THE TWISTED CARTESIAN MODEL FOR THE DOUBLE PATH SPACE FIBRATION

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Abstract. The paper introduces the notion of a truncating twisting function from a cubical set to a permutahedral set and the corresponding notion of twisted Cartesian product of these sets. The latter becomes a permutocubical set that models in particular the path space fibration on a loop space. The chain complex of this twisted Cartesian product in fact is a comultiplicative twisted tensor product of cubical chains of base and permutahedral chains of fibre. This construction is formalized as a theory of twisted tensor products for Hirsch algebras.

1. introduction

The paper continues [12] in which a combinatorial model for a fibration was constructed based on the notion of a *truncating twisting function* from a simplicial set to a cubical set and on the corresponding notion of twisted Cartesian product of these sets being a cubical set. Applying the cochain functor we obtain a *multiplicative* twisted tensor product modeling the corresponding fibration.

There arises a need to iterate this construction for fibrations over loop or path spaces the bases of which are modeled by cubical sets. A cubical base naturally requires a permutahedral fibre; this really agrees with the first usage of the permutahedra (the Zilchgons) as modeling polytopes for loops on the standard cube due to R.J. Milgram [15] (see also [7]).

For this here we introduce the notion of a truncating twisting function $\vartheta : Q_* \to P_{*-1}$ from a 1-reduced cubical set Q to a monoidal permutahedral set P([17]). For a permutahedral set L with a given P-action, ϑ defines the corresponding twisted Cartesian product $Q \times_\vartheta L$. The latter becomes a *permutocubical set.* The *permutocube* is defined as a polytope which is obtained from the standard cube by the specific truncation procedure due to N. Berikashvili [5], see also bellow. The permutocube can be thought of as a modeling polytope for paths on the cube.

The general theory of the truncating twisting functions here goes almost parallel to that of [12]. Namely, we construct a functor assigning to a cubical set Q a monoidal permutahedral set $\mathbf{\Omega} Q$ together with the canonical inclusion $\vartheta_U : Q \to \mathbf{\Omega} Q$ of degree -1 being an universal example of a truncating twisting function: any

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$\vartheta : Q_* \to P_{*-1}$ factors as $\vartheta : Q \xrightarrow{\vartheta_U} \mathbf{\Omega} Q \xrightarrow{f_\vartheta} P$ where f_ϑ is a map of monoidal permutahedral sets.

The twisted Cartesian product $\mathbf{P}Q = Q \times_{\vartheta_U} \mathbf{\Omega} Q$ is a permutocubical set functorially depending on Q. Note that $\mathbf{\Omega} Q$ models the loop space $\Omega|Q|$ and $\mathbf{P}Q$ models the path space fibration on $|Q|$.

The chain complex $C_*^{\diamondsuit}(\mathbf{\Omega} Q)$ coincides with the cobar construction $\Omega C_*^{\square}(Q)$. Furthermore, $C_*^{\boxminus}(Q \times_{\vartheta_U} \mathbf{\Omega} Q)$ coincides with the acyclic cobar construction $\Omega(C_*^{\square}(Q); C_*^{\square}(Q))$.

Moreover, applying the chain functor to $\vartheta : Q_* \to P_{*-1}$ we obtain a twisting cochain $\vartheta_* = C_*(\vartheta) : C_*^{\square}(Q) \to C_{*-1}^{\diamondsuit}(P)$ and then $C_*^{\boxminus}(Q \times_\vartheta L)$ coincides with the twisted tensor product $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(L)$.

We construct the explicit diagonal for the permutocube B_n which agrees with that of P_n [17] by means of the natural embedding $P_n \to B_n$. The equalities $C_*^{\diamondsuit}(\mathbf{\Omega} Q) = \Omega C_*^{\square}(Q)$ and $C_*^{\boxminus}(Q \times_\vartheta L) = C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(L)$ allow us to transport these diagonals to the cobar construction $\Omega C_*^{\square}(Q)$ and the twisted tensor product $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(L)$ respectively. Thus, finally, we obtain *comultiplicative* models for the loop space $\Omega|Q|$ and the twisted Cartesian product $Q \times_\vartheta L$.

In fact the diagonal $\Omega C_*^{\square}(Q) \to \Omega C_*^{\square}(Q) \otimes \Omega C_*^{\square}(Q)$ is determined by higher order chain operations

$$\{E^{p,q} : C_*^{\square}(Q) \to C_*^{\square}(Q)^{\otimes p} \otimes C_*^{\square}(Q)^{\otimes q}\}_{p+q>0};$$

in particular, the cooperation $E^{1,1}$ is the dual operation of the cubical version of Steenrod's cochain $\smile_1$-operation and all operations $\{E^{p,q}\}$ define on $C_*^{\square}(Q)$ the structure which we call a *Hirsch coalgrebra*. This structure together with the action $C_*^{\diamondsuit}(P) \otimes C_*^{\diamondsuit}(L) \to C_*^{\diamondsuit}(L)$ and the twisting cochain ϑ_* describes the above mentioned comultiplication on the twisted tensor product $C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(L)$.

Dually, the permutahedral $\smile$-product of $C^*_{\diamondsuit}(\mathbf{\Omega} Q)$ induces a product on $BC^*_{\square}(Q) \subset C^*_{\diamondsuit}(\mathbf{\Omega} Q)$ which, in fact, is determined by higher order cochain operations

$$\{E_{p,q} : C^*_{\square}(Q)^{\otimes p} \otimes C^*_{\square}(Q)^{\otimes q} \to C^*_{\square}(Q)\}_{p+q>0}; \tag{1}$$

in particular, the operation $E_{1,1}$ is the cubical version of Steenrod's cochain $\smile_1$-operation and all operations $\{E_{p,q}\}$ define on $C^*_{\square}(Q)$ the structure which we call a *Hirsch algrebra*. Again, this structure together with the coaction $C^*_{\diamondsuit}(L) \to C^*_{\diamondsuit}(P) \otimes C^*_{\diamondsuit}(L)$ and the twisting cochain $\vartheta^* : C^*_{\diamondsuit}(P) \to C^{*+1}_{\square}(Q)$ describes the multiplication on the twisted tensor product $C^*_{\square}(Q) \otimes_{\vartheta^*} C^*_{\diamondsuit}(L)$ induced by the permutocubical multiplication of $C^*_{\boxminus}(Q \times_\vartheta L)$. Note that this multiplication is not strictly associative but could be extended to an A_∞-algebra structure.

We formalize this construction by developing the general theory of multiplicative twisted tensor products for Hirsch algebras instead of dga's. A Hirsch algebra we define as an object $(A, d, \cdot, \{E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A\}_{p+q>0})$, i.e., $(A, d, \cdot)$ is an associative dga and the sequence of operations $\{E_{p,q}\}$ determines a product on the bar construction BA turning it into a dg Hopf algebra (this multiplication can be viewed as a perturbation of the shuffle product and is not necessarily associative). In particular $E_{1,1}$ has properties similar to $\smile_1$ product, so a Hirsch algebra can be considered as to have a structure measuring the lack of commutativity of A. Let C be a dg Hopf algebra and M be a dga and a dg C-comodule simultaneously. A twisting element $\phi : C \to A$ wa call *multiplicative* if the induced map $C \to BA$ is a dg Hopf algebra map. In this case we introduce on $A \otimes_\phi M$ a twisted multiplication

μ_ϕ in terms of ϕ and the Hirsch algebra structure of A by the same formulas as in the case $A = C^*_\square(Q)$, $C = C^*_\diamondsuit(P)$ and $M = C^*_\diamondsuit(L)$ where $\phi = \vartheta^* : C^*_\diamondsuit(P) \to C^{*+1}_\square(Q)$ is automatically multiplicative.

Furthermore, we apply the above machinery for a fibration $F \to E \to Z$ on 1-connected space Z associated with a principal G-fibration $G \to E' \to Z$ by an action $G \times F \to F$ to obtain the following combinatorial model. Let $Q = Sing^{1^I} Z \subset Sing^I Z$ be the Eilenberg 1-subcomplex generated by singular cubes sending the 1-skeleton of the standard n-cube I^n into the base point of Z, and let $P = Sing^P G$ and $L = Sing^P F$, where $Sing^I$ and $Sing^P$ denote the singular cubical and the permutahedral complex of a space respectively (see [17] and Section 2). We construct the Adams-Milgram map

$$\omega_* : \Omega C^\square_*(Q) \to C^\diamondsuit_*(\Omega Z)$$

which in fact is realized by a monoidal permutahedral map $\omega : \mathbf{\Omega} Q \to Sing^P \Omega Z$. On the other hand, one has a map of monoidal permutahedral sets $Sing^P \Omega Z \to Sing^P G = P$ induced by the canonical map $\Omega Z \to G$ of monoids. The composition of these two maps immediately yields a truncating twisting function $\vartheta : Q \to P$. The resulting twisted Cartesian product $Sing^{1^I} Z \times_\vartheta Sing^P F$, being a permutocubical set, just provides the required model of E: there exists a permutocubical weak equivalence $Sing^{1^I} Z \times_\vartheta Sing^P F \to Sing^B E$, where $Sing^B$ denotes the singular permutocubical complex of a space. Applying the cochain functor we obtain a certain multiplicative twisted tensor product for the fibration.

In particular, we can obtain a combinatorial model for the path space fibration $\Omega^2 Y \to P\Omega Y \to \Omega Y$ in the following way. Taking for the base $Z = \Omega Y$ the cubical model $Q = \mathbf{\Omega} Sing^2 Y$ from [12] the above theory yields the twisted Cartesian model $\mathbf{\Omega} Sing^2 Y \times_{\vartheta_U} \mathbf{\Omega}\mathbf{\Omega} Sing^2 Y$ being a *permutocubical* set.

Consequently, we introduce on the acyclic bar construction $B(BC^*(Y); BC^*(Y))$ the multiplication whose restriction to the double bar construction $BBC^*(Y)$ is just the one constructed in [17].

To summarize we observe the following. In [12] it is indicated the homotopy G-algebra structure on $C^*(Y)$ consisting of cochain operations

$$\{E_{k,1} : C^*(Y)^{\otimes k} \otimes C^*(Y) \to C^*(Y)\}_{k\geq 1},$$

defining a multiplication on $BC^*(Y)$. Here we extend this multiplication to the structure of Hirsch algebra on $BC^*(Y)$, i.e., to operations (1)

$$\{E_{p,q} : (BC^*(Y))^{\otimes p} \otimes (BC^*(Y))^{\otimes q} \to BC^*(Y)\}_{p+q>0},$$

which actually are cochain operations of type $C^*(Y)^{\otimes m} \to C^*(Y)^{\otimes n}$. This two sets of operations including in particular $\smile$, $\smile_1$ and $\smile_2$ operations, allow us to construct multiplicative models for ΩY, $\Omega^2 Y$ and multiplicative twisted tensor products for path space fibrations on Y and ΩY as well as for fibrations associated with them.

Finally, we mention that the geometric realization $|\mathbf{\Omega}\mathbf{\Omega} Sing^2 Y|$ of $\mathbf{\Omega}\mathbf{\Omega} Sing^2 Y$ is homeomorphic to the cellular model for the double loop space due to G. Carlsson and R. J. Milgram [7] and is homotopically equivalent to the cellular model due to H.-J. Baues [3].

The paper is organized as follows. We adopt the notions and the terminology from [12]; note that here a (co)algebra need not have a (co)associative (co)multiplication if it is not specially emphasized. In Section 2 we construct the functor

$\mathbf{\Omega}$ from the category of cubical sets to the category of permutahedral sets; Section 3 introduces the permutocubes; in Section 4 we introduce the notion of a permutocubical set; Section 5 introduces the notion of a truncating twisting function and the resulting twisted Cartesian product; in Section 6 we define an explicit diagonal on the permutocubes; in Section 7 we build the permutocubical set model for the double path space fibration; in Section 8 a permutocubical model and the corresponding multiplicative twisted tensor product for a fibration are constructed, and, finally, in Section 9 the twisted tensor product theory for Hirsch algebras is developed.

2. The permutahedral set functor $\mathbf{\Omega}Q$

For completeness we first recall some basic facts about permutahedral sets from [17] (compare, [13]).

2.1. **Permutahedral sets.**

Permutahedral sets are combinatorial objects generated by permutahedra and equipped with the appropriate face and degeneracy operators. Naturally occurring examples include the double cobar construction, i.e., the cobar construction on Adams' cobar construction [1] with coassociative coproduct [3], [7], [12] . Permutahedral sets are similar in many ways to simplicial or cubical sets with one crucial difference: Permutahedral sets have higher order structure relations, whereas structure relations in simplicial or cubical sets are strictly quadratic. We note that the exposition on polyhedral sets by D.W. Jones [11] makes no mention of structure relations.

Let S_{n+1} denote the symmetric group on $\underline{n+1} = \{1, 2, \ldots, n+1\}$ and recall that the permutahedron (the Zilchgon) P_{n+1} is the convex hull of $(n+1)!$ vertices $(\sigma(1), \ldots, \sigma(n+1)) \in \mathbb{R}^{n+1}$, $\sigma \in S_{n+1}$ [8], [15]. As a cellular complex, P_{n+1} is an n-dimensional convex polytope whose $(n-k)$-faces are indexed by all (ordered) partitions $M_1|\cdots|M_{k+1}$ of $\underline{n+1}$. For $1 \leq j \leq k$, let $M_{2j-1}|M_{2j}$ be a partition of $\underline{n-j+2}$; then each $(n-k)$-face corresponds to a composition of face operators $d_{M_{2k-1}|M_{2k}} \cdots d_{M_1|M_2}$ acting on P_{n+1}, where $M_{2j-1}|M_{2j}$ is a special partition of $\underline{n-j+2}$ for $1 \leq j \leq k$ (see Theorem 2.1). Since a partition $A|B$ of $\underline{n+1}$ denotes the same $(n-1)$-face as $d_{A|B}$, we use the two symbols interchangeably (see figure 1).

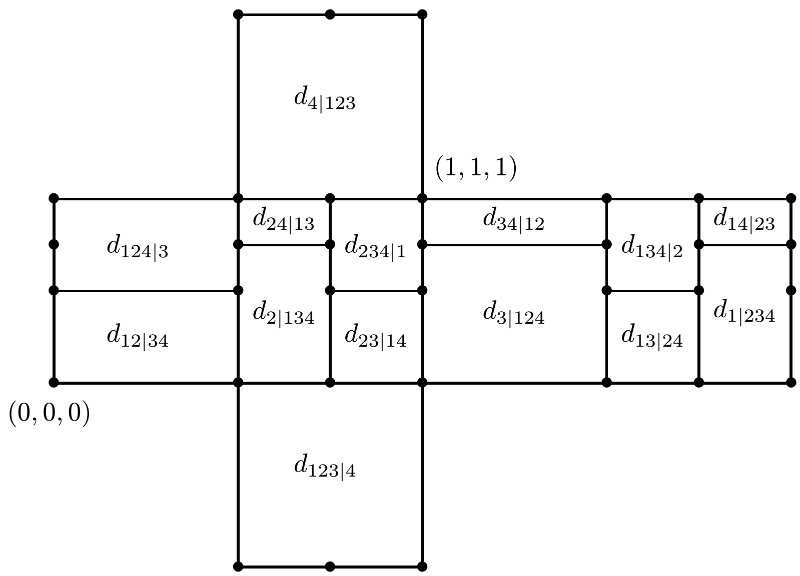

Figure 1: P_4 as a subdivision of $P_3 \times I$.

Labels $A|B$ for general $(n-1)$-faces of P_{n+1} can be obtained in purely set-theoretic terms. For $\epsilon = 0, 1$ and $1 \leq i \leq n$, let $e_{i,\epsilon}^{n-1}$ denote the $(n-1)$-face $(x_1, \ldots, x_{i-1}, \epsilon, x_{i+1}, \ldots, x_n) \subset I^n$. For $0 \leq i \leq j \leq \infty$, let $I_{i,j} = \left[1 - 2^{-i}, 1 - 2^{-j}\right] \subset I$, where $2^{-\infty}$ is defined to be 0, and for M a non-empty set, let $\aleph M$ denote its cardinality and define $\aleph\emptyset = 0$. When $n = 1$, label the vertices of P_2 by $e_{1,0}^0 \leftrightarrow 1|2$ and $e_{1,1}^0 \leftrightarrow 2|1$. Inductively, if P_n has been constructed, $n \geq 1$, obtain P_{n+1} by subdividing and labeling the $(n-1)$-faces of $P_n \times I$ as indicated below:

Face of P_{n+1}	Label or subscript
$e_{n,0}^{n-1}$	$\underline{n}\vert n+1$
$e_{n,1}^{n-1}$	$n+1\vert\underline{n}$
$A\vert B \times I_{0,\aleph B}$	$A\vert B \cup \{n+1\}$
$A\vert B \times I_{\aleph B,\infty}$	$A \cup \{n+1\}\vert B.$

Interestingly, some (but not all) compositions $d_{C|D}d_{A|B}$ act on P_{n+1}. This situation is quite different from the simplicial or cubical cases in which all compositions $\partial_i\partial_j$ or $d_i^\epsilon d_j^\epsilon$ act on the standard n-simplex Δ^n or the standard n-cube I^n, respectively. The conditions under which $d_{C|D}d_{A|B}$ acts on P_{n+1} can be stated in terms of set operations defined as follows.

Given a non-empty ordered set $A = \{a_1 < \cdots < a_m\} \subseteq \mathbb{Z}$, let $I_A : A \to \underline{\aleph A}$ be the index map $a_i \mapsto i$; for $z \in \mathbb{Z}$ let $A + z = \{a_1 + z < \cdots < a_m + z\}$ with the understanding that addition takes preference over set operations. For $1 \leq p \leq n$, let $\overline{p}$ denote the set containing the last p elements of $\underline{n}$, i.e., $\overline{p} = \{n-p+1 < \cdots < n\}$; in particular, $\overline{p} = \{q < \cdots < n\}$ when $p + q = n + 1$.

Definition 2.1. *Given non-empty disjoint subsets $A, B \subset \underline{n}$, define the lower and upper disjoint unions*

$$A\underline{\sqcup}B = \begin{cases} I_{\underline{n}\setminus A}(B) + \aleph A - 1 \cup \underline{\aleph A}, & \textit{if } \min B = \min(\underline{n}\setminus A) \\ I_{\underline{n}\setminus A}(B) + \aleph A - 1, & \textit{if } \min B > \min(\underline{n}\setminus A)\,; \end{cases}$$

and

$$A\overline{\sqcap}B = \begin{cases} I_{\underline{n}\setminus B}(A) \cup \overline{\aleph B} - 1, & \textit{if } \max A = \max(\underline{n}\setminus B) \\ I_{\underline{n}\setminus B}(A)\,, & \textit{if } \max A < \max(\underline{n}\setminus B)\,. \end{cases}$$

If either A or B is empty, define $A\underline{\sqcup}B = A\overline{\sqcap}B = A\cup B$.

In particular, if $A|B$ is a partition of $\underline{n}$, then

$$A\underline{\sqcup}B = A\overline{\sqcap}B = \underline{n-1}.$$

Given a partition $A_1|\cdots|A_{k+1}$ of $\underline{n+1}$, define $A^{(0)} = A^{[k+2]} = \emptyset$; inductively, given $A^{(j)}$, $0 \le j \le k$, let

$$A^{(j+1)} = A^{(j)}\underline{\sqcup}A_{j+1};$$

and given $A^{[j]}$, $2 \le j \le k+2$, let

$$A^{[j-1]} = A_{j-1}\overline{\sqcap}A^{[j]}.$$

And finally, for $1 \le j \le k+1$, let

$$A_{(j)} = A_1 \cup \cdots \cup A_j.$$

Now to a given $(n-k)$-face $A_1|\cdots|A_{k+1}$ of P_{n+1}, assign the compositions of face operators

$$d_{A^{(k)}|A^{(k-1)}\underline{\sqcup}\left(\underline{n+1}\setminus A_{(k)}\right)} \cdots d_{A^{(1)}|A^{(0)}\underline{\sqcup}\left(\underline{n+1}\setminus A_{(1)}\right)}$$

$$= d_{A_{(1)}\overline{\sqcap}A^{[3]}|A^{[2]}} \cdots d_{A_{(k)}\overline{\sqcap}A^{[k+2]}|A^{[k+1]}} \tag{2}$$

and denote either composition by $d_{A_1|\cdots|A_{k+1}}$.

Note that both sides of relation (2) are identical when $k=1$, reflecting the fact that each $(n-1)$-face is a boundary component of exactly one higher dimensional face (the top cell of P_{n+1}). On the other hand, each $(n-2)$-face $A|B|C$ is a boundary component shared by exactly two $(n-1)$-faces. Consequently, $A|B|C$ can be realized as a quadratic composition of face operators in two different ways given by (2) with $k=2$:

$$d_{A\underline{\sqcup}B|A\underline{\sqcup}C}\; d_{A|B\cup C} = d_{A\overline{\sqcap}C|B\overline{\sqcap}C}\; d_{A\cup B|C} \tag{3}$$

(see Figure 2). Relation (3) reminds us of the quadratic relation $\partial_i\partial_j = \partial_{j-1}\partial_i$ $(i<j)$ for face operators in a simplicial set.

Example 2.1. *In P_8, the 5-face $A|B|C = 12|345|678 = 12|345678 \cap 12345|678$. Since $A\underline{\sqcup}B = \{1234\}$, $A\underline{\sqcup}C = \{567\}$, $A\overline{\sqcap}C = \{12\}$ and $B\overline{\sqcap}C = \{34567\}$, we obtain the following quadratic relation on $12|345|678$:*

$$d_{1234|567}d_{12|345678} = d_{12|34567}d_{12345|678};$$

similarly, on $345|12|678$ we have

$$d_{1234|567}d_{345|12678} = d_{34567|12}d_{12345|678}.$$

Similar relations on the six vertices of P_3 appear in Figure 2 below.

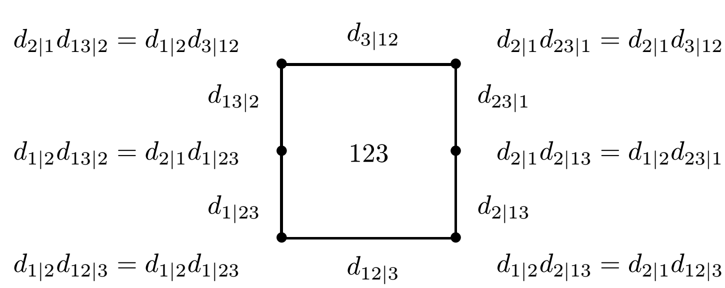

Figure 2: Codimension 2 relations on P_3.

For $1 \leq p < n$, let

$$\begin{aligned}
\mathcal{Q}_p(n) = &\ \left\{\text{partitions } A|B \text{ of } \underline{n} \mid \underline{p} \subseteq A \text{ or } \underline{p} \subseteq B\right\}, \\
\mathcal{Q}^p(n) = &\ \left\{\text{partitions } A|B \text{ of } \underline{n} \mid \overline{p} \subseteq A \text{ or } \overline{p} \subseteq B\right\}, \\
\mathcal{Q}_p^q(n) = &\ \mathcal{Q}_p(n) \cup \mathcal{Q}^q(n), \text{ where } p+q=n+1.
\end{aligned}$$

Given a sequence of (not necessarily distinct) positive integers $\{n_j\}_{1\leq j\leq k}$ such that $n = \sum n_j$, let

$$\mathcal{P}_{n_1,\ldots,n_k}(n) = \{\text{partitions } A_1|\cdots|A_k \text{ of } \underline{n} \mid \aleph A_j = n_j\}.$$

Theorem 2.1. *Let* $A|B \in \mathcal{P}_{p,q}(n+1)$ *and* $C|D \in \mathcal{P}_{**}(n)$. *Then* $d_{C|D}d_{A|B}$ *denotes an* $(n-2)$*-face of* P_{n+1} *if and only if* $C|D \in \mathcal{Q}_p^q(n)$.

Proof. If $d_{C|D}d_{A|B}$ denotes an $(n-2)$-face, say $X|Y|Z$, then according to relation (3) we have either

$$A|B = X|\ Y\cup Z \text{ and } C|D = X\underline{\sqcup}Y|X\underline{\sqcup}Z$$

or

$$A|B = X\cup Y|Z \text{ and } C|D = X\overline{\sqcap}Z|\,Y\overline{\sqcap}Z.$$

Hence there are two cases.

<u>*Case 1:*</u> $A|B = X|Y\cup Z$. If $\min Y = \min Y\cup Z$, then $\underline{p} \subseteq X\underline{\sqcup}Y$; otherwise $\min Y \cup Z = \min Z$ and $\underline{p} \subseteq X\underline{\sqcup}Z$. In either case, $C|D{=}X\underline{\sqcup}Y|X\underline{\sqcup}Z \in \mathcal{Q}_p(n)$.

<u>*Case 2:*</u> $A|B = X\cup Y|Z$. If $\max X = \max X\cup Y$, then $\overline{q} \subseteq X\overline{\sqcap}Z$; otherwise $\max(X\cup Y) = \max Y$ and $\overline{q} \subseteq Y\overline{\sqcap}Z$. In either case, $C|D{=}\ X\overline{\sqcap}Z|Y\overline{\sqcap}Z \in \mathcal{Q}^q(n)$.

Conversely, given $A|B \in \mathcal{P}_{p,q}(n+1)$ and $C|D \in \mathcal{Q}_p^q(n)$, let

$$[A|B;\, C|D] = \begin{cases} A|S(C)\,|S(D), & C|D \in \mathcal{Q}_p(n) \\ T(C)\,|T(D)\,|B, & C|D \in \mathcal{Q}^q(n), \end{cases}$$

where

$$S(X) = I_B^{-1}\left(\underline{q}\cap X - p + 1\right) \text{ and } T(X) = I_A^{-1}\left(\underline{p}\cap X\right).$$

A straightforward calculation shows that

$$[X|Y\cup Z;\ X\underline{\sqcup}Y|X\underline{\sqcup}Z] = X|Y|Z = [X\cup Y|Z;\ X\overline{\sqcap}Z|\,Y\overline{\sqcap}Z].$$

Consequently, if $X|Y|Z = [A|B;\, C|D]$, either

$$A|B = X|\ Y\cup Z \text{ and } C|D = X\underline{\sqcup}Y|X\underline{\sqcup}Z$$

when $C|D \in \mathcal{Q}_p(n)$ or

$$A|B = X\cup Y|Z \text{ and } C|D = X\overline{\sqcap}Z|\,Y\overline{\sqcap}Z$$

when $C|D \in \mathcal{Q}^q(n)$. $\square$

On the other hand, if $C|D \notin \mathcal{Q}_p^q(n)$, higher order structure relations involving both face and degeneracy operators appear. This rich structure distinguishes "permutahedral sets" from simplicial or cubical sets whose structure relations are strictly quadratic.

To motivate the definition of an abstract permutahedral set, we first construct the universal example–singular permutahedral sets. Define $\underline{0} = \overline{0} = \emptyset$. For $1 \leq r \leq n$ and $r + s = n + 1$, define canonical projections

$$\Delta_{r,s} : P_n \to P_r \times P_s,$$

mapping each face $A|B \in \mathcal{Q}_r^s(n)$ homeomorphically onto the $(n-2)$-product cell

$$\begin{cases} A \setminus \overline{s-1} \mid B \setminus \overline{s-1} \times \overline{s} & A|B \in \mathcal{Q}^s(n), \\ \underline{r} \times A \setminus \underline{r-1} \mid B \setminus \underline{r-1} & A|B \in \mathcal{Q}_r(n), \end{cases}$$

and each face $A|B \notin \mathcal{Q}_r^s(n)$ onto the $(n-3)$-product cell

$$A \setminus \overline{s-1} \mid B \setminus \overline{s-1} \times A \setminus \underline{r-1} \mid B \setminus \underline{r-1},$$

where $A \setminus \overline{s-1} \mid B \setminus \overline{s-1}$ is a particular partition of $\underline{r}$ and $A \setminus \underline{r-1} \mid B \setminus \underline{r-1}$ is a particular partition of $\overline{s}$ (see Figure 3).

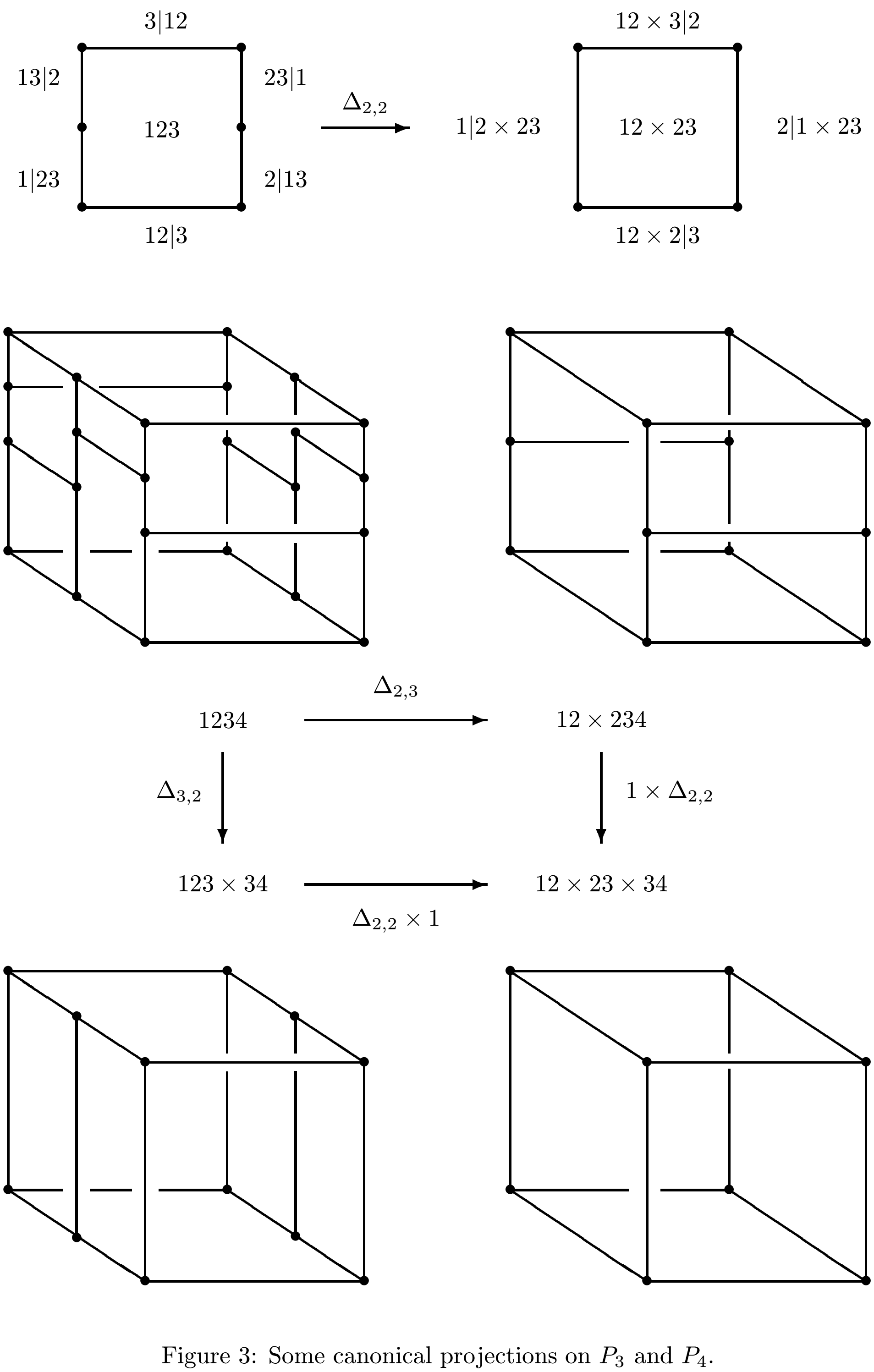

Figure 3: Some canonical projections on P_3 and P_4.

Now each $A|B \in \mathcal{P}_{r,s}\,(n+1)$ is an $(n-1)$-face of P_{n+1} homeomorphic to $P_r \times P_s$, so choose a homeomorphism $\delta_{A|B} : P_r \times P_s \to A|B$. In singular permutahedral sets, face operators pullback along the cellular projection $P_n \xrightarrow{\Delta_{r,s}} P_r \times P_s \xrightarrow{\delta_{A|B}} A|B$ and degeneracy operators pullback along the cellular projections

$$\alpha_i, \beta_j : P_n \to P_{n-1},$$

where α_i identifies the faces $\underline{i}|\underline{n} \setminus \underline{i}$ and $\underline{n} \setminus \underline{i}|\underline{i}$, $1 \leq i \leq n-1$, and β_j identifies the faces $j|\underline{n} \setminus j$ and $\underline{n} \setminus j|j$, $1 \leq j \leq n$. Note that $\alpha_1 = \beta_1$ and $\alpha_{n-1} = \beta_n$; the projections β_j were first defined by R.J. Milgram in [15] and denoted by D_j.

Example 2.2. *Let Y be a topological space. The singular permutahedral set of Y is a tuple $(Sing_*^P Y,\ d_{A|B}, \varrho_i, \varsigma_j)$, where*

$$Sing_{n+1}^P Y = \{\textit{continuous maps } P_{n+1} \to Y\},\ n \geq 0,$$

face operators

$$d_{A|B} : Sing_{n+1}^P Y \to Sing_n^P Y$$

are defined by

$$d_{A|B}(f) = f \circ \delta_{A|B} \circ \Delta_{r,s}$$

for each $A|B \in \mathcal{P}_{r,s}(n+1)$ and degeneracy operators

$$\varrho_i, \varsigma_j, : Sing_n^P Y \to Sing_{n+1}^P Y$$

are defined by

$$\varrho_i(f) = f \circ \beta_i \textit{ and } \varsigma_j(f) = f \circ \alpha_j$$

for each $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

It is easy to check that singular permutahedral sets are in fact permutahedral sets per Definition 2.2 below. For example, for the presence of a higher order structure relation see Figure 4.

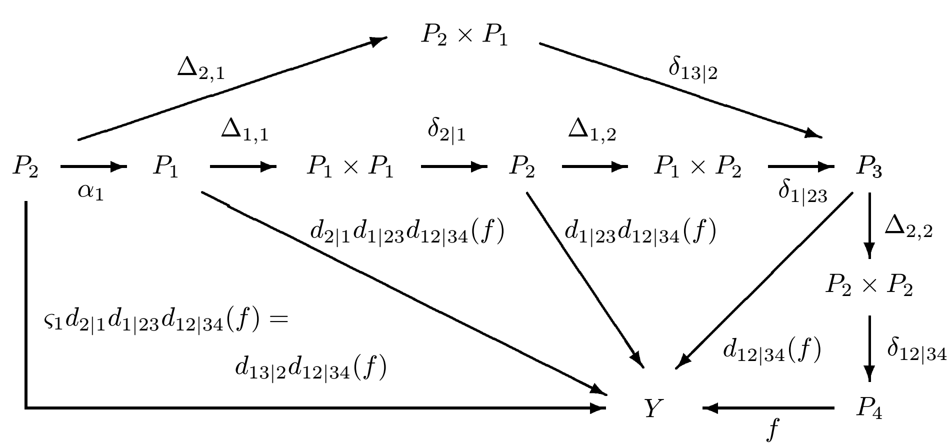

Figure 4: The quartic relation $\varsigma_1 d_{2|1}d_{1|23}d_{12|34} = d_{13|2}d_{12|34}$.

Now $Sing^P Y$ determines the singular (co)homology of Y in the following way: Form the "chain complex" $(C_*(Sing^P Y), d)$ of $Sing^P Y$ with

$$d = \sum_{A|B \in \mathcal{P}_{r,s}(n+1)} (-1)^r sgn(A;B)\ d_{A|B},$$

where $sgn(A;B)$ denotes the sign of the shuffle. Note that if $f \in C_*(Sing_4^P Y)$ and $d_{13|2}d_{12|34}(f) \neq 0$, the component $d_{13|2}d_{12|34}(f)$ of $d^2(f) \in C_*(Sing_2^P Y)$ is not cancelled and $d^2 \neq 0$ (see Figure 4). Thus d is not a differential and $(C_*(Sing^P Y), d)$ is not a complex in the classical sense. So form the quotient

$$C_*^{\diamondsuit}(Y) = C_*\left(Sing^P Y\right)/D,$$

where D is the submodule generated by the degeneracies; then $\left(C_*^{\diamondsuit}(Y), d\right)$ is the complex of singular permutahedral chains on Y. The sequence of cellular projections

$$P_{n+1} \xrightarrow{\chi} I^n \xrightarrow{\psi} \Delta^n,$$

$\chi = (1 \times \Delta_{2,2}) \cdots (1 \times \Delta_{2,n-1})\Delta_{2,n}$, ψ is defined in [18](see also [12]), induces a sequence of homomorphisms

$$C_*(SingY) \to C_*(Sing^I Y) \to C_*(Sing^P Y) \to C_*^{\diamondsuit}(Y)$$

whose composition is a chain map that induces a natural isomorphism

$$H_*(Y) \approx H_*^{\diamondsuit}(Y) = H_*(C_*^{\diamondsuit}(Y), d).$$

Although the first two terms in the sequence above are non-normalized chain complexes of singular simplicial and cubical sets, the map between them is not a chain map. In general, a cellular projection between polytopes induces a chain map between corresponding singular complexes if one uses normalized chains in the target. Finally, we note that $Sing^P Y$ also determines the singular cohomology ring of Y since the diagonal on the permutahedra and the Alexander-Whitney diagonal on the standard simplex commute with projections.

We are ready to define the notion of an abstract permutahedral set. For purposes of applications, only relation (4) in the definition below is essential; the other relations may be assumed modulo degeneracies.

Definition 2.2. *Let $P = \{P_{n+1}\}_{n \geq 0}$ be a graded set together with face operators*

$$d_{A|B} : P_{n+1} \to P_n$$

for each $A|B \in \mathcal{P}_{**}(n+1)$ and degeneracy operators

$$\varrho_i, \varsigma_j : P_n \to P_{n+1}$$

for each $1 \leq i \leq n+1$, $1 \leq j \leq n$ such that $\varrho_1 = \varsigma_1$ and $\varrho_{n+1} = \varsigma_n$. Then $\left(P, d_{A|B}, \varrho_i, \varsigma_j\right)$ is a permutahedral set if the following structure relations hold:

For all $A|B|C \in \mathcal{P}_{***}(n+1)$

(4) $$d_{A \underline{\sqcup} B | A \underline{\sqcup} C}\ d_{A|B \cup C} = d_{A \overline{\sqcup} C | B \overline{\sqcup} C}\ d_{A \cup B | C}.$$

For all $A|B \in \mathcal{P}_{r,s}(n+1)$ and $C|D \in \mathcal{P}_{**}(n) \setminus \mathcal{Q}_r^s(n)$

(5) $$d_{C|D} d_{A|B} = \varsigma_j d_{M|N} d_{K|L} d_{A|B} \quad \textit{where}$$

$$\text{either}\quad\begin{cases} K|L=\underline{n}\diagdown(\underline{r}\cap D)\,|\underline{r}\cap D, \\ M|N=C\overline{\sqcup}(\underline{r}\cap D)\,|\,(D\diagdown(\underline{r}\cap D))\overline{\sqcup}(\underline{r}\cap D), \\ j=\aleph C+\aleph(\underline{r}\cap D)-1 \text{ when } r\in C \\ \qquad\text{or} \\ K|L=\underline{r}\cap C|\underline{n}\diagdown(\underline{r}\cap C), \\ M|N=(\underline{r}\cap C)\underline{\sqcup}(C\diagdown(\underline{r}\cap C))\,|\,(\underline{r}\cap C)\underline{\sqcup}D, \\ j=\aleph(\underline{r}\cap C) \text{ when } r\in D. \end{cases}$$

For all $A|B\in\mathcal{P}_{**}(n+1)$ and $1<j<n$ (for $j=1,n$ see (7) below)

(6)

$$d_{A|B}\varsigma_j=\begin{cases} 1, & \text{if } A=\underline{j} \text{ or } B=\underline{j}, \\ \varsigma_j d_{\underline{j}|\underline{n}\diagdown\underline{j}}, & \text{if } A|B\in\mathcal{Q}_j(n+1),\ A\neq\underline{j} \text{ or } B\neq\underline{j}, \\ \varsigma_{j-1}d_{\underline{j-1}|\underline{n}\diagdown\underline{j-1}}, & \text{if } A|B\in\mathcal{Q}^{n+1-j}(n+1),\ A\neq\underline{j} \text{ or } B\neq\underline{j}, \\ \varsigma_j\varsigma_j d_{M|N}d_{K|L}, & \text{if } A|B\notin\mathcal{Q}_j^{n+1-j}(n+1) \text{ where} \end{cases}$$

$$\text{either}\quad\begin{cases} K|L=A\overline{\sqcup}\left(\underline{j}\cap B\right)|\left[B\diagdown\left(\underline{j}\cap B\right)\right]\overline{\sqcup}\left(\underline{j}\cap B\right), \\ M|N=\underline{\aleph\left(\underline{j}\cap A\right)}|\underline{n-1}\diagdown\underline{\aleph\left(\underline{j}\cap A\right)}, \\ \text{when } j\in A \\ \qquad\text{or} \\ K|L=\left(\underline{j}\cap A\right)\underline{\sqcup}\left(\underline{j}\cap B\right)|\left(\underline{j}\cap A\right)\underline{\sqcup}B, \\ M|N=\underline{j-1}|\underline{n-1}\diagdown\underline{j-1}, \\ \text{when } j\in B. \end{cases}$$

For all $A|B\in\mathcal{P}_{**}(n+1)$ and $1\leq i\leq n+1$

$$d_{A|B}\varrho_i=\begin{cases} 1, & \text{if } A=\{i\} \text{ or } B=\{i\}, \\ \varrho_j d_{C|D}, & \text{where} \end{cases} \tag{7}$$

$$\text{either}\quad\begin{cases} C|D=I_{\underline{n+2}\diagdown i}(A\diagdown i)\,|I_{\underline{n+2}\diagdown i}(B), \\ j=I_A(i) \text{ when } \{i\}\subsetneqq A \\ \qquad\text{or} \\ C|D=I_{\underline{n+2}\diagdown i}(A)\,|I_{\underline{n+2}\diagdown i}(B\diagdown i), \\ j=I_B(i)+\aleph A \text{ when } \{i\}\subsetneqq B. \end{cases}$$

For all $i\leq j$

$$\begin{aligned} \varrho_i\varrho_j&=\varrho_{j+1}\varrho_i, \\ \varsigma_i\varsigma_j&=\varsigma_{j+1}\varsigma_i, \\ \varsigma_i\varrho_j&=\varrho_{j+1}\varsigma_i, \\ \varrho_i\varsigma_j&=\varsigma_{j+1}\varrho_i. \end{aligned} \tag{8}$$

2.2. **The Cartesian product of permutahedral sets.**

Let $P'=\{P'_r,d'_{A|B},\varsigma'_i,\varrho'_j\}$ and $P''=\{P''_s,d''_{A|B},\varsigma''_i,\varrho''_j\}$ be permutahedral sets and let

$$P'\times P''=\left\{(P'\times P'')_n=\bigcup_{r+s=n+1}P'_r\times P''_s\right\}_{n\geq1}\Big/\sim\ ,$$

where $(a,b)\sim(c,d)$ if and only if $a=\varsigma'_r(c)$ and $d=\varsigma''_1(b)$, i.e.,

$$(\varsigma'_r(c),b)=(c,\varsigma''_1(b)) \text{ for all } (c,b)\in P'_r\times P''_s.$$

Definition 2.3. *The product of P' and P'', denoted by $P' \times P''$, is the permutahedral set*

$$\{P' \times P'', d_{A|B}, \varsigma_i, \varrho_j\}$$

with face and degeneracy operators defined by

$$d_{A|B}(a,b) = \begin{cases} \left(d'_{\underline{r}\cap A|\underline{r}\cap B}(a), b\right), & \text{if } A|B \in \mathcal{Q}^s(n), \\ \left(a, d''_{\underline{s}\cap(A-n+s)|\underline{s}\cap(B-n+s)}(b)\right), & \text{if } A|B \in \mathcal{Q}_r(n), \\ \varsigma_i d_{M|N} d_{K|L}(a,b), & \text{otherwise, where} \end{cases} \tag{9}$$

$$\text{either} \begin{cases} K|L = \underline{r} \cap A|(\underline{r} \cap B) \cup \underline{s-1} + r \\ M|N = (\underline{r} \cap A) \sqcup (B \setminus (\underline{r} \cap B)) \,|\, (\underline{r} \cap A) \sqcup B \\ i = \aleph(\underline{r} \cap A) \ \text{ when } r \in B, \\ \qquad \text{or} \\ K|L = A \cup (B \setminus (\underline{r} \cap B)) \,|\, \underline{r} \cap B \\ M|N = A \,\bar{\sqcup}\, (\underline{r} \cap B) \,|\, (B \setminus (\underline{r} \cap B)) \,\bar{\sqcup}\, (\underline{r} \cap B) \\ i = \aleph A + \aleph(\underline{r} \cap B) - 1 \ \text{ when } r \in A; \end{cases}$$

$$\varsigma_i(a,b) = \begin{cases} (\varsigma'_i(a), b), & 1 \leq i < r, \\ \left(a, \varsigma''_{i-r+1}(b)\right), & r \leq i \leq n; \end{cases} \tag{10}$$

$$\varrho_j(a,b) = \begin{cases} \left(\varrho'_j(a), b\right), & 1 \leq j \leq r, \\ \left(a, \varrho''_{j-r+1}(b)\right), & r < j \leq n+1. \end{cases} \tag{11}$$

Remark 2.1. *Note that the right-hand side of the third equality in (9) reduces to the first two; indeed, if $r \in B$, then $K|L \in \mathcal{Q}^s(n)$ and $M|N \in \mathcal{Q}_r(n)$; if $r \in A$, $K|L \in \mathcal{Q}^s(n)$ and $M|N \in \mathcal{Q}_r(n)$ if $\underline{m_2} + r - 1 \subset A \setminus (\underline{r-1} \cap A)$, $m_2 = \aleph(\underline{r} \cap B)$, while for $\underline{m_2} + r - 1 \not\subset A \setminus (\underline{r-1} \cap A)$ one has $K|L \in \mathcal{Q}^s(n)$, $M|N \notin \mathcal{Q}_r(n)$ and $r - 1 \in L$.*

Example 2.3. *The canonical map $\iota : Sing^P X \times Sing^P Y \to Sing^P(X \times Y)$ defined for $(f,g) \in Sing^P_r X \times Sing^P_s Y$ by*

$$\iota(f,g) = (f \times g) \circ \Delta_{r,s}$$

is a map of permutahedral sets. Consequently, if X is a topological monoid, the singular permutahedral complex $Sing^P X$ inherits a canonical monoidal structure.

Definition 2.4. *A monoidal permutahedral set is a permutahedral set P with a map $\mu : P \times P \to P$ of permutahedral sets which is associative and has the unit $e \in P_1$.*

Clearly, for a monoidal permutahedral set P its chain complex $(C^{\diamondsuit}_*(P;R), d)$ is a dg Hopf algebra.

For a permutahedral set L a *P-module* structure on it we define as a permutahedral map $P \times L \to L$ being associative and with the unit of P acting on L as identity. In this case $C^*_{\diamondsuit}(L;R)$ is a dga comodule over dg Hopf algebra $(C^*_{\diamondsuit}(P;R), d)$.

2.3. **The permutahedral set functor $\mathbf{\Omega}Q$.**

Let $Q=(Q_n,d^0_i,d^1_i,\eta_i)_{n\geq 0}$ be a cubical set. Recall that the diagonal

$$\Delta : C^{\square}_*(Q)\to C^{\square}_*(Q)\otimes C^{\square}_*(Q)$$

of Q is defined on $a\in Q_n$ by

$$\Delta(a)=\sum sgn(A;B)\, d^0_B(a)\otimes d^1_A(a),$$

where $d^0_B=d^0_{j_1}...d^0_{j_q}$, $d^1_A=d^1_{i_1}...d^1_{i_p}$, the summation is over all shuffles $\{A,B\}=\{i_1<...<i_q,j_1<...<j_p\}$ of the set $\underline{n}$. In particular the extreme cases $A=\emptyset$ and $B=\emptyset$ give the primitive part of the diagonal with $sgn(\emptyset;B)=sgn(A;\emptyset)=+$.

First, for Q let define the graded set $\mathbf{\Omega}'Q$ as follows. Let Q^c_* be the graded set of formal expressions

$$Q^c_{n+k}=\{\varsigma_{i_k}\cdots\varsigma_{i_1}\varsigma_{i_0}(a)|\, a\in Q_n\}_{n\geq 0;k\geq 0},$$

where

$$i_1\leq\cdots\leq i_k,\ 1\leq i_j\leq n+j-1,\ 1\leq j\leq k,\ \varsigma_{i_0}=1,$$

and let $\bar{Q}^c=s^{-1}(Q^c_{>0})$ denote the desuspension of Q^c. Then define $\mathbf{\Omega}''Q$ as the free graded monoid (without unit) generated by $\bar{Q}^c$. Let $\mathbf{\Omega}'Q$ be the monoid obtained from $\mathbf{\Omega}''Q$ via

$$\mathbf{\Omega}'Q=\mathbf{\Omega}''Q/\sim\ ,$$

where $\overline{\varsigma_{p+1}(a)}\cdot\bar{b}\sim\bar{a}\cdot\overline{\varsigma_1(b)}$ for $a,b\in Q^c$, $|a|=p+1$. Clearly, we have the inclusion $MQ\subset\mathbf{\Omega}'Q$ of graded monoids where MQ denotes the free monoid generated by $\bar{Q}=s^{-1}(Q_{>0})$.

Then we introduce the canonical structure of a permutahedral set on $\mathbf{\Omega}'Q$ as follows. First define the degeneracy operator ς_i by $\varsigma_i(\bar{a})=\overline{\varsigma_i(a)}$ for a monoidal generator $\bar{a}\in\bar{Q}$; next, for $\bar{a}\in\bar{Q}\subset\bar{Q}^c$ define $\varrho_j(\bar{a})=\overline{\eta_j(a)}$; and finally, if $\bar{a}$ is any other element of $\bar{Q}^c$ define its degeneracy accordingly to (8). Use formulas (10) and (11) to extend both degeneracy operators on decomposables. Now for $\bar{a}\in\bar{Q}_{n+1}\subset\bar{Q}^c{}_{n+1}$, define the face operator $d_{M_1|M_2}$ by

$$d_{M_1|M_2}(\bar{a})=\overline{d^0_{M_2}(a)}\cdot\overline{d^1_{M_1}(a)},\quad M_1|M_2\in\mathcal{P}_{*,*}(n+1),$$

while for other elements of $\bar{Q}^c$ and for decomposables in $\mathbf{\Omega}'Q$ use formulas (5)-(7) and (9) to define $d_{M_1|M_2}$ by induction on grading.

Now suppose Q has a fixed vertex $*$. Then $\overline{\eta_1(*)}$ is declared as a unit, e, of $\mathbf{\Omega}'Q$. This relation converts $\mathbf{\Omega}'Q$ into a (unital) graded monoidal permutahedral set denoted by $(\mathbf{\Omega}Q,\, d_{M_1|M_2},\, \varsigma_i,\, \varrho_j)$.

In particular, we have the following identities:

$$d_{i|\underline{n+1}\setminus i}\,(\overline{a})=\overline{d^1_i(a)},\quad 1\leq i\leq n,$$

$$d_{\underline{n+1}\setminus i|i}\,(\overline{a})=\overline{d^0_i(a)},\quad 1\leq i\leq n.$$

Thus, for a 1-reduced cubical set Q all its face operators are involved in the definition of $\mathbf{\Omega}Q$.

Remark 2.2. *Note that the definition of $\mathbf{\Omega}Q$ uses all cubical degeneracies. This is justified geometrically by the fact that a degenerate singular n-cube in the base of a path space fibration lifts to a singular $(n-1)$-permutahedron in the fibre, which is degenerate with respect to Milgram's projections. On the other hand, we must formally adjoin the other degeneracies to achieve relations (5) (c.f., the definition of the cubical set $\mathbf{\Omega}X$ on a simplicial set X [12]).*

3. The permutocubes

The pertmutocube B_n is an n-dimensional polytope discovered by N. Berikashvili which can be thought of as a "twisted Cartesian product" of the cube and the permutahedron. Originally the permutocube B_n has been obtained from I^n by the following truncation procedure: First the n-cube is truncated at the minimal vertex $a_0=(0,...,0)$, then it is truncated along those $n-1$-faces that contained a_0, and continuing so the last truncation is along those 1-faces (edges) of the n-cube that contained a_0. Hence, B_2 is a pentagon (Figure 6), for B_3 see Figure 8. In particular at a_0 one obtains the permutahedron P_n. So that we get the natural cellular embedding (see Figures 5,7)

$$\delta_{\emptyset]\underline{n}}: P_n \to B_n. \tag{12}$$

The notation for the above inclusion map is motivated by the following combinatorial description of B_n. First remark that the faces of B_n are in one-to-one correspondence with partitions $A]M_1|...|M_m$ of all subsets of the set $\underline{n}$ in which only A is allowed to be the empty set $\emptyset$. Since faces of P_n correspond to all (non-empty) partitions of $\underline{n}$ the canonical bijection $\underline{n} \xrightarrow{\approx} \emptyset]\underline{n}$ is thought of as a combinatorial analog of $\delta_{\emptyset]\underline{n}}$.

Let $\mathcal{A}(n)$ be the set of all (ordered) subsets of $\underline{n}$ including the empty set $\emptyset$ too. In particular, $\aleph\mathcal{A}(n)=2^n$. For $\lambda\in\mathcal{A}(n)$ let $\mathcal{A}_\lambda$ denote its corresponding subset in $\underline{n}$. First we introduce a face operator d_i which is thought of as deliting i-th element of $\underline{n}$; so that it resembles the simplicial operator ∂_{i-1}. We have the one-to-one correspondence between the set $\mathcal{A}(n)$ and the set of formal compositions of d_i's defined by

$$\mathcal{A}_\lambda=\{1,...,\hat{i}_k,...,\hat{i}_1,...,n\}\longleftrightarrow d_{i_k}\cdots d_{i_1}.$$

Then to a face of B_n corresponding to the subset $\mathcal{A}_\lambda\subset\underline{n}$ we assign the composition of face operators $d_{i_k}\cdots d_{i_1}$.

Now for a set $\mathcal{A}_\lambda$ let

$$\mathcal{P}^0_{r,m_1,...,m_q}(\mathcal{A}_\lambda)=\{\text{partitions } A_0]\bar{M}_1|...|\bar{M}_q \text{ of } \mathcal{A}_\lambda\,|\aleph A_0=r\geq 0, \aleph\bar{M}_j=m_j\geq 1\},$$

$1\leq j\leq q,\ 1\leq q\leq\aleph\mathcal{A}_\lambda$. For example, $q=\aleph\mathcal{A}_\lambda$ if and only if $A_0=\emptyset$ and each $\bar{M}_j$ consists of a single element. Such partitions just correspond to the vertices of B_n. For $A_\lambda=\underline{m}$ we simply denote $\mathcal{P}^0(\underline{m}\,)$ by $\mathcal{P}^0(m)$.

Next introduce the second type of a face operator $d_{A]M}$ for those $(n-1)$-faces of B_n which correspond to partitions $A_0]\bar{M}\in\mathcal{P}^0_{r,m}(\mathcal{A}_\lambda)$ where $A=I_{\mathcal{A}_\lambda}(A_0)$ and $M=I_{\mathcal{A}_\lambda}(\bar{M})$; in particular the face operator $d_{\emptyset]\underline{n}}$ just denotes the single $(n-1)$-permutahedral face $\delta_{\emptyset]\underline{n}}(P_n)\subset B_n$.

Then any $(n-k-q)$-face u of B_n corresponding to a partition $A_0]\bar{M}_1|...|\bar{M}_q\in\mathcal{P}^0_{r,m_1,...,m_q}(\mathcal{A}_\lambda)$ can be expressed as the composition of face operators

$$d_{A_q]M_q}\cdots d_{A_1]M_1}\,d_{i_k}\cdots d_{i_1},$$

with $A_j = I_{B_j}(B_{j+1})$, $M_j = I_{B_j}(\bar{M}_{q-j+1})$, $B_j = \mathcal{A}_\lambda \setminus (\bar{M}_{q-j+2} \cup \cdots \cup \bar{M}_q), B_1 = \mathcal{A}_\lambda$, $1 \leq j \leq q$, and let denote this composition by $d_{A_0]\bar{M}_1|\ldots|\bar{M}_q}$ or by d_u.

For example, for $n = 9$ if $\{i_2 < i_1\} = \{2 < 5\}$, then $\mathcal{A}_\lambda = \{1, \hat{2}, 3, 4, \hat{5}, 6, 7, 8, 9\}$, and for the 4-face u of B_9 corresponding to $38]14|6|79 \in \mathcal{P}^0_{2,2,1,2}(\mathcal{A}_\lambda)$, one gets $u = d_{38]14|6|79}(B_9) = d_{24]13}d_{1235]4}d_{12346]57}d_2d_5(B_9)$.

We have that B_n also admits a realization as a subdivision of the standard n-cube I^n compatible with inclusion (12) (see, Figures 6,8). Indeed, let $B_0 = *$ and label the endpoints of $B_1 = [0,1]$ via $e^0_{1,0} \leftrightarrow d_{\emptyset]1}$ and $e^0_{1,1} \leftrightarrow d_1$. Inductively, if B_{n-1} has been constructed, obtain B_n as a subdivision of $B_{n-1} \times I$ in the following way:

Face of B_n	Label	
$e^{n-1}_{n,0}$	$d_{\underline{n-1}]n}$	
$e^{n-1}_{i,1}$	$d_i,$	$i \in \underline{n}$
$d_{A]M} \times I_{0,\aleph M}$	$d_{A]M\cup\{n\}}$	
$d_{A]M} \times I_{\aleph M,\infty}$	$d_{A\cup\{n\}]M}.$	

From this we evidently see that that each proper m-cell e_m of B_n has the form $e_m = e_p \times e_{q+1}$, $m = p + q$, where e_p and e_{q+1} are top cells of B_p and P_{q+1} respectively. Consequently, on proper cells of the permutocube we have the action of a permutahedral face operator $d_{M_1|M_2}$ as $d_{M_1|M_2}(e_m) = e_p \times d_{M_1|M_2}(e_{q+1})$.

These operators are connected together with d_i and $d_{A]M}$ by the canonical relations. Namely, combinatorially the relations between $d_{A]M}$ and $d_{M_1|M_2}$ reflect the associativity of the partition procedure, while the relations between d_i and either $d_{A]M}$ or $d_{M_1|M_2}$ reflect the commutativity of the deleting and the partition procedures.

These relations together with those involving degeneracies incorporated in the singular permutocubes (see Example 4.1) motivates the notion of a permutocubical set given in the next section.

4. Permutocubical sets

Definition 4.1. *A permutocubical set is a graded set*

$$B = \{B^{p,q}_n \,|\, p, q \geq 0;\, p + q = n\}_{n\geq 0}$$

together with face and degeneracy operators

$$\begin{aligned}
&d_i &&: B^{p,q}_n \to B^{p-1,q}_{n-1}, && i \in \underline{p}\,,\\
&d_{A]M} &&: B^{p,q}_n \to B^{p-r,q+r-1}_{n-1}, && A]M \in \mathcal{P}^0_{p-r,r}(p),\\
&d_{M_1|M_2} &&: B^{p,q}_n \to B^{p,q-1}_{n-1}, && M_1|M_2 \in \mathcal{P}_{*,*}(q+1),\\
&\eta_j &&: B^{p,q}_n \to B^{p+1,q}_{n+1}, && j \in \underline{p+1},\\
&\varsigma_i, \varrho_j &&: B^{p,q}_n \to B^{p,q+1}_{n+1}, && i \in \underline{q+1},\ j \in \underline{q+2},
\end{aligned}$$

that satisfy the following relations:

For each $p \geq 0$ the graded set

$$\{B_n^{p,q};\, d_{M_1|M_2},\, \varsigma_i, \varrho_j\}_{q\geq 0;\, p+q=n}$$

is a permutahedral set and

$$\begin{array}{lll}
d_i d_j & = d_{j-1} d_i, & i<j, \\
d_i d_{A]M} & = d_{A\setminus j]M} d_j, & j = I_A^{-1}(i),\ \ i\in \underline{p-r}, \\
d_i d_{M_1|M_2} & = d_{M_1|M_2} d_i, & \\
d_{M_1|M_2} d_{A]M} & = d_{A]M} d_{M_3|M_4}, & M_1|M_2 \in \mathcal{Q}_r(q+r), \\
 & & M_3|M_4 = M_1+1-r\cap \underline{q+1}|M_2+1-r\cap \underline{q+1}, \\
d_{M_1|M_2} d_{A]M} & = d_{A_2]L_2} d_{A_1]L_1}, & A_1|L_1 = A\cup I_M^{-1}(M_1\cap \underline{r})|\, I_M^{-1}(M_2\cap \underline{r}), \\
 & & A_2|L_2 = A|\, I_M^{-1}(M_1\cap \underline{r}),\ \ M_1|M_2 \notin \mathcal{Q}_r(q+r),
\end{array}$$

$$\begin{array}{lll}
d_i\eta_j & = \eta_j d_i, & i<j; \\
d_i\eta_j & = 1, & i=j; \\
d_i\eta_j & = \eta_j d_{i-1}, & i>j; \\
d_{A]M}\eta_j & = \eta_i d_{A_1]M_1}, & A_1|M_1 = I_{\underline{p+1}\setminus j}(A\setminus j)|\, I_{\underline{p+1}\setminus j}(M), \\
 & & i = I_A(j),\, j\in A, \\
d_{A]M}\eta_j & = \varrho_i d_{A_1]M_1}, & A_1|M_1 = I_{\underline{p+1}\setminus j}(A)|\, I_{\underline{p+1}\setminus j}(M\setminus\{j\}), \\
 & & i = I_M(j), j\in M, r>1, \\
d_{A]M}\eta_j & = 1, & A]M = \underline{p+1}\setminus j|j, \\
d_{M_1|M_2}\eta_j & = \eta_j d_{M_1|M_2}, & \\
d_i\zeta_j & = \zeta_j d_i, & \zeta = \varsigma, \varrho, \\
d_{A]M}\zeta_j & = \zeta_{j+r-1} d_{A]M}, & \zeta = \varsigma, \varrho, \\
\eta_i\eta_j & = \eta_{j+1}\eta_i, & i\leq j, \\
\zeta_i\eta_j & = \eta_j\zeta_i, & \zeta = \varsigma, \varrho.
\end{array}$$

Example 4.1. *For a topological space Y define the singular permutocubical complex $Sing^B Y$ as follows: Let*

$$(Sing^B Y)_n^{p,q} = \{\textit{continuous maps } B_p\times P_{q+1}\to Y\}_{p,q\geq 0;\, p+q=n},$$

$B_p\times P_{q+1}$ is a Cartesian product of the permutocube B_p and the permutohedron P_{q+1}. Let

$$\begin{array}{lll}
\delta_i\times 1 & : B_{p-1}\times P_{q+1}\to B_p\times P_{q+1}, & 1\leq i\leq p, \\
\bar{\delta}_{A]M} & : B_{p-r}\times P_{q+r}\xrightarrow{1\times\Delta_{r,q+1}} B_{p-r}\times P_r\times P_{q+1}\xrightarrow{\delta_{A]M}\times 1} B_p\times P_{q+1}, & \\
1\times\delta_{M_1|M_2} & : B_p\times P_q\to B_p\times P_{q+1}, &
\end{array}$$

be the maps in which δ_i *and* $\delta_{A]M}$ *are the canonical inclusions, while* $\delta_{M_1|M_2}$ *is defined in Example 2.2. Consider also the maps*

$$\begin{array}{ll} \gamma_j\times 1: B_{p+1}\times P_{q+1}\rightarrow B_p\times P_{q+1}, & j\in \underline{p+1},\\ 1\times\alpha_j: B_p\times P_{q+2}\quad\rightarrow B_p\times P_{q+1}, & j\in\underline{q+1},\\ 1\times\beta_j: B_p\times P_{q+2}\quad\rightarrow B_p\times P_{q+1}, & j\in\underline{q+2}, \end{array}$$

where $\gamma_j: B_{p+1}\rightarrow B_p$ *is the projection that identifies the faces* $d_{\underline{p+1}\setminus j|j}$ *and* d_j*.*

Then for $f\in (Sing^BX)^{p,q}_n$ *define*

$$\begin{array}{ll} d_i & : (Sing^BY)^{p,q}_n\rightarrow (Sing^BY)^{p-1,q}_{n-1},\\ d_{A]M} & : (Sing^BY)^{p,q}_n\rightarrow (Sing^BY)^{p-r,q+r-1}_{n-1},\\ d_{M_1|M_2} & : (Sing^BY)^{p,q}_n\rightarrow (Sing^BY)^{p,q-1}_{n-1}, \end{array}$$

and

$$\begin{array}{ll} \eta_j & : (Sing^BY)^{p,q}_n\rightarrow (Sing^BY)^{p+1,q}_{n+1},\\ \varsigma_i,\varrho_j & : (Sing^BY)^{p,q}_n\rightarrow (Sing^BY)^{p,q+1}_{n+1}, \end{array}$$

as compositions

$$\begin{array}{ll} d_i(f) & = f\circ(\delta_i\times 1),\\ d_{A]M}(f) & = f\circ\bar{\delta}_{A]M},\\ d_{M_1|M_2}(f) & = f\circ(1\times\delta_{M_1|M_2}),\\ \eta_i(f) & = f\circ(\gamma_i\times 1),\\ \varsigma_i(f) & = f\circ(1\times\alpha_i),\\ \varrho_i(f) & = f\circ(1\times\beta_i). \end{array}$$

It is easy to check that $(Sing^BY,\ d_i,\ d_{A]M}, d_{M_1|M_2},\ \eta_i,\ \varsigma_i,\ \varrho_i)$ *is a permutocubical set.*

The singular permutocubical complex $Sing^BY$ *determines the singular (co)homology of* Y *in the following way: Form the "chain complex"* $(C_*(Sing^BY),d)$ *of* $Sing^BY$ *with*

$$d=\sum(-1)^{i+1}d_i - sgn(A;M)(-1)^{\aleph A}d_{A]M}+sgn(M_1;M_2)(-1)^{\aleph M_1}d_{M_1|M_2},$$

where the summation is over all $i\in\underline{n}$*,* $A]M\in\mathcal{P}^0_{**}(p)$ *and* $M_1|M_2\in\mathcal{P}_{**}(q+1)$*.*

Then consider the quotient being a chain complex in the classical sense (i.e., $d^2=0$*)*

$$C^{\boxminus}_*(Y)=C_*(Sing^BY)/D,$$

where D *is the submodule of* $C_*(Sing^BY)$ *generated by the degenerate elements of* $Sing^BY$*.*

Now let $\varphi: B_n\rightarrow I^n$ *be the cellular projection defined by the property that it maps homeomorphically the faces* $d_{\underline{n}\setminus i]i}(B_n)$ *and* $d_i(B_n)$ *onto the faces* $d^0_i(I^n)$ *and* $d^1_i(I^n)$ *respectively,* $1\leq i\leq n$*. Then the composition of maps*

$$B_p\times P_{q+1}\xrightarrow{\phi} I^p\times I^q=I^{p+q}\xrightarrow{\psi}\Delta^{p+q},\quad \phi=\varphi\times\chi,$$

clearly induces a composition of maps of graded sets

$$SingY\xrightarrow{\psi} Sing^IY\xrightarrow{\phi} Sing^BY$$

denoted by the same symbols. After the passage on the non-normalized chains (unless the last one) one gets a sequence of homomorphisms

$$C_*(SingY) \to C_*(Sing^I Y) \to C_*(Sing^B Y) \to C_*^{\boxminus}(Y),$$

whose composition is a chain map inducing a natural isomorphism

$$H_*(Y) \approx H_*^{\boxminus}(Y) = H_*(C_*^{\boxminus}(Y), d).$$

Since the diagonal on the permutocube constructed in Section 6 is compatible with the AW diagonal on the standard simplex under the above cellular projections, $H_*^{\boxminus}(Y)$ determines the singular cohomology ring of Y as well.

Basic examples of a permutocubical set are provided in the next section.

5. Truncating twisting functions and twisted Cartesian products

An universal example of truncating twisting function is just the canonical inclusion function $\vartheta_U : Q \to \mathbf{\Omega} Q$, $x \to \bar{x}$, of degree -1, where $\mathbf{\Omega} Q$ is the permutahedral set for a cubical set Q constructed above.

The geometrical interpretation of ϑ_U answers to the truncation procedure that converts I^n into B_n mentioned in Section 3. By this the permutocube is thought of as a "twisted Cartesian product" of the cube and the permutohedron (see Fig. 5,7).

Motivated by this here we give the general formalism for such functions and then the corresponding notion of twisted Cartesian product.

Definition 5.1. *Let $Q = (Q_n, d_i^0, d_i^1, \eta_i)$ be a 1-reduced cubical set and $P = (P_{n+1}, d_{M_1|M_2}, \varsigma_i, \varrho_i)$ be a monoidal permutahedral set. A sequence $\vartheta = \{\vartheta_n\}_{n\geq 1}$ of degree -1 functions $\vartheta_n : Q_n \to P_n$ is called a truncating twisting function if*

$$\begin{array}{llll} \vartheta(a) & = e, & & a \in Q_1, \\ d_{M_1|M_2}\vartheta(a) & = \vartheta d^0_{M_2}(a)\cdot \vartheta d^1_{M_1}(a), & M_1|M_2 \in \mathcal{P}_{*,*}(n), & a \in Q_n, \\ \varrho_i\vartheta(a) & = \vartheta\eta_i(a), & i \in \underline{n}. & \end{array}$$

Note that since the first condition above we in particular get

$$\begin{aligned} d_{i|\underline{n}\setminus i}\vartheta(a) &= \vartheta d_i^1(a), \quad i \in \underline{n}, \\ d_{\underline{n}\setminus i|i}\vartheta(a) &= \vartheta d_i^0(a), \quad i \in \underline{n}, \end{aligned}$$

for any $a \in Q_{n>0}$.

Remark 5.1. *By definition a truncation twisting function involves only the permutahedral degeneracy operator ϱ_i, since it is in fact arisen by the cubical degeneracy operator η_i (cf. Remark 2.2).*

We have the following

Proposition 5.1. *Let Q be a 1-reduced cubical set and P be a monoidal permutahedral set. A sequence $\vartheta = \{\vartheta_n\}_{n\geq 1}$ of degree -1 functions $\vartheta_n : Q_n \to P_n$ is a truncating twisting function if and only if the monoidal map $f : \mathbf{\Omega} Q \to P$ defined by $f(\bar{a}_1 \cdots \bar{a}_k) = \vartheta(a_1)\cdots\vartheta(a_k)$ is a map of permutahedral sets.*

Proof. Obvious. □

Definition 5.2. *Let $Q=(Q_n,d_i^0,d_i^1,\eta_i)$ be a 1-reduced cubical set and $P=(P_{n+1},d_{M_1|M_2},\varsigma_i,\varrho_i)$ be a monoidal permutahedral set and L be a permutahedral set with P-module structure. Let $\vartheta=\{\vartheta_n\}_{n\geq 1}$, $\vartheta_n:Q_n\to P_n$ be a truncating twisting function. The twisted Cartesian product $Q\times_\vartheta L$ is the Cartesian product of sets*

$$Q\times L=\{(Q\times L)_n^{p,q}=\bigcup_{n=p+q} Q_p\times L_{q+1}\}$$

endowed with the face and degeneracy operators $d_i, d_{A]M}, d_{M_1|M_2}, \eta_j,\varsigma_j,\varrho_j$ defined for $(a,b)\in Q_p\times L_{q+1}$ by :

$$\begin{array}{lcll}
d_i(a,b) & = & (d_i^1(a),\ b), & i\in \underline{p},\\
d_{A]M}(a,b) & = & (d_M^0(a),\ \vartheta d_A^1(a)\cdot b), & A]M\in \mathcal{P}^0_{*,*}(p),\\
d_{M_1|M_2}(a,b) & = & (a,\ d_{M_1|M_2}(b)), & M_1|M_2\in \mathcal{P}_{*,*}(q+1),\\
\eta_j(a,b) & = & (\eta_j(a),\ b), & j\in \underline{p+1},\\
\varsigma_j(a,b) & = & (a,\ \varsigma_j(b)), & j\in \underline{q+1},\\
\varrho_j(a,b) & = & (a,\ \varrho_j(b)), & j\in \underline{q+2}.
\end{array}$$

It is easy to check that $(Q\times_\vartheta L, d_i, d_{A]M}, d_{M_1|M_2},\eta_j,\varsigma_j,\varrho_j)$ is a permutocubical set.

Remark 5.2. *Note that to a twisted Cartesian product $Q\times_\vartheta L$ in fact corresponds the sequence of graded sets*

$$L\xrightarrow{\iota} Q\times_\vartheta L\xrightarrow{\xi} Q$$

with $\iota(b)=(a_0,b)$ and $\xi(a,b)=a,\ a_0\in Q_0,\ a\in Q,\ b\in L.$

Example 5.1. *Let $M=\{e_k\}_{k\geq 0}$ be the free minoid on a single generator $e_1\in M_1$ with trivial permutahedral set structure and let $\vartheta:Q\to M$ be the sequence of constant maps $\vartheta_n:Q_n\to M_{n-1}$, $n\geq 1$. Then the twisted Cartesian product $Q\times_\vartheta M$ can be thought of as a permutocubical resolution of a 1-reduced cubical set Q.*

5.1. **The permutocubical set functor $\mathbf{P}Q$.**

For the universal truncating twisting function ϑ_U the corresponding twisted Cartesian product implies the following

Definition 5.3. *A functor from the category of 1-reduced cubical sets to the category of permutocubical sets defined by $Q\to Q\times_{\vartheta_U}\mathbf{\Omega}Q$ is the universal permutocubical functor and is denoted by $\mathbf{P}$.*

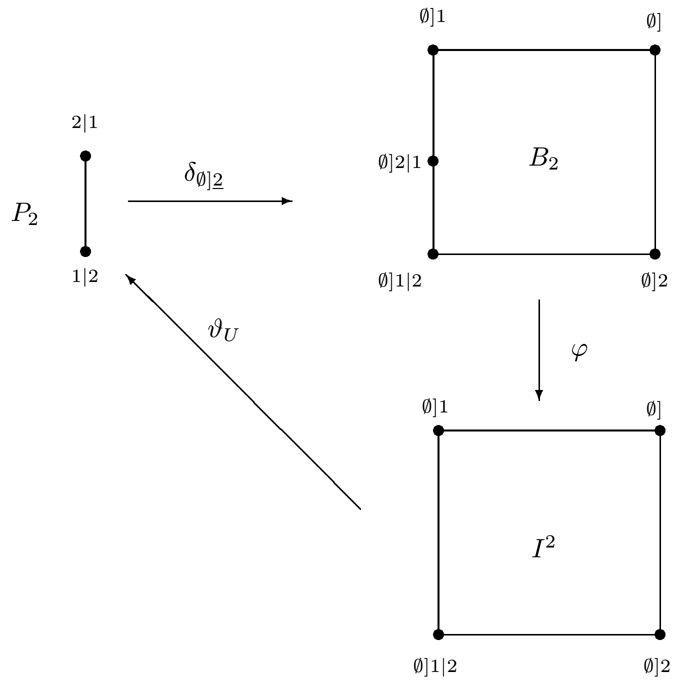

Figure 5: The universal truncating twisting function ϑ_U.

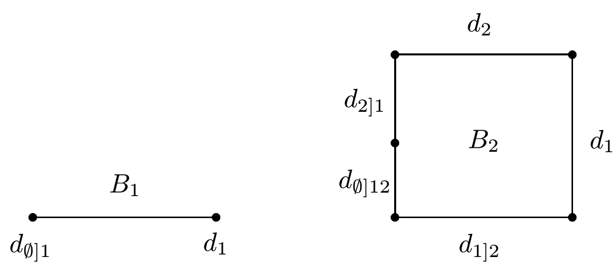

Figure 6: B_2 as a subdivision of $B_1 \times I$.

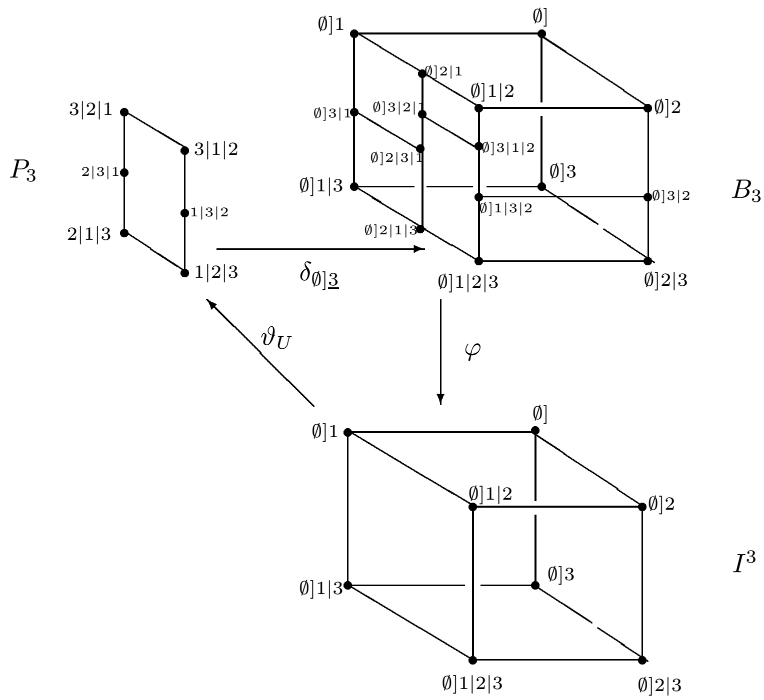

Figure 7: The universal truncating twisting function ϑ_U.

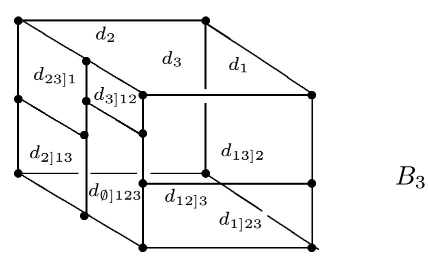

Figure 8: B_3 as a subdivision of $B_2 \times I$.

6. The diagonal of permutocubes

Here we construct the explicit diagonal $\Delta_B : C_*(B_n) \to C_*(B_n) \otimes C_*(B_n)$ for permutocubes which induces a diagonal for a permutocubical set too.

6.1. **The orthogonal stream.**

Suppose that an n-dimensional polytope X is realized as a subdivision of the cube I^n so that each m-dimensional cell $e_m \subset X,\ 0 \leq m \leq n$, is itself a subdivision of I^m (I^m need not to be a face of I^n, cf. B_n).

In particular, we have an induced partial ordering on the set of all vertices of e_m defined by $x \leq y$ if there is an oriented polygonal line from x to y.

Let e_m be a cell. For a cell $e_k \subset e_m$ let $I^{m(e_k)} \subset I^m$ be the face of I^m of the minimal dimension $m(e_k)$ that contains e_k. Then we introduce the following

Definition 6.1. *Let $e_m \subset X$ be a cell and $x \in e_m$ be a vertex. An orthogonal stream $OS_x(e_m)$ of x with support e_m is a pair (U_x, V_x) of collections of those faces $U_x = \{u_1, ..., u_r\}$ and $V_x = \{v_1, ..., v_s\}$ of e_m which satisfy the following conditions:*
1. $\max u_r = x = \min v_1$ *and* $\dim u_r + \dim v_1 = m$;
2. $I^{m(u_i)} = I^{m(u_r)}$, $\dim u_i = \dim u_r$ *and* $\max u_i \leq x,\ 1 \leq i \leq r$;
3. $I^{m(v_j)} = I^{m(v_1)}$, $\dim v_j = \dim v_1$ *and* $\min v_j \geq x,\ 1 \leq j \leq s$.
The union $\cup_{x\in e_m} SO_x(e_m)$ is denoted by $SO(e_m)$.

A pair $(u_i, v_j) \in OS_x(e)$ is referred to as a complementary pair (CP), while the pair $(u_r, v_1) \in OS_x(e)$ to as a strong complementary pair (SCP) (compare, [17]) and will be denoted by (u_x, v_x).

Clearly, any vertex x of $e_m \subset B_n$ uniquely defines (u_x, v_x) in $OS_x(e_m)$, and, consequently, the whole $OS_x(e_m)$ is uniquely determined by the vertex x. In particular, if x coincides with a vertex of I^m then $\dim u_x = m(u_x)$ and $\dim v_x = m(v_x)$, so that U_x and V_x actually lay on orthogonal faces of I^m at the vertex x.

For B_n, an orthogonal stream $OS_x(B_n)$ admits the specific combinatorial description. First, let B a linearly ordered (finite) set and let $y = \{b_1, b_2, ..., b_m\},\ m \geq 1$, be any (non-ordered) sequence formed by its elements (i.e., corresponding to some element of S_m). Then it corresponds two sequences with ordered blocks $u_y = A_1|...|A_p$ and $v_y = C_1|...|C_q$ defined as follows: $A_1 = \{b_{j_1} < ... < b_1\}$ is the first maximal block of decreasing elements (i.e., $b_{j_1} < b_{j_1+1}$), $A_2 = \{b_{j_2} < ... < b_{j_1+1}\}$ is the next such a block, and so on, while $C_1 = \{b_1 < ... < b_{k_1}\}$ is the first maximal block of increasing elements (i.e., $b_{k_1} > b_{k_1+1}$), $C_2 = \{b_{k_1+1} < ... < b_{k_2}\}$ is the next such a block, and so on.

For example, for $B = \underline{5}$ and $y = \{2, 1, 4, 3, 5\}$ one gets $u_y = 12|34|5$ and $v_y = 2|14|35$.

Now let $x = \emptyset]a_1|...|a_{n-k},\ 0 \leq k \leq n$, be a vertex of B_n, i.e., the set $\{a_1, ..., a_{n-k}\}$ is the same as $d_{i_k} \cdots d_{i_1}(\underline{n})$ with $\{a_1, ..., a_{n-k}\} = \underline{n} \setminus A_0,\ A_0 = \{i_k < \cdots < i_1\}$. For the sequence $x_0 = \{a_1, ..., a_{n-k}\}$ let $(u_{x_0}, v_{x_0}) = (A_1|...|A_p\,, C_1|...|C_q)$ be the corresponding pair determined above. Then for the SCP (u_x, v_x) we get the equality

$$(u_x, v_x) = (A_0]A_1|...|A_p\,,\ C_1]C_2|...|C_q\,).$$

For example, $x = \emptyset]2|1|3|6|5$, then $(u_x, v_x) = (4]12|3|56\,, 2]136|5)$.

Next for a partition $a = A_0]A_1|...|A_\ell$ of an ordered (finite) set we define the right-shift R and the left-shift L operators respectively as follows (compare, [17]): Let $M_i \subset A_i$ and $N_j \subset A_j$, $0 \leq i < \ell$, $0 < j \leq \ell$, be proper subsets, while $M_0 = A_0$

is also allowed, as well as all the subsets to be the $\emptyset$. Let

$$R_{M_i}(a) = A_0]A_1|\cdots|A_i\setminus M_i|A_{i+1}\cup M_i|\cdots|A_\ell \quad \text{for} \quad \min M_i > \max A_{i+1},$$
$$L_{N_j}(a) = A_0]A_1|\cdots|A_{j-1}\cup N_j|A_j\setminus N_j|\cdots|A_\ell \quad \text{for} \quad \min N_j > \max A_{j-1},$$

where $R_\emptyset = Id = L_\emptyset$. Then each CP $(u,v)\in(U_x,V_x)$ can be obtained from the SCP (u_x,v_x) by successive application of the above operators as

$$(u,v) = (R_{M_{\ell-1}}\cdots R_{M_1}R_{M_0}(u_x)\,,\, L_{N_1}\cdots L_{N_\ell}(v_x))$$

for some $\{M_i\}_{0\leq i<\ell}$ and $\{N_j\}_{0<j\leq\ell}$. Thus, the whole orthogonal stream $OS_x(B_n)$ is recovered by all successive applications of the operators R and L on the SCP (u_x,v_x).

For example, for the point $x=\emptyset]2|1|3|6|5$ we obtain

$$OS_x(B_6) = (U_x,V_x) =$$
$$(\{\emptyset]\,12|34|56,\ \emptyset]\,124|3|56,\ 4]\,12|3|56\},\ \{2]\,136|5],\ 23]\,16|5,\ 26]13|5,\ 236]\,1|5\}).$$

It is clear to describe $SO_x(e_m)$ by the same manner for each proper cell $e_m\subset B_n$ too. In particular, for the permutahedron P_n regarded as the codim 1 cell $d_{\emptyset]\underline{n}}(B_n)$ of B_n such a description simply ignores the symbol $\emptyset]$.

Thus, for an arbitrary sequence of natural numbers corresponding to a vertex either in B_n or P_n (more precisely, for B_n the sequence begins with the symbol $\emptyset$), it has sense to regard an orthogonal stream to be a pair of collections of partitions of $\underline{n}$. To emphasize such a purely combinatorial description of an orthogonal stream we use the notations $OS(n])$ and $OS(n)$ instead of $OS(B_n)$ and $OS(P_n)$ respectively.

6.2. **The signs of $SO_x(B_n)$ and $SO_x(P_n)$.**

a) The sign for a pair $(u,v)\in OS_x(B_n)$ is deduced by motivation that the cellular projection $\varphi: B_n\to I^n$ preserves diagonals.

First let y be a vertex of the cube and $(u_y,v_y)\in OS_y(B_n)$ be the SCP. Then let

$$\epsilon_0(y) = sgn\, u_y\cdot sgn(y)\cdot sgn\, v_y,$$

where $sgn(y)$ is the sign of the shuffle for the orthogonal pair at y in the diagonal of the cube. Next let

$$f:\{\text{vertices of } B_n\}\to\{\text{vertices of } B_n \text{ that coincide with those of } I^n\}$$

be a function defined for a vertex $x\in B_n$ by $f(x)=\max I^{n(x)}$. Then for a pair $(u,v)\in OS_x(B_n)$ define

$$sgn(u,v) = sgn\ u\cdot\epsilon_0(f(x))\cdot sgn\ v,$$

where for an $(n-q)$-face $u=A]M_1|\cdots|M_q\in B_n$,

$$sgn\, u = \prod_{1\leq i\leq q}(-1)^{\aleph A_i+q}sgn(A_i;M_i),\ \ A_i = A\cup M_1\cup\ldots\cup M_{i-1}.$$

b) The sign for a pair $(u,v)\in OS_x(P_{n+1})$ is defined analogously but with the following modifications:

1. $\varepsilon_0(y) = sgn\, u_1\cdot sgn(y)\cdot sgn\, v_y,\quad \max\, u_1\leq\max\, u,\ u\in U_y,$ and
2. f is replaced by

$$g:\{\text{vertices of } P_{n+1}\}\to\{\text{vertices of } P_{n+1} \text{ that coincide with those of } I^n\}$$

defined by $g(x)=\min I^{n(x)}$. Consequently, we get

$$sgn(u,v) = sgn\ u\cdot\varepsilon_0(g(x))\cdot sgn\ v,$$

where for an $(n-q)$-face $u = M_1|\cdots|M_{q+1} \in P_{n+1}$,

$$sgn\, u = \prod_{1\le i\le q} (-1)^{\aleph M^{(i)}} sgn(M^{(i)}; M^{(i-1)} \sqcup (\underline{n+1} \setminus M_i)).$$

6.3. **The diagonal of the permutocube.**

It appears that the notion of an orthogonal stream is essential to produce explicit diagonals for polytopes in question.

Theorem 6.1. *The explicit diagonal of* B_n

$$\Delta_B : C_*(B_n) \to C_*(B_n) \otimes C_*(B_n)$$

is defined for a cell $e \subset B_n$ *by*

$$\Delta_B(e) = \sum_{(e_1,e_2)\in OS(e)} sgn(e_1, e_2)\, e_1 \otimes e_2.$$

Proof. The proof is straightforward and analogous to that of Theorem 1 in [17]. □

In particular, in terms of orthogonal streams the diagonal Δ_P for permutahedra established in [17] can be formulated as follows.

Theorem 6.2. *The explicit diagonal of* P_n

$$\Delta_P : C_*(P_n) \to C_*(P_n) \otimes C_*(P_n)$$

is defined for a cell $e \subset P_n$ *by*

$$\Delta_P(e) = \sum_{(e_1,e_2)\in OS(e)} sgn(e_1, e_2)\, e_1 \otimes e_2.$$

Below all components of Δ_B for the top cell of B_n are written down for $n = 1, 2, 3$ in which rows correspond to the orthogonal streams.

Example 6.1.

$$\begin{array}{llcll} \Delta_B\,(1]) = & & & & \\ & \emptyset]1 & \otimes & 1] & x=\emptyset]1 \\ & +\ 1] & \otimes & \emptyset] & x=\emptyset] \end{array}$$

Example 6.2.

$$\begin{array}{llcll} \Delta_B\,(12]) = & & & & \\ & \emptyset]1|2 & \otimes & 12] & x=\emptyset]1|2 \\ & -\ \emptyset]12 & \otimes & 2]1 & x=\emptyset]2|1 \\ & -\ (\emptyset]12 + 2]1) & \otimes & 1] & x=\emptyset]1 \\ & +\ 1]2 & \otimes & 2] & x=\emptyset]2 \\ & +\ 12] & \otimes & \emptyset] & x=\emptyset] \end{array}$$

Example 6.3. *Up to sign, we have*

$$
\Delta_B\,(123]) =
\begin{array}{llll}
 & \emptyset]1|2|3 & \otimes\ 123] & x=\emptyset]1|2|3\\
 & +\ \emptyset]12|3 & \otimes\ 2]13 & x=\emptyset]2|1|3\\
 & +\ \emptyset]1|23 & \otimes\ 13]2 & x=\emptyset]1|3|2\\
 & +\ (\emptyset]12|3+\emptyset]1|23) & \otimes\ 3]12 & x=\emptyset]3|1|2\\
 & +\ \emptyset]12|3 & \otimes\ (2]13+23]1) & x=\emptyset]2|1|3\\
 & +\ \emptyset]2|13 & \otimes\ 23]1 & x=\emptyset]2|3|1\\
 & +\ (\emptyset]12|3+2]1|3) & \otimes\ 13] & x=\emptyset]1|3\\
 & +\ 2]13 & \otimes\ 3]1 & x=\emptyset]3|1\\
 & +\ (\emptyset]1|23+\emptyset]13|2+3]1|2) & \otimes\ 12] & x=\emptyset]1|2\\
 & +\ (\emptyset]123+3]12) & \otimes\ 2]1 & x=\emptyset]2|1\\
 & +\ 1]2|3 & \otimes\ 23] & x=\emptyset]2|3\\
 & +\ 1]23 & \otimes\ 3]2 & x=\emptyset]3|2\\
 & +\ (\emptyset]123+3]12+2]13+23]1) & \otimes\ 1] & x=\emptyset]1\\
 & +\ (1]23+13]2) & \otimes\ 2] & x=\emptyset]2\\
 & +\ 12]3 & \otimes\ 3] & x=\emptyset]3\\
 & +\ 123] & \otimes\ \emptyset] & x=\emptyset].
\end{array}
$$

6.4. **The diagonal on a permutocubical set.**

Now we use the combinatorial description of an orthogonal stream to define the explicit diagonal for a permutocubical set $B=\{B_n^{p,q}\,|\,p,q\geq 0;\,p+q=n\}_{n\geq 0}$. For partitions $u\in\mathcal{P}^0(p)$ and $v\in\mathcal{P}(q+1)$ let d_u and d_v be the corresponding compositions of face operators according to the notations in Sections 3 and 2 respectively.

Then

$$\Delta : C_*(B)\to C_*(B)\otimes C_*(B)$$

is defined for $a\in B_n^{p,q}$ by

$$\Delta(a)=\sum_{\substack{(u_1,u_2)\in OS(p])\\ (v_1,v_2)\in OS(q+1)}} sgn(u_1,u_2)\cdot sgn(v_1,v_2)\cdot(-1)^{\epsilon}\, d_{u_1}d_{v_1}(a)\otimes d_{u_2}d_{v_2}(a),$$

$\epsilon=|d_{u_2}(a)||d_{v_1}(a)|$.

7. The permutocubical model for the path space fibration

Let $\Omega Y\xrightarrow{i}PY\xrightarrow{\pi}Y$ be the Moore path space fibration on a topological space Y. In [1] Adams constructed a dga map

$$\Omega C_*(Y)\to C_*^{\square}(\Omega Y)$$

being a weak equivalence for a simply connected Y, where C_* denotes the singular simplicial chain complex, while in [2] Adams and Hilton constructed a model for the path space fibration using the singular cubical complex for each term of the fibration. Here we obtain a natural combinatorial model for the path space fibration where for the base the singular cubical complex and for the fibre the singular permutohedral complex are taken; the total space in this case is modeled by the permutocubical set being a twisted Cartesian product described in Section

5. This model is naturally mapped into the singular permutocubical complex of the total space. The chain complex of the obtained model is a (comultiplicative) twisted tensor product, while the Adams-Hilton model is not. In particular, the acyclic cobar construction $\Omega(C_*^{\square}(Y);C_*^{\square}(Y))$ coincides with the chain complex of the permutocubical set (compare, Theorem 5.1 in [12]).

For a space Y let $\iota_0 : \mathrm{Sing}^P Y \to \mathrm{Sing}^B Y$ be an inclusion of sets induced by the identification $P_{q+1} = B_0 \times P_{q+1}$. Let denote $\iota_* = \iota_0 \circ i_* : \mathrm{Sing}^P \Omega Y \to \mathrm{Sing}^B PY$. Let $\phi : \mathrm{Sing}^I Y \to \mathrm{Sing}^B Y$ be a map of graded sets from Example 4.1. Then we have the following theorem (compare, [15], [7], [3]).

Theorem 7.1. *(i) For the fibration* $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ *there is a commutative diagram of graded sets*

$$\begin{array}{ccccc} \mathrm{Sing}^P \Omega Y & \xrightarrow{\iota_*} & \mathrm{Sing}^B PY & \xrightarrow{\pi_*} & \mathrm{Sing}^B Y \\ {\scriptstyle\omega}\uparrow & & {\scriptstyle p}\uparrow & & {\scriptstyle\phi}\uparrow \\ \mathbf{\Omega}\mathrm{Sing}^{1^I} Y & \xrightarrow{\iota} & \mathbf{P}\mathrm{Sing}^{1^I} Y & \xrightarrow{\xi} & \mathrm{Sing}^{1^I} Y \end{array} \tag{13}$$

which is natural in Y, and p and ω are maps of permutocubical and permutahedral sets respectively; moreover, they are homotopy equivalences provided Y is simply connected.

(ii) The chain complex $C_*^{\diamondsuit}(\mathbf{\Omega}\mathrm{Sing}^{1^I} Y)$ coincides with the cobar construction $\Omega C_*^{\square}(Y)$.

(iii) The chain complex $C_*^{\boxminus}(\mathbf{P}\mathrm{Sing}^{1^I} Y)$ coincides with the acyclic cobar construction $\Omega(C_*^{\square}(Y);C_*^{\square}(Y))$.

Proof. (i). The constructions of the p and ω are simultaneous by induction on the dimension of singular cubes in $Sing^{1^I}Y$. For $i = 0,1$ and $(\sigma, e) \in \mathbf{P}Sing^{1^I}Y$ with $\sigma \in Sing^{1^I}_i Y$, define $p\,(\sigma, e)$ as the constant map $B_i \to PY$ at the base point of PY, where e denotes the unit of the monoid $\mathbf{\Omega}Sing^{1^I}Y$ (and of the monoid $Sing^P\Omega Y$ too). Put $\omega(e) = e$.

Denote by $\mathbf{P}Sing^{1^I}_{(i,j)}Y$ the subset in $\mathbf{P}Sing^{1^I}Y$ consisting of the elements (σ,τ) with $|\sigma| \leq i$ and $\tau \in \mathbf{\Omega}Sing^{1^I}_{(j)}Y$, a submonoid in $\mathbf{\Omega}Sing^{1^I}Y$ having (monoidal) generators $\bar{\sigma} = \vartheta_U(\sigma)$ of degree $\leq j$.

Suppose by induction that we have constructed p and ω on $\mathbf{P}Sing^{1^I}_{(n-1,n-2)}Y$ and $\mathbf{\Omega}Sing^{1^I}_{(n-2)}Y$ respectively such that

$$p\,(\sigma,\tau) = p\,(\sigma, e)\cdot\omega(\tau),\ (\iota_*\circ\omega)(\bar{\sigma}) = p\,(d_{\emptyset]\underline{r}}(\sigma, e)),\ |\sigma| = r,\ 1 \leq r < n, \quad \text{and}\quad \pi_*\circ p = \phi\circ\xi,$$

where the $\cdot$ product is determined by the action $PY \times \Omega Y \to PY$. Let $\bar{B}_n \subset B_n$ be the union of the all $(n-1)$-faces of B_n except the face $d_{\emptyset]\underline{n}}(B_n)$, and then for a singular cube $\sigma : I^n \to Y$ define the map $\bar{p} : \bar{B}_n \to PY$ by

$$\bar{p}\,|_{d_i(B_n)} = p\,(d_i(\sigma, e)),\ 1 \leq i \leq n,\ \text{and}\ \ \bar{p}\,|_{d_{A]M}(B_n)} = p\,(d_{A]M}(\sigma, e)),\ A, M \neq \emptyset.$$

Then we obtain the following commutative diagram

$$\begin{array}{ccccc} \bar{B}_n & \xrightarrow{\bar{p}_\sigma} & P_\sigma Y & \xrightarrow{g_\sigma} & PY \\ {\scriptstyle \bar{i}}\downarrow & & {\scriptstyle \pi_\sigma}\downarrow & & {\scriptstyle \pi}\downarrow \\ B_n & \xrightarrow{\varphi} & I^n & \xrightarrow{\sigma} & Y. \end{array}$$

Clearly, $\bar{i}$ is a strong deformation retract and we define $p\,(\sigma,e) : B_n \to PY$ as a lift of $\sigma\circ\varphi$. Define $p\,(d_{\emptyset]\underline{n}}(\sigma,e)) = p(\sigma,e)|_{d_{\emptyset]\underline{n}}(B_n)}$, and then $\omega(\bar{\sigma})$ is determined by $(\iota_*\circ\omega)(\bar{\sigma}) = p(\sigma,e)\circ\delta_{\emptyset]\underline{n}} : P_n \to B_n \to PY$.

The proof that p and ω are homotopy equivalences (after the geometric realizations) immediately follows, for example, by observation that ξ induces a long exact sequence for the homotopy groups. The last statement itself can be deduced from the two facts: 1. $|\mathbf{P}Sing^{1^I}Y|$ is contractible, 2. The projection ξ induces an isomorphism $\pi_*(|\mathbf{P}Sing^{1^I}Y|, |\mathbf{\Omega}Sing^{1^I}Y|) \xrightarrow{\xi_*} \pi_*(|Sing^{1^I}Y|)$.

(ii). It is straightforward to check (cf. [17]).

(iii). It is straightforward to check. □

Remark 7.1. *We have that p in fact preserves the obvious actions of* $\mathrm{Sing}^P\,\Omega Y$ *and* $\mathbf{\Omega}\mathrm{Sing}^{1^I}Y$ *on* $\mathrm{Sing}^B\,PY$ *and* $\mathbf{P}\mathrm{Sing}^{1^I}Y$ *respectively.*

Thus, by passing on chain complexes in diagram (13) one obtains the following comultiplicative model of π formed by dgc's (not necessarily coassociative ones).

Corollary 7.1. *For the path space fibration $\Omega Y \xrightarrow{i} PY \xrightarrow{\pi} Y$ there is a comultiplicative model formed by dgc's*

$$\begin{array}{ccccc} C_*^{\diamondsuit}(\Omega Y) & \xrightarrow{\iota_*} & C_*^{\boxminus}(PY) & \xrightarrow{\pi_*} & C_*^{\boxminus}(Y) \\ {\scriptstyle \omega_*}\uparrow & & {\scriptstyle p_*}\uparrow & & {\scriptstyle \phi_*}\uparrow \\ \Omega C_*^{\square}(Y) & \longrightarrow & \Omega(C_*^{\square}(Y); C_*^{\square}(Y) & \xrightarrow{\xi_*} & C_*^{\square}(Y) \end{array}$$

which is natural in Y.

8. Permutocubical models for fibrations

Let $F \to E \xrightarrow{\zeta} Z$ be the fibration associated with a principal G-fibration $G \to E' \xrightarrow{\pi} Z$ by the action $G\times F \to F$. Let $Q = Sing^{1^I}Z$, $P = Sing^P G$ and $L = Sing^P F$. The group operation $G\times G \to G$ induces on P a structure of monoidal permutahedral set (cf. Example 2.3), and the action $G\times F \to F$ induces the structure of P-module $P\times L \to L$ on L.

Theorem 8.1. *Let $F \to E \xrightarrow{\zeta} Z$ be a fibration with 1-connected base Z associated with a principal G-fibration $G \to E' \xrightarrow{\pi} Z$ by an action $G\times F \to F$. Then the principal fibration determines a truncating twisting function $\vartheta : Sing^{1^I}Z \to Sing^P G$ such that twisted Cartesian product $Sing^{1^I}Z \times_\vartheta Sing^P F$ models E, that is, there exists a permutocubical map*

$$Sing^{1^I}Z \times_\vartheta Sing^P F \to Sing^B E$$

inducing a homology isomorphism.

Proof. Let $\omega : \mathbf{\Omega} Q \to Sing^P \Omega Z$ be the map of monoidal permutahedral sets from Theorem 7.1. Then by Proposition 5.1 it corresponds to a truncating twisting function $\vartheta' : Q = Sing^{1^I} Z \xrightarrow{\vartheta_U} \mathbf{\Omega} Q = \mathbf{\Omega} Sing^{1^I} Z \xrightarrow{\omega} Sing^P \Omega Z$. Composing ϑ' with the map of monoidal permutahedral sets $Sing^P \Omega Z \to Sing^P G = P$ induced by the canonical map $\Omega Z \to G$ of monoids we obtain a truncating twisting function $\vartheta : Q \to P$. The resulting twisted Cartesian product $Sing^{1^I} Z \times_\vartheta Sing^P F$ is a permutocubical model of E. Indeed, we have the canonical equality

$$Q \times_\vartheta L = (Q \times_\vartheta P) \times L/\sim,$$

where $((a, bz), c) \sim ((a, b), zc),\ a \in Q,\ b, z \in P,\ c \in L$. Next the argument of the proof of Theorem 7.1 gives a permutocubical map $f' : Q \times_{\vartheta_U} \mathbf{\Omega} Q \to Sing^B E'$ preserving the actions of $\mathbf{\Omega} Q$ and P. Hence, this map extents to a permutocubical map $f : Q \times_\vartheta P \to Sing^B E'$ by $f(a, b) = f'(a, e)b$. Then it is easy to see that the composition

$$(Q \times_\vartheta P) \times L \xrightarrow{f \times 1} Sing^B E' \times L \xrightarrow{\lambda} Sing^B (E' \times F),$$
$$\lambda(g, h) = (g \times h) \circ (1 \times \Delta_{r,s}),$$

induces the map of permutocubical sets

$$Sing^{1^I} Z \times_\vartheta Sing^P F \to Sing^B E$$

as desired. □

For convenience, assume that Q, P and L are as in the Definition 5.2. We have that a truncating twisting function ϑ induces on chain level the twisting cochains $\vartheta_* : C_*^{\square}(Q) \to C_{*-1}^{\diamondsuit}(P)$ and $\vartheta^* : C^*_{\diamondsuit}(P) \to C^{*+1}_{\square}(Q)$ in the standard sense ([6],[4],[10]). It is straightforward to verify that we have the equality

$$C_*^{\boxminus}(Q \times_\vartheta L) = C_*^{\square}(Q) \otimes_{\vartheta_*} C_*^{\diamondsuit}(L) \tag{14}$$

and, consequently, the inclusion

$$C^*_{\boxminus}(Q \times_\vartheta L) \supset C^*_{\square}(Q) \otimes_{\vartheta^*} C^*_{\diamondsuit}(L) \tag{15}$$

of dg modules (where we have an equality too if the graded sets are of finite type).

The permutocubical structure of $Q \times_\vartheta L$ induces a dgc sturcture on $C_*^{\boxminus}(Q \times_\vartheta L)$ which after transporting on the right side of (14) gives a *comultiplicative* model of $L \to Q \times_\vartheta L \to Q$. Analogously it arises the multiplication on the right hand side of (15). To describe these structures, first we need some (co)chain operations on the (co)chain complex of Q.

8.1. **The canonical Hirsch algebra structure on $C^*_{\square}(Q)$.**

First let consider the equality

$$C_*^{\diamondsuit}(\mathbf{\Omega} Q) = \Omega C_*^{\square}(Q).$$

We have that $\mathbf{\Omega} Q$ is a permutahedral set and the diagonal Δ_P of P_n induces on the cobar construction $\Omega C_*^{\square}(Q)$ a comultiplication Δ which converts it into a dg Hopf algebra.

To describe Δ combinatorially let first recall that each face of the n-cube can be expressed as a sequence of blocks $[0, ..., b_1] \cdots [b_k, ..., n+1]$ (cf. [12]). Let the dimension of this cube be $q,\ 1 \le q \le n$. Then by overlapping this sequence of blocks by braces $\{[0, ..., b_1] \cdots [b_k, ..., n+1]\}$ we regard it as a $(q-1)$-permutahedron P_q.

Then each $(n-p)$-face $u = A_1|\cdots|A_p,\quad A_j = \{a_{1,j} < \cdots < a_{n_j,j}\}$, of P_n can be expressed as a sequence of blocks

$$\begin{array}{l}\{[0,a_{1,1},...,b_{1,1}][b_{1,1},...,b_{s_1,1}]...[b_{m_1,1},...,a_{n_1,1},n+1]\}\\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\\ \{[0,a_{1,p-1},...,b_{1,p-1}][b_{1,p-1},...,b_{s_{p-1},p-1}]...[b_{m_{p-1},p-1},...,a_{n_{p-1},p-1},n+1]\}\\ \hfill\{[0,b_{1,p-1},...,b_{m_{p-1},p-1},n+1]\}\end{array}$$

where $\{b_{1,i} < ... < b_{s_i,i} < ... < b_{m_i,i}\} = A_{i+1}\cup ...\cup A_p, 1\le i < p$. Note that a block containing in brackets only two elements from b's, i.e., without a's ($=$0-cube), is also regarded. In particular, for the last block we have $\{b_{1,p-1} < ... < b_{m_{p-1},p-1}\} = \{a_{1,p} < ... < a_{n_p,p}\}$. For example, the sequence of blocks

$$\{[01][123][3456]\}\{[013][36]\}\{[036]\}$$

corresponds to the 2-face $245|1|3$ of P_5.

Now let $v = A'_1|\cdots|A'_q,\quad A'_j = \{a'_{1,j} < \cdots < a'_{n'_j,j}\}, 1\le j\le q$. Then for the diagonal Δ_P we have

$$\begin{array}{l}\{[0,1,...,n+1]\}\xrightarrow{\Delta_P}\\ \quad\sum_{(u,v)\in OS(n)}\{[0,a_{1,1},...,b_{1,1}][b_{1,1},...,b_{s_1,1}]...[b_{m_1,1},...,a_{n_1,1},n+1]\}\\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\\ \{[0,a_{1,p-1},...,b_{1,p-1}][b_{1,p-1},...,b_{s_{p-1},p-1}]...[b_{m_{p-1},p-1},...,a_{n_{p-1},p-1},n+1]\}\\ \hfill\{[0,b_{1,p-1},...,b_{m_{p-1},p-1},n+1]\}\otimes\\ \hfill\{[0,a'_{1,1},...,b'_{1,1}][b'_{1,1},...,b'_{s'_1,1}]...[b_{m'_1,1},...,a'_{n'_1,1},n+1]\}\\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\\ \{[0,a'_{1,q-1},...,b'_{1,q-1}][b'_{1,q-1},...,b'_{s'_{q-1},q-1}]...[b'_{m'_{q-1},q-1},...,a'_{n'_{q-1},q-1},n+1]\}\\ \hfill\{[0,b'_{1,q-1},...,b'_{m'_{q-1},q-1},n+1]\},\end{array}$$

where

$$\{[012][23]...[n,n+1]\}\{[023][34]...[n,n+1]\}\cdots\{[0,n,n+1]\}\otimes\{[01...n+1]\}$$

and

$$\begin{array}{l}\{[01...n+1]\}\otimes\\ \quad\{[01]...[n-2,n-1][n-1,n,n+1]\}\{[01]...[n-3,n-2][n-2,n-1,n+1]\}\\ \hfill\cdots\{[012]\}\end{array}$$

form the primitive part of the diagonal.

Then regarding the blocks of natural numbers above as faces of the standard n-cube we obtain the following formula for the coproduct $\Delta:\Omega C^{\square}_*(Q)\to\Omega C^{\square}_*(Q)\otimes\Omega C^{\square}_*(Q)$: for a generator $\sigma\in C^{\square}_n(Q)\subset\Omega C^{\square}_*(Q)$ let $\sigma([0,...,j_1]\cdots[j_k,...,n+1][0,j_1,...,j_k,n+1])$ denote its suitable face; then

$$\Delta([\sigma]) = \sum(-1)^{\epsilon}[\sigma([0,a_{1,1},...,b_{1,1}][b_{1,1},...,b_{s_1,1}]...[b_{m_1,1},...,a_{1,n_1},n+1])|$$
$$......................$$
$$|\sigma([0,a_{1,p-1},...,b_{1,p-1}][b_{1,p-1},...,b_{s_{p-1},p-1}]...[b_{m_{p-1},p-1},...,a_{n_{p-1},n-1},n+1])|$$
$$\sigma([0,b_{1,p-1},...,b_{m_{p-1},p-1},n+1])]\otimes$$
$$[\sigma([0,a'_{1,1},...,b'_{1,1}][b'_{1,1},...,b'_{s'_1,1}]...[b_{m'_1,1},...,a'_{n'_1,1},n+1])|$$
$$......................$$
$$|\sigma([0,a'_{1,q-1},...,b'_{1,q-1}][b'_{1,q-1},...,b'_{s'_{q-1},q-1}]...[b'_{m'_{q-1},q-1},...,a'_{n'_{q-1},q-1},n+1])|$$
$$\sigma([0,b'_{1,q-1},...,b'_{m'_{q-1},q-1},n+1])].$$

Since Q is assumed to be 1-reduced, for each 1-dimensional face

$$\sigma([0,1]\cdots[k-1,k,k+1]\cdots[n,n+1]) \text{ or } \sigma([0,k,n+1]),\ 1\leq k\leq n,$$

its image

$$[\sigma([0,1]\cdots[k-1,k,k+1]\cdots[n,n+1])] \text{ or } [\sigma([0,k,n+1])] \text{ in } \Omega C_*^{\square}(Q)$$

is the unit and so will be omitted.

Dualizing we obtain a multiplication on the bar construction

$$BC^*_{\square}(Q)\otimes BC^*_{\square}(Q)\to BC^*_{\square}(Q),$$

or, equivalently, the sequence of cochain operations

$$\{E_{p,q}: C^*_{\square}(Q)^{\otimes p}\otimes C^*_{\square}(Q)^{\otimes q}\to C^*_{\square}(Q)\}_{p,q\geq 1}.$$

These cochain operations just form on $C^*_{\square}(Q)$ the structure of a *Hirsch algebra* (see the next section). They can be viewed as the restriction of some cochain operations which naturally arise on $\bar{C}^*_{\square}(Q)$ (the non-normalized chains) for an arbitrary Q without assuming it to be 1-reduced. Namely, we have

$$\{E_{p,q}: \bar{C}^*_{\square}(Q)^{\otimes p}\otimes \bar{C}^*_{\square}(Q)^{\otimes q}\to \bar{C}^*_{\square}(Q)\}_{p,q\geq 1}$$

written down by the following explicit formulas. For $a_i\in\bar{C}^{m_i}(Q)$, $b_j\in\bar{C}^{r_j}(Q)$, $m_i,\ r_j\geq 2$, $1\leq i\leq p$, $1\leq j\leq q$, let

$$E_{p,q}(a_1,...,a_p;\,b_1,...,b_q)=\sum_{s\geq p;\,t\geq q}\bar{E}_{s,t}(\epsilon^1,a_1,\epsilon^1,...,\epsilon^1,a_p,\epsilon^1;\,\epsilon^1,b_1,\epsilon^1,...,\epsilon^1,b_q,\epsilon^1),$$

$\epsilon^1\in\bar{C}^1(Q)$ is the generator represented by the constant map at the base point and the operations $\bar{E}_{s,t}$ are themselves defined for $a_i\in\bar{C}^{m_i}_{\square}(Q),\ b_j\in\bar{C}^{r_j}_{\square}(Q),\ u=M_1|...|M_s,\ v=L_1|...|L_t,\ (u,v)\in OS(n),\ m_i=\aleph M_i\geq 1,\ r_j=\aleph L_j\geq 1,\ 1\leq i\leq s,\ 1\leq j\leq t,\ \sigma\in Q_n$, by

$$\bar{E}_{s,t}(a_1,...,a_s\,;\,b_1,...,b_t)=c\in\bar{C}^n_{\square}(Q),$$

$$\begin{aligned} c(\sigma)= \quad & sgn(u,v)a_1(\sigma_1)\cdots a_s(\sigma_s)\cdot b_1(\sigma'_1)\cdots b_t(\sigma'_t),\\ & \sigma_i=d^0_{M_{i+1}\cup\cdots\cup M_s}d^1_{M_{i-1}}\cdots d^1_{M_1}(\sigma), && 1\leq i\leq s,\\ & \sigma'_j=d^0_{L_{j+1}\cup\cdots\cup L_t}d^1_{L_{j-1}}\cdots d^1_{L_1}(\sigma), && 1\leq j\leq t,\end{aligned}$$

and $\bar{E}_{s,t}(a_1,...,a_s;b_1,...,b_t)=0$ otherwise.

Remark 8.1. *1. Note that the above formula for $k=1$ defines $E_{1,1}$ as being the cubical version of Steenrod's cochain $\smile_1$-operation without any restriction on Q.*
2. The operations $\{E^{p,q}\}$ on $C_*^{\square}(Q)=\Omega C_*(X)$, $Q=\mathbf{\Omega} Sing^2X$ (cf. [12]), in fact have the form

$$E^{p,q}=\sum \Delta_E^{p-1}\otimes\Delta_E^{q-1}$$

where $\Delta_E^k:\Omega C_*(X)\to \Omega C_*(X)^{\otimes k+1}$ is the k-th iteration of the comultiplication $\Delta_E:\Omega C_*(X)\to\Omega C_*(X)\otimes\Omega C_*(X)$ being itself induced by the homotopy G-coalgebra structure $\{E^{k,1}\}$ on $C_*(X)$ (cf. [12]).

8.2. **Twisted multiplicative model for a fibration.**

Now we again turn to the twisted Cartesian product $Q\times_\vartheta L$. To describe the corresponding coproduct and product on the right sides of (14) and (15) respectively it is very convenient to express this diagonal using the following combinatorics of B_n. Let assign to an $(n-p)$-face $u=A_0]A_1|\cdots|A_p$, $A_j=\{a_{1,j}<\cdots<a_{n_j,j}\}$, of B_n a sequence of blocks

$$\begin{array}{l}[0,a_{1,0},...,b_{1,0}][b_{1,0},...,b_{s_0,0}]...[b_{m_0,0},...,a_{n_0,0},n+1]\}\\ \qquad\qquad\{[0,a_{1,1},...,b_{1,1}][b_{1,1},...,b_{s_1,1}]...[b_{m_1,1},...,a_{n_1,1},n+1]\}\\ \qquad\ldots\ldots\ldots\ldots\\ \{[0,a_{1,p-1},...,b_{1,p-1}][b_{1,p-1},...,b_{s_{p-1},p-1}]...[b_{m_{p-1},p-1},...,a_{n_{p-1},p-1},n+1]\}\\ \qquad\qquad\qquad\{[0,b_{1,p-1},...,b_{m_{p-1},p-1},n+1]\}\end{array}$$

where $\{b_{1,i}<...<b_{s_i,i}<...<b_{m_i,i}\}=A_{i+1}\cup...\cup A_p, 0\le i<p$. Here again a block containing in brackets only two elements from b's, i.e., without a's ($=0$-cube), is regarded. In particular, $[0,1,...,n+1]\}$ is assigned to whole B_n.

For example, the sequence of blocks

$$[01][123][3456]\}\{[013][36]\}\{[036]\}$$

corresponds to the 3-face $245]1|3$ of B_5.

Then Δ_B can be expressed as

$$\begin{array}{l}[0,1,...,n+1]\}\xrightarrow{\Delta_B}\\ \quad\sum_{(u,v)\in OS(n])}[0,a_{1,0},...,b_{1,0}][b_{1,0},...,b_{s_0,0}]...[b_{m_0,0},...,a_{n_0,0},n+1]\}\\ \qquad\qquad\{[0,a_{1,1},...,b_{1,1}][b_{1,1},...,b_{s_1,1}]...[b_{m_1,1},...,a_{n_1,1},n+1]\}\\ \qquad\ldots\ldots\ldots\ldots\\ \{[0,a_{1,p-1},...,b_{1,p-1}][b_{1,p-1},...,b_{s_{p-1},p-1}]...[b_{m_{p-1},p-1},...,a_{n_{p-1},p-1},n+1]\}\\ \qquad\qquad\qquad\{[0,b_{1,p-1},...,b_{m_{p-1},p-1},n+1]\}\otimes\\ \qquad\qquad[0,a'_{1,1},...,b'_{1,1}][b'_{1,1},...,b'_{s'_1,1}]...[b_{m'_1,1},...,a'_{n'_1,1},n+1]\}\\ \qquad\qquad\{[0,a'_{1,2},...,b'_{1,2}][b'_{1,2},...,b'_{s'_2,2}]...[b_{m'_2,2},...,a'_{n'_2,2},n+1]\}\\ \qquad\ldots\ldots\ldots\ldots\\ \{[0,a'_{1,q-1},...,b'_{1,q-1}][b'_{1,q-1},...,b'_{s'_{q-1},q-1}]...[b'_{m'_{q-1},q-1},...,a'_{n'_{q-1},q-1},n+1]\}\\ \qquad\qquad\qquad\{[0,b'_{1,q-1},...,b'_{m'_{q-1},q-1},n+1]\},\end{array}$$

where

$$[01][12]...[n,n+1]\}\{[012][23]...[n,n+1]\}\{[023][34]...[n,n+1]\} \\ \cdots\{[0,n,n+1]\}\otimes[01...n+1]\}$$

and

$$[01...n+1]\}\otimes[0,n+1]\}$$

form the primitive part of the diagonal. Note that by removing the component

$$[0,a_{1,0},...,b_{1,0}][b_{1,0},...,b_{s_0,0}]...[b_{m_0,0},...,a_{n_0,0},n+1]\}$$

and by replacing the block

$$[0,a'_{1,1},...,b'_{1,1}][b'_{1,1},...,b'_{s'_1,1}]...[b_{m'_1,1},...,a'_{n'_1,1},n+1]\}$$

by

$$\{[0,a'_{1,1},...,b'_{1,1}][b'_{1,1},...,b'_{s'_1,1}]...[b_{m'_1,1},...,a'_{n'_1,1},n+1]\}$$

in Δ_B we just obtain Δ_P acting on $\{[01...n+1\setminus A_0]\}$.

Now, using this diagonal, it is not hard to see that by means of $\{E_{p,q}\}_{p+q>0}$ and the induced comodule structure $\Delta_L: C^*_\diamondsuit(L)\to C^*_\diamondsuit(P)\otimes C^*_\diamondsuit(L)$ by the action $P\times L\to L$ the permutocubical multiplication of the left side of (15) can be expressed by the following formula. Let $a_1\otimes m_1, a_2\otimes m_2\in C^*_\square(Q)\otimes_{\vartheta^*}C^*_\diamond(L)$ and $\Delta^k_L: C^*_\diamondsuit(L)\to C^*_\diamondsuit(P)^{\otimes k}\otimes C^*_\diamondsuit(L)$ be the iterated Δ_L with $\Delta^0_L=\mathrm{Id}: C^*_\diamondsuit(L)\to C^*_\diamondsuit(L)$, and let $\Delta^p_L(m_1)=\sum c^1_1\otimes\ldots\otimes c^p_1\otimes m^{p+1}_1,\ \ \Delta^{q-1}_L(m_2)=\sum c^1_2\otimes\ldots\otimes c^{q-1}_2\otimes m^q_2$. Then

$$\text{(16)}\quad \mu((a_1\otimes m_1)\otimes(a_2\otimes m_2))= \\ \sum_{p\geq 0;\ q\geq 1}(-1)^{\epsilon}a_1E_{p,q}(\vartheta(c^1_1),\ldots,\vartheta(c^p_1);a_2,\vartheta(c^1_2),...,\vartheta(c^{q-1}_2))\otimes m^{p+1}_1m^q_2,$$

$$\epsilon=|m^{p+1}_1|(|a_2|+|c^1_2|+\cdots+|c^{q-1}_2|).$$

Corollary 8.1. *Let $F\to E\xrightarrow{\zeta}Z$ be the fibration associated with G-fibration $G\to E'\xrightarrow{\pi}Z$ by the action $G\times F\to F$. Then the tensor product $C^*_\square(Z)\otimes C^*_\diamondsuit(F)$ becomes a dga $(C^*_\square(Z)\otimes C^*_\diamondsuit(F),d_\vartheta,\mu)$ with both twisted differential d_ϑ and the multiplication μ.*

In particular, letting $P=L=\mathbf{\Omega}Q$ in (16) we deduce the following explicit formula for the multiplication on the acyclic bar construction $B(C^*_\square(Z);C^*_\square(Z))$ converting it into a dga. For $a=a_0\otimes[\bar a_1|\cdots|\bar a_n],\ \ b=b_0\otimes[\bar b_1|\cdots|\bar b_m],\ \ a_i,b_j\in C^*_\square(Z),\ 0\leq i\leq n,\ 0\leq j\leq m$, let

(17)

$$ab=\sum_{p\geq 0;\,q\geq 1}(-1)^{\epsilon}a_0E_{p,q}(a_1,...,a_p\,;b_0,b_1,...,b_{q-1})\otimes[\bar a_{p+1}|\cdots|\bar a_n]\circ[\bar b_q|\cdots|\bar b_m],$$

$\epsilon=(|\bar a_{p+1}|+\cdots+|\bar a_n|)(|b_0|+|\bar b_1|+\cdots+|\bar b_{q-1}|)$.

Using the fact that $BC^*(Y)$ has an associative multiplication [12] we canonically introduce on the acyclic bar construction $B(BC^*(Y);BC^*(Y))$ the multiplication by (17) that agrees with the one on the double bar construction $BBC^*(Y)$ [17].

9. Twisted tensor products for Hirsch algebras

The notion of a Hirsch (co)algebra naturally generalizes the one of a homotopy G-(co)algebra. Again the structure such a (co)algebra on the cubical (co)chain complex of a topological space defined by the diagonal of permutahedra became the motivation for the material of this section and that formulas (16) and (17) established in the previous section are valid in a purely algebraic situation.

Let for a dga A

$$(\mathrm{Hom}(BA\otimes BA,A),\nabla)$$

be the canonical dga with $\smile$-product, where $BA\otimes BA$ has the standard tensor coalgera structure.

We have the following definition

Definition 9.1. *A Hirsch algebra is a 1-reduced associative dga A with multilinear maps*

$$E_{p,q}:A^{\otimes p}\otimes A^{\otimes q}\to A,\ \ p,q\geq 0,\ \ p+q>0,$$

satisfying the following conditions:
(i) $E_{p,q}$ is of degree $1-p-q$;
(ii) $E_{1,0}=Id=E_{0,1}$ and $E_{k>0,0}=0=E_{0,k>0}$;
(iii) The homomorphism $E:BA\otimes BA\to A$ defined by

$$E([\bar a_1|\cdots|\bar a_p]\otimes[\bar b_1|\cdots|\bar b_q])=E_{p,q}(a_1,...,a_p;b_1,...,b_q)$$

is a twisting element in the dga $(Hom(BA\otimes BA,A),\nabla)$, i.e., it satisfies $\nabla E=-E\smile E$.

Entirely dually one can formulate the notion of a Hirsch coalgebra.

The condition (i) guarantees that the comultiplicative coextension $\mu_E:BA\otimes BA\to BA$ is a map of degree 0, the condition (ii) guarantees that the empty bracket $[\,]\in BA$ is a unit for μ_E, and the condition (iii) guarantees that μ_E is a chain map; thus BA becomes a dg Hopf algebra with not necessarily associative multiplication μ_E (cf. [9], [19]).

The condition (iii) can be rewritten in terms of components $E_{p,q}$. In particular the operation $E_{1,1}$ satisfies the conditions similar to that of Steenrod's $\smile_1$ product:

$$dE_{1,1}(a;b)-E_{1,1}(da;b)+(-1)^{|a|}E_{1,1}(a;db)=(-1)^{|a|}ab-(-1)^{|a|(|b|+1)}ba,$$

so it measures the non-commutativity of the product of A (thus, a Hirsch algebra with $E_{p,q}=0$ for $p,q\geq 1$ is just a commutative dga).

Main examples of Hirsch (co)algebras are: $C^*_{\square}(Q)$ (see previous section), in particular, Adams' cobar construction $\Omega C_*(X)$ ([17]), and the singular simplicial cochain complex $C^*(X)$: in [14] a twisting element $E:BC^*(X)\otimes BC^*(X)\to C^*(X)$ satisfying (i)-(iii) is constructed and these conditions determined E uniquely up to the standard equivalence of twisting elements.

9.1. Multiplicative twisted tensor products.

Let A be a Hirsch algebra, C be a dg Hopf algebra, and M be a dga being a dg comodule over C.

Definition 9.2. *A twisting element $\vartheta:C\to A$ in $Hom(C,A)$ we call multiplicative if its comultiplicative coextension $C\to BA$ is an algebra map.*

It is clear that if $\vartheta:C\to A$ is a multiplicative twisting element and if $g:B\to C$ is a map of dg Hopf algebras then the composition $\vartheta g:B\to A$ is again a multiplicative twisting element.

The canonical projection $BA \to A$ provides an example of the universal multiplicative element.

We have that the argument of the proof of formula (16) immediately yields

Theorem 9.1. *Let $\vartheta^* : C \to A$ be a multiplicative twisting element. Then the tensor product $A \otimes M$ with the canonical twisting differential $d_{\vartheta^*} = d \otimes 1 + 1 \otimes d + \vartheta^* \cap\_$ becomes a dga $(A \otimes M, d_{\vartheta^*}, \mu)$ with the twisted multiplication μ determined by formula (16).*

Thus the above theorem includes the twisted tensor product theory both for homotopy G-algebras [12] and for commutative algebras ([16]).

Corollary 9.1. *For a Hirsch algebra A the acyclic bar construction $B(A; A)$ canonically becomes a dga with the twisted multiplication determined by formula (17).*

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