mapping cone $C_B(-)$, fiberwise suspension $\Sigma_B(-)$, fiberwise wedge $-\bigvee_B -$, etc. We refer to [18] for the definitions. Two other constructions play a role in what follows:

1. If $(X; x_0)$ is a pointed space, $(X; x_0)_B$ will denote the sectionned space over B
$$B \xrightarrow{s_0} X \times B \xrightarrow{pr_2} B$$
where pr_2 is the projection on the second factor and $s_0(b) = (x_0, b)$.

2. Let $X \xrightarrow{p} B$ be a space over B and A a closed subspace of X. The fiberwise collapse of X with respect to A is the sectionned space
$$B \xrightarrow{\bar{s}} X/_B A \xrightarrow{\bar{p}} B$$
defined by:
   - $X/_B A = (X \coprod B)/a \sim p(a) \quad \forall a \in A$
   - $\bar{p}(x) = p(x), \forall x \in X$ and $\bar{p}(b) = b \quad \forall b \in B$
   - $\bar{s}(b) = b, \quad \forall b \in B$.

In order to simplify the notation, we write X/A for $X/_B A$ and X^+ for $X/_B \emptyset = X \coprod B$ (base point adjunction). We mention in passing that X/A is fiberwise homotopy equivalent to the fiberwise cone of the obvious inclusion $A^+ \hookrightarrow X^+$.

We turn to an important fact, also treated in [18] and [11]: for any map $\Phi : X_1 \longrightarrow X_2$ in $(Top/B)_*$ there is a Puppe sequence of maps in $(Top/B)_*$

$$X_1 \xrightarrow{\Phi} X_2 \xrightarrow{i(\Phi)} C_B(\Phi) \xrightarrow{j(\Phi)} \Sigma_B X_1 \xrightarrow{\Sigma_B(\Phi)} \Sigma_B X_2 \longrightarrow \dots \qquad (1)$$

with the following properties:

1. For any Z in $(Top/B)_*$, the Puppe sequence of the map $\Phi \wedge_B id$ is obtained from (1) by smashing with Z.

2. If s_1, s_2, s respectively denote the sections of X_1, X_2 and $C_B(\Phi)$ then

$$X_1/s_1(B) \xrightarrow{\bar{\Phi}} X_2/s_2(B) \xrightarrow{i(\bar{\Phi})} C_B(\Phi)/s(B)$$

is a cofibration sequence in the category Top_*.

Definition 1.1 *A* cohomology theory *on* $(Top/B)_*$ *consists of a sequence of contravariant functors* $\{\widetilde{h}^i : (Top/B)_* \longrightarrow Ab\}_{i\in\mathbb{Z}}$ *(Ab is the category of abelian groups) and a sequence of natural transformations* $\{\delta^i\}$ *such that:*

1. $\delta^i : \widetilde{h}^i(X_1) \longrightarrow \widetilde{h}^{i+1}(C_B(\Phi))$ *is defined for any cofibration sequence* $X_1 \xrightarrow{\Phi} X_2 \xrightarrow{i(\Phi)} C_B(\Phi)$ *and the following sequence is exact:*

$$\ldots \longrightarrow \widetilde{h}^{i-1}(X_1) \xrightarrow{\delta^{i-1}} \widetilde{h}^i(C_B(\Phi)) \xrightarrow{i(\Phi)^*} \widetilde{h}^i(X_2) \xrightarrow{\Phi^*} \widetilde{h}^i(X_1) \longrightarrow \ldots$$

2. *(Dold's cylinder axiom) For any space* X *and any* $\theta : X \times [0,1] \longrightarrow B$, *we have* $\widetilde{h}^*(X \times [0,1]/X) = 0$, *where the inclusion of* X *into* $X \times [0,1]$ *is given by* $x \mapsto (x, 0)$.

3. *(Strong additivity) For any family* $\{X_\beta\}_{\beta\in J}$ *of sectionned spaces over* B, *we have*

$$\widetilde{h}^*(\bigvee_B X_\beta) \cong \prod \widetilde{h}^*(X_\beta)\ .$$

It can be checked that these axioms imply that a cohomology theory $\widetilde{h}^*(-)$ on $(Top/B)_*$ is homotopy invariant and possesses a natural suspension isomorphism, that is

$$\widetilde{h}^{*+1}(\Sigma_B X) \cong \widetilde{h}^*(X)$$

for any X in $(Top/B)_*$.

Before giving examples, we would like to discuss product structures. A cohomology theory $\widetilde{h}^*(-)$ on $(Top/B)_*$ is called *multiplicative* if it is equipped with a natural pairing of graded abelian groups

$$\bar{\kappa} : \widetilde{h}^*(X) \otimes \widetilde{h}^*(Y) \longrightarrow \widetilde{h}^*(X \wedge_B Y), \qquad X, Y \in (Top/B)_*$$

which is associative and has a unit $1 \in \widetilde{h}^0((S^0, *)_B)$. The sectionned space $(S^0, *)_B$ plays the role of a point in our category, since $(S^0, *)_B \wedge_B X = X$ for any X in $(Top/B)_*$. Consequently $\widetilde{h}^*((S^0,*)_B)$ is a ring, called the *coefficient ring* of $\widetilde{h}^*(-)$, and $\widetilde{h}^*(X)$ is a two sided module over it. In [11, Lemma 4.1] it is shown that the pairing $\bar{\kappa}$ factors through:

$$\kappa : \widetilde{h}^*(X) \otimes_{\bar{h}} \widetilde{h}^*(Y) \longrightarrow \widetilde{h}^*(X \wedge_B Y), \qquad X, Y \in (Top/B)_*$$

where $\bar{h}$ is a short hand for $\widetilde{h}^*((S^0,*)_B)$.

Example. Let $\widetilde{\mathcal{E}}^*(-)$ be a multiplicative cohomology theory on the category of pointed spaces and $\mathcal{E}^*(-)$ the associated unreduced theory. Then

$$\widetilde{\mathcal{E}}^*_B(-) : (Top/B)_* \longrightarrow Ab, \quad X \mapsto \mathcal{E}^*(X/s(B))$$

defines a multiplicative cohomology theory on $(Top/B)_*$. The coefficient ring is $\widetilde{\mathcal{E}}^*_B((S^0,*)_B) \cong \mathcal{E}^*(B)$, so that any $\widetilde{\mathcal{E}}^*_B(X)$ is an $\mathcal{E}^*(B)$-module. More generally, for any fibration $X \longrightarrow B$ the functor

$$(Top/B)_* \longrightarrow Ab, \quad Y \mapsto \widetilde{\mathcal{E}}^*_B(X^+ \wedge_B Y)$$

is also a multiplicative cohomology theory on $(Top/B)_*$, whose coefficient ring is $\mathcal{E}^*(X)$. The interested reader might consult the details in [5, Section 3.4].

From now on, $\widetilde{\mathcal{E}}^*(-)$ will denote a multiplicative cohomology theory on the category Top_* and $\mathcal{E}^*(-)$ the associated unreduced theory. We will always assume that $\mathcal{E}^*(pt)$ is a graded field; that is, its non zero homogeneous elements are invertible. This assumption implies that $\widetilde{\mathcal{E}}^*(-)$ satisfies the Künneth isomorphism (see [3]).

After all these preliminaries, we are now ready to present the *geometric cobar resolution*. To this end, we suppose that we are given a space X in $(Top/B)_*$. From this datum, we perform the following construction:

1. First we set $X_0 = X$.
2. For each integer $i \leq 0$ we suppose that $B \xrightarrow{s} X_i \xrightarrow{p} B$ has been constructed and we successively define:
   - $\widetilde{X}_i = (X_i/s(B))_B$
   - $\phi_i : X_i \longrightarrow \widetilde{X}_i$, $\phi_i(x) = ([x], p(x))$
   - $X_{i-1} = C_B(\phi_i)$.

The most important properties of this construction are given in

Lemma 1.2 *With the notations above and for any integer $i \leq 0$, the following statements are true:*

1. *For any space $Y \in (Top/B)_*$, the spaces $(\widetilde{X}_i \wedge_B Y)/s(B)$ and $X_i/s(B) \wedge Y/s(B)$ are homotopy equivalent in Top_*.*
2. *$\widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \cong \widetilde{\mathcal{E}}^*_B(X_i) \widehat{\otimes}_{\mathcal{E}^*(pt)} \mathcal{E}^*(B)$ as $\mathcal{E}^*(B)$-modules, where the $\mathcal{E}^*(B)$-action on the right hand side is given by right multiplication.*
3. *This induced morphism $\phi_i^* : \widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \longrightarrow \widetilde{\mathcal{E}}^*_B(X_i)$ is surjective.*

Proof. The first part can be safely left to the reader. For the second part, take $Y = (S^0, *)_B$ in the first part to obtain $\widetilde{X}_i/s(B) = X_i/s(B) \wedge B_+$. It suffices then to apply the Künneth isomorphism. Concerning the third part, let

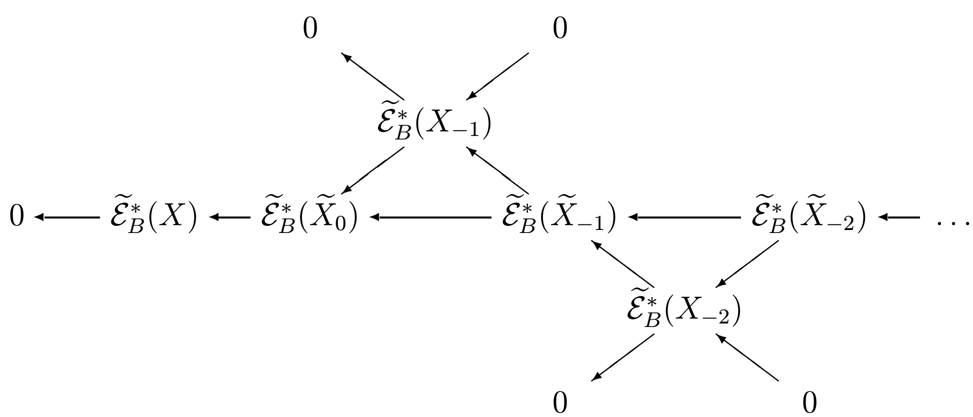
$$\psi_i : X_i/s(B) \wedge B_+ \longrightarrow X_i/s(B)$$

denote the collapsing of B to a point. Then $\psi_i \circ \bar{\phi}_i = id$, where $\bar{\phi}_i : X_i/s(B) \longrightarrow \widetilde{X}_i/s(B)$ is induced by ϕ_i. Consequently ψ_i^* is a right inverse for ϕ_i^* and we are done. $\square$

Thanks to this Lemma, the long exact sequences of the ϕ_i's break into short exact sequences:

$$0 \longrightarrow \widetilde{\mathcal{E}}^*_B(X_{i-1}) \longrightarrow \widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \longrightarrow \widetilde{\mathcal{E}}^*_B(X_i) \longrightarrow 0\ .$$

As usual in homological algebra, we splice all these sequences together as follows:



As a result, we obtain a long exact sequence of $\mathcal{E}^*(B)$-modules:

$$0 \longleftarrow \widetilde{\mathcal{E}}^*_B(X) \longleftarrow \widetilde{\mathcal{E}}^*_B(\widetilde{X}_0) \longleftarrow \widetilde{\mathcal{E}}^*_B(\widetilde{X}_{-1}) \longleftarrow \dots \qquad (2)$$

Fix $i \leq 0$ and write $\widetilde{\mathcal{E}}^*_B(X_i) = \widetilde{\mathcal{E}}^*(X_i/s(B)) = \underset{\longleftarrow}{lim}\ V_i^\alpha$, where each V_i^α is a finite dimensional $\mathcal{E}^*(pt)$-module (see Example 1 in the Appendix). By Lemma 1.2 and Theorem 6.3.ii), we have

$$\widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \cong (\underset{\longleftarrow}{lim}\ V_i^\alpha)\widehat{\otimes}_{\mathcal{E}^*(pt)}\mathcal{E}^*(B) \cong \underset{\longleftarrow}{lim}\ (V_i^\alpha\widehat{\otimes}_{\mathcal{E}^*(pt)}\mathcal{E}^*(B)) .$$

Since each $V_i^\alpha\widehat{\otimes}_{\mathcal{E}^*(pt)}\mathcal{E}^*(B)$ is obviously a free $\mathcal{E}^*(B)$-module, we can invoke Theorem 6.1.ii) to conclude that $\widetilde{\mathcal{E}}^*_B(\widetilde{X}_i)$ is a projective $\mathcal{E}^*(B)$-module. Consequently the sequence (2) is a projective resolution of the $\mathcal{E}^*(B)$-module $\widetilde{\mathcal{E}}^*_B(X)$.

2 The spectral sequence

The aim of this section is to construct the spectral sequence announced in the introduction, to identify its E_2-term and to discuss its convergence. This spectral sequence will be defined by means of derived exact couples. In order to simplify the presentation, we will always replace finite sequences of maps by inclusions in $(Top/B)_*$, via the fibrewise mapping cylinder construction. With this convention, the cofibre of a map $f : A \longrightarrow C$ will simply be written C/A.

Let X, Y be spaces in $(Top/B)_*$ and let $\{X_i \longrightarrow \widetilde{X}_i \longrightarrow X_{i-1}\}_{i\leq 0}$ be the geometric cobar resolution of X. Set

$$W_i(X,Y) = W_i = X_i \wedge_B Y\ , \qquad \text{for all i} \leq 0\ .$$

For a fixed integer $i \leq 0$, we iterate the Puppe construction of the ϕ_j's to obtain the sequence $X_{i-1} \longrightarrow \Sigma_B X_i \longrightarrow \Sigma_B^{-i+1}X_0$. The latter induces a cofibration

$$\Sigma_B W_i/W_{i-1} \xrightarrow{\beta_i} \Sigma_B^{-i+1}W_0/W_{i-1} \xrightarrow{\alpha_i} \Sigma_B^{-i+1}W_0/\Sigma_B W_i\ . \tag{3}$$

To introduce the spectral sequence we define, for $i \leq 0$

$$D_1^{i,j} = \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B^{-i+1}W_0/W_{i-1})$$
$$E_1^{i,j} = \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B W_i/W_{i-1})\ ,$$

and for $i > 0$

$$D_1^{i,j} = E_1^{i,j} = 0\ .$$

The cofibration (3) induces a long exact sequence in $\widetilde{\mathcal{E}}^*_B$-cohomology which can be written as

$$\ldots \longrightarrow D_1^{i+1,j-1} \xrightarrow{\alpha_i^*} D_1^{i,j} \xrightarrow{\beta_i^*} E_1^{i,j} \xrightarrow{\gamma_i^*} D_1^{i+1,j} \longrightarrow \ldots$$

where γ_i^* is induced by the Puppe map

$$\gamma_i : \Sigma_B^{-i+1} W_0 / \Sigma_B W_i \longrightarrow \Sigma_B(\Sigma_B W_i / W_{i-1}) \ .$$

We splice all these exact sequences to form the following unraveled exact couple (in the sense of Boardman [2])

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & D_1^{i+1,*} & & \longrightarrow & D_1^{i,*} & & \longrightarrow & D_1^{i-1,*} & \longrightarrow \ \cdots \\
 & & \nwarrow & \swarrow & & \nwarrow & \swarrow & & \\
 & & & E_1^{i,*} & & & E_1^{i-1} & &
\end{array}$$

Observe that the bidegree of α_i^* (respectively β_i^*, γ_i^*) is $(-1,1)$ (respectively $(0,0)$, $(1,0)$). By standard techniques this yields a spectral sequence $\{E_r^{*,*}(X,Y); d_r\}_{r\geq 1}$ (or simply $\{E_r^{*,*}; d_r\}_{r\geq 1}$ when X and Y are understood), with differentials $d_r : E_r^{i,j} \longrightarrow E_r^{i+r,j+1-r}$. As usual, for any $r \geq 2$, the cycles and boundaries are defined as

$$Z_r^{*,*} = Ker(d_{r-1}), \qquad B_r^{*,*} = Im(d_{r-1}) \ .$$

Since $E_r^{*,*} = Z_r^{*,*}/B_r^{*,*}$, it is important to have a geometric description of the groups $Z_r^{*,*}$ and $B_r^{*,*}$. For this purpose, we write

- ι for the natural inclusion $(W_{i-1}; \Sigma_B W_i) \subset (W_{i-1}; \Sigma_B^r W_{i+r-1})$
- $\delta = \delta_1 \circ \sigma$, where δ_1 is the coboundary of the triple $(W_{i-r}, \Sigma_B^{r-1} W_{i-1}, \Sigma_B^r W_i)$ and σ the suspension isomorphism

$$\widetilde{\mathcal{E}}_B^{j+r}(\Sigma_B^r W_i / \Sigma_B^{r-1} W_{i-1}) \cong \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B W_i / W_{i-1}) \ .$$

Then we have

$$\begin{aligned}
Z_r^{i,j} &\cong Im\{\iota^* : \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B^r W_{i+r-1}/W_{i-1}) \longrightarrow \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B W_i / W_{i-1})\} \\
B_r^{i,j} &\cong Im\{\delta : \widetilde{\mathcal{E}}_B^{j+r-1}(\Sigma_B^{r-1} W_{i-1}/W_{i-r}) \longrightarrow \widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B W_i / W_{i-1})\} \ .
\end{aligned}$$

With these identifications the differentials d_r are induced by the composite

$$\begin{array}{ccc}
\widetilde{\mathcal{E}}_B^{j+1}(\Sigma_B^r W_{i+r-1}/W_{i-1}) & \xrightarrow{\delta_2} & \widetilde{\mathcal{E}}_B^{j+2}(\Sigma_B^{-i+1} W_0 / \Sigma_B^r W_{i+r-1}) \\
 & & \downarrow \beta_{i+r}^* \\
 & & \widetilde{\mathcal{E}}_B^{j+2}(\Sigma_B^{r+1} W_{i+r} / \Sigma_B^r W_{i+r-1}) \\
 & & \cong \downarrow \sigma \\
 & & \widetilde{\mathcal{E}}_B^{j-r+2}(\Sigma_B W_{i+r} / W_{i+r-1}) \ ,
\end{array}$$

where δ_2 is the coboundary of the triple $(W_{i+1}, \Sigma_B^r W_{i+r+1}, \Sigma_B^{-i+1} W_0)$ and β^*_{i+r} is as above. The proof of these facts is standard; the interested reader might want to perform it by chasing on diagrams like the one on the top of page 661 in [22].

Everything is now in place for the identification of the E_2-term of that spectral sequence. This is based on the following

Lemma 2.1 *For any $i \leq 0$ and any space Y in $(Top/B)_*$, the product in $\widetilde{\mathcal{E}}^*_B$-cohomology induces an isomorphism*

$$\widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \widehat{\otimes}_{\mathcal{E}^*(B)} \widetilde{\mathcal{E}}^*_B(Y) \cong \widetilde{\mathcal{E}}^*_B(\widetilde{X}_i \wedge_B Y) \ .$$

Proof. Consider the diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{E}}^*_B(\widetilde{X}_i) \widehat{\otimes}_{\mathcal{E}^*(B)} \widetilde{\mathcal{E}}^*_B(Y) & \longrightarrow & \widetilde{\mathcal{E}}^*_B(\widetilde{X}_i \wedge_B Y) \\
\downarrow \cong & & \downarrow \cong \\
(\widetilde{\mathcal{E}}^*(X_i/s(B)) \widehat{\otimes}_{\mathcal{E}^*(pt)} \mathcal{E}^*(B)) \widehat{\otimes}_{\mathcal{E}^*(B)} \widetilde{\mathcal{E}}^*(Y/s(B)) & \longrightarrow & \widetilde{\mathcal{E}}^*(X_i/s(B) \wedge Y/s(B))
\end{array}$$

We proceed as in [11, page 49] to show that it commutes. By Lemma 1.2, the two vertical maps are isomorphism. The lower horizontal map is also an isomorphism (this is the ordinary Künneth isomorphism in $\mathcal{E}^*$-cohomology). This complete the proof of the lemma. □

Let us now deal with the differential $d_1 : E_1^{i-1,*} \longrightarrow E_1^{i,*}$. By definition it is the composite $\beta_i^* \circ \gamma_{i-1}^*$. As easily checked, the following diagram commutes

$$\begin{array}{ccccc}
 & & \Sigma_B^{-i+2} W_0 & & \\
 & & \uparrow & & \\
\Sigma_B^2 W_i / \Sigma_B W_{i-1} & \longrightarrow & \Sigma_B^2 W_{i-1} & \longrightarrow & \Sigma_B(\Sigma_B W_{i-1}/W_{i-2}) \\
 & \searrow^{\beta_i} & \uparrow & \nearrow_{\gamma_{i-1}} & \\
 & & \Sigma_B^{-i+2} W_0 / \Sigma_B W_{i-1} & &
\end{array}$$

The horizontal composition is equivalent to

$$\Sigma_B \widetilde{X}_{i+1} \wedge_B Y \stackrel{\phi_{i+1} \wedge Y}{\longrightarrow} \Sigma_B^2 \widetilde{X}_i \wedge_B Y \stackrel{\phi_i \wedge Y}{\longrightarrow} \Sigma_B^2 \widetilde{X}_i \wedge_B Y \ .$$

This observation and Lemma 2.1 imply that the complex

$$0 \longleftarrow E_1^{0,*} \xleftarrow{d_1} E_1^{-1,*} \xleftarrow{d_1} E_1^{-2,*} \xleftarrow{d_1} \dots$$

is isomorphic to the one obtained by applying the functor $-\widehat{\otimes}_{\mathcal{E}^*(B)}\widetilde{\mathcal{E}}_B^*(Y)$ to the complex

$$0 \longleftarrow \widetilde{\mathcal{E}}_B^*(\widetilde{X}_0) \longleftarrow \widetilde{\mathcal{E}}_B^*(\widetilde{X}_{-1}) \longleftarrow \dots$$

We have thus shown that

$$E_2^{i,*} \cong \mathcal{T}or^{-i}_{\mathcal{E}^*(B)}(\widetilde{\mathcal{E}}_B^*(X), \widetilde{\mathcal{E}}_B^*(Y)) ,$$

where $\mathcal{T}or^{-i}_{\mathcal{E}^*(B)}(-,-)$ is the i-th derived functor of the completed tensor product (see the Appendix).

We now turn to the convergence questions. Let us introduce the graded group $\mathcal{H}^*(X,Y)$ by taking $\mathcal{H}^r(X,Y)$ as the direct limit of the sequence

$$D_1^{0,r} \longrightarrow D_1^{-1,r+1} \longrightarrow \dots \longrightarrow D_1^{i,r-i} \longrightarrow \dots$$

The group $\mathcal{H}^*(X,Y)$ is filtered (as graded group) by

$$F_i\mathcal{H}^r(X,Y) = Im\{D_1^{i,r-i} \longrightarrow \mathcal{H}^r(X,Y)\} .$$

According to Boardman ([2, Section 6]), this filtration is Hausdorff and complete and the spectral sequence constructed in the previous section converges to $\mathcal{H}^*(X,Y)$; that is,

$$E_\infty^{i,r-i} \cong F_i\mathcal{H}^r(X,Y)/F_{i+1}\mathcal{H}^r(X,Y) , \quad \text{for all } i .$$

The expected target group is $\widetilde{\mathcal{E}}_B^*(W_0) = \widetilde{\mathcal{E}}_B^*(X \wedge_B Y)$. In this section we will give conditions under which $\mathcal{H}^*(X,Y)$ is isomorphic to $\widetilde{\mathcal{E}}_B^*(X \wedge_B Y)$.

For each $i \leq 0$, we consider the exact triangle induced in $\mathcal{E}^*$-cohomology by the cofibration $W_{i-1} \longrightarrow \Sigma_B^{-i+1}W_0 \longrightarrow \Sigma_B^{-i+1}W_0/W_{i-1}$. The direct limit of these triangles is the exact triangle

$$\begin{array}{ccc} \mathcal{H}^*(X,Y) & \xrightarrow{\Phi} & \widetilde{\mathcal{E}}_B^*(W_0) \\ & \nwarrow \quad \swarrow & \\ & \mathcal{G}^*(X,Y) & \end{array} \qquad (4)$$

where $\mathcal{G}^r(X,Y)$ is the direct limit of the sequence

$$\widetilde{\mathcal{E}}_B^{r+1}(W_{-1}) \longrightarrow \widetilde{\mathcal{E}}_B^{r+2}(W_{-2}) \longrightarrow \dots \longrightarrow \widetilde{\mathcal{E}}_B^{r-i+1}(W_{i-1}) \longrightarrow \dots$$

If the group $\mathcal{G}^*(X, Y)$ vanishes, then the map Φ of the triangle (4) will be an isomorphism so that our spectral sequence will have the expected abutment. The following proposition describes the main property of this obstruction group and gives a sufficient condition for the convergence to $\widetilde{\mathcal{E}}^*_B(X \wedge_B Y)$.

Proposition 2.2 *Let X be an element of $(Top/B)_*$ such that the projection $p : X \longrightarrow B$ is a fibration. The functor $Y \mapsto \mathcal{G}^*(X, Y)$ is a cohomology theory on $(Top/B)_*$. Consequently, if $\mathcal{G}^*(X, pt^+) = 0$, then $\mathcal{G}^*(X, Y) = 0$ for all spaces Y in $(Top/B)_*$.*

Proof. The first axiom of Definition 1.1 is easily checked using the exactness of the direct limits and the properties of the smash product and cofibrations in the category $(Top/B)_*$. The cylinder axiom requires the extra assumption on the projection p. By Proposition 4.8 of [18] and Section 3.4 of [5], the functors $Y \mapsto \widetilde{\mathcal{E}}^*_B(X_i \wedge_B Y)$ are cohomology theories for each $i \leq 0$. We invoke one more time the exactness of the direct limits to conclude. We are left to show that $\mathcal{G}^*(X, Y)$ is strongly additive. Let $\{Y_\beta\}_{\beta \in J}$ be a family of spaces in $(Top/B)_*$. We want to prove that the natural injections $Y_\beta \hookrightarrow \bigvee_B Y_\beta$ induce an isomorphism

$$\mathcal{G}^*(X, \textstyle\bigvee_B Y_\beta) \cong \prod \mathcal{G}^*(X, Y_\beta) .$$

We start with three observations:

1. $W_0(X, \bigvee_B Y_\beta)$ is homotopy equivalent to $\bigvee_B W_0(X, Y_\beta)$.
2. $\widetilde{\mathcal{E}}^*_B(\bigvee_B W_0(X, Y_\beta))$ is naturally isomorphic to $\prod \widetilde{\mathcal{E}}^*_B(W_0(X, Y_\beta))$.
3. The following triangle is exact (the direct product preserves exactness)

$$\begin{array}{ccc} \prod \mathcal{H}^*(X, Y_\beta) & \xrightarrow{\quad\Phi\quad} & \prod \widetilde{\mathcal{E}}^*_B(W_0(X, Y_\beta)) \\ & \nwarrow \qquad \swarrow & \\ & \prod \mathcal{G}^*(X, Y_\beta) & \end{array}$$

Because of these observations, it is sufficient to prove that

$$\mathcal{H}^*(X, \textstyle\bigvee_B Y_\beta) \cong \prod \mathcal{H}^*(X, Y_\beta) .$$

From the definition of the spectral sequence and exactness of the direct product, one checks easily that $\{\prod E_r^{*,*}(X, Y_\beta); \prod d_r\}_{r \geq 1}$ is a spectral sequence. Moreover, the natural injections $Y_\beta \hookrightarrow \bigvee_B Y_\beta$ induce a morphism of spectral sequences

$$\psi_r : E_r^{*,*}(X, \textstyle\bigvee_B Y_\beta) \longrightarrow \prod E_r^{*,*}(X, Y_\beta) . \tag{5}$$

As the completed tensor product commutes with the direct products, the morphism

$$\psi_2 : \mathcal{T}or^*_{\mathcal{E}^*(B)}(\widetilde{\mathcal{E}}^*_B(X), \widetilde{\mathcal{E}}^*_B(\bigvee_B Y_\beta)) \longrightarrow \prod \mathcal{T}or^*_{\mathcal{E}^*(B)}(\widetilde{\mathcal{E}}^*_B(X), \widetilde{\mathcal{E}}^*_B(Y_\beta))$$

is an isomorphism. Thus the spectral sequences of (5) are isomorphic.
We filter the group $\prod \mathcal{H}^*(X, Y_\beta)$ by setting $F_i \prod \mathcal{H}^*(X, Y_\beta) = \prod F_i \mathcal{H}^*(X, Y_\beta)$. This filtration is Hausdorff and complete. We observe that the natural injections $Y_\beta \hookrightarrow \bigvee_B Y_\beta$ induce a continuous homomorphism

$$\psi_0 : \mathcal{H}^*(X, \bigvee_B Y_\beta) \longrightarrow \prod \mathcal{H}^*(X, Y_\beta) \ . \tag{6}$$

Its associated graded homomorphism is nothing but the isomorphism

$$\psi_\infty : E^{*,*}_\infty(X, \bigvee_B Y_\beta) \longrightarrow \prod E^{*,*}_\infty(X, Y_\beta) \ .$$

As the two groups involved in (6) are Hausdorff and complete, ψ_0 has to be an isomorphism. The last assertion follows from the comparison theorem 4.1 of [5]. □

3 The multiplicative structure

The discussion of the multiplicative structure of the Eilenberg-Moore spectral sequence requires more general definitions than those given in Section 2. Since this material will be used only in the present section, we have chosen to introduce it here.

Definition 3.1 *Let $m \geq 1$ and $X \in (Top/B)_*$. A* negative m-filtration *of X is a sequence U_* of spaces $\{U_i\}_{i \leq 0}$ and maps $\psi_i : U_{i-1} \longrightarrow \Sigma^m_B U_i$ in $(Top/B)_*$, with $U_0 = X$. A morphism $f_* : U_* \longrightarrow V_*$ of negative m-filtrations is a sequence of maps $f_i : U_i \longrightarrow V_i$ making the obvious diagrams commutative.*

Examples.

1. For X in $(Top/B)_*$ and $m \geq 1$ the geometric cobar resolution of degree $m-1$ of X is inductively defined by setting $X_0 = X$ and for $i \leq 0$,
   - $\widetilde{X}_i = (X_i/s(B))_B$
   - $\phi_i : X_i \longrightarrow \widetilde{X}_i$, $\phi_i(x) = ([x], p(x))$
   - $X_{i-1} = C_B(\Sigma^{m-1}_B \phi_i)$.

We set $X_i(m) := X_i$ and take $\psi_i : X_{i-1}(m) \longrightarrow \Sigma^m_B X_i(m)$ to be the next map in the Puppe sequence of the cofibration

$$\Sigma^{m-1}_B X_i(m) \longrightarrow \Sigma^{m-1}_B \widetilde{X}_i(m) \longrightarrow X_{i-1}(m) .$$

The resulting negative m-filtration will be denoted $X_*(m)$ and called the *cobar m-filtration*. When $m = 1$, we recover the 1-filtration associated to the geometric cobar resolution of Section 1; we will then simply write $X_* := X_*(1)$.

2. Let U_* be a negative m-filtration of $X \in (Top/B)_*$ and let $Y \in (Top/B)_*$. One constructs a negative m-filtration $U_* \wedge Y$ by smashing all the constituents of U_* by Y. In particular, we suspend negative filtrations by smashing them with $Y = (S^k; *)_B$.

We recall here a construction of Hodgkin, since it will play a central role in our discussion. It might be illuminating to view this construction as the geometric counterpart of the tensor product of chain complexes in homological algebra. To get into work, we fix $X \in (Top/B)_*$ and let U_* be a negative m-filtration of $U_0 = X$. For any integer $i \leq 0$ we consider the sequence

$$U_i \xrightarrow{\psi_{i+1}} \Sigma^m_B U_{i+1} \xrightarrow{\psi_{i+2}} \dots \xrightarrow{\psi_0} \Sigma^{-im}_B U_0 . \tag{7}$$

We may and will assume (in accordance with our convention) that all the maps in this sequence are inclusions. Let Z_i be the subspace in $(Top/B)_*$ of $\Sigma^{-im}_B U_0 \bigwedge_B \Sigma^{-im}_B U_0$ defined by

$$Z_i = \bigcup_{k+l=i} \Sigma^{-km}_B U_{i-k} \wedge_B \Sigma^{-lm}_B U_{i-l} .$$

By comparing the sequence (7) for the index i and $i-1$, Hodgkin constructed a natural map $\chi_i : Z_{i-1} \longrightarrow \Sigma^{2m}_B Z_i$ and showed (see [11, page 24]) that

$$\Sigma^{2m}_B Z_i/Z_{i-1} \simeq \bigvee_{k+l=i} \Sigma^{-(i-k)m}_B (\Sigma^m_B U_k/U_{k-1}) \wedge_B \Sigma^{-(i-l)m}_B (\Sigma^m_B U_l/U_{l-1}) . \tag{8}$$

As in [11], the negative $2m$-filtration $\{Z_i; \chi_i\}_{i\leq 0}$ will be noted $U_* \otimes U_*$.

We continue our recollection of Hodgkin's work, by explaining how negative filtrations give rise to spectral sequences. Let U_* be a negative m-filtration and $\widetilde{\mathcal{E}}^*_B(-)$ a cohomology theory on $(Top/B)_*$. We fix an integer

$i \leq 0$ and extract the part of the sequence (7) given by the inclusions $U_{i-1} \hookrightarrow \Sigma_B^m U_i \hookrightarrow \Sigma_B^{(-i+1)m} U_0$. This sequence induces a cofibration

$$\Sigma_B^m U_i/U_{i-1} \longrightarrow \Sigma_B^{(-i+1)m} U_0/U_{i-1} \longrightarrow \Sigma_B^{(-i+1)m} U_0/\Sigma_B^m U_i \ .$$

With respect to this cofibration we define

$$\begin{aligned} D_1^{i,j} &= \widetilde{\mathcal{E}}_B^{j+1+(-i+1)(m-1)}(\Sigma_B^{(-i+1)m} U_0/U_{i-1}) \\ E_1^{i,j} &= \widetilde{\mathcal{E}}_B^{j+1+(-i+1)(m-1)}(\Sigma_B^m U_i/U_{i-1}) \ . \end{aligned}$$

We extend to all integers by setting, for $i > 0$

$$D_1^{i,j} = E_1^{i,j} = 0 \ .$$

As in Section 2 we obtain an unravelled exact couple; the associated spectral sequence is written $\{E_r^{i,j}(U_*); d_r\}_{r \geq 1}$. Here also, the result of Boardman [2] implies that the spectral sequence strongly converges to $\mathcal{H}^*(U_*) := \underrightarrow{lim}\ (D_1^{0,*} \to D_1^{-1,*+1} \to \dots)$. The latter is related to the desired abutment via a natural map $\Phi : \mathcal{H}^*(U_*) \longrightarrow \widetilde{\mathcal{E}}_B^*(U_0)$.

Lemma 3.2 *Let $X, Y \in (Top/B)_*$ and $m \geq 1$. Write $X_*(m)$ (respectively X_*) for the cobar m-filtration (respectively 1-filtration) of X. Then there is a natural morphism of spectral sequences*

$$E_r^{*,*}(X_*(m) \wedge Y) \longrightarrow E_r^{*,*}(X_* \wedge Y)$$

which is an isomorphism for $r \geq 2$.

Proof. We construct inductively maps $\phi_i : X_i(m) \longrightarrow \Sigma_B^{-i(m-1)} X_i$ by taking $\phi_0 = id$ and requiring the commutativity of the diagram

$$\begin{array}{ccccc} \Sigma_B^{m-1} X_i(m) & \longrightarrow & \Sigma_B^{m-1} \widetilde{X}_i(m) & \longrightarrow & X_{i-1}(m) \\ \downarrow \phi_i & & \downarrow \widetilde{\phi}_i & & \downarrow \phi_{i-1} \\ \Sigma_B^{(-i+1)(m-1)} X_i & \longrightarrow & \Sigma_B^{(-i+1)(m-1)} \widetilde{X}_i & \longrightarrow & \Sigma_B^{(-i+1)(m-1)} X_{i-1} \end{array}$$

where $\widetilde{\phi}_i$ is the composite

$$\Sigma_B^{m-1} \widetilde{X}_i(m) \longrightarrow \Sigma_B^{m-1}(\Sigma_B^{-i(m-1)} X_i/s(B) \times B) \longrightarrow \Sigma_B^{(-i+1)(m-1)} \widetilde{X}_i \ .$$

Then the map $\phi_i \wedge id : X_i(m) \wedge_B Y \longrightarrow \Sigma_B^{-i(m-1)} X_i \wedge_B Y$ induces the desired homomorphism of spectral sequences. As in Section 1, one constructs a projective $\mathcal{E}^*(B)$-resolution

$$0 \longleftarrow \widetilde{\mathcal{E}}_B^*(X) \longleftarrow \widetilde{\mathcal{E}}_B^*(\widetilde{X}_0(m)) \longleftarrow \widetilde{\mathcal{E}}_B^*(\widetilde{X}_{-1}(m)) \longleftarrow \dots$$

The comparison theorem of projective resolutions now implies that $\phi_i \wedge id$ induces an isomorphism of E_2-terms and the lemma follows. $\square$

This is the right place to start the discussion of the multiplicative properties of the spectral sequence.

Proposition 3.3 *Let $X, Y \in (Top/B)_*$ and set $W_* = X_* \wedge Y$, where X_* is the cobar 1-filtration of X. For $1 \leq r \leq \infty$, there exist associative pairings*

$$\psi_r : E_r^{i,j}(W_*) \otimes E_r^{p,q}(W_*) \longrightarrow E_r^{i+p,j+q}(W_* \otimes W_*),\ a \otimes b \mapsto a \cdot b$$

satisfying the following properties:

1. *ψ_1 is induced by the multiplication of $\widetilde{\mathcal{E}}_B^*(-)$.*
2. *ψ_{r+1} is induced by ψ_r, via the isomorphism $E_{r+1} \cong H(E_r)$.*
3. *For all $a \in E_r^{i,j}(W_*)$ and $b \in E_r^{p,q}(W_*)$, we have*

$$d_r(a \cdot b) = d_r(a) \cdot b + (-1)^{j+1} a \cdot d_r(b) \ .$$

Proof. We begin with some notation. For any negative m-filtration U_*, we set:

- $A_r^{i,j}(U_*) = \widetilde{\mathcal{E}}_B^{j+1+(-i+1)(m-1)}((\Sigma_B^{rm} U_{i+r-1}/U_{i-1}))$.
- For $s \leq r$, $\alpha_{r,s} : A_r^{i,j}(U_*) \longrightarrow A_s^{i,j}(U_*)$ is the morphism induced by the inclusion $(U_{i-1}, \Sigma_B^{sm} U_{i+s-1}) \hookrightarrow (U_{i-1}, \Sigma_B^{rm} U_{i+r-1})$.
- $\Delta_r^{i,j} : A_r^{i,j}(U_*) \longrightarrow A_r^{i+r,j-r+1}(U_*)$ is the coboundary operator of the triple $(U_{i-1}, \Sigma_B^{rm} U_{i+r-1}, \Sigma_B^{2rm} U_{i+2r-1})$.

Let us observe that $A_1^{i,j}(U_*) = E_1^{i,j}(U_*)$, the first term of the spectral sequence associated to U_*. By proceding as in the case $m = 1$ (see Section 2), we see that

$$Z_r^{i,j} \cong Im\{\alpha_{r,1} : A_r^{i,j}(U_*) \longrightarrow A_1^{i,j}(U_*)\}$$
$$B_r^{i,j} \cong Im\{\delta : A_{r-1}^{i-r+1,j+r-2}(U_*) \longrightarrow A_1^{i,j}(U_*)\} \ ,$$

where δ is defined as in Section 2. We set $\nu = j + 1 + (-i + 1)(m - 1)$ and consider the commutative diagram (obtained by playing around with adequate triples):

$$
\begin{array}{ccccc}
 & & \widetilde{\mathcal{E}}_B^{\nu+1}((\Sigma_B^{(-i+1)m} U_0 / \Sigma_B^{rm} U_{i+r-1})) & & \\
 & \overset{\delta}{\nearrow} & & \overset{\beta_{i+r}^*}{\searrow} & \\
\widetilde{\mathcal{E}}_B^{\nu}((\Sigma_B^{rm} U_{i+r-1} / U_{i-1})) & & \Big\downarrow & & \widetilde{\mathcal{E}}_B^{\nu+1}((\Sigma_B^{m} U_{i+r} / U_{i+r-1})) \\
 & \underset{\Delta}{\searrow} & & \underset{\alpha}{\nearrow} & \\
 & & \widetilde{\mathcal{E}}_B^{\nu+1}((\Sigma_B^{2rm} U_{i+2r-1} / \Sigma_B^{rm} U_{i+r-1})) & &
\end{array}
$$

As in Section 2 and with the identifications above, the differential d_r is equal $\beta_{i+r}^* \circ \delta$. Consequently, d_r is induced by the morphism $\Delta_r^{*,*}$.

We now go back to our data and define pairings

$$\phi_r : A_r^{i,j}(W_*) \otimes A_r^{p,q}(W_*) \longrightarrow A_r^{i+p,j+q}(W_* \otimes W_*)$$

in the following manner. First we use the product of $\widetilde{\mathcal{E}}_B^*(-)$ and the suspension isomorphism to send $A_r^{i,j} \otimes A_r^{p,q}$ into

$$\widetilde{\mathcal{E}}_B^*(\Sigma_B^{-p+1}(\Sigma_B^r W_{i+r-1}/W_{i-1}) \wedge_B \Sigma_B^{-i+1}(\Sigma_B^r W_{p+r-1}/W_{p-1}))\ .$$

Then we proceed as in [11, page 30] to send the latter into $A_r^{i+p,j+q}(W_* \otimes W_*)$. We observe that the arguments in Hilfsatz 13 and 14 of [13] apply verbatim to our situation. To be entirely honest, we should mention that Kulze's proofs are based on two hypothesis (the axioms M1 and M2 (page 290) of [13]). Fortunately, these hypotheses are satisfied in our case (see Theorem 9.10 (page 238) of [1]). □

As in the classical case, the next step consists in comparing the spectral sequences of the negative filtrations $W_* \otimes W_*$ and W_*, via the diagonal map. Unfortunately we have not been able to proceed directly. However, as we will see in a moment the suspensions of these filtrations can be compared. This will be sufficient for our purpose. The reason is the following: For any negative m-filtration U_*, the suspension isomorphism induces an isomorphism of spectral sequences (This is easily checked at the level of exact couples)

$$\sigma :\ E_r^{i,j}(U_*) \xrightarrow{\cong} E_r^{i,j+1}(\Sigma U_*). \tag{9}$$

Lemma 3.4 *Let U_* be a negative m-filtration of $U_0 = V$. As above, we write $V_*(m)$ for the cobar m-filtration of V. There exists a negative m-filtration Z_* of V and two morphisms of negative filtrations*

$$\Sigma V_*(m) \xleftarrow{g^*} Z_* \xrightarrow{f^*} \Sigma U_*$$

such that $f_0 = g_0 = id$.

Proof. To begin with we set $Z_0 = V$ and $f_0 = g_0 = id$. Given $f_i : Z_i \longrightarrow \Sigma_B U_i$ and $g_i : Z_i \longrightarrow \Sigma_B V_i(m)$, let

$$U_i' = \Sigma_B^m U_i / U_{i-1},\ V_i' = \Sigma_B^m \widetilde{V}_i(m) \text{ and } Z_i' = \Sigma_B^{m-1} \widetilde{Z}_i / s(B) \times (U_i' \times_B V_i')\ .$$

We define Z_{i-1} as the fiberwise cone of the obvious map $\Sigma_B^{m-1} Z_i \longrightarrow Z_i'$ and we construct f_{i-1} and g_{i-1} by requiring that the following diagram commutes:

$$\begin{array}{ccccc}
\Sigma_B^m U_i & \longrightarrow & U_i' & \longrightarrow & \Sigma_B U_{i-1} \\
\uparrow{\scriptstyle f_i} & & \uparrow{\scriptstyle f_i'} & & \uparrow{\scriptstyle f_{i-1}} \\
\Sigma_B^{m-1} Z_i & \longrightarrow & Z_i' & \longrightarrow & Z_{i-1} \\
\downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle g_i'} & & \downarrow{\scriptstyle g_{i-1}} \\
\Sigma_B^m V_i(m) & \longrightarrow & V_i' & \longrightarrow & \Sigma_B V_{i-1}(m)
\end{array}$$

Here f_i' and g_i' are the obvious projections. □

Theorem 3.5 *Let $X, Y \in (Top/B)_*$ and set $W_* = X_* \wedge Y$, where X_* is the cobar 1-filtration of X. There exist associative pairings, for $2 \leq r \leq \infty$,*

$$\mu_r : E_r^{i,j}(W_*) \otimes E_r^{p,q}(W_*) \longrightarrow E_r^{i+p,j+q}(W_*),\ a \otimes b \mapsto ab$$

satisfying the following properties:

1. *If $E_2^{*,*}(W_*)$ is identified with $\mathcal{T}or^*_{\mathcal{E}^*(B)}(\widetilde{\mathcal{E}}_B^*(X), \widetilde{\mathcal{E}}_B^*(Y))$, then μ_2 becomes the usual internal product of $\mathcal{T}or^*_{\mathcal{E}^*(B)}(-,-)$.*

2. *μ_{r+1} is induced by μ_r, via the isomorphism $E_{r+1} \cong H(E_r)$.*

3. *For all $a \in E_r^{i,j}(W_*)$, $b \in E_r^{p,q}(W_*)$ we have*

$$d_r(ab) = d_r(a)b + (-1)^{j+1} a d_r(b)\ .$$

4. *Let $\mathcal{H}^r(X,Y)$ be the limit of the spectral sequence $\{E_r^{i,j}(W_*); d_r\}_{r \geq 1}$. There is a product $\mu : \mathcal{H}^*(X,Y) \otimes \mathcal{H}^*(X,Y) \longrightarrow \mathcal{H}^*(X,Y)$ which induces μ_∞. In addition, the natural homomorphism*

$$\Phi : \mathcal{H}^*(X,Y) \longrightarrow \widetilde{\mathcal{E}}_B^*(X \wedge_B Y)$$

respects the products.

Proof. For the construction of the μ_r's, we already have the composite

$$\begin{array}{ccc} E_r^{*,*}(W_*) \otimes E_r^{*,*}(W_*) & \xrightarrow{\psi_r} & E_r^{*,*}(W_* \otimes W_*) \\ & & \cong \downarrow \tau_r \\ & & E_r^{*,*}((X_* \otimes X_*) \wedge (Y \wedge_B Y)) \\ & & \cong \downarrow \sigma_r \\ & & E_r^{*,*}(\Sigma_B (X_* \otimes X_*) \wedge (Y \wedge_B Y)) \end{array} \tag{10}$$

where ψ_r has been defined in Proposition 3.3, τ_r is induced by the twists $X \wedge_B X_j \simeq X_j \wedge_B X$ and σ_r is the suspension isomorphism discussed above (see (9)). Next we apply Lemma 3.4 with $U_* := X_* \otimes X_*$ and $V := X \wedge_B X$. This yields a negative 2-filtration Z_* and two morphisms of spectral sequences

$$\begin{array}{ccc} E_r^{*,*}(\Sigma_B U_* \wedge (Y \wedge_B Y)) & \xrightarrow{f^*} & E_r^{*,*}(Z_* \wedge (Y \wedge_B Y)) \\ & & g^* \uparrow \\ & & E_r^{*,*}(\Sigma_B V_*(2) \wedge (Y \wedge_B Y)) \; . \end{array} \tag{11}$$

We claim that g_* is an isomorphism for any $r \geq 2$. This can be checked as in Lemma 3.2, using property (8) and the construction of Z_* (see Lemma 3.4). To obtain the pairing μ_r, we compose the diagrams (10) and (11) with the sequence

$$\begin{array}{ccc} E_r^{*,*}(\Sigma_B V_*(2) \wedge (Y \wedge_B Y)) & \xrightarrow{\sigma^{-1}} & E_r^{*,*}(V_*(2) \wedge (Y \wedge_B Y)) \\ & & \Delta^* \downarrow \\ & & E_r^{*,*}(X_*(2) \wedge Y) \\ & & \cong \downarrow \\ & & E_r^{*,*}(W_*) \end{array} \tag{12}$$

where σ^{-1} is the inverse of the suspension isomorphism (9), Δ^* is induced by the diagonals $X \longrightarrow X \wedge_B X$ and $Y \longrightarrow Y \wedge_B Y$ and the last arrow is the isomorphism of Lemma 3.2. Even though suspensions appear in the construction, we note that the μ_r's are bigraded morphisms.

We now go through the claimed properties of the pairings. The first one follows from the definition of the internal product of $\mathcal{T}or^*_{\mathcal{E}^*(B)}(-,-)$ (see [21, page 65]). The next two properties are consequences of Proposition 3.3.

Let us deal with the multiplicative structure of $\mathcal{H}^*(X,Y) = \mathcal{H}^*(W_*)$. First we proceed as in Proposition 3.3 to define morphisms

$$D_1^{i,j}(W_*) \otimes D_1^{p,q}(W_*) \longrightarrow D_1^{i+p,j+q}(W_* \otimes W_*).$$

The direct limit of these morphisms yields a pairing

$$\mathcal{H}^r(W_*) \otimes \mathcal{H}^s(W_*) \longrightarrow \mathcal{H}^{r+s}(W_* \otimes W_*).$$

To obtain the desired product on $\mathcal{H}^*(W_*)$, we compose this pairing with a sequence of morphisms following the same pattern as in diagrams (10), (11) and (12). Of course one needs to check that

$$g^* : \mathcal{H}^*(\Sigma_B V_*(2) \wedge (Y \wedge_B Y))) \longrightarrow \mathcal{H}^*(Z_* \wedge (Y \wedge_B Y)))$$

is an isomorphism, but this is true because the corresponding spectral sequences are isomorphic (see the claim after diagram (11)). The naturality of these constructions implies that Φ is multiplicative and μ induces μ_∞. $\square$

4 An example of convergence

Let B be a pointed space and $p : EB \longrightarrow B$ be the path space fibration. The adjoint of the identity of ΩB will be denoted $e : \Sigma\Omega B \longrightarrow B$. Assume that $\mathcal{E}^*(\Omega B) \cong \Lambda(\xi_1, \dots, \xi_n)$ with $\xi_i \in \mathcal{E}^{odd}(B)$ for $i = 1, \dots, n$. An easy calculation with the Rothenberg-Steenrod spectral sequence implies that $\mathcal{E}^*(B) \cong \mathcal{E}^*[[\rho_1, \dots, \rho_n]]$ where $\rho_i \in \mathcal{E}^{even}(B)$ is chosen so that $e^*(\rho_i) = \sigma\xi_i$ for $i = 1, \dots, n$.

Theorem 4.1 *Let B be a connected and pointed space, $p : EB \longrightarrow B$ the path space fibration and assume that $\mathcal{E}^*(\Omega B) \cong \Lambda(\xi_1, \dots, \xi_n)$ with $\xi_i \in \mathcal{E}^{odd}(B)$. Then the Eilenberg-Moore spectral sequence of the pull-back diagram*

$$\begin{array}{ccc} \Omega B & \longrightarrow & EB \\ \downarrow & & \downarrow \\ pt & \hookrightarrow & B \end{array}$$

converges strongly to $\mathcal{E}^*(\Omega B)$.

Following the notation of Section 1, we write $\{X_i \longrightarrow \widetilde{X}_i \longrightarrow X_{i-1}\}_{i\le 0}$ for the cobar resolution of $X_0 = EB^+$ and we set

$$W_i := X_i \wedge_B pt^+ \ , \qquad (i \le 0) \ .$$

The first steps of this construction are summarized in the following commutative diagram (where the vertical maps are induced by the inclusion $pt^+ \hookrightarrow B^+ = (S^0, *)_B$):

$$\begin{array}{ccccccc}
W_0 & \longrightarrow & EB^+ & \longrightarrow & W_{-1} & \xrightarrow{i_0} & \Sigma_B W_0 \\
\downarrow{\scriptstyle j_0} & & \downarrow & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} \\
X_0 & \xrightarrow[\phi_0]{} & \widetilde{X}_0 & \longrightarrow & X_{-1} & \longrightarrow & \Sigma_B X_0 \ .
\end{array}$$

By definition $\widetilde{\mathcal{E}}^*_B(W_0) \cong \mathcal{E}^*(\Omega B)$ and the contractibility of EB implies that the composite $\widetilde{\mathcal{E}}^*(\Sigma\Omega B) \hookrightarrow \mathcal{E}^*(\Sigma\Omega B) \cong \widetilde{\mathcal{E}}^*_B(\Sigma_B W_0) \xrightarrow{i_0^*} \widetilde{\mathcal{E}}^*_B(W_{-1})$ is an isomorphism. By abuse of notation, we will identify the generators $\sigma\xi_i \in \widetilde{\mathcal{E}}^*(\Sigma\Omega B)$ to their images in $\widetilde{\mathcal{E}}^*_B(W_{-1})$. A similar argument shows that $\widetilde{\mathcal{E}}^*_B(X_{-1}) \cong \widetilde{\mathcal{E}}^*(B)$; here also we will identify the ρ_i's with their images in $\widetilde{\mathcal{E}}^*_B(X_{-1})$.

Lemma 4.2 *With the notation and identifications above, the morphism induced in $\widetilde{\mathcal{E}}^*_B$-cohomology by the map $j_1 : W_{-1} \longrightarrow X_{-1}$ satisfies: $j_1^*(\rho_i) = -\sigma\xi_i$, for $i = 1, \dots, n$.*

Proof. Let $\varphi : \Sigma\Omega B \longrightarrow B$ be the adjoint of the map $inv : \Omega B \longrightarrow \Omega B$ which sends a loop α to its inverse α^{-1}. We leave as an exercise to check that, in $\mathcal{E}^*$-cohomology, $\varphi^*(\rho_i) = -\sigma\xi_i$, for $i = 1, \dots, n$.

We will construct two maps (in Top_*) $F : X_{-1}/s(B) \longrightarrow B$ and $f : W_{-1}/s(B) \longrightarrow \Sigma\Omega B$ making the following diagram commutative:

$$\begin{array}{ccccc}
W_{-1} & \longrightarrow & W_{-1}/s(B) & \xrightarrow{f} & \Sigma\Omega B \\
\downarrow{\scriptstyle j_{-1}} & & \downarrow & & \downarrow{\scriptstyle \varphi} \\
X_{-1} & \longrightarrow & X_{-1}/s(B) & \xrightarrow[F]{} & B \ .
\end{array} \qquad (13)$$

The space X_{-1} is by definition the cofiber of

$$\phi_0 : X_0 = EB^+ \longrightarrow \widetilde{X}_0 = X_0/s(B) \times B \ .$$

More precisely, $X_{-1} = C_B(X_0) \coprod \widetilde{X}_0/(\alpha;0) \sim \phi_0(\alpha)$, where $C_B(X_0)$ denotes the reduced cone over B of X_0 (see [18] for its definition). We construct a map $X_{-1} \longrightarrow B$ by sending

$$\begin{array}{lcll} (\alpha;t) & \mapsto & \left\{ \begin{array}{l} \alpha(1-t) \\ \alpha \end{array} \right. & \begin{array}{l} \text{if } \alpha \in EB \\ \text{if } \alpha \in B \end{array} \\ ([x];b) & \mapsto & \quad b & \text{if } ([x];b) \in \widetilde{X}_0 \,. \end{array}$$

We leave as an exercise to check that this map is well defined and induces $F : X_{-1}/s(B) \longrightarrow B$. It is also easily checked that the composite

$$(EB \times B) \coprod pt = \widetilde{X}_0/s(B) \longrightarrow X_{-1}/s(B) \xrightarrow{F} B$$

is the projection onto the second factor. This shows that $F^*(\rho_i) = \rho_i$ for $i = 1, \ldots, n$.

Similarly W_{-1} is the cofiber of the map $W_0 = \Omega B^+ \longrightarrow EB^+$. Thus $W_{-1} = C_B(W_0) \coprod EB^+/(\alpha;0) \sim \alpha$. We construct a map $W_{-1} \longrightarrow \Sigma\Omega B$ by sending

$$\begin{array}{lcll} (\alpha;t) & \mapsto & \left\{ \begin{array}{l} [(\alpha,t)] \\ [(\epsilon_0,0)] \end{array} \right. & \begin{array}{l} \text{if } \alpha \in \Omega B \\ \text{if } \alpha \in B \end{array} \\ \alpha & \mapsto & \quad [(\epsilon_0,0)] & \text{if } \alpha \in EB^+ \,, \end{array}$$

where $\epsilon_0 \in EB^+$stands for the trivial loop. It is easily checked that this map is well defined and induces $f : W_{-1}/s(B) \longrightarrow \Sigma\Omega B$. We also leave as an exercice that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} W_{-1}/s(B) & \xrightarrow{i_0} & \Sigma_B W_0/s(B) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle =} \\ \Sigma\Omega B & \xleftarrow[\Sigma pr]{} & \Sigma(\Omega B \coprod pt) \end{array}$$

This implies that $f^*(\sigma\xi_i) = \sigma\xi_i$ for $i = 1, \ldots, n$.

The commutativity of the diagram (13) is straightforward and implies the lemma. $\square$

To proceed further we need to consider one more step in the cobar resolution. More precisely, we will study the inclusions $W_{-1} \hookrightarrow \Sigma_B W_0$ and $W_{-2} \hookrightarrow \Sigma_B W_{-1}$. For simplicity, we will use the following identifications

$$\widetilde{\mathcal{E}}^*_B(\Sigma_B W_{-1}/W_{-2}) \cong \widetilde{\mathcal{E}}^*_B(\Sigma_B \widetilde{X}_{-1} \wedge_B pt^+) \cong \widetilde{\mathcal{E}}^*(\Sigma B) \cong \widetilde{\mathcal{E}}^{*-1}(B) \,.$$

The second isomorphism follows from Lemma 1.2 and the others are obvious.

Lemma 4.3 *Let q_1 be the projection $\Sigma_B W_{-1} \longrightarrow \Sigma_B W_{-1}/W_{-2}$. With the identifications above, we have $q_1^*(\rho_i) = -\sigma\xi_i$,* for $i = 1, \ldots, n$.

Proof. We consider the following commutative diagram in $(Top/B)_*$, obtained from the natural map $X_{-2} \hookrightarrow \Sigma_B X_{-1}$:

$$\begin{array}{ccc} \Sigma_B W_{-1} & \xrightarrow{q_1} & \Sigma_B W_{-1}/W_{-2} \\ {\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ \Sigma_B X_{-1} & \xrightarrow{q_2} & \Sigma_B X_{-1}/X_{-2} \ . \end{array}$$

With our usual identifications, we have

$$\widetilde{\mathcal{E}}_B^*(\Sigma_B X_{-1}/X_{-2}) \cong \widetilde{\mathcal{E}}_B^*(\Sigma_B \widetilde{X}_{-1}) \cong \widetilde{\mathcal{E}}_B^{*-1}(\widetilde{X}_{-1}) \cong \widetilde{\mathcal{E}}^*(B) \otimes \mathcal{E}^*(B) \ .$$

The two homomorphisms j_2^* and q_2^* send $\rho_i \bigotimes 1$ to ρ_i. We note that these homomorphisms behave differently, for example on the elements $\rho_i \bigotimes \rho_j$. Lemma 4.2 and the commutative diagram above now implies the assertion. □

Proof (of Theorem 4.1). We will show, as claimed, that the map $\Phi : \mathcal{H}^*(EB^+, pt^+) \longrightarrow \widetilde{\mathcal{E}}_B^*(W_0)$ is a ring isomorphism. In our situation, the E_2-term of the Eilenberg-Moore spectral sequence is isomorphic to

$$\mathcal{T}or^*_{\mathcal{E}^*(B)}(\mathcal{E}^*(pt), \mathcal{E}^*(pt)) \cong \Lambda(y_1, \ldots, y_n) \ ;$$

the argument is standard and involves the Koszul resolution (see [21]). For dimensional reasons, the generators y_i are permanent cycles. The multiplicative properties of the spectral sequence imply that it collapses; that is, $E_2 \cong E_\infty$. Hence, $\mathcal{H}^*(EB^+, pt^+)$ is a free $\mathcal{E}^*(pt)$-module of rank 2^n. We next consider the diagram

$$\begin{array}{ccccccc} \widetilde{\mathcal{E}}_B^*(W_0) & \xrightarrow{i_0^*} & \widetilde{\mathcal{E}}_B^{*+1}(W_{-1}) & \xrightarrow{\qquad i_1^* \qquad} & \widetilde{\mathcal{E}}_B^{*+2}(W_{-2}) & \longrightarrow & \ldots \\ & & & \nwarrow {\scriptstyle q_1^*} & & & \\ & & & & \widetilde{\mathcal{E}}_B^*(\Sigma_B W_{-1}/W_{-2}) & & \end{array}$$

Since $Im(q_1^*) = Ker(i_1^*)$, Lemma 4.3 implies that the image of the generators $\xi_i \in \widetilde{\mathcal{E}}_B^*(W_0) \cong \mathcal{E}^*(\Omega B)$ are zero in the obstruction group $\mathcal{G}^*(EB^+, pt^+) = \underrightarrow{lim}\ \widetilde{\mathcal{E}}_B^*(W_i)$. Therefore the generators ξ_i lie in the image of Φ, showing the surjectivity of the latter. A dimension count now implies the claim. □

Theorem 4.4 *Let B be a connected and pointed space such that $\mathcal{E}^*(\Omega B) \cong \Lambda(\xi_1, \dots, \xi_n)$, with $\xi_i \in \mathcal{E}^{odd}(B)$. Let X, Y in $(Top/B)_*$ and assume that the projection $p : X \longrightarrow B$ is a fibration. Then the Eilenberg-Moore spectral sequence $\{E_r^{*,*}(X,Y); d_r\}_{r\geq 1}$ converges strongly to $\widetilde{\mathcal{E}}_B^*(X \wedge_B Y)$.*

Proof. Recall that $\mathcal{G}^*(-,-)$ denotes the obstruction to the "good behaviour" of the Eilenberg-Moore spectral sequence (see Section 2). By Theorem 4.1, we have $\mathcal{G}^*(EB^+, pt^+) = 0$, where $p : EB \longrightarrow B$ is the path space fibration. It follows from Proposition 2.2 that $\mathcal{G}^*(EB^+, X)$ is also trivial. The argument of Hodgkin (see pages 40-41 of [11]) shows that $0 = \mathcal{G}^*(EB^+, X) = \mathcal{G}^*(X, EB^+)$. Since $p : X \longrightarrow B$ is a fibration, its homotopy fiber is homotopy equivalent to the fiber over the base point; hence $\mathcal{G}^*(X, EB^+) = \mathcal{G}^*(X, pt^+) = 0$. We invoke one more time Proposition 2.2 to obtain $\mathcal{G}^*(X,Y) = 0$ and this concludes the proof. □

Remark. If $\mathcal{E}^*(\Omega B)$ is not an exterior algebra, the spectral sequence may not converge to the desired target. For instance, let $\mathcal{E}^*(-) = K^*(-;\mathbb{F}_2)$ be the mod 2 complex K-theory and $B = BSO(3)$, the classifying space of the Lie group $SO(3)$. As well known, we have

$$K^*(B;\mathbb{F}_2) \cong \mathbb{F}_2[[\rho]] \quad \text{and} \quad K^*(\Omega B;\mathbb{F}_2) \cong \mathbb{F}_2[\xi]/(\xi^4) .$$

For the path space fibration $\Omega B \longrightarrow EB \longrightarrow B$, The limit of the Eilenberg-Moore spectral sequence is $\mathcal{H}^*(EB^+, pt^+) \cong \Lambda(y)$, which is obvously different from $K^*(\Omega B;\mathbb{F}_2)$.

5 Applications

In this section, we will use our main theorem to describe the K-theory of certain spaces associated to p-compact groups. The basic references for the theory of p-compact groups are [7] and [15]; we refer to these papers for the relevant facts about these objects.

Throughout this section, p is a fixed prime and R denotes either the field $\mathbb{F}_p$ of order p or the ring $\mathbb{Z}_{\hat{p}}$ of p-adic integers. The complex K-theory with coefficient R will be denoted $K^*(-;R)$.

Our first result provides a large class of p-compact groups satisfying the hypothesis of the third point in the main theorem. The result might be well known to the experts, but we haven't found any explicit reference.

Theorem 5.1 *Let X be a connected p-compact group. Then $K^*(X;R)$ is an exterior algebra on odd degrees generators if and only if $\pi_1(X)$ is torsion free.*

Proof. A Bockstein spectral sequence argument shows that $K^*(X;\mathbb{F}_p)$ is an exterior algebra on odd degrees generators if and only $K^*(X;\mathbb{Z}_{\hat{p}})$ is an exterior algebra, if and only if $K^*(X;\mathbb{Z}_{\hat{p}})$ is torsion free. This reduces our problem to the case $R = \mathbb{Z}_{\hat{p}}$.

Observe that $\pi_1(X)$ is a finitely generated $\mathbb{Z}_{\hat{p}}$-module, hence $\pi_1(X) \cong {\mathbb{Z}_{\hat{p}}}^r \oplus \pi$ where π is a finite abelian p-group. We combine the arguments of Theorem 4.3 (page 63) in [14] and corollary 3.3 of [15] to show that X is homotopy equivalent (only as a space) to $Y \times (S^1_{\hat{p}})^r$, where Y is a connected p-compact group with $\pi_1(Y) \cong \pi$.

If $\pi_1(X)$ is torsion free, then Y is a 1-connected p-compact group and we can invoke a theorem of Kane and Lin (see [12, Theorem 1.2]) to conclude that $K^*(X;\mathbb{Z}_{\hat{p}})$ is an exterior algebra. The argument for the converse is implicit in [10]. Suppose that π is non trivial and choose a cyclic subgroup $i : \mathbb{Z}/p \hookrightarrow \pi$. Let $Y<1>$ be the universal cover of Y and $M(p) = S^1 \cup_p e^2$ be the 2-skeleton of $B\mathbb{Z}/p$. Consider the pullback diagram

$$\begin{array}{ccccc}
Y<1> & = & Y<1> & = & Y<1> \\
\downarrow & & \downarrow & & \downarrow \\
Y(p) & \longrightarrow & Y' & \longrightarrow & Y \\
\Big\downarrow{\scriptstyle \widetilde{q}} & & \downarrow & & \Big\downarrow{\scriptstyle q} \\
M(p) & \xrightarrow{\ j\ } & B\mathbb{Z}/p & \xrightarrow{\ Bi\ } & B\pi\ .
\end{array}$$

Since $Y<1>$ is 2-connected, obstruction theory tells us that the fibration $\widetilde{q}$ is trivial, hence $\widetilde{q}$ induces an injection in K-theory. As well known, the map Bi induces a surjection in K-theory; an easy computation shows that the same is true for the map j. All these facts imply that there exists a class $\xi \neq 0$ in the image of $q^* : K^*(B\pi;\mathbb{Z}_{\hat{p}}) \longrightarrow K^*(Y;\mathbb{Z}_{\hat{p}})$. A Chern character argument shows that ξ is a non trivial torsion class in $K^*(Y;\mathbb{Z}_{\hat{p}})$ and this implies that $K^*(X;\mathbb{Z}_{\hat{p}})$ has non trivial torsion. □

Proposition 5.2 *Let X be a connected p-compact group such that $\pi_1(X)$ is torsion free and let $i : T \longrightarrow X$ be a maximal torus. The ring homomorphism $Bi^* : K^*(BX;R) \longrightarrow K^*(BT;R)$ is injective and makes $K^*(BT;R)$ into a free and finitely generated $K^*(BX;R)$- module.*

Proof. The arguments of the proof of Theorem 2.7 in [12] are valid in this more general situation. The injectivity is due to the equality of the Krull dimensions. □

Everything is now in place for the main result of this section.

Theorem 5.3 *Let X be a connected p-compact group such that $\pi_1(X)$ is torsion free. Let $i : T \longrightarrow X$ be a maximal torus and X/T the associated homogeneous space; that is, X/T is the homotopy fibre of $Bi : BT \longrightarrow BX$. Then the inclusion $X/T \hookrightarrow BT$ induces an isomorphism*

$$K^*(X/T; R) \cong K^*(BT; R) \otimes_{K^*(BX;R)} K^*(pt; R) ,$$

where the $K^*(BX; R)$-module structure on $K^*(BT; R)$ (respectively $K^*(pt; R)$) is given by the induced map Bi^* (respectively the augmentation map).

Proof. Let us first deal with the case $R = \mathbb{F}_p$. We may and we will assume that the map $Bi : BT \longrightarrow BX$ is a fibration. Thanks to Theorem 5.1, we can apply our main theorem to the pullback diagram

$$\begin{array}{ccc} X/T & \hookrightarrow & BT \\ \downarrow & & \downarrow Bi \\ pt & \hookrightarrow & BX \end{array} .$$

Consequently, there is a strongly convergent Eilenberg-Moore spectral sequence

$$E_2^{i,*} = Tor^i_{K^*(BX;\mathbb{F}_p)}(K^*(BT; \mathbb{F}_p); K^*(pt; \mathbb{F}_p)) \Rightarrow K^*(X/T; \mathbb{F}_p) .$$

Proposition 5.2 above implies that the spectral sequence is trivial, i.e. $E_2^{*,*} = E_2^{0,*} = E_\infty^{*,*}$ and the claim follows.

The case $R = \mathbb{Z}_{\hat{p}}$ is a consequence of the preceding one, the universal coefficients theorem and Nakayama's lemma. □

For a general p-compact group and even if the Eilenberg-Moore spectral sequence does not behave as expected, we still have the following qualitative result.

Corollary 5.4 *Let X be a connected p-compact group, $i : T \longrightarrow X$ a maximal torus and W the corresponding Weyl group. Then $K^1(X/T; R) = 0$ and $K^0(X/T; R)$ is a free R-module of rank $|W|$.*

Proof. Let $X < 1 >$ be the universal cover of X. As we have seen above, $X<1>$ is a p-compact group. In the proof of corollary 5.6 of [15], it is shown that X/T is homotopy equivalent to $X < 1 > /S$, where $S \longrightarrow X < 1 >$ is maximal torus for $X < 1 >$. Since the latter is 1-connected, Theorem 5.3

applies. We invoke Proposition 9.5 of [7] to obtain the assertion about the rank. □

One of the main conjecture in the theory of p-compact groups states that $H^*(X/T;\mathbb{Z}_{\hat{p}})$ is torsion free and concentrated in even degrees. The interested reader is refered to [16] for some partial results about this conjecture. The corollary above can be viewed as a positive solution of its K-theoretical version.

As a second application, we will now give a slightly different proof of the main result in [12]. With this new method, we obtain an analoguous result for mod p K-theory. Even in the case of Lie groups, we are not aware of this mod p K-theory statement in the litterature.

Corollary 5.5 *Let X be a connected p-compact group, $i : T \longrightarrow X$ a maximal torus and W the corresponding Weyl group. The map Bi induces a ring isomorphism*

$$K^*(BX;R) \cong K^*(BT;R)^W.$$

Proof. By proceeding as in Section 3 of [12], we may and we will assume that X is 1-connected. Theorem 5.1 and a Rothenberg-Steenrod spectral sequence argument imply that $K^*(BX;R) \cong R[[\rho_1,\dots,\rho_n]]$ with $\rho_i \in K^0(BX;R)$ and n equal to the rank of X. Similarly, $K^*(BT;R) \cong R[[\tau_1,\dots,\tau_n]]$ with $\tau_i \in K^0(BT;R)$. For simplicity we set

$$S_X = R[[\rho_1,\dots,\rho_n]],\ S_T = R[[\tau_1,\dots,\tau_n]]$$

and we will identify S_X with its image in S_T (this is justified because Bi^* is injective by Proposition 5.2).

By construction S_X is contained in the ring of invariants S_T^W. To show the equality, we consider the following diagram

$$\begin{array}{ccc} S_T & \hookrightarrow & Frac(S_T) \\ \uparrow & & \uparrow \\ S_T^W & \hookrightarrow & Frac(S_T^W) \\ \uparrow & & \uparrow \\ S_X & \hookrightarrow & Frac(S_X) \end{array}$$

where $Frac(-)$ stands for the fractions field. By Proposition 5.2 and Corollary 5.4, S_T is a free S_X-module of rank $|W|$; it follows that $Frac(S_X) \hookrightarrow Frac(S_T)$ is a field extension of degree $|W|$. By Galois theory, $Frac(S_T^W) =$

$Frac(S_T)^W \hookrightarrow Frac(S_T)$ is a field extension of degree $|W|$. As consequence, the fields $Frac(S_X)$ and $Frac(S_T^W)$ coincide. Since S_T is integrally closed (it is a power series ring over R) and the ring extension $S_X \hookrightarrow S_T$ is integral, we obtain that $S_X = S_T^W$. □

Let us close this section by describing how our results extend to Morava K-theories. For $n \geq 1$, $K(n)^*(-)$ denotes the n-th Morava K-theory, its coefficient ring is the graded field $\mathbb{F}_p[v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$.

Let X be a connected p-compact group, $i : T \longrightarrow X$ a maximal torus and W the corresponding Weyl group. If $K(n)^*(X)$ is an exterior algebra on odd degrees generators, then the preceding arguments apply and we have:

1. $K(n)^*(X/T) \cong K(n)^*(BT) \bigotimes_{K(n)^*(BX)} K(n)^*(pt)$.
2. $K(n)^*(BX) \cong K(n)^*(BT)^W$.

These statements naturally give rise to the following question:

> *For each integer $n \geq 1$, find all the p-compact groups X such that $K(n)^*(X)$ is an exterior algebra.*

As well-known, $K(n)^*(X)$ is an exterior algebra when $H^*(X; \mathbb{Z}_{\hat{p}})$ is torsion free. Hence we recover Theorem 3.1 of [20]. In contrast to the paper just quoted, we can treat many spaces with torsion. For instance, our Theorem 5.1 provides a complete answer to the question when $n = 1$. More interestingly, Maria Santos (private communication) has observed that $K(2)^*(DI(4))$ is an exterior algebra; here $DI(4)$ is the exotic 2-compact group constructed by Dwyer and Wilkerson [6]. Are there any other examples of this type?

6 Appendix

Let R be a graded ring with 1 and denote by $Mod(R)$ the category of graded R-modules, where the morphisms are R-modules homomorphisms of degree 0. This category is abelian and possesses arbitrary direct and inverse limits (perform all the relevant contructions degreewise). If R is also commutative, then the graded tensor product yields a biadditive functor which is associative, commutative and has R as a unit (up to coherence). Moreover, for any $N \in Mod(R)$, the functor

$$- \otimes_R N : Mod(R) \longrightarrow Mod(R), \; M \mapsto M \otimes_R N$$

is right exact. In our applications, we will be dealing with particular graded rings and special subcategories of their modules categories. In the sequel, all inverse systems and limits are taken over direct sets.

A *profinite graded ring* is an inverse limit of graded rings of finite length (i.e nœtheriean and artinian). We emphasize that, according to this definition, every graded field is profinite; recall that a graded field is a graded ring whose non zero homogeneous elements are all invertible. If $R = \underleftarrow{lim}\ R_\alpha$ is a profinite ring and $\pi_\alpha : R \longrightarrow R_\alpha$ are the canonical projections, the family of graded ideals $\{Ker(\pi_\alpha)\}$ equipp R with a topology which is complete, Hausdorff and compatible with the graded ring structure.

Example 1 Let $\mathcal{E}^*(-)$ be a multiplicative (unreduced) cohomology theory such that $\mathcal{E}^*(pt)$ is a graded field. In other words, all graded $\mathcal{E}^*(pt)$-modules are free. Given a CW-complex X, let $\{X_\alpha\}$ be the direct system of all finite CW-subcomplexes of X. Then we have ([3, Theorem 4.14])

$$\mathcal{E}^*(X) \cong \underleftarrow{lim}\ \mathcal{E}^*(X_\alpha)\ ;$$

the corresponding topology will be called the *profinite topology* of $\mathcal{E}^*(X)$. Since the $\mathcal{E}^*(X_\alpha)$ are finitely generated free $\mathcal{E}^*(pt)$-modules, they are rings of finite length; hence $\mathcal{E}^*(X)$ is a profinite graded ring.

Let R be a profinite graded ring. We consider the full subcategory $\mathcal{F}(R)$ of $Mod(R)$ consisting of the objects M which are discrete topological R-modules and have finite length. Since the discrete topology is the only linear topology that an R-module of finite length can carry, the morphisms of $\mathcal{F}(R)$ are automatically continuous.

A *profinite* R-module is an inverse limit of of objects in $\mathcal{F}(R)$. Thus it carries a natural topology which makes it into a complete Hausdorff topological R-modules. Let $Mod^{prof}(R)$ be the subcategory of $Mod(R)$ whose objects are profinite R-modules and whose morphisms are continuous R-module homomorphisms of degree 0.

Example 2 $\mathcal{E}^*(-)$ is as in example 1 above. Let $f : X \longrightarrow B$ be a map of CW-complexes. By the CW-approximation theorem, the induced map $\mathcal{E}^*(f) : \mathcal{E}^*(B) \longrightarrow \mathcal{E}^*(X)$ is a continuous ring homomorphism (with respect to the profinite topologies). It follows that $\mathcal{E}^*(f)$ induces a profinite $\mathcal{E}^*(B)$-module structure on $\mathcal{E}^*(X)$.

Example 3 $\mathcal{E}^*(-)$ is as above and $B \xrightarrow{s} X \xrightarrow{p} B$ is a sectionned space over B. Fix a finite subcomplex X_α of X, set $B_\alpha = s^{-1}(s(B) \cap X_\alpha)$ and denote the image of X_α in $X/s(B)$ by $(X/s(B))_\alpha$. In the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{s} & X & \longrightarrow & X/s(B) \\ \uparrow & & \uparrow & & \uparrow \\ B_\alpha & \xrightarrow{s} & X_\alpha & \longrightarrow & (X/s(B))_\alpha \end{array}$$

the rows are cofibrations. Since $p \circ s = id$, the long exact sequences in $\mathcal{E}^*$-cohomology of these cofibrations reduce to the following commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\mathcal{E}}^*(X/s(B)) & \longrightarrow & \mathcal{E}^*(X) & \xrightarrow{s^*} & \mathcal{E}^*(B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\mathcal{E}}^*((X/s(B))_\alpha) & \longrightarrow & \mathcal{E}^*(X_\alpha) & \xrightarrow{s^*} & \mathcal{E}^*(B_\alpha) & \longrightarrow & 0\ . \end{array} \tag{14}$$

It follows that $\mathcal{E}^*(X) \cong \widetilde{\mathcal{E}}^*(X/s(B)) \oplus \mathcal{E}^*(B)$ as profinite $\mathcal{E}^*(B)$-modules, showing that $\widetilde{\mathcal{E}}^*_B(X) := \widetilde{\mathcal{E}}^*(X/s(B))$ is a profinite $\mathcal{E}^*(B)$-module with respect to the topology induced by $\mathcal{E}^*(X)$. Using diagram (14), one checks that this topology is the same as the profinite topology. Hence we have shown that $\widetilde{\mathcal{E}}^*_B(X)$ is always a profinite $\mathcal{E}^*(B)$-module with respect to the profinite topology.

Theorem 6.1 *Let R be a profinite graded ring.*

i) The category $Mod^{prof}(R)$ of profinite R-modules is abelian. It has enough projective objects and exact inverse limits.

ii) Every inverse limit of projective objects of $Mod^{prof}(R)$ is projective. The profinite R-module R is projective.

Proof. As easily checked $\mathcal{F}(R)$ is abelian, artinian (i.e, every descending chain of subobjects of any object of $\mathcal{F}(R)$ stabilizes) and equivalent to a small category. Let $Pro(\mathcal{F}(R))$ be the category of inverse systems in $\mathcal{F}(R)$ (see [19, page 21] for the definition). By [17] and [9, page 356]), $Pro(\mathcal{F}(R))$ is an abelian category with enough projective objects and exact inverse limits. Moreover every inverse system is isomorphic to a strict one (i.e, whose transition morphisms are epimorphisms). Let us now consider the functor

$$\Lambda : Pro(\mathcal{F}(R)) \longrightarrow Mod^{prof}(R), \quad (M_\alpha)_{\alpha \in I} \longmapsto \varprojlim M_\alpha\ .$$

The argument of Section 2.6 in [19] are easily adapted to our situation to show that Λ is an equivalence of categories. Consequently $Mod^{prof}(R)$ enjoys all the properties of $Pro(\mathcal{F}(R))$ mentionned above, and we are done with the first point of the theorem.
For the second part, we note that both $Pro(\mathcal{F}(R))$ and $Mod^{prof}(R)$ are proartinian in the sense of [4, page 563]. The first assertion follows from Corollaire 3.4 (page 567) in [4]. Finally R is projective because every morphism from R into a profinite R-module M is of the form $r \mapsto r \cdot m$ for some $m \in M$. $\square$

For the rest of this section, R denotes a profinite graded ring which is commutative. Our next aim is to study the topological tensor product in $Mod^{prof}(R)$. We start with

Lemma 6.2 *If M and M' are in $\mathcal{F}(R)$, so is $M \otimes_R M'$.*

Proof. Since M is finitely generated, its annihilator $Ann(M) = \{r \in R;\ r \cdot m = 0,\ \forall m \in M\}$ is an open ideal in R (it is the intersection of the annihilators of the members of a finite generating set of M). Similarly $Ann(M')$ is an open ideal of R. We write $R = \varprojlim R_\alpha$, with each projection $\pi_\alpha : R \longrightarrow R_\alpha$ surjective (this is always possible).Then there exists α such that $Ker(\pi_\alpha) \subset Ann(M) \cap Ann(M')$. Consequently M, M' and $M \otimes_R M'$ can be regarded as R_α-modules. These new structures are strongly related to the former since any subset of M (respectively M', $M \otimes_R M'$) is a R-submodule if and only if it is a R_α-submodule. We observe now that R_α is a ring of finite length and $M \otimes_R M'$ is a finitely generated R_α-module; this implies that $M \otimes_R M'$ is a R-module of finite length, that is, an object of $\mathcal{F}(R)$. $\square$

Let $M = \varprojlim M_\alpha$ and $N = \varprojlim N_\beta$ be two modules in $Mod^{prof}(R)$. Their *completed tensor product* is defined as

$$M \widehat{\otimes}_R N = \varprojlim\ (M_\alpha \otimes_R N_\beta)\ .$$

The preceding lemma insures that $M \widehat{\otimes}_R N$ lies in $Mod^{prof}(R)$. It can be shown that $M \widehat{\otimes}_R N$ is the completion of the ordinary tensor product $M \otimes N$ with respect to a certain filtration (see [3, page 603]). This last assertion is very useful in proving that $M \widehat{\otimes}_R N = M \otimes_R N$ when N is a finitely generated R-module.

Theorem 6.3 *Let N be a profinite R-module.*

i) The functor $-\widehat{\otimes}_R N : Mod^{prof}(R) \longrightarrow Mod^{prof}(R)$ is right exact and commutes with inverse limits.

ii) Let $\mathcal{T}or^j_R(-,N)$ ($j = 0,1,2,\dots$) denote the j-th left derived functor of $-\widehat{\otimes}_R N$ (their existence follows from the preceding point). Then $\mathcal{T}or^j_R(-,N)$ commutes with inverse limits. Moreover it coincides with the usual $Tor^j_R(-,N)$ when N is a finitely generated R-module.

Proof.

i) This point is true because of the right exactness of $-\otimes_R N$ and because inverse limits are exact and commute with each other in $Mod^{prof}(R)$.

ii) The first assertion is standard homological algebra (see for example Corollaire 3.8 (page 568) in [4]). The last assertion is a consequence of the fact stated before the theorem. □

References

[1] J.F. Adams, Stable homotopy and generalized homology, University of Chicago Press (1974).

[2] J.M. Boardman, Conditionally convergent spectral sequences, preprint 1981.

[3] J.M. Boardman, Stable operations in generalized cohomology, Handbook of Algebraic Topology (I.M. James, ed.), Chap. 14, North Holland (1995).

[4] M. Demazure & P. Gabriel, Groupes algébriques (tome I), Masson & Cie - North Holland (1970).

[5] A. Dold, Chern classes in general cohomology, Symposia Math. vol. V (Geometria), Instituto Naz. di Alta Matematica, Roma (1971).

[6] W.G. Dwyer & C.W. Wilkerson, A new finite loop space at the prime 2, J. Amer. Math. Soc. 6 (1993), 37-64.

[7] W.G. Dwyer & C.W. Wilkerson, Homotopy fixed point methods for Lie groups and finite loop spaces, Ann. of Math. (2) 139 (1994), 395-442.

[8] W.G. Dwyer & C.W. Wilkerson, The center of a p-compact group, Proceedings of the Čech centennial homotopy theory conference, AMS Contemporary Mathematics, vol. 181 (M. Cenkl and H. Miller, eds), 1995, pp.119-157.

[9] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France 90 (1962), 323-448.

[10] R.P. Held & U. Suter, On the unitary K-theory of compact Lie groups with finite fundamental group, Quart. J. Math. 24 (1973), 343-356.

[11] L. Hodgkin, The equivariant Künneth theorem in K-theory, Springer Lecture Notes in Math. No 496 (1975), 1-101.

[12] A. Jeanneret & A. Osse, The K-Theory of p-Compact Groups, preprint (1996), *to appear in* Comment. Math. Helv.

[13] R. Kultze, Über multiplikative Eigenschaften von spektralen Sequenzen, Math. Annalen 158 (1965), 233-268.

[14] N. Mahammed, R. Piccinini & U. Suter, Some applications of topological K-theory, North-Holland Mathematical Studies No 45, North-Holland (1980).

[15] J.M. Møller & D. Notbohm, Centers and finite coverings of finite loop spaces, J. reine angew. math. 456 (1994), 99-133.

[16] F. Neumann, On the cohomology of homogeneous spaces of finite loop spaces and the Eilenberg-Moore spectral sequence, preprint 1997.

[17] F. Oort, Commutative group schemes, Lecture Notes in Math. No 15, Springer (1966).

[18] L. Smith, Lectures on the Eilenberg-Moore spectral sequence, Lecture Notes in Math. No 134, Springer (1970).

[19] J.-P. Serre, Groupes proalgébriques, Publ. Math. IHES 7 (1960), 5-68.

[20] M. Tanabe, On Morava K-theories of Chevalley groups, Amer. J. Math. 117 (1995), 263-278.

[21] C.A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics No 38, Cambridge University Press (1994).

[22] G.W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics No 61, Springer (1979).

[23] U. Würgler, Cobordism theories of unitary manifolds with singularities and formal group laws, Math. Zeit. 150 (1976), 239-260.

Mathematisches Institut
Sidlerstrasse 5, CH-3012 BERN (Switzerland)
email: jeanner@math-stat.unibe.ch

Institut de Mathématiques
Rue Emile Argand 11, CH-2007 NEUCHATEL (Switzerland)
email: Akimou.Osse@maths.unine.ch