A CUBICAL MODEL FOR A FIBRATION

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Abstract. In the paper the notion of a truncating twisting function from a simplicial set to a cubical set and the corresponding notion of twisted Cartesian product of these sets are introduced. The latter becomes a cubical set. This construction together with the theory of twisted tensor products for homotopy G-algebras allows to obtain multiplicative models for fibrations.

1. Introduction

For a fibration $F \to E \to Y$ on the tensor product $C^*(Y) \otimes C^*(F)$ E. Brown [8] has introduced a *twisted* differential d_τ such that the homology of the cochain complex $(C^*(Y) \otimes C^*(F), d_\tau)$ is isomorphic to the cohomology $H^*(E)$ but just additively. So there arises the problem of introducing of an *associative multiplication* on that complex to describe $H^*(E)$ as an algebra too. In this way various multiplicative models were constructed in which either the associativity was abolished or a differential was not a derivation (see, for example, L. Lambe and J. Stasheff [21] for references).

The standard notion of a twisting function $\tau : X_* \to G_{*-1}$ from a simplicial set to a simplicial group and the notion of corresponding twisted Cartesian product $X \times_\tau G$ does not allow to introduce directly a desired comultiplication on the twisted tensor product $C_*(X) \otimes_{C_*(\tau)} C_*(G)$ of the simplicial chain complexes since it does not coincide with the simplicial chain complex $C_*(X \times_\tau G)$. The situation radically changes if we replace simplicial group G by a *monoidal cubical set* and suitably modify the notion of a twisting function.

This idea comes from recent results of N. Berikashvili: In [5] a multiplicative model with associative multiplication in the case when the fiber F is the *cubical* version of the Eilenberg-MacLane space is constructed; in the next paper [6] a multiplicative model $C^*(Y) \otimes_\phi C^*_\square(F)$, $\phi : C^*_\square(G) \to C^{*+1}(Y)$ is constructed where $C^*(Y)$ is the singular *simplicial* cochain complex of the base and $C^*_\square(G)$ and $C^*_\square(F)$ are the singular *cubical* cochain complexes of the structure group and the fiber respectively.

In this paper we begin to develop the general theory of twisting functions to form twisted Cartesian products of abstract sets of *different* kind. The continuation will follow in [19] where twisted functions from cubical sets to permutahedral sets will be considered.

Here we introduce the notion of a *truncating twisting function* $\tau : X_* \to Q_{*-1}$ where X is a 1-reduced *simplicial* set and Q is a monoidal cubical set (that is there is given a cubical map $Q \times Q \to Q$ with $(Q \times Q)_n = \bigcup_{p+q=n} Q_p \times Q_q$). For a cubical set L with a given Q-action such a twisting function τ determines the *twisted Cartesian product* $X \times_\tau L$ being a *cubical set*.

To present an universal example of a truncating twisting function we construct a functor assigning to a simplicial set X a monoidal cubical set ΩX together with the canonical truncating twisting function $\tau : X \to \Omega X$ in such a way that any truncating function $\tau' : X_* \to Q_{*-1}$ factors through it, that is, $\tau' : X \xrightarrow{\tau} \Omega X \to Q$ where the second map is a monoidal cubical map.

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The twisted Cartesian product $\mathbf{P}X = X \times_\tau \Omega X$ is a cubical set functorially depending on X. Note that ΩX models the loop space $\Omega|X|$ and PX models the path space fibration on $|X|$. Moreover, the normalized chain complex $C_*^\square(\Omega X)$ coincides with Adams' cobar construction $\Omega C_*(X)$ and $C_*^\square(\mathbf{P}X)$ coincides with the acyclic cobar construction $\Omega(C_*(X); C_*(X))$.

Applying the chain functor to $\tau : X_* \to Q_{*-1}$ we obtain a twisting cochain $\tau_* = C_*(\tau) : C_*(X) \to C_{*-1}^\square(Q)$ and then $C_*^\square(X \times_\tau L)$ coincides with the twisted tensor product $C_*(X) \otimes_{\tau_*} C_*^\square(L)$. Furthermore, the cubical structure of $X \times_\tau L$ evidently determines on $C_*^\square(X \times_\tau L) = C_*(X) \otimes_{\tau_*} C_*^\square(L)$ an *associative comultiplication.* Dually on $C^*(X) \otimes_{\tau^*} C^*_\square(L) \subset C^*_\square(X \times_\tau L)$ (we have here the equality too provided the graded sets in question are of finite type) one immediately obtains a desired multiplication.

From computational point of view the next question is how to express this (co)multiplication in terms of tensor factors. It appears that the translation of the cubical diagonal of $C_*^\square(X \times_\tau L)$ describes the comultiplication on $C_*(X) \otimes_{\tau_*} C_*^\square(L)$ by canonical higher order chain operations of degree k

$$E^{k,1} : C_*(X) \to C_*(X)^{\otimes k} \otimes C_*(X),\ k \geq 0,$$

on $C_*(X)$, which agree with operations constructed by Baues [2], [3] on the normalized complex $C_*^N(X)$, and by the action $C_*^\square(Q) \otimes C_*^\square(L) \to C_*^\square(L)$ involving also τ_*. Interestingly, τ_* takes place in both definition of a twisted differential and of a twisted comultiplication !

Moreover, the cooperation $E^{1,1}$ is the dual operation of Steenrod's cochain $\smile_1$-operation and all $\{E^{k,1}\}$ operations introduce on $C_*(X)$ the structure of a *homotopy G-coalgebra* just being the dual notion of a *homotopy G-algebra* in the sense of [12].

Of course, algebraically a reason of difficulty in obtaining the associative multiplication on twisted tensor products was hidden in the non-commutativity of the Alexander-Whitney diagonal. The classical tool which measures the lack of commutativity, Steenrod's $\smile_1$-product, as we now can clearly see, is not enough to compensate it. The above fact justifies the structure of a homotopy G-algebra as a good notion for a dga with $\smile_1$. Note also that the specification of this structure on a dg (co)algebra A is equivalent to a dg Hopf algebra structure on the (co)bar construction $(\Omega A)BA$.

We develop the theory of *multiplicative* twisted tensor products for homotopy G-algebras in the following sense. In [24] it is shown that if a twisting element $\phi : C \to A$ from a dg Hopf algebra C to a *commutative* dga A is *coprimitive*, that is if the induced map $C \to BA$ is a map of dg Hopf algebras (where on BA the standard shuffle multiplication is meant), and M is dga being also a comodule over C then the twisted tensor product $A \otimes_\phi M$ is a dga with respect to standard multiplication of the tensor product $A \otimes M$ of dga's.

Now we replace the commutative A by a homotopy G-algebra. By definition this structure determines on BA a strictly associative multiplication converting BA into a dg Hopf algebra (this multiplication can be viewed as a perturbation of the shuffle one). A twisting element $\phi : C \to A$ wa call *multiplicative*, if the induced map $C \to BA$ is a dg Hopf algebra map. In this case we introduce on $A \otimes_\phi M$ a twisted associative multiplication μ_ϕ in terms of ϕ and homotopy G-algebra structure of A by the same formulas as in the case $A = C^*(X)$, $C = C^*_\square(Q)$ and $M = C^*_\square(L)$. It remains only to remark that $\tau^* : C^*_\square(Q) \to C^*(X)$ provides a basic example of a multiplicative twisting element.

Thus, the above theory unifies commutative and homotopically commutative cases, in particular, the cases of singular and Sullivan de-Ram cochain complexes for topological spaces.

Now for a fibration $F \to E \to Y$ on a 1- connected space Y associated with a principal G-fibration $G \to E' \to Y$ by an action $G \times F \to F$ we obtain a combinatorial model, being a cubical set, in the following way. Let $X = Sing^1 Y \subset Sing Y$ be the Eilenberg 1-subcomplex generated by singular simplices sending the 1-skeleton of the standard n-simplex Δ^n to the base point of Y, and let $Q = Sing^I G$ and $M = Sing^I F$. We have that Adams' map $\omega_* : \Omega C_*(Y) = C_*(\Omega X) \to C_*^\square(\Omega Y)$ is in fact realized by a monoidal cubical map $\omega : \Omega X \to Sing^I \Omega Y$. Composing this map with the map of monoidal cubical sets $Sing^I \Omega Y \to Sing^I G = Q$ induced by the canonical map $\Omega Y \to G$ of monoids we immediately obtain a truncating twisting function $\tau : X \to Q$.

The resulting twisted Cartesian product $Sing^1Y \times_\tau Sing^IF$ just provides the required cubical model of E: there exists a cubical weak equivalence $Sing^1Y \times_\tau Sing^IF \to Sing^IE$. Applying cochain functor we obtain the above multiplicative twisted tensor product [6]. Note also that in this multiplicative model it could be introduced Steenrod's (co)chain operations as they are defined for cubical sets [16].

By using the standard triangulation of the cubes one obtains a map of dg Hopf algebras $C^*_N(G) \to C^*_\square(G)$ and then it is possible to obtain again multiplicative twisted tensor product $C^*(Y) \otimes_{\tau'^*} C^*_N(F)$ but now with respect to the multiplicative twisting cochain $\tau'^* : C^*_N(G) \to C^*_\square(G) \xrightarrow{\tau^*} C^*(Y)$. In other words, for a special twisting cochain one introduces on Brown's model a required multiplication.

Finally, we mention that the geometric realization $|\Omega Sing^1Y|$ of $\Omega Sing^1Y$ is homeomorphic to the cellular model for the loop space observed by G. Carlsson and R. J. Milgram [9]. In [2], [3] H.-J. Baues has defined a geometric coassociative and homotopy cocommutative diagonal on the cobar construction $\Omega C^N_*(Y)$ by means of a certain cellular model for the loop space (homotopically equivalent to $|\Omega Sing^1Y|$) the cellular chains of which coincide with the $\Omega C^N_*(Y)$; consequently, one obtains a homotopy G-coalgebra structure on $C^N_*(Y)$. Another modification of Adams' cobar construction is considered by Y. Felix, S. Halperin and J.-C. Thomas [10].

The paper is organized as follows: Section 2 contains some background material; in Section 3 we construct two functors Ω and $\mathbf{P}$ from the category of simplicial sets to the category of cubical sets, so in fact here we construct the universal twisted Cartesian product $\mathbf{P}X = X \times_\tau \Omega X$; based on this universal example we introduce the notion of a truncating twisting function and the corresponding twisted Cartesian product in the next Section 4; in Section 5 we build the cubical set model for the path space fibration; in Section 6 a cubical model and the corresponding multiplicative twisted tensor product for a fibration are constructed, and, finally, in the Section 7 the twisted tensor product theory for homotopy G-algebras is developed containing an application to Brown's model too.

2. Notation and preliminaries

Let R be a commutative ring with unit 1. A *differential graded algebra* (dga) is a graded R-module $C = \{C^i\},\ i \in \mathbb{Z}$, with an associative multiplication $\mu : C^i \otimes C^j \to C^{i+j}$ and a homomorphism (*a differential*) $d : C^i \to C^{i+1}$ with $d^2 = 0$ and satisfying the Leibniz rule $d(xy) = d(x)y + (-1)^{|x|}xd(y)$, where $xy \in C^{i+j}$ is the element $\mu(x \otimes y),\ x \in C^i,\ y \in C^j,\ |x| = i$. We assume that a dga contains a unit $1 \in C^0$. A non-negatively graded dga C is *connected* if $C^0 = R$. A connected dga C is *n-reduced* if $C^i = 0, 1 \le i \le n$. A dga is *commutative* if $\mu = \mu T$, where $T(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$.

A *differential graded coalgebra* (dgc) is a graded R-module $C = \{C_i\},\ i \in \mathbb{Z}$, with an coassociative comultiplication $\Delta : C \to C \otimes C$ and a homomorphism (*a differential*) $d : C_i \to C_{i-1}$ with $d^2 = 0$ and satisfying $\Delta d = (d \otimes 1 + 1 \otimes d)\Delta$. A dgc C is assumed to have a counit $\epsilon : C \to R,\ \ (\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta = 1$. A non-negatively graded dgc C is *connected* if $C_0 = R$. A connected dgc C is *n-reduced* if $C_i = 0, 1 \le i \le n$. A dgc is *cocommutative* if $\Delta = \Delta T$.

A (connected) *differential graded Hopf algebra* (dgha) (C, μ, Δ) is a connected dga (C, μ) and a connected dgc (C, Δ) simultaneously such that $\Delta : C \to C \otimes C$ is an algebra map.

A dga M is a (left) *comodule* over a dgha C if $\Delta : M \to C \otimes M$ is a dga map.

2.1. **Cobar and Bar constructions.** For an R-module M let $T(M) = \oplus_{i=0}^\infty T^i(M)$, $T^i(M) = M^{\otimes i}$ where $T^0(M) = R$. Denote the tensor product of elements $a_j \in M$ by $[a_1|\cdots|a_n] \in T^n(M)$. By $s^{-1}M$ we denote the desuspension of M, i.e. $(s^{-1}M)_i = M_{i+1}$.

Let (C_*, d_C, Δ) be a 1-reduced dgc. Let $\Delta' : C \xrightarrow{\Delta} C \otimes C \xrightarrow{pr} C_{>0} \otimes C_{>0}$. The (reduced) cobar construction ΩC on C is the tensor algebra $T(\bar{C})$, $\bar{C} = s^{-1}(C_{>0})$, with differential $d = d_1 + d_2$ defined for $\bar{c} \in \bar{C}_{>0}$ by

$$d_1[\bar{c}] = -[\overline{d_C(c)}]$$

and

$$d_2[\bar{c}] = \sum (-1)^{|c'|}[\bar{c'}|\bar{c''}] \text{ for } \Delta'(c) = \sum c' \otimes c'',$$

and extended as a derivation.

The acyclic cobar construction $\Omega(C;C)$ is the twisted tensor product $C \otimes \Omega C$ in which the tensor differential is twisted by the canonical twisting map $C \to \Omega C$ being an inclusion of degree -1.

Let (A, d_A, μ) be a 1-reduced dga. The (reduced) bar construction BA on A is the tensor coalgebra $T(\bar{A})$, $\bar{A} = s^{-1}(A_{>0})$, with differential $d = d_1 + d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n] \in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n] = -\sum_{i=1}^{n}(-1)^{\varepsilon_i}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n],$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n] = -\sum_{i=2}^{n}(-1)^{\varepsilon_i}[\bar{a}_1|\cdots|\overline{a_{i-1}a_i}|\cdots|\bar{a}_n],$$

$\varepsilon_i = \sum_{j<i} |\bar{a_j}|$.

The acyclic bar construction $B(A;A)$ is the twisted tensor product $A \otimes BA$ in which the tensor differential is twisted by the canonical twisting map $BA \to A$ being a projection of degree 1.

2.2. **Adams' cobar construction.** For a 1-reduce simplicial complex X with a base point $*$ (i.e. $X_i = *_i, i = 0, 1,$) let $\tilde{C}_*(X)$ be its chain complex in the ordinary sense. Then define the chain complex $C_*(X)$ as to be

$$C_*(X) = \tilde{C}_*(X)/\tilde{C}_{>0}(*).$$

Clearly it is a 1-reduced dgc with respect to the AW diagonal. Let $\operatorname{Sing} Y$ be the singular simplicial complex of a topological space Y and $X = \operatorname{Sing}^1 Y \subset \operatorname{Sing} Y$ be the (Eilenberg) 1-subcomplex generated by those singular simplices which send the 1-skeleton of the standard simplex Δ^n, $n \geq 0$, at the base point of Y. Let define the dgc $C_*(Y)$ as $C_*(Y) = C_*(X)$. Then Adams' cobar construction $\Omega C_*(Y)$ is the cobar construction of the dgc $C_*(Y)$.

2.3. **Cubical sets.** A cubical set is a sequence of sets $Q = \{Q_n\}_{n \geq 0}$ with boundary operators $d_i^\epsilon : Q_n \to Q_{n-1}$, $\epsilon = 0, 1$, $1 \leq i \leq n$, and degeneracy operators $\eta_i : Q_n \to Q_{n+1}, 1 \leq i \leq n+1$, satisfying the standard conditions [15].

An example of a cubical set is the singular cubical set $\operatorname{Sing}^I Y = \{\operatorname{Sing}_n^I Y\}_{n \geq 0}$ of a space Y, where $\operatorname{Sing}_n^I Y$ is the set of all continuous maps $I^n \to Y$ [22].

Analogously to a simplicial set for a cubical set Q and an R-module A its chain complex in coefficients A is defined which will be denoted by $(\bar{C}_*^\square(Q;A), d)$. The normalized chain complex $(C_*^\square(Q;A), d)$ of Q is defined as the quotient $C_*^\square(Q;A) = \bar{C}_*^\square(Q;A)/D_*(Q)$, where $D_*(Q)$ is the subcomplex of $(\bar{C}_*^\square(Q;A), d)$ generated by the degenerate elements of Q. Note also that both $\bar{C}_*^\square(Q)$ and $C_*^\square(Q)$ are DG-coalgebras with respect to the canonical comultiplication determined by the Cartesian product decomposition of the n-cube $I^n = I \times \cdots \times I$[27]. For a space Y we will denote $C_*^\square(\operatorname{Sing}^I Y; \mathbb{Z})$ by $C_*^\square(Y)$.

The *(tensor) product* of two cubical sets Q and Q' is

$$Q \times Q' = \{(Q \times Q')_n = \bigcup_{p+q=n} Q_p \times Q'_q\}/\sim$$

where $(\eta_{p+1}(a), b) \sim (a, \eta_1(b))$, $(a, b) \in Q_p \times Q'_q$, and endowed with the obvious face and degeneracy operators [15].

A *graded monoidal cubical set* we define as a cubical set Q with a cubical map $\mu : Q \times Q \to Q$, which is associative and has the unit $e \in Q_0$. Clearly, for a monoidal cubical set its chain complex $C_*^\square(Q;R)$ is a dg Hopf algebra as well as $C^*_\square(Q;R)$.

For a graded monoidal cubical set Q a Q-module we define as a cubical set L together with associative action $Q \times L \to L$ and the unit of Q acting on it as identity. In this case $C^*_\square(L;R)$ is a dga comodule over dg Hopf algebra $(C^*_\square(Q;R), d)$.

3. The cubical set functors ΩX and $\mathbf{P}X$

3.1. **The cubical set functor ΩX.** First, for a simplicial set $X=\{X_n,\partial_i,s_i\}_{n\geq 0}$, let define the graded set $\Omega' X$ as follows. Let X^c be the graded set of formal expressions

$$X^c_{n+k}=\{\eta_{i_k}\cdots\eta_{i_1}\eta_{i_0}(x)|\, x\in X_n\}_{n\geq 0;k\geq 0},$$

where

$$i_1\leq\cdots\leq i_k,\ 1\leq i_j\leq n+j-1,\ 1\leq j\leq k,\ \eta_{i_0}=1,$$

and let $\bar X^c=s^{-1}(X^c_{>0})$ denote the desuspension of X^c. Then define $\Omega'' X$ as the free graded monoid (without unit) generated by $\bar X^c$. Elements of $\Omega'' X$ we denote by $\bar x_1\cdots\bar x_k$ for $x_j\in X^c_j,\ 1\leq j\leq k$. The product of two elements $\bar x_1\cdots\bar x_k$ and $\bar y_1\cdots\bar y_\ell$ is defined by concatenation $\bar x_1\cdots\bar x_k\bar y_1\cdots\bar y_\ell$. This only subject to the associativity relation and no other relations whatsoever between the expressions. The total degree of an element $\bar x_1\cdots\bar x_k$ is the sum $m_{(k)}=m_1+\cdots+m_k,\ m_j=|\bar x_j|$, and we write $\bar x_1\cdots\bar x_k\in(\Omega' X)_{m_{(k)}}$.

Let $\Omega' X$ be the monoid obtained from $\Omega'' X$ via

$$\Omega' X=\Omega'' X/\sim\ ,$$

where $\overline{\eta_{p+1}(x)}\cdot\bar y\sim\bar x\cdot\overline{\eta_1(y)}$ for $x,y\in X^c$, $|x|=p+1$. Clearly, we have the inclusion $MX\subset\Omega' X$ of graded monoids where MX denotes the free monoid generated by $\bar X=s^{-1}(X_{>0})$.

It appears that $\Omega' X$ canonically admits the structure of a cubical set. Namely, denote by

$$\nu_i:X_n\to X_i\times X_{n-i},\quad \nu_i(x)=\partial_{i+1}\cdots\partial_n(x)\times\partial_0\cdots\partial_{i-1}(x),\quad 0\leq i\leq n,$$

the components of AW diagonal, and let x^n denote an n-simplex, i.e. $x^n\in X_n$. Then

$$\nu_i(x^n)=((x')^i,\ (x'')^{n-i})\in X_i\times X_{n-i}$$

for all $n\geq 0$. Define the face operators $d^0_i,d^1_i:(\Omega' X)_{n-1}\to(\Omega' X)_{n-2}$ on a (monoidal) generator $\overline{x^n}\in(\bar X)_{n-1}\subset(\overline{X^c})_{n-1}$ by

$$\begin{aligned}d^0_i(\overline{x^n})&=\overline{(x')^i}\cdot\overline{(x'')^{n-i}},\quad && i=1,...,n-1,\\ d^1_i(\overline{x^n})&=\overline{\partial_i(x^n)}, && i=1,...,n-1,\end{aligned}$$

and extend to elements $\bar x_1\cdots\bar x_k\in MX$ by

$$\begin{aligned}d^0_i(\bar x_1\cdots\bar x_k)&=\bar x_1\cdots\overline{(x'_q)^{j_q}}\cdot\overline{(x''_q)^{m_q+1-j_q}}\cdots\bar x_k,\\ d^1_i(\bar x_1\cdots\bar x_k)&=\bar x_1\cdots\overline{\partial_{j_q}(x_q)}\cdots\bar x_k,\end{aligned}$$

where $m_{(q-1)}<i\leq m_{(q)},\ j_q=i-m_{(q-1)},\ 1\leq q\leq k,\ 1\leq i\leq n-1$.

Then for $d^0_i,\ d^1_i$ the defining identities of a cubical set can be easily checked on MX. In particular, the simplicial relations between ∂_i's imply the cubical relations between d^1_i's; the associativity relations between ν_i 's imply the cubical relations between d^0_i's, and the commutativity relations between ∂_i's and ν_j's imply the cubical relations between d^1_i's and d^0_j's.

Before we extend the face operators on the whole $\Omega' X$ define a degeneracy operator η_i : $(\Omega' X)_{n-1}\to(\Omega' X)_n$ on a (monoidal) generator $\overline{x}\in(\overline{X^c})_{n-1}$ by

$$\eta_i(\overline{x})=\overline{\eta_i(x)}$$

and extend to elements $\bar x_1\cdots\bar x_k\in\Omega' X$ by

$$\begin{aligned}\eta_i(\bar x_1\cdots\bar x_k)&=\bar x_1\cdots\eta_{j_q}(\overline{x_q})\cdots\bar x_k,\\ \eta_n(\bar x_1\cdots\bar x_k)&=\bar x_1\cdots\bar x_{m_{k-1}}\cdot\eta_{m_k+1}(\overline{x_k}),\end{aligned}$$

where $m_{(q-1)}<i\leq m_{(q)},\ j_q=i-m_{(q-1)},\ 1\leq q\leq k,\ 1\leq i\leq n-1$.

Finally, inductively extend the face operators on these degenerate elements so that the defining identities of a cubical set are satisfied. It is easy to see that the cubical set $\{\Omega' X,d^0_i,d^1_i,\eta_i\}$ depends functorially on X.

It is convenient to verify the above cubical relations by the following combinatorics of the standard cube (compare, [4]).

Remark 3.1. *Motivated by the combinatorial description of the standard $(n+1)$-simplex Δ^{n+1} we denote the set $\{0,1,...,n+1\}$ by $[0,1,...,n+1]$ to assign to whole I^n. Next for the face operators*

$$d_i^0 \leftrightarrow x_1,...,x_{i-1},0,x_{i+1},...,x_n, \quad i=1,...,n$$
$$d_i^0 \leftrightarrow x_1,...,x_{i-1},1,x_{i+1},...,x_n, \quad i=1,...,n$$

(here faces of I^n are written in the Euclidean coordinates) put the correspondence

$$d_i^0 \leftrightarrow [0,1,...,i][i,...,n+1], \quad i=1,...,n$$
$$d_i^1 \leftrightarrow [0,1,...,\hat{\imath},...,n+1], \quad i=1,...,n.$$

In general, any q-dimensional face of I^n is expressed as a sequence of blocks

$$[0,i_1...,i_{k_1}][i_{k_1},...,i_{k_2}][i_{k_2},...,i_{k_3}]...[i_{k_{p-1}},...,i_{k_p},n+1], \quad 0<i_1<\ldots<i_{k_p}<n+1, \quad q=k_p-p+1.$$

In this combinatorics a cubical degeneracy operator η_i is thought of as to add a formal element $*$ to the set $[0,1,...,n+1]$ at the $i+1$-th place: $\eta_i[0,1,...,n+1]=[0,1,...,i-1,*,i,...,n+1]$ with the following convention: $[0,1,...,i-1,*][*,i,...,n+1]=[0,1,...,n+1]$ that guarantees the equality $d_i^0\eta_i=1=d_i^1\eta_i$. Geometrically this just agrees with the standard projections $I^{n+1}\to I^n$.

Now suppose X has a fixed vertex $*$. Then declare $\overline{s_0(*)}$ as a unit of $\Omega' X$ and denote it by e. Moreover, for each $x\in X_n$, $n>0$, we put the relation $\eta_n(\bar{x})=\overline{s_n(x)}$. Let $(\Omega X, d_i^0, d_i^1, \eta_i)$ be the resulting (unital) graded monoidal cubical set.

In particular, for a 1-reduced simplicial set X we will have the following identities

$$d_1^0(\overline{x^n})=\overline{(x')^1}\cdot\overline{(x'')^{n-1}}=e\cdot\overline{(x'')^{n-1}}=\overline{(x'')^{n-1}}=\overline{\partial_0(x^n)},$$
$$d_{n-1}^0(\overline{x^n})=\overline{(x')^{n-1}}\cdot\overline{(x'')^1}=\overline{(x')^{n-1}}\cdot e=\overline{(x')^{n-1}}=\overline{\partial_n(x^n)}, \quad x^n\in X_n.$$

Thus, all the face operators ∂_i of X are included in the definition of ΩX.

Remark 3.2. *Note that in the definition of ΩX we have to add formally the degeneracies, since simplicial degeneracies are not applicable unless the last one. This is also justified by the geometrical fact that in the path space fibration a degenerate singular n-simplex of base lifts to a singular $(n-1)$-cube of the fibre which need not to be degenerate in general (cf. the proof of Theorem 5.1).*

3.2. **The cubical set functor $\mathbf{P}X$.** First, for a simplicial set X we define the cubical set $\mathbf{P}'X$ as follows. Consider the Cartresian product $X^c{\times}\Omega' X=\{(X^c\times\Omega' X)_n=\bigcup_{p+q=n}X_p^c\times(\Omega' X)_q\}$ of the graded sets X^c and $\Omega' X$ (ignoring for the moment the underlying structures). Let

$$X^c\widetilde{\times}\Omega' X=X^c{\times}\Omega' X/\sim,$$

where $(\eta_{p+1}(x),y)\sim(x,\eta_1(y))$, $(x,y)\in X_p^c\times(\Omega' X)_q$. Then introduce on $X^c\widetilde{\times}\Omega' X$ the face operators d_i^0,d_i^1 and the degeneracy operators η_i as follows. For an element $(x,y)\in X_p\times(\Omega' X)_q\subset X_p^c\times(\Omega' X)_q$, $p+q=n$, let

$$d_i^0(x,y)=\begin{cases}((x')^{i-1},\overline{(x'')^{p+1-i}}\cdot y), & 1\le i\le p,\\ (x,d_{i-p}^0(y)), & p<i\le n,\end{cases}$$

$$d_i^1(x,y)=\begin{cases}(\partial_{i-1}(x),y), & 1\le i\le p,\\ (x,d_{i-p}^1(y)), & p<i\le n,\end{cases}$$

$$\eta_i(x,y)=(\eta_i(x),y), \quad 1\le i\le p,$$
$$\eta_i(x,y)=(x,\eta_{i-p}(y)), \quad p<i\le n+1.$$

It is easy to check that the face operators satisfy the canonical cubical identities. These data uniquely extend to the structure of a cubical set on whole $X^c\widetilde{\times}\Omega' X$. The resulting cubical set is denoted by $\mathbf{P}'X$, and then obtain the cubical set $\mathbf{P}X$ from $\mathbf{P}'X$ by replacing $\Omega' X$ by ΩX.

It is convenient to verify the cubical relations in $\mathbf{P}'X$ by the following combinatorics of the standard cube.

Remark 3.3. *To $\mathbf{P}'X$ it corresponds the following combinatorial model of the standard cube (compare Remark 3.1). Now denote the set $\{0,1,...,n+1\}$ by $0,1,...,n+1]$ and assign to whole I^{n+1}, while assign to its any proper q-face a sequence of blocks*

$$j_1...j_{s_1}][j_{s_1}...j_{s_2}][j_{s_2}...j_{s_3}]...[j_{s_{t-1}}...j_{s_t},n+1],\quad 0\leq j_1<\ldots<j_{s_t}<n+1,\quad q=s_t-t.$$

In particular the dimension of the block $0...j]$ is greater by one than the dimension of the block $[0...j]$. The face and degeneracy operators are acting on blocks as in Remark 3.1, but for the first block we have the following correspondence

$$\begin{array}{ll} d_i^0\leftrightarrow j_1...j_{i-1}][j_{i-1}...j_{s_1}], & 1\leq i<s_1,\\ d_i^1\leftrightarrow j_1...\widehat{j_i}...j_{s_1}], & 1\leq i<s_1.\end{array}$$

Obviously one has the inclusion of graded sets $\Omega X\to\mathbf{P}X$ defined by $y\to(*,y),\ *\in X_0$, and the projection $\mathbf{P}X\xrightarrow{\xi}X$ defined by $(x,y)\to x$. On the other hand, the canonical cellular map $\psi:I^{n+1}\to\Delta^{n+1}$ ([27]) admits the following combinatorial description

$$j_1...j_{s_1}][j_{s_1}...j_{s_2}][j_{s_2}...j_{s_3}]...[j_{s_{t-1}}...j_{s_t},n+1]\to\{j_1...j_{s_1}\}$$

(see, Fig. 1); in particular, the face $0][0,1,...,n+1]$ of I^{n+1}, i.e. $d_1^0(I^{n+1})$, is mapped onto the minimal vertex (the base point) $0\in\Delta^{n+1}$. So that ψ can be thought of as a combinatorial model for the projection ξ.

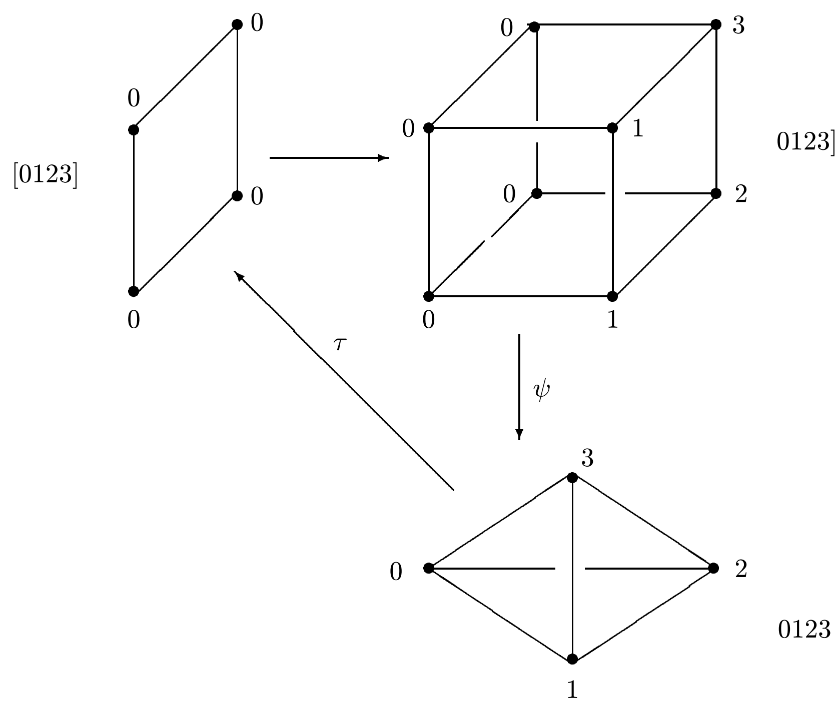

Figure 1: The universal truncating twisting function τ.

4. Truncating twisting functions and twisted Cartesian products

It appears that the cubical set $\mathbf{P}X$ can be viewed as a twisted Cartesian product determined by the inclusion function $\tau : X \to \Omega X$, $x \to \bar{x}$, being of degree -1. It will be referred to as the *universal truncating function.* Here we give the general formalism for description such functions.

Definition 4.1. *Let X be a 1-reduced simplicial set and Q be a monoidal cubical set. A sequence $\tau = \{\tau_n\},\ \tau_n : X_n \to Q_{n-1},\ n \geq 1,$ of degree -1 functions is a truncating twisting function if it satisfies:*

$$\begin{array}{llll} \tau(x) & = e, & & x \in X_1, \\ d_i^0\tau(x) & = \tau\partial_{i+1}\cdots\partial_n(x)\cdot\tau\partial_0\cdots\partial_{i-1}(x), & 1 \leq i \leq n-1, & x \in X_n,\ n \geq 1 \\ d_i^1\tau(x) & = \tau\partial_i(x), & 1 \leq i \leq n-1, & x \in X_n,\ n \geq 1 \\ \eta_n\tau(x) & = \tau s_n(x), & & x \in X_n,\ n \geq 1. \end{array}$$

Remark 4.1. *Note that by definition a truncating twisting function commutes only with the last degeneracy operators (compare [27]), since it is so for the universal truncating function.*

Proposition 4.1. *Let X be a 1-reduced simplicial set and Q be a monoidal cubical set. A sequence $\tau = \{\tau_n\}_{n\geq 1}$ of degree -1 functions $\tau_n : X_n \to Q_{n-1}$ is a truncating twisting function if and only if the monoidal map $f : \Omega X \to Q$ defined by $f(\bar{x}_1 \dots \bar{x}_k) = \tau(x_1)\dots\tau(x_k)$ is a map of cubical sets.*

Given a truncating twisting function τ and a cubical set L with cubical map $Q \times L \to L$ the corresponding twisted Cartesian product $X \times_\tau L$ is defined by replacing ΩX by L in the definition of $\mathbf{P}X$. Namely, we get the following

Definition 4.2. *Let X be a 1-reduced simplicial set, Q be a monoidal cubical set and L be a Q-module $Q \times L \to L$. Let $\tau = \{\tau_n\}_{n\geq 1},\ \tau_n : X_n \to Q_{n-1}$, be a truncating twisting function. The twisted Cartesian product $X \times_\tau L$ is the graded set*

$$X \times_\tau L = X^c \times L/\sim,$$

where $(\eta_{p+1}(x), y) \sim (x, \eta_1(y))$, $(x, y) \in X_p^c \times L_q$, and endowed with face and degeneracy operators d_i^0, d_i^1, and η_i defined for $(x, y) \in X_p \times L_q \subset X_p^c \times L_q$ by

$$d_i^0(x,y) = \begin{cases} (\partial_1\cdots\partial_p(x), \tau(x)\cdot y), & i = 1, \\ (\partial_i\cdots\partial_p(x), \tau\partial_0\cdots\partial_{i-2}(x)\cdot y), & 1 < i \leq p, \\ (x, d_{i-p}^0(y)), & p < i \leq n, \end{cases}$$

$$d_i^1(x,y) = \begin{cases} (\partial_{i-1}(x), y), & 1 \leq i \leq p, \\ (x, d_{i-p}^1(y)), & p < i \leq n, \end{cases}$$

$$\begin{array}{ll} \eta_i(x,y) = (\eta_i(x), y), & 1 \leq i \leq p, \\ \eta_i(x,y) = (x, \eta_{i-p}(y)), & p < i \leq n+1. \end{array}$$

For any $(x, y) \in X \times_\tau L$ the operators uniquely extend to form the cubical set $(X \times_\tau L,\ d_i^0, d_i^1, \eta_i)$.

The geometrical interpretation of $\tau : X \to \Omega X$ is the following: The standard n-simplex (the base) is converted into the $(n-1)$-cube (the fiber) by the canonical truncating procedure; this truncation at this time yields n-cube (the total space) too which is thought of as the "twisted Cartesian product" of the simplex and the cube (see Fig. 1).

Example 4.1. *Let $M = \{e_k\}_{k\geq 0}$ be the free graded monoid on a single generator $e_1 \in M_1$ with trivial cubical set structure and let $\tau : X \to M$ be the sequence of constant maps $\tau_n : X_n \to M_{n-1}$, $n \geq 1$. Then the twisted Cartesian product $X \times_\tau M$ can be thought of as a cubical resolution of a 1-reduced simplicial set X.*

5. The cubical model of the path space fibration

Let $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ be the Moore path space fibration on a topological space Y. In [1] Adams constructed a dga map

$$\omega_* : \Omega C_*(Y) \to C_*^{\square}(\Omega Y) \tag{1}$$

being a weak equivalence for simply connected Y. It appears that Adams' cobar construction and the above map ω_* as well as the acyclic cobar construction $\Omega(C_*(Y); C_*(Y))$ have explicit combinatorial interpretations by means of cubical sets. Namely, we have the following theorem (compare, [23], [9], [2], [3] [10]).

Theorem 5.1. *(i) For the fibration $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ there is the following commutative diagram*

$$\begin{array}{ccccc} \operatorname{Sing}^I \Omega Y & \xrightarrow{i_*} & \operatorname{Sing}^I PY & \xrightarrow{\pi_*} & \operatorname{Sing}^I Y \\ {\scriptstyle\omega}\uparrow & & {\scriptstyle p}\uparrow & & {\scriptstyle\psi}\uparrow \\ \Omega \operatorname{Sing}^1 Y & \longrightarrow & \mathbf{P}\operatorname{Sing}^1 Y & \xrightarrow{\xi} & \operatorname{Sing}^1 Y \end{array} \tag{2}$$

which is natural in Y, $\psi : \operatorname{Sing}^1 Y \to \operatorname{Sing}^I Y$ is a map of graded sets induced by the $\psi : I^n \to \Delta^n$, while p and ω are maps of cubical sets; moreover, the cubical maps are homotopy equivalences provided Y is simply connected.

(ii) The chain complex $C_*^{\square}(\Omega \operatorname{Sing}^1 Y)$ coincides with Adams' cobar construction $\Omega C_*(Y)$; consequently, for a simply connected Y Adams' weak equivalence (1) is induced by the map of monoidal cubical sets ω.

(iii) The chain complex $C_*^{\square}(\mathbf{P}\operatorname{Sing}^1 Y)$ coincides with the acyclic cobar construction $\Omega(C_*(Y); C_*(Y))$.

Proof. (i). The constructions of the p and ω are simultaneous by induction on the dimension of singular simplices in $\operatorname{Sing}^1 Y$. For $i = 0, 1$ and $(\sigma, e) \in \mathbf{P}\operatorname{Sing}^1 Y$, $\sigma \in (\operatorname{Sing}^1 Y)_i$, define $p(\sigma, e)$ as the constant map $I^i \to PY$ to the base point of PY, where e denotes the unit of the monoid $\Omega \operatorname{Sing}^1 Y$ (and of the monoid $\operatorname{Sing}^I \Omega Y$ too). Put $\omega(e) = e$.

Denote by $P\operatorname{Sing}^1 Y_{(i,j)}$ the subset in $\mathbf{P}\operatorname{Sing}^1 Y$ consisting of the elements (σ, σ') with $|\sigma| \le i$, $\sigma' \in \Omega \operatorname{Sing}^1 Y_{(j)}$, a submonoid in $\Omega \operatorname{Sing}^1 Y$ having (monoidal) generators $\bar\sigma = \tau(\sigma)$ of degree $\le j$.

Suppose by induction that we have constructed p and ω on $\mathbf{P}\operatorname{Sing}^1 Y_{(n-1,n-2)}$ and $\Omega \operatorname{Sing}^1 Y_{(n-2)}$ respectively such that

$$p(\sigma, \tau) = p(\sigma, e) \cdot \omega(\sigma'), \quad i_* \circ \omega(\bar\sigma) = p(d_1^0(\sigma, e)) \;\text{ and }\; \pi_* \circ p = \psi \circ \xi,$$

where the $\cdot$ product is determined by the action $PY \times \Omega Y \to PY$. Let $\bar I^n \subset I^n$ be the union of the $(n-1)$-faces $d_i^{\epsilon}(I^n)$ of I^n except the $d_1^0(I^n) = (0, x_2, ..., x_n)$, and then for a singular simplex $\sigma : \Delta^n \to Y$ define the map $\bar p : \bar I^n \to PY$ by

$$\bar p|_{d_i^{\epsilon}(I^n)} = p(d_i^{\epsilon}(\sigma, e)), \;\epsilon = 0, 1, \;\text{ and }\; i \ne 1 \;\text{ for }\; \epsilon = 0.$$

Then we get the following commutative diagram

$$\begin{array}{ccccc} \bar I^n & \xrightarrow{\bar p_\sigma} & P_\sigma Y & \xrightarrow{g_\sigma} & PY \\ {\scriptstyle\bar i}\downarrow & & {\scriptstyle\pi_\sigma}\downarrow & & {\scriptstyle\pi}\downarrow \\ I^n & \xrightarrow{\psi} & \Delta^n & \xrightarrow{\sigma} & Y. \end{array}$$

Clearly, $\bar i$ is a strong deformation retract and we define $p(\sigma, e) : I^n \to PY$ as a lift of $\sigma \circ \psi$. Define $p\left(d_1^0(\sigma, e)\right) = p(\sigma, e)|_{d_1^0(I^n)}$, and then $\omega(\bar\sigma)$ is determined by $(i_* \circ \omega)(\bar\sigma) = p(\sigma, e) \circ \delta_1^0 : I^{n-1} \to I^n \to PY$.

The proof of p and ω being homotopy equivalences (after the geometric realizations) immediately follows, for example, by observation that ξ induces a long exact sequence for the homotopy groups.

The last statement itself can be deduced from the two facts: 1. $|P\,\mathrm{Sing}^1 X|$ is contractible, 2. The projection ξ induces an isomorphism $\pi_*(|P\,\mathrm{Sing}^1 Y|, |\Omega\,\mathrm{Sing}^1 Y|) \xrightarrow{\xi_*} \pi_*(|\,\mathrm{Sing}^1 Y|)$.

(ii)-(iii). It is straightforward to check. □

Thus, by passing on chain complexes in diagram (2) one obtains the following comultiplicative model of π formed by dgc's.

Corollary 5.1. *For the path space fibration $\Omega Y \to PY \xrightarrow{\pi} Y$ there is a comultiplicative model formed by coassociative dgc's*

$$\begin{array}{ccccc} C_*^{\square}(\Omega Y) & \longrightarrow & C_*^{\square}(PY) & \xrightarrow{\pi_*} & C_*^{\square}(Y) \\ {\scriptstyle \omega_*}\uparrow & & {\scriptstyle p_*}\uparrow & & {\scriptstyle \psi_*}\uparrow \\ \Omega C_*(Y) & \longrightarrow & \Omega(C_*(Y);C_*(Y)) & \xrightarrow{\xi_*} & C_*(Y) \end{array}$$

which is natural in Y.

6. Cubical models for fibrations

Here we prove the main result of this article.

Let $F \to E \xrightarrow{\zeta} Y$ be a fibration associated with a principal G-fibration $G \to E' \xrightarrow{\pi} Y$ by an action $G \times F \to F$. Let $X = Sing^1 Y$, $Q = Sing^I G$ and $L = Sing^I F$. The group operation $G \times G \to G$ induces on Q a structure of monoidal cubical set and the action $G \times F \to F$ induces the structure of Q-module $Q \times L \to L$ on L.

Theorem 6.1. *Let $F \to E \xrightarrow{\zeta} Y$ be a fibration with 1-connected base Y associated with a principal G-fibration $G \to E' \xrightarrow{\pi} Y$ by an action $G \times F \to F$. Then the principal fibration determines a truncating twisting function $\tau : Sing^1 Y \to Sing^I G$ such that twisted Cartesian product $Sing^1 Y \times_\tau Sing^I F$ models E, that is, there exists a cubical map*

$$Sing^1 Y \times_\tau Sing^I F \to Sing^I E$$

inducing homology isomorphism.

Proof. Let $\omega : \Omega X \to Sing^I \Omega Y$ be the map of monoidal cubical sets from Theorem 5.1. Then by Proposition 4.1 it corresponds to a truncating twisting function $\tau' : X = Sing^1 Y \xrightarrow{\tau_0} \Omega X = \Omega Sing^1 Y \xrightarrow{\omega} Sing^I \Omega Y$ (here by τ_0 we denote the universal truncating twisting function). Composing τ' with the map of monoidal cubical sets $Sing^I \Omega Y \to Sing^I G = Q$ induced by the canonical map $\Omega Y \to G$ of monoids we obtain a truncating twisting function $\tau : X \to Q$. The resulting twisted Cartesian product $Sing^1 Y \times_\tau Sing^I F$ is a cubical model of E. Indeed, we have the canonical equality

$$X \times_\tau L = (X \times_\tau Q) \times L / \sim,$$

where $((a, bz), c) \sim ((a, b), zc), a \in X,\ b, z \in Q,\ c \in L$. Next the argument of the proof of Theorem 5.1 gives a cubical map $f' : X \times_{\tau_0} \Omega X \to Sing^I E'$ preserving the actions of ΩX and Q. Hence, this map extents to a cubical map $f : X \times_\tau Q \to Sing^I E'$ by $f(a, b) = f'(a, e)b$. Then it is easy to see that the composition

$$(X \times_\tau Q) \times L \xrightarrow{f \times 1} Sing^I E' \times L \to Sing^I (E' \times F)$$

induces the map of cubical sets

$$Sing^1 Y \times_\tau Sing^I F \to Sing^I E$$

as desired. □

For convenience, assume that X, Q and L are as in the Definition 4.2. We have that a truncating twisting function τ induces on chain level the twisting cochains $\tau_* : C_*(X) \to C_{*-1}(Q)$ and $\tau^* : C^*(Q) \to C^{*+1}(X)$ in the standard sense ([8],[7],[14]). It is straightforward to verify that we have the equality

$$C_*^{\square}(X \times_\tau L) = C_*(X) \otimes_{\tau_*} C_*^{\square}(L) \tag{3}$$

and, consequently, the inclusion

$$C^*_{\square}(X \times_\tau L) \supset C^*(X) \otimes_{\tau^*} C^*_{\square}(L) \tag{4}$$

of dg modules (where we have an equality too if the graded sets are of finite type).

The cubical structure of $X \times_\tau L$ induces a dgc sturcture on $C_*^{\square}(X \times_\tau L)$ which after transporting on the right side of (3) gives a *comultiplicative* model of our fibration. Analogously it arises the multiplication on the right hand side of (4). To describe these structures first we need some (co)chain operations on the (co)chain complex of X.

6.1. **The canonical homotopy G-algebra structure on $C^*(X)$.** To describe these structures in more details first we turn to the equality

$$C_*^{\square}(\Omega X) = \Omega C_*(X).$$

Again, the cubical structure of ΩX induces on the cobar construction $\Omega C_*(X)$ a comultiplication which converts it into a dg Hopf algebra. Actually such comultiplication was defined on the normalized complex $C_*^N(X)$ by Baues in [2], [3]. This comultiplication is the result of the standard cubical diagonal

$$\Delta = \Sigma(-1)^\epsilon\ d^0_{j_1} \cdots d^0_{j_p} \otimes d^1_{i_1} \cdots d^1_{i_q}, \tag{5}$$

where the summation is over all shuffles $\{i_1 < ... < i_q, j_1 < ... < j_p\}$ of the set $\{1, ..., n\}$ and ϵ is the shuffle sign. Then in the combinatorics of Remark 3.1 this diagonal is expressed as

$$\begin{aligned}\Delta[0,1,...,n+1] = \ & \Sigma(-1)^\epsilon\ [0,1,...,j_1][j_1,...,j_2][j_2,...,j_3]...[j_p,...,n+1]\otimes \\ & [0,j_1,j_2,...,j_p,n+1]\end{aligned}$$

where the components $[01...n+1] \otimes [0, n+1]$ and $[01][12][23]...[n, n+1] \otimes [01...n+1]$ form the primitive part of the diagonal.

Now regarding the blocks of natural numbers above as faces of the standard $(n+1)$-simplex we obtain Baues' formula for the coproduct $\Delta : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$: for a generator $\sigma \in C_{n+1}(X) \subset \Omega C_*(X)$

$$\begin{aligned}\Delta[\sigma] = \ & \Sigma(-1)^\epsilon\ [\sigma(0,1,...,j_1)|\sigma(j_1,...,j_2)|\sigma(j_2,...,j_3)|...|\sigma(j_p,...,n+1)]\otimes \\ & [\sigma(0,j_1,j_2,...,j_p,n+1)],\end{aligned}$$

where $\sigma(i_1, ..., i_k)$ denotes the suitable face of σ. Note that since X is assumed to be 1-reduced for each 1-dimensional face $\sigma(k, k+1)$ its image $[\sigma(k, \bar{k}+1)]$ in $\Omega C_*(X)$ is the unit and so will be omitted. Note also that the formula is highly asymmetric, the left hand factors of $\Omega C_*(X) \otimes \Omega C_*(X)$ have the length ≥ 1 and the right hand factors have the length 1; this is a result of (5) and the structure of d^0_i, d^1_i from Remark 3.1.

Actually this diagonal consists of *components* $E^{k,1} = pr \circ \Delta : C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X) \to C_*(X)^{\otimes k} \otimes C_*(X),\ k \geq 1$, where *pr* is the clear projection. The basic component $E^{1,1}$ looks as

$$\begin{aligned}E^{1,1}(\sigma) = \ & \Sigma_{s,t}(-1)^\epsilon\ (\sigma(0,1) \otimes \sigma(1,2) \otimes ... \otimes \sigma(s-1,s) \otimes \sigma(s,s+1,...,t) \otimes \sigma(t,t+1)\otimes \\ & ... \otimes \sigma(n,n+1)) \otimes \sigma(0,1,...,s-1,s,t,t+1,...,n+1) = \\ & \Sigma_{s,t}(-1)^\epsilon\ \sigma(s,s+1,...,t) \otimes \sigma(0,1,...,s-1,s,t,t+1,...,n+1),\end{aligned}$$

so it is a chain operation dual to Steenrod's $\smile_1$-product.

Dualizing the operations $E^{k,1}$ we obtain a multiplication on the bar construction $BC^*(X) \otimes BC^*(X) \to BC^*(X)$, or, equivalently, the sequence of cochain operations

$$\{E_{k,1} : C^*(X)^{\otimes k} \otimes C^*(X) \to C^*(X)\}_{k \geq 1}.$$

These cochain operations just form on $C^*(X)$ a structure of *homotopy G-algebra* (see the next section). These operations are restriction of some more general cochain operations which naturally arise on $\tilde{C}^*(X)$ for an arbitrary X without assuming it to be 1-reduced. These are the operations

$$\{E_{k,1}:\tilde{C}^*(X)^{\otimes k}\otimes\tilde{C}^*(X)\to\tilde{C}^*(X)\}_{\,k\geq 0},$$

written down by the following explicit formulas. For $a_i\in\tilde{C}^{m_i}(X),\ m_i\geq 2,\ 1\leq i\leq k$, let

$$E_{k,1}(a_1,...,a_k;a_0)=\sum_{j\geq k}\tilde{E}_{j,1}(\epsilon^1,a_1,\epsilon^1,...,\epsilon^1,a_k,\epsilon^1;a_0),$$

$\epsilon^1\in\tilde{C}^1(X)$ is the generator represented by the constant map at the base point and the operations $\tilde{E}_{k,1}$ are themselves defined for $c_j\in\tilde{C}^{m_j}(X),\ m_j\geq 1,\ 1\leq j\leq k,\ n=m_1+\cdots+m_k,\sigma\in X_n$, $c_0\in\tilde{C}^k(X)$, by

$$\begin{array}{rl} & \tilde{E}_{k,1}(c_1,c_2,...,c_k;c_0)=c\in\tilde{C}^n(X),\\ c(\sigma)= & (-1)^{\varepsilon}c_1(\partial_{i_1+1}\cdots\partial_n\sigma)c_2(\partial_0\cdots\partial_{i_1-1}\partial_{i_2+1}\cdots\partial_n\sigma)\cdots\\ & c_k(\partial_0\cdots\partial_{i_{k-1}-1}\sigma)c_0(\hat{\partial}_0\partial_1\hat{\partial}_{i_1}\cdots\hat{\partial}_{i_{k-1}}\cdots\partial_{n-1}\hat{\partial}_n\sigma)\\ & \varepsilon=\sum_{j=1}^k(j-1)(m_j-1),\end{array}$$

$i_q=m_1+\cdots+m_q,\ 1\leq q\leq k-1$, and $\tilde{E}_{k,1}(c_1,c_2,...,c_k;c_0)=0$ otherwise.

Remark 6.1. *Though each $\tilde{E}_{k,1}$, and in particular $\tilde{E}_{1,1}$ had only one component the formula for $k=1$ defines $E_{1,1}$ as being exactly Steenrod's cochain $\smile_1$-operation without any restriction on X. This fact evidently indicates a difference between topological and algebraic interpretation of the operations $\{E_{k,1}\}_{k\geq 1}$ in terms of 1-reduced algebras.*

6.2. **Twisted multiplicative model for a fibration.** Now we again turn to the twisted Cartesian product $X\times_\tau L$. To describe the corresponding coproduct and product on the right sides of (3) and (4) respectively it is very convenient to express cubical diagonal (5) using the combinatorics of Remark 3.3. Namely, we have

$$\begin{array}{rl} 01...n]\xrightarrow{\Delta} & \Sigma(-1)^{\epsilon}\ 0...j_1][j_1...j_2][j_2...j_3]...[j_k...n]\otimes\\ & \hat{0},...,\widehat{j_1-1},j_1,\widehat{j_1+1},...,\widehat{j_2-1},j_2,...,j_k,\widehat{j_{k+1}},...,\widehat{n-1},n],\end{array}$$

$0\leq j_1<\cdots<j_k<n$, where the components $01...n]\otimes n]$ and $0][01][12][23]...[n-1,n]\otimes 01...n]$ form the primitive part of the diagonal.

Now, using this diagonal, it is not hard to see that by means of $\{E_{k,1}\}_{k\geq 1}$ and the induced comodule structure $\Delta_L:C^*_\square(L)\to C^*_\square(Q)\otimes C^*_\square(L)$ by the action $Q\times L\to L$ the cubical product of the left side of (4) can be expressed by the following formula. Let $a_1\otimes m_1,a_2\otimes m_2\in C^*(X)\otimes_{\tau^*}C^*_\square(L)$ and $\Delta^k_L:C^*_\square(L)\to C^*_\square(Q)^{\otimes k}\otimes C^*_\square(L)$ be the iterated Δ_L with $\Delta^0_L=1:C^*_\square(L)\to C^*_\square(L)$, and let $\Delta^k_L(m_1)=\sum c^1\otimes\ldots\otimes c^k\otimes m_1^{k+1}$. Then

$$\mu((a_1\otimes m_1)\otimes(a_2\otimes m_2))=\sum_{k\geq 0}(-1)^{|a_2||m_1^{k+1}|}a_1E_{k,1}(\tau^*(c^1),\ldots,\tau^*(c^k);a_2)\otimes m_1^{k+1}m_2. \tag{6}$$

Corollary 6.1. *Let $F\to E\xrightarrow{\zeta}Y$ be the associated fibration with G-fibration $G\to E'\xrightarrow{\pi}Y$ by the action $G\times F\to F$. Then the tensor product $C^*(Y)\otimes C^*_\square(F)$ becomes a dga $(C^*(Y)\otimes C^*_\square(F),d_\tau,\mu)$ with both twisted differential d_τ and the multiplication μ.*

Note that in [6] such a multiplicative model is constructed without explicit formulas for the multiplication.

In particular, letting $Q=L=\Omega X$ in (6) we deduce the following explicit formula for the associative product on the acyclic bar construction $B(C^*(Y);C^*(Y))$ converting it into a dga. For $a=a_0\otimes[\bar{a}_1|\cdots|\bar{a}_n],\quad b=b_0\otimes[\bar{b}_1|\cdots|\bar{b}_m],\quad a_i,b_j\in C^*(Y),\ 0\leq i\leq n,\ 0\leq j\leq m$, let

$$ab=\sum_{k=0}^{n}(-1)^{|b_0|(|\bar{a}_{k+1}|+\cdots+|\bar{a}_n|)}a_0E_{k,1}(a_1,\ldots,a_k;b_0)\otimes[\bar{a}_{k+1}|\cdots|\bar{a}_n]\circ[\bar{b}_1|\cdots|\bar{b}_m]. \tag{7}$$

7. Twisted tensor products for homotopy G-algebras

The notion of a homotopy G-(co)algebra naturally generalizes the one of a (co)commutative (co)algebra. Just the structure such a (co)algebra on the (co)chain complex of a topological space became the basic motivation for the material of this section. It appears that the formulas (namely (6) and (7)) established in the previous section have an universal character in the sense that they are valid in a purely algebraic situation. Moreover, to see these formulas directly, without help of these (combinatorial) topological examples, seems to be not easy question. In turn, the end of this section evidently demonstrates an useful application of the algebraic formalism we develop here to geometry.

First we recall the definition of homotopy G-algebra (hga) that differs only by grading from [12] (see also [13]).

Let for a dga A

$$(\mathrm{Hom}(BA\otimes BA,A),\nabla)$$

be the canonical dga with $\smile$-product, where $BA\otimes BA$ has the standard tensor coalgera structure.

Definition 7.1. *A homotopy G-algebra is a 1-reduced associative dga A with multilinear maps*

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\ \ p,q\geq 0,\ \ p+q>0,$$

with the following properties:
(i) $E_{p,q}$ is of degree $1-p-q$;
(ii) $E_{p,q}=0$ except $E_{1,0}=1=E_{0,1}$ and $E_{k,1},\ k\geq 1$;
(iii) The homomorphism $E:BA\otimes BA\to A$ defined by

$$E([\bar a_1|\cdots|\bar a_p]\otimes[\bar b_1|\cdots|\bar b_q])=E_{p,q}(a_1,...,a_p;b_1,...,b_q)$$

is a twisting element in the dga $(Hom(BA\otimes BA,A),\nabla)$, i.e. satisfies $\nabla E=-E\smile E$ (this condition implies that the comultiplicative coextension $\mu_E:BA\otimes BA\to BA$ is a chain map);
(iv) The multiplication μ_E is associative, i.e. it turns BA into a dg Hopf algebra.

Entirely dually one can formulate the notion of a homotopy G-coalgebra.

The conditions (iii) and (iv) can be rewritten in terms of components $E_{p,q}$ (see [12]). In particular the operation $E_{1,1}$ satisfies the conditions similar to that of Steenrod's $\smile_1$ product: the condition (iii) gives

$$dE_{1,1}(a;b)-E_{1,1}(da;b)+(-1)^{|a|}E_{1,1}(a;db)=(-1)^{|a|}ab-(-1)^{|a|(|b|+1)}ba,$$

so it measures the non-commutativity of the product of A (thus, a hga with $E_{k,1}=0$ for $k\geq 1$ is just a commutative dga). We denote $E_{1,1}(a,b)=a\smile_1 b$. This notation is justified also by the other condition which follows from (iii):

$$c\smile_1 ab=(c\smile_1 a)b+(-1)^{|a|(|c|-1)}a(c\smile_1 b). \tag{8}$$

This formula means that the map $a\smile_1 -:A\to A$ is a derivation which in the case of $C^*(X)$ is called as the Hirsch formula. As for the map $-\smile_1 c:A\to A$, it follows from (iii) that it is a derivation only *up to homotopy* and for the suitable homotopy it just serves the operation $E_{2,1}$:

$$\begin{aligned}dE_{2,1}(a,b;c)-\ & E_{2,1}(da,b;c)-(-1)^{|a|}E_{2,1}(a,db;c)-(-1)^{|a|+|b|}E_{2,1}(a,b;dc)=\\ & (-1)^{|a|+|b|}ab\smile_1 c-(-1)^{|a|+|b||c|}(a\smile_1 c)b-(-1)^{|a|+|b|}a(b\smile_1 c).\end{aligned} \tag{9}$$

Main examples of hga's are: $C^*(X)$ (see [2], [3],[13] and previous section) and the Hochschild cochain complex of an associative algebra where $E_{1,1}$ and $E_{2,1}$ were defined by Gerstenhaber in [11] and the higher operations were described in ([17], [13]). One more example is the cobar construction of a dg Hopf algebra [18]. Note also that certain algebras (including polynomial ones), which are realized as the cohomology of topological spaces, admit a non-trivial hga structure too [26].

Remark 7.1. *Note that the $E_{2,1}$ measures also the lack of associativity of $E_{1,1}=\smile_1$; in particular, condition (iv) yields*

$$a\smile_1(b\smile_1 c)-(a\smile_1 b)\smile_1 c=E_{2,1}(a,b;c)+(-1)^{(|a|+1)(|b|+1)}E_{2,1}(b,a;c).$$

The last condition implies that the commutator $[a,b] = a \smile_1 b - (-1)^{(|a|+1)(|b|+1)} b \smile_1 a$ satisfies the Jacobi identity. Together with (8) it implies on the $H(A)$ a Lie bracket of degree -1. Besides (8) and (9) imply that $[a,-] : H(A) \to H(A)$ is a derivation, so $H(A)$ becomes a Gerstenhaber algebra [11]*. Note that this notion is not a particular case of hga. We have that the induced Gerstenhaber algebra structure on $H(C^*(X)) = H^*(X)$ is trivial since of the existence of $\smile_2$ product.*

7.1. **Multiplicative twisted tensor products.** Let C be a dgc, A be a dga and M be a dg comodule over C. Brown's twisting element $\tau : C \to A$ determines the following maps: a dga map (the multiplicative extension of τ) $f_\tau : \Omega C \to A$, a dgc map (the comultiplicative coextension of τ) $g_\tau : C \to BA$ and the twisted differential $d_\tau = d \otimes 1 + 1 \otimes d + \tau \cap_- : A \otimes M \to A \otimes M$.

Suppose now that C is a dg Hopf algebra and M is a dga in addition with $M \to C \otimes M$ beeng a dga map. In general d_τ is not a derivation with respect to the multiplication of the tensor product $A \otimes M$. But if A is a *commutative* dga (in this case BA is a dg Hopf algebra with respect to the shuffle product μ_{sh}) and $g_\tau : C \to BA$ is a map of dg Hopf algebras, then the twisted tensor product $A \otimes_\tau C$ is a dga ([24]).

Suppose now that A is a homotopy G-algebra. In this case BA is again a dg Hopf algebra with respect to the multiplication μ_E.

Definition 7.2. *A twisting element $\tau : C \to A$ in $Hom(C, A)$ we call multiplicative if the comultiplicative coextension $C \to BA$ is an algebra map.*

It is clear that if $\tau : C \to A$ is a multiplicative twisting element and if $g : B \to C$ is a map of dg Hopf algebras then the composition $\tau g : B \to A$ is again a multiplicative twisting element.

The canonical projection $BA \to A$ provides an example of the universal multiplicative element.

For a commutative dga A one has the equality $\mu_E = \mu_{sh}$, so Proute's twisting element is multiplicative (see, for example, [25]).

We have that the argument of the proof of formula (6) immediately yields

Theorem 7.1. *Let $\tau^* : C \to A$ be a multiplicative twisting element. Then the tensor product $A \otimes M$ with the canonical twisting differential $d_{\tau^*} = d \otimes 1 + 1 \otimes d + \tau^* \cap_-$ becomes a dga $(A \otimes M, d_{\tau^*}, \mu_{\tau^*})$ with the twisted multiplication μ_{τ^*} determined by formula (6).*

Thus the above theorem includes the twisted tensor product theory for commutative algebras ([24]).

Corollary 7.1. *For a homotopy G-algebra A the acyclic barconstruction $B(A; A)$ becomes a dga with the twisted multiplication determined by formula (7).*

7.2. **Brown's model as a dga.** Now it is possible to replace in Corollary 6.1 the cubical cochains by the simplicial ones to introduce on Brown's model an associative multiplication. Indeed, let $C_N^*(F)$ denote the normalized singular *simplicial* cochain complex of F.

Corollary 7.2. *Let $F \to E \xrightarrow{\zeta} Y$ be a fibration as in Corollary 6.1. Then on the tensor product $C^*(Y) \otimes C_N^*(F)$ there are both twisted differential $d_{\tau'}$ and the multiplication $\mu_{\tau'}$, with $\tau' : C_N^*(G) \to C^*(Y)$ being multiplicative twisting element, such that $(C^*(Y) \otimes C_N^*(F), d_{\tau'}, \mu_{\tau'})$ is a dga with cohomology algebra isomorphic to $H^*(E)$.*

Proof. It is sufficient to observe that the map $\varphi : C_N^*(G) \to C_\square^*(G)$ induced by triangulation of the cubes (see, for example, [10]) is a map of dgha's, so that the composition of $\varphi \otimes 1$ with $\Delta : C_N^*(F) \to C_N^*(G) \otimes C_N^*(F)$ defines on $C_N^*(F)$ a comodule structure over the dgha $C_\square^*(G)$. Then we take $\tau' = \tau^* \varphi$. □

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