

On a Theorem of Ossa

David Copeland Johnson
University of Kentucky
Lexington, Kentucky 40506
johnson@ms.uky.edu

and

W. Stephen Wilson
Johns Hopkins University
Baltimore, Maryland 21218
wsw@math.jhu.edu

21 December 1995

Abstract

If V is an elementary abelian 2-group, Ossa proved that the connective K -theory of BV splits into copies of $\mathbf{Z}/2$ and of the connective K -theory of the infinite real projective space. We give a brief proof of Ossa's theorem.

Introduction. We have been asked whether our work, [1] and [2] on the Brown-Peterson homology of BV , V an elementary p -group, gives a nice structure of the connective K -theory of BV . The answer is that the approach of [1] leads to the elegant structure theorem of Ossa [3]. Although the approach is motivated by our [1] and [2], the proof is independent of that work. In this reproof of an established theorem we shall limit our exposition to the $p = 2$ case. For us, the notation makes this the easiest case, but for Ossa, it was the more difficult one. With obvious modifications, the odd-primary version of our argument follows the same outline.

Notation. Let bu be the connective K -theory spectrum and let P denote $B\mathbf{Z}/2$ (also known as infinite real projective space). Let $H\mathbf{Z}/2$ be the $\mathbf{Z}/2$ Eilenberg-MacLane spectrum.

Theorem 1 (Ossa) *With the above notation, there is a homotopy equivalence of spectra*

$$bu \wedge P \wedge P \simeq \left[\bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbf{Z}/2 \right] \vee [\Sigma^2 bu \wedge P].$$

Note that $H\mathbf{Z}/2 \wedge P \simeq \bigvee_{0 < i} \Sigma^i H\mathbf{Z}/2$. (The proof of this is like that of Lemma 3.) Thus the theorem can be used inductively to split $bu \wedge P \wedge \cdots \wedge P$ into suspended copies of $H\mathbf{Z}/2$ and one suspended copy of $bu \wedge P$. Since $bu \wedge BV = bu \wedge (P \times \cdots \times P)$ is a wedge sum of $bu \wedge P \wedge \cdots \wedge P$'s, we get the following corollary.

Corollary 2 *Let V be an elementary abelian p -group, then $bu_*(BV)$ is isomorphic to a sum of suspended copies of $\mathbf{Z}/2$ and of $bu_*(P)$. \square*

Lemma 3 *There is a homotopy equivalence $bu \wedge \mathbf{C}P^\infty \simeq \bigvee_{0 < n} bu \wedge S^{2n}$. In particular, there is a projection $\rho : bu \wedge \mathbf{C}P^\infty \longrightarrow bu \wedge S^2$.*

Proof. Choose $f_n : S^{2n} \longrightarrow bu \wedge \mathbf{C}P^\infty$ representing the bu_* generators of $bu_*(\mathbf{C}P^\infty)$. Define $f : \bigvee_{0 < n} S^{2n} \longrightarrow bu \wedge \mathbf{C}P^\infty$ by $f|_{S^{2n}} = f_n$. We have the composition

$$F : \bigvee_{0 < n} bu \wedge S^{2n} \xrightarrow{bu \wedge f} bu \wedge bu \wedge \mathbf{C}P^\infty \xrightarrow{\mu \wedge \mathbf{C}P^\infty} bu \wedge \mathbf{C}P^\infty$$

where μ is the pairing of the bu spectrum. F induces an isomorphism in homotopy and thus is an equivalence. \square

The Proof of Theorem 1. Let $\pi : P \longrightarrow \mathbf{C}P^\infty$ represent the nonzero second dimensional integral homology class of P . Define g_1 to be the composition

$$g_1 : bu \wedge P \wedge P \xrightarrow{bu \wedge \pi \wedge P} bu \wedge \mathbf{C}P^\infty \wedge P \xrightarrow{\rho \wedge P} bu \wedge S^2 \wedge P.$$

Let $H^*(P \wedge P; \mathbf{Z}/2) \cong \mathbf{Z}/2[s, t](st)$ be the mod 2 cohomology of $P \wedge P$. For $b = s^{2i-1} \wedge t^{2j-1} \in H^{2i+2j-2}(P \wedge P; \mathbf{Z}/2)$, let $g_b : P \wedge P \rightarrow \Sigma^{\dim(b)} H\mathbf{Z}/2$. Now construct the map g_0 by the following composition

$$g_0 : bu \wedge P \wedge P \xrightarrow{bu \wedge \vee_i g_b} bu \wedge \left[\bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbf{Z}/2 \right] \xrightarrow{\vee_b \nu} \left[\bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbf{Z}/2 \right]$$

where $\nu : bu \wedge H\mathbf{Z}/2 \rightarrow H\mathbf{Z}/2$ is the standard pairing making mod 2 homology a module theory over connective K -theory. The map

$$g = g_0 \vee g_1 : bu \wedge P \wedge P \rightarrow \left[\bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbf{Z}/2 \right] \vee [\Sigma^2 bu \wedge P]$$

is our candidate for the equivalence.

Let A be the mod 2 Steenrod algebra and $E = E[Q_0, Q_1]$ ($Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2 Sq^1$). Then $H^*(bu; \mathbf{Z}/2) \cong A//E$. In $H^*(P \wedge P; \mathbf{Z}/2)$, the classes $\{s^2 \wedge t^i : i > 0\}$ give a basis for an E -module M^* isomorphic to $H^*(S^2 \wedge P; \mathbf{Z}/2)$. Modulo M^* , $H^*(P \wedge P; \mathbf{Z}/2)$ is isomorphic to a free E -module with basis $\{s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$. Clearly in dimension 2,

$$(bu \wedge \pi)^* \circ \rho^* : H^2(bu \wedge S^2; \mathbf{Z}/2) \rightarrow H^2(bu \wedge P; \mathbf{Z}/2)$$

is an isomorphism. Thus g_1^* takes $H^2(bu \wedge S^2 \wedge P; \mathbf{Z}/2)$ isomorphically onto $A//E \otimes M^*$. By the construction of g_0 , we see that g_0^* takes $H^*(\bigvee_{0 < i, j} \Sigma^{2i+2j-2} H\mathbf{Z}/2; \mathbf{Z}/2)$ onto the free $A//E \otimes E \cong A$ -module with basis $\{s^{2i-1} \wedge t^{2j-1} : i, j > 0\}$. Thus g induces an isomorphism in mod 2 cohomology and thus is an equivalence. \square

References

- [1] D. C. Johnson and W. S. Wilson, The Brown-Peterson homology of elementary p -groups, *Amer. J. Math.* **102** (1982), 427-454.
- [2] D. C. Johnson, W. S. Wilson, and D. Y. Yan, Brown-Peterson homology of elementary p -groups II, *Topology and its Applications* **59** (1994) 117-136.
- [3] E. Ossa, Connective K -theory of elementary abelian groups, *Transformation Groups, Osaka 1987*, K. Kawakubo (ed.), Springer Lecture Notes in Mathematics **1375** (1989) 269-275.