QUANTUM METHODS IN ALGEBRAIC TOPOLOGY

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In this paper, we present a new version of cochains in Algebraic Topology, starting with **"quantum differential forms"**. This version provides many examples of modules over the braid group, together with a control of the non commutativity of cup-products on the cochain level. If the quantum parameter q is equal to 1, we essentially recover the commutative differential graded algebra of de Rham-Sullivan forms on a simplicial set [1][12]. For topological applications, we may take either q = 1 if we are dealing with rational coefficients or q = 0 in the general case. In both cases, the quantum formulas are simpler (if q = 0 for instance, the quantum exponential $e_q(x)$ is just the function 1/1-x).

From this viewpoint, we extract a new structure of **"neo-algebra"**[1]. This structure is detailed in section III of this paper. To a simplicial set X we can associate in a functorial way a neo-algebra $\hat{\Omega}^*(X)$, which cohomology is canonically isomorphic to the usual one with coefficients in k (k might be an arbitrary commutative ring). As a differential graded algebra, $\hat{\Omega}^*(X)$ is related to the usual algebra of cochains C*(X) by a (zigzag) sequence of quasi-isomorphisms.

Using in an essential way some recent results of M.-A. Mandell [8] [9], one may then show that $\hat{\Omega}^*(X)$ (up to quasi-isomorphisms of neo-algebras) determines the homotopy type[2] of X. The proof relies on the basic fact that $\hat{\Omega}^*(X)$ may be provided with an E_∞-algebra structure which is related to the classical one on C*(X) by a sequence of quasi-isomorphisms.

On a more practical level, we can show how to compute Steenrod operations in mod. p cohomology, as well as homotopy groups of X from the neo-algebraic data on $\hat{\Omega}^*(X)$.

Finally in the fourth section of this paper, we see how all the theory can be dualized in the framework of "**neo-coalgebras**".

This paper is mainly expository, although some proofs are sketched. Details will be published elsewhere, as well as applications to homotopy theory (closed model categories, homotopy groups of Moore spaces...). The following URL address :

http://www.math.jussieu.fr/~karoubi/

contains already many complementary informations.

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[1] This is closely related to the notion of partial algebra and E_∞-algebra of I. Kriz and P. May [6]. As a matter of fact, a neo-algebra is just a special case of a partial algebra.

[2] One has to assume that the spaces involved are connected, nilpotent, p-complete and of finite type.

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I. Braided differential graded algebras and q-cohomology.

1.1. Let k be a commutative ring and A be a k-algebra (with unit). A braiding [5] on A is given by a k-module endomorphism $R : A \otimes_k A \longrightarrow A \otimes_k A$. Let us consider the following properties of R :

α) In the set of endomorphisms of $A^{\otimes 3} = A \otimes_k A \otimes_k A$, the "Yang-Baxter equations" $R_{12}.R_{23}.R_{12} = R_{23}.R_{12}.R_{23}$ are satisfied[3]. If R is an automorphism, this implies that the braid group $\mathfrak{B}_n$ acts on $A^{\otimes n}$ (the generators of the braid group are mapped to the automorphisms $R_{i,i+1}$)

β) If 1 is the unit element of A, we have the relations $R(1 \otimes a) = a \otimes 1$ and $R(a \otimes 1) = 1 \otimes a$

γ) If $\mu : A^{\otimes 2} \longrightarrow A$ is the multiplication, we have the following relations among the morphisms from $A^{\otimes 3}$ to $A^{\otimes 2}$

$$R.\mu_{12} = \mu_{23}.R_{12}.R_{23} \quad \text{and} \quad R.\mu_{23} = \mu_{12}.R_{23}.R_{12}$$

δ) The algebra is called R-commutative (or commutative in the quantum sense) if $\mu = \mu.R$ as morphisms from $A^{\otimes 2}$ to A

ε) Finally, if A is a differential graded algebra (DGA in short), R is a morphism of complexes (of degree 0) for the usual differential on $A \otimes_k A$.

If all these properties are satisfied, the differential graded algebra A is called **braided R-commutative** (or simply braided commutative).

1.2. Fundamental example. Let Λ be a commutative k-algebra provided with an endomorphism $a \mapsto \bar{a}$. We denote by $\overline{\Omega}^1(\Lambda)$ the cokernel of the morphism $\bar{b} : \Lambda^{\otimes 3} \longrightarrow \Lambda^{\otimes 2}$ defined by $\bar{b}(a_0 \otimes a_1 \otimes a_2) = a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + \bar{a}_2a_0 \otimes a_1$ (twisted Hochschild boundary). As a left Λ-module, $\overline{\Omega}^1(\Lambda)$ is generated by elements of the type u.dv (class of $u \otimes v$) with the following relation which is a variant of the Leibniz formula :

[3] For i < j, $R_{ij} = R_{i,j}$ denotes in general the endomorphism of $A^{\otimes n}$ deduced from R carried to the (i, j)-component of the tensor product. In the same way, $\mu_{i,i+1}$ denotes the morphism from $A^{\otimes n}$ to $A^{\otimes (n-1)}$, obtained from the multiplication μ restricted to the (i, i+1)-component.

$$u.d(vw) = uv.dw + u\overline{w}\,.dv$$

We put now $\overline{\Omega}^0(\Lambda) = \Lambda$ and $\overline{\Omega}^i(\Lambda) = 0$ for $i > 1$. The direct sum $\overline{\Omega}^*(\Lambda) = \overline{\Omega}^0(\Lambda) \oplus \overline{\Omega}^1(\Lambda)$ is obviously a DGA, where the following braiding is defined (u, v, w and t being elements of Λ)

$$R(u \otimes v) = v \otimes u$$
$$R(udv \otimes w) = \overline{w} \otimes udv$$
$$R(u \otimes vdw) = vdw \otimes u + v(w - \overline{w}) \otimes du$$
$$R(udv \otimes wdt) = - \overline{w}d\overline{t} \otimes udv$$

1.3. THEOREM. *With the above braiding, the differential graded algebra* $A = \overline{\Omega}^*(\Lambda)$ *satisfies the axioms* α, β, γ, δ *and* ε. *Therefore, it is a braided commutative* DGA (*in the quantum sense*).

Proof. Easy, but tedious (about 10 pages…).

1.4. An important case of the previous theorem is when $\Lambda = k[t]$, the endomorphism $a \mapsto \bar{a}$ being given by $t \mapsto qt$, with $q \in k$. The braided differential graded algebra A, denoted by $\Omega(t)$ or $\Omega^*(t)$, is well known to the experts (see [7] for instance). It is generated by the symbols t and dt, with the relations dt.dt = 0 and $(t^n\, dt)\, t^m = q^m\, t^{n+m}.dt$. If we assume $1 + q + ... + q^n$ to be invertible[4] in k for all n (q = 0 for instance), Poincaré's lemma is true for $\Omega^*(t)$: the complex

$$0 \longrightarrow \Omega^0(t) \longrightarrow \Omega^1(t) \longrightarrow 0$$

has trivial cohomology, except in degree 0, in which case it is isomorphic to k .

On the other hand, let A and B be two braided DGA's with braidings R and S respectively. The graded tensor product $A \hat{\otimes} B$ may be provided with the braiding given by the following composition of morphisms :

$$A_1 \hat{\otimes} B_1 \hat{\otimes} A_2 \hat{\otimes} B_2 \cong (A_1 \hat{\otimes} A_2) \hat{\otimes} (B_1 \hat{\otimes} B_2) \xrightarrow{R \otimes S}$$
$$(A_2 \hat{\otimes} A_1) \hat{\otimes} (B_2 \hat{\otimes} B_1) \cong A_2 \hat{\otimes} B_2 \hat{\otimes} A_1 \hat{\otimes} B_1$$

(subscripts indicate the selected copy of A or B), where we assume that elements of A and B commute (in the graded sense). It is easy to check the properties listed in 1.1, if R and S satisfy them.

1.5. Of course, these remarks may be applied to an arbitrary number of braided DGA's. In particular, the graded tensor product $\Omega(y_1, ..., y_r) = \Omega(y_1) \hat{\otimes} \ ... \ \hat{\otimes}\, \Omega(y_r)$ is provided with a structure of braided commutative DGA and Poincaré's lemma is still true. This last fact can be

[4] From now on, we shall always assume this hypothesis. One should note however that if k is any commutative ring, we may replace k by a suitable localization k' of k[q] : it is obtained by making invertible the multiplicative set generated by the polynomials $1 + q + ... + q^n$ for all n. This localization process is faithful on the level of the cohomology (and also on the level of the homotopy type).

checked directly - as in the classical case - by introducing an auxiliary parameter t and making the substitution $y_i \mapsto t\, y_i$; however, the resulting homotopy operator depends of the order of the variables, because the variables dt and t^m do not commute.

After these general preliminaries, we define for all m a <u>cosimplicial</u> DGA by the following formula

$$A^{(r)} = \prod_{i_0 < \ldots < i_r} \Omega(x_0, \ldots, \hat{x}_{i_0}, \ldots, \hat{x}_{i_r}, \ldots, x_m)$$

In particular, the two coface operators

$$A^{(0)} = \prod_{i} \Omega(x_0, \ldots, \hat{x}_i, \ldots, x_m) \rightrightarrows A^{(1)} = \prod_{i<j} \Omega(x_0, \ldots, \hat{x}_i, .., \hat{x}_j, \ldots, x_m)$$

are obvious (we let the variables x_i or $x_j = 0$). The equalizer of these two morphisms defines a commutative braided DGA called $\Omega^*(\Delta_m)$.

1.6. THEOREM. *The correspondence* $m \mapsto \Omega^*(\Delta_m)$ *defines a braided* <u>*simplicial*</u> DGA (*for instance, the face operators are defined by the relations* $x_i = 1$). *In other words, for any non-decreasing map* $[s] \longrightarrow [r]$, *the following diagram commutes* :

$$\begin{array}{ccc} \Omega^*(\Delta_r) \otimes \Omega^*(\Delta_r) & \xrightarrow{R} & \Omega^*(\Delta_r) \otimes \Omega^*(\Delta_r) \\ \downarrow & & \downarrow \\ \Omega^*(\Delta_s) \otimes \Omega^*(\Delta_s) & \xrightarrow{R} & \Omega^*(\Delta_s) \otimes \Omega^*(\Delta_s) \end{array}$$

1.7. Let $X = X_\natural$ be now a simplicial set. We define the differential graded algebra $\underline{\Omega}^*(X)$ of quantum differential forms on X as the algebra of simplical maps from $X_\natural$ to $\Omega^*(\Delta_\natural)$.

1.8. THEOREM. *The functors* $X \mapsto H^n(\underline{\Omega}^*(X))$ *are the elements of a (multiplicative) cohomology theory on* X *which is naturally isomorphic to the usual cohomology with coefficients in* k.

Proof (compare with [1]). According to what has been said before, Poincaré's lemma is true for the algebra $\Omega^*(\Delta_r)$: the following complex (where $\Omega^i(\Delta_r) = 0$ for $i > r$)

$$0 \longrightarrow \Omega^0(\Delta_r) \xrightarrow{d_0} \Omega^1(\Delta_r) \xrightarrow{d_1} \ldots$$

is acyclic, except in degree 0. The kernel of d_0 is isomorphic to k and is simplicially trivial. On the other hand, for a fixed s, it is easy to see that the homotopy groups of the simplicial abelian group $r \mapsto \Omega^s(\Delta_r)$ are equal to 0. The theorem then follows from a classical result on cohomology theories [1].

1.9. *Remark.* This theory is very closely related to the one sketched by my student C. Mouet in [10].

1.10. *Remark***.** In the above theorem, we may replace the simplicial k-module $r \mapsto \Omega(\Delta_r)$ by a "stabilized" version $r \mapsto \operatorname{colim}_p \Omega^{\otimes p}(\Delta_r)$, which we shall denote by $\hat{\Omega}(\Delta_r)$ (the inductive system is given by $\omega \mapsto \omega \otimes 1$). This will be necessary in the next sections.

1.11. *Remark***.** It is easy to define a "quantum integral"

$$\int_{\Delta_r} : \Omega^r(\Delta_r) \longrightarrow k$$

starting with the definition of $\Omega^r(\Delta_r)$ given in 1.5. This integral generalizes the well-known formula

$$\int_0^1 t^n \, dt = \frac{1}{1 + q + ... + q^n}$$

In this context, "Stokes' formula" can be written as follows

$$\int_{\Delta_r} d\omega = \sum_{i=0}^{r} (-1)^i \int_{\partial_i \Delta_r} \omega$$

Here ω is of degree r-1 and the $\partial_i \Delta_r$ run through all the faces of Δ_r. This quantum integral defines a (non multiplicative) quasi-isomorphism between $\underline{\Omega}^*(X)$ and the complex of classical cochains on X with coefficients in k. In order to define a zigzag sequence of quasi-isomorphisms respecting the multiplicative structure, one has to use the DGA of non commutative differential forms which is detailed in [3].

1.12. There is a variant of $\underline{\Omega}^*(X)$ (called $\Omega^*(X)$ in order to avoid confusion) which is more adapted to infinite complexes, thanks to a well-known notion : the "reduced product" of simplicial and cosimplicial modules. This has been shown to me by M. Zisman and is for instance (in a much more general form) in the book of A.-K. Bousfield and D.-M. Kan[5]). More precisely, let C^* (resp. S_*) be a cosimplicial k-module (resp. a simplicial k-module). Their "reduced product" $C^* \nabla S_*$ is defined as the quotient of the direct sum $\bigoplus_n C^n \otimes S_n$ by relations of the type $\sum (u^* \otimes 1)(\theta) - \sum (1 \otimes u_*)(\theta)$, $\theta \in C^p \otimes S_n$, for any non-decreasing map $u : [p] \longrightarrow [n]$, with the associated morphisms $u_* : S_n \longrightarrow S_p$ and $u^* : C^p \longrightarrow C^n$. On the other hand, we may consider as well the normalized k-module $\overline{S}_*$ (resp. $\overline{C}^*$) regarded

[5] A.-K.BOUSFIELD and D.-M. KAN. Homotopy limits, completions and localizations, Springer Lecture Notes in Mathematics 304 (1972).

as a chain complex (resp. a cochain complex) and may take the same type of quotient, also denoted by $\overline{C}^* \nabla \overline{S}_*$. More precisely, in the direct sum of the $\overline{C}^n \otimes \overline{S}_n$ we take the cokernel of $d \otimes 1 - 1 \otimes d'$, where $d : \overline{C}^n \longrightarrow \overline{C}^{n+1}$ (resp. $d' : \overline{S}_n \longrightarrow \overline{S}_{n-1}$) is the differential of the cochain complex (resp. the chain complex).

The general fact about these reduced products is then the following : there exists a canonical isomorphism $C^* \nabla S_* \longrightarrow \overline{C}^* \nabla \overline{S}_*$. This follows simply from the observation that for any k-module M, one has

$$\mathrm{Hom}(C^* \nabla S_*, M) = \mathrm{Hom}_\Delta(C^*, \mathrm{Hom}(S_*, M))$$

and the same type of identity for the Hom functor between cochain complexes

$$\mathrm{Hom}(\overline{C}^* \nabla \overline{S}_*, M) = \mathrm{Hom}(\overline{C}^*, \mathrm{Hom}(\overline{S}_*, M))$$

Since $\mathrm{Hom}(\overline{C}^*, \mathrm{Hom}(\overline{S}_*, M)) \cong \mathrm{Hom}_\Delta(C^*, \mathrm{Hom}(S_*, M))$ according to the Dold-Kan theorem, the result follows : choose $M = \overline{C}^* \nabla \overline{S}_*$.

1.13. THEOREM. *Let us assume that the complex* $\overline{S}_*$ *has trivial homology. Then an exact sequence of cosimplicial* _flat_ k-*modules*

$$0 \longrightarrow C'^* \longrightarrow C^* \longrightarrow C''^* \longrightarrow 0$$

induces an exact sequence of the associated reduced products

$$0 \longrightarrow C'^* \nabla S_* \longrightarrow C^* \nabla S_* \longrightarrow C''^* \nabla S_* \longrightarrow 0$$

Proof. Since $\overline{C}^*$ is naturally a direct factor in C^* in general, we have also an exact sequence of normalized complexes

$$0 \longrightarrow \overline{C}'^* \longrightarrow \overline{C} \longrightarrow \overline{C}''^* \longrightarrow 0$$

Let us put in general $C_n = C^{-n}$ and consider the total complex associated to the tensor product of _homology_ complexes $\overline{C}_{-*} \otimes \overline{S}_*$. The reduced product $\overline{C}^* \nabla \overline{S}_*$ is just the quotient module $\mathrm{Tot}^0/d(\mathrm{Tot}_1)$ in the previous Tot complex. Let us prove first that the homology of this Tot complex is 0. For this, we may assume without loss of generality that $\overline{C}^*$ is bounded (since we start with a cycle lying in a direct sum). We then prove the statement by induction on the size of $\overline{C}'^*$, using Künneth's theorem.

This last result shows that $\mathrm{Tot}_0/d(\mathrm{Tot}_1)$ is also $Z_0\mathrm{Tot}$, the k-module of 0-cycles in the Tot complex. On the other hand, since C''^* is flat, we have an exact sequence

$$0 \longrightarrow \mathrm{Tot}(\overline{C}'_{-*} \otimes \overline{S}_*) \longrightarrow \mathrm{Tot}(\overline{C}_{-*} \otimes \overline{S}_*) \longrightarrow \mathrm{Tot}(\overline{C}''_{-*} \otimes \overline{S}_*) \longrightarrow 0$$

The exact sequence required

$$0 \longrightarrow Z_0\mathrm{Tot}(\overline{C}'_{-*} \otimes \overline{S}_*) \longrightarrow Z_0\mathrm{Tot}(\overline{C}_{-*} \otimes \overline{S}_*) \longrightarrow Z_0\mathrm{Tot}(\overline{C}''_{-*} \otimes \overline{S}_*) \longrightarrow 0$$

is then a consequence of the vanishing of $H_{-1}(\mathrm{Tot}(\overline{C}'_{-*} \otimes \overline{S}_*))$.

1.14. Let us apply these general considerations to the case where S_* is the simplicial module $\Omega^p(\Delta_*)$. Since $\pi_p(Z^p(\Delta_*)) \cong k$, we can pick a representative $\chi_p \in Z^p(\Delta_p)$ (which vanishes on all the faces). As it is well known (cf. [4] § 3 for instance), these forms χ_p may be choosen by induction on p, starting with the obvious choice of χ_0 ; we write χ_p as the restriction to the last face Δ_p of a form ω_{p+1} belonging to $\Omega^p(\Delta_{p+1})$, vanishing on all the faces except Δ_p (this is in fact the definition of the normalization $\overline{\Omega}(\Delta_{p+1})$). We then choose χ_{p+1} as $d\omega_{p+1}$. We define a morphism

$$\theta_p : \quad \overline{C}^p \longrightarrow \overline{C}^* \nabla\ \overline{\Omega}^p(\Delta_*)$$

by the formula

$$\theta_p(c) = c \otimes \chi_p + (-1)^{p+1}\, dc \otimes \omega_{p+1}$$

(we write the elements of $\overline{C}^* \nabla\ \overline{\Omega}^p(\Delta_*)$ as 0-cycles of the Tot complex ; cf. 1.13).

Since the diagram

$$\begin{array}{ccc} \overline{C}^p & \longrightarrow & \overline{C}^* \nabla\ \overline{\Omega}^p(\Delta_*) \\ \downarrow & & \downarrow \\ \overline{C}^{p+1} & \longrightarrow & \overline{C}^* \nabla\ \overline{\Omega}^{p+1}(\Delta_*) \end{array}$$

commutes, the θ_p's define a morphism of cochain complexes.

1.15. THEOREM. *Let us assume that* $\overline{C}^*$ *is flat. Then, the morphism* θ *above defines a quasi-isomorphism between the complexes* $\overline{C}^\natural$ *and* $\overline{C}^* \nabla\ \overline{\Omega}^\natural(\Delta_*)$.

Proof. Without loss of generality, we may assume that the normalized complex $\overline{C}^*$ is bounded. As a direct consequence of 1.13, it is enough to prove the statement when the complex $\overline{C}^*$ is concentrated in a single degree, say n. In this case, it follows from the fact that $\overline{C}^* \nabla\ \overline{\Omega}^\natural(\Delta_*)$ is the complex $\overline{C}^n \otimes \overline{\overline{\Omega}}^\natural(\Delta_n)$, where $\overline{\overline{\Omega}}^\natural(\Delta_n)$ is the space of differential forms on Δ_n which vanish on <u>all</u> the faces. Since $\overline{C}^n$ is flat, its cohomology is $\overline{C}^n \otimes H^n(\Sigma^n)$, where Σ^n is the sphere of dimension n (viewed as the quotient of Δ_n by its boundary).

1.16. THEOREM. *Let* $C^*(X)$ *be the cochain complex associated to a simplicial set* X, *with coefficients in* **Z** *or a field and let us denote by* $\Omega^\natural(X)$ *the complex of* k*-modules* $C^*(X) \nabla \Omega^\natural(\Delta_*)$ *Then we have a natural commutative triangle of quasi-isomorphisms of complexes*

$$\begin{array}{ccc} \Omega^\natural(X) = C^*(X) \nabla \Omega^\natural(\Delta_*) & \longrightarrow & \mathrm{Hom}(X_*, \Omega^\natural(\Delta_*)) = \underline{\Omega}^\natural(X_*) \\ \nwarrow & & \nearrow \\ & C^\natural(X) & \end{array}$$

Proof. It follows immediately from the previous considerations. It is also easy to notice that both $\Omega^{\natural}(X)$ and $\underline{\Omega}^{\natural}(X)$ are DGA's. The canonical morphism $C^*(X) \longrightarrow \Omega^{\natural}(X)$ defined above is a map of differential graded modules (but <u>not</u> an algebra map). If k contains **Q**, we may choose the quantum parameter $q = 1$. In this case, $\Omega^{\natural}(\Delta_*)$ is a commutative DGA, as well as $\Omega^{\natural}(X)$ and $\underline{\Omega}^{\natural}(X)$.

1.17. ***Remark*****.** More generally, we might consider a sheaf $\mathfrak{F}$ of free k-modules over a space X where k is a field or **Z**. If $\mathfrak{F}^p$ denotes the Godement (cosimplicial) resolution of the sheaf $\mathfrak{F}$, $\Omega^{\natural}(X ; \mathfrak{F}) = \mathfrak{F}^* \nabla\, \Omega^{\natural}(\Delta_*)$ is an acyclic resolution of $\mathfrak{F}$, which might call the (abstract) de Rham resolution of $\mathfrak{F}$. The complex of sections $\Gamma(\mathfrak{F}^* \nabla\, \Omega^{\natural}(\Delta_*)) = \Gamma(\mathfrak{F}^*) \nabla\, \Omega^{\natural}(\Delta_*)$ computes the cohomology of X with values in $\mathfrak{F}$. The same type of remark applies to the (cosimplicial) Čech complex associated to a covering of the space X. If k contains **Q** and if choose the quantum parameter $q = 1$, this resolution is a commutative DGA if $\mathfrak{F}$ is a sheaf of commutative k-algebras.

1.18. ***Remark*****.** As in 1.10, we may replace $\Omega^{\natural}(\Delta_*)$ by its "stabilized" version $\hat{\Omega}^{\natural}(\Delta_*)$ and define in the same way $\hat{\Omega}(X)$ or more generally $\hat{\Omega}(X ; \mathfrak{F})$ if $\mathfrak{F}$ is a sheaf : the interest of these definitions will be explained in the next section.

II. Symmetric kernel of braided differential graded algebras.

2.1. Let A be a braided DGA with braiding R. For $i < j$, we recall that $R_{i,j} = R_{ij}$ is the endomorphism R acting on the (i, j)-components of the tensor product $A^{\otimes n}$ (and the identity on the others). We put $R_{j,i} = \sigma_{i,j} R_{i,j} \sigma_{i,j}$, where $\sigma_{i,j}$ is the obvious transposition (taking into account the signs for the gradation). By definition, the <u>symmetric kernel</u> of $A^{\otimes n}$ is the k-submodule of $A^{\otimes n}$ consisting of elements ω such that $R_{u,v}\omega = \sigma_{u,v}\omega$ for all couples[6] (u, v). This symmetric kernel is denoted by $A^{\overline{\otimes} n}$; it is clearly invariant under the action of the symmetric group $\mathfrak{S}_n$.

2.2. Example. Let us suppose that $1 - q^{\alpha}$ is invertible for all α and consider the braided algebra $A = \Omega(t)$ of 1.4. If we identify $A^{\otimes n}$ with $\Omega(x_1, \ldots x_n)$, its symmetric kernel is concentrated[7] in degrees 0 and 1 : we have ${}^0(A^{\overline{\otimes} n}) = {}^0(A^{\otimes n}) = k[x_1, \ldots, x_n]$ and ${}^1(A^{\overline{\otimes} n}) = d(k[x_1, .., x_n])$. In particular, the inclusion of $A^{\overline{\otimes} n}$ in $A^{\otimes n}$ is a quasi-isomorphism.

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[6] As a matter of fact, the $R_{u,v}$ for $u < v$ are sufficient for the applications we have in mind.

[7] In general, ${}^iC$ denotes the submodule of elements of degree i in the graded module C.

2.3. Example. Let us assume moreover that k is a field or the ring of integers **Z**. Let A and B be two braided commutative DGA's which are modules over k[q] such that the eigenvalues of σR are powers of q in $A^{\otimes 2}$ and $B^{\otimes 2}$. Then, the symmetric kernel of $(A \hat{\otimes} B)^{\otimes n}$ may be identified with $A^{\overline{\otimes} n} \hat{\otimes} B^{\overline{\otimes} n}$, taking into account the canonical isomorphism $(A \hat{\otimes} B)^{\otimes n} \cong A^{\otimes n} \hat{\otimes} B^{\otimes n}$.

From these examples, we deduce the following theorem (with the definitions of 1.5) :

2.4. THEOREME. *Let us suppose that* $1 - q^{\alpha}$ *is invertible for all* α *and that* k *is* **Z** *or a field. Then the inclusion of* $\Omega^*(\Delta_r)^{\overline{\otimes} n}$ *in* $\Omega^*(\Delta_r)^{\otimes n}$ *is a quasi-isomorphism.*

2.5. The braided structure of $\Omega^*(\Delta_r)^{\otimes n}$ does not extend to a n-simplicial structure on the k-module of all $\Omega^*(\Delta_{r_1}) \otimes ... \otimes \Omega^*(\Delta_{r_n})$ for $r_1, .., r_n$ belonging to **N**. However, we can give a n-simplicial meaning to the symmetric kernel if we replace $\Omega^*(\Delta_\bullet)$ by its stabilized version $\hat{\Omega}^*(\Delta_\bullet)$ with the notations of 1.10. More precisely, let us consider the restriction morphism

$$r : \hat{\Omega}^*(\Delta_r) \otimes \hat{\Omega}^*(\Delta_s) \longrightarrow \hat{\Omega}^*(\Delta_t) \otimes \hat{\Omega}^*(\Delta_t)$$

where t = Inf(r, s). The symmetric kernel of $\hat{\Omega}^*(\Delta_r) \otimes \hat{\Omega}^*(\Delta_s)$, denoted by $\hat{\Omega}^*(\Delta_r) \overline{\otimes} \hat{\Omega}^*(\Delta_s)$, is defined as the graded k-submodule of $\hat{\Omega}^*(\Delta_r) \otimes \hat{\Omega}^*(\Delta_s)$ consisting of the elements ω such that $r(\omega) \in \hat{\Omega}^*(\Delta_t)^{\overline{\otimes} 2}$. The "symmetric kernel" of $\hat{\Omega}^*(\Delta_{r_1}) \otimes ... \otimes \hat{\Omega}^*(\Delta_{r_n})$, also denoted by $\hat{\Omega}^*(\Delta_{r_1}) \overline{\otimes} \ldots \overline{\otimes} \hat{\Omega}^*(\Delta_{r_n})$, is the intersection of the n(n-1)/2 partial symmetric kernels obtained by considering all (i, j)-components of the tensor product[8].

2.6. Let us consider now a simplicial set X and the associated differential graded algebra $\Omega^{\natural}(X)$, written simply $\Omega(X)$, defined at the end of § 1 as the reduced product $C^*(X) \nabla \Omega(\Delta_*)$. More precisely, we should also consider the "stabilized" version of it, defined by $\hat{\Omega}(X) = C^*(X) \nabla \hat{\Omega}(\Delta_*)$ (cf. 1.10). This notion of reduced product ∇, which we used already many times, can be easily extended to multisimplicial and multicosimplicial-modules. In particular, one might consider $[C^*(X) \otimes C^*(X)] \nabla [\Omega(\Delta_*) \otimes \Omega(\Delta_*)]$ as well as $[C^*(X) \otimes C^*(X)] \nabla [\hat{\Omega}(\Delta_*) \otimes \hat{\Omega}(\Delta_*)]$. Since $C^*(X)$ is flat (as a **Z**-module), we can identify these various reduced products to $\Omega(X) \otimes \Omega(X)$ and $\hat{\Omega}(X) \otimes \hat{\Omega}(X)$ respectively. By the

[8] Note that the n-complex associated to the n-simplicial k-module $(r_1, ..., r_n) \mapsto \hat{\Omega}^*(\Delta_{r_1}) \overline{\otimes} ... \overline{\otimes} \hat{\Omega}^*(\Delta_{r_n})$ is n-acyclic.

same method, we can write the n^{th} tensor product as a reduced product of n factors. These identifications enable us to define the symmetric kernel of $\hat{\Omega}(X)^{\otimes n}$, denoted $\hat{\Omega}(X)^{\overline{\otimes} n}$, as the reduced product of $C^*(X)^{\otimes n}$ and $\hat{\Omega}(\Delta_*)^{\overline{\otimes} n}$, where $\hat{\Omega}(\Delta_*)^{\overline{\otimes} n}$ is defined above. This symmetric kernel $\hat{\Omega}(X)^{\overline{\otimes} n}$ has two essential properties :

1. The canonical inclusion of $\hat{\Omega}(X)^{\overline{\otimes} n}$ in $\hat{\Omega}(X)^{\otimes n}$ is a quasi-isomorphism ; it is equivariant for the natural action of the symmetric group $\mathfrak{S}_n$ on both factors.

2. A map α from {1, ..., n} to {1, ..., p} induces in a functorial way a morphism of k-modules $\alpha_* : \hat{\Omega}(X)^{\overline{\otimes} n} \longrightarrow \hat{\Omega}(X)^{\overline{\otimes} p}$ by the formula

$$\alpha_*(a_1 \otimes \ldots \otimes a_n) = b_1 \otimes \ldots \otimes b_p$$

with $b_j = \prod_{\alpha(i)=j} a_i$ (this product is independant of the order). In particular, the product map $\hat{\Omega}(X)^{\overline{\otimes} n} \longrightarrow \hat{\Omega}(X)$ is equivariant (with the trivial action of the symmetric group on $\hat{\Omega}(X)$).

2.7. Using the previous considerations and some elementary homological algebra, it is easy to define cup i-products and Steenrod operations <u>on the level of quantum differential forms</u>. The sequence

$$\hat{\Omega}(X)^{\otimes n} \hookleftarrow \hat{\Omega}(X)^{\overline{\otimes} n} \longrightarrow \hat{\Omega}(X)$$

defines an equivariant morphism from $\hat{\Omega}(X)^{\otimes n}$ to $\hat{\Omega}(X)$ in the derived category of $\mathfrak{S}_n$-complexes as we have seen above. From this fact, we deduce a morphism of $k[\mathfrak{S}_n]$-complexes which is well defined up to homotopy[9]

$$B_\natural(\mathfrak{S}_n) \longrightarrow \mathrm{Hom}_\natural(\hat{\Omega}(X)^{\otimes n}, \hat{\Omega}(X))$$

where $B_\natural(\mathfrak{S}_n)$ is any projective resolution of k as a $k[\mathfrak{S}_n]$-module. Let us suppose now that n = p is a prime number and let us replace the symmetric group by the cyclic group C_p. We may choose for $B_\natural(C_p)$ the classical acyclic resolution of k by $k[C_p]$-modules of rank 1 (with k = $\mathbf{F}_p$ and the quantum parameter q = 0 in order to fix the ideas ; other choices are possible). From the previous observations, we deduce morphisms of degree -i, which we might call “cup i-products” :

$$\mu_i : \hat{\Omega}(X)^{\otimes p} \longrightarrow \hat{\Omega}(X)$$

They are well defined up to homotopy (μ_0 is the usual cup-product map). As it is well known, Steenrod operations can be deduced from the μ_i as morphisms from $H^m(X)$ to $H^{mp-i}(X)$, by

[9] Hom_0 denotes the k-module of morphisms of degree 0 which are homotopic to the multiplication μ. For i > 0, Hom_i is the k-module of <u>all</u> morphisms of degree -i.

taking the composition μ_i with the p^{th} power operation $P : \hat{\Omega}(X) \longrightarrow \hat{\Omega}(X)^{\otimes p}$ which is also equivariant[10]. This can be proved, using for instance the method described in [4].

III. Neo-algebras : towards an algebraic description of the homotopy type.

3.1. A "neo-algebra" is given by the following data (1, 2 and 3), subject to the conditions α, β, γ and δ explained below (*this definition will imply that our neo-algebras are just particular cases of* *<u>partial</u>* *DGA's, as defined in* [6] p. 40) :

1. A differential graded k-module[11] A with a "unit element" $1 \in {}^0A$

2. A differential graded k-submodule A_2 of $A^2 = A^{\otimes 2}$, stable under the action of the group **Z**/2 acting naturally on A^2 and containing $k.1 \otimes A$ (and therefore $A \otimes k.1$)

3. A "partial multiplication"

$$\mu : A_2 \longrightarrow A$$

which defines a morphism of complexes

We call μ_{12} (resp. μ_{23}) the partial multiplication on $A_2 \otimes A$ (resp. $A \otimes A_2$) with values in $A^2 = A \otimes A$. On the other hand, for i and j belonging to the set P = {1, ..., n}, we denote by $A_{i,j}$ the image of $A_2 \otimes A^{n-2}$ in A^n under the permutation $(1, 2) \mapsto (i, j)$ of the factors. If S is a subset of P x P, the k-module A_S is the intersection of all the $A_{i,j}$ where $(i, j) \in S$. In particular, A_n is defined as the module obtained when S = P x P.

Here are the properties α, β, γ and δ which characterize a neo-algebra (if $A_2 = A^2$, we just recover the definition of a commutative DGA) :

α**)** The inclusion of A_n in A^n is a quasi-isomorphism

β) We have the identity $\mu(1 \otimes a) = \mu(1 \otimes a) = a$ for any element a of A **(*unital axiom*)**

γ**)** The partial multiplication $\mu : A_2 \longrightarrow A$ is equivariant, the group **Z**/2 acting trivially on A (***commutativity axiom***)

δ**)** The k-module $\mu_{12}(A_3)$ is included[12] in A_2. Moreover, we assume that the following diagram commutes (***associativity axiom***) :

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[10] On the first factor, the action of the symmetric group is induced by the signature of the permutations if the differential forms are of odd degree and is the identity otherwise.

[11] We recall again that ${}^rC$ denotes the k-module of elements of degree r in the graded k-module C, and that C^r is the tensor product $C^{\otimes r}$ of r copies of C.

[12] By symmetry, this implies the same property for $\mu_{23}(A_3)$

$$\begin{array}{ccc} A_3 & \xrightarrow{\mu_{12}} & A_2 \\ \mu_{23}\downarrow & & \downarrow\mu \\ A_2 & \xrightarrow{\mu} & A \end{array}$$

As for commutative algebras, these properties imply that any set map from {1, ..., n} to {1, ..., p} induces a functorial morphism $\alpha_* : A_n \longrightarrow A_p$. This property is closely related to the theory of Γ-spaces introduced by G. Segal [11].

3.2. Example. The braided differential graded algebra $A = \Omega(\Delta_n)$ defined in 1.5 is a neo-algebra with the symmetric kernel playing the role of A_2 (cf. 2.4 where we assume that k = **Z** or a field).

3.3. On the other hand, according to 2.6, $\Omega(X_1) \otimes \ldots \otimes \Omega(X_n)$ may be identified with the reduced product $[C^*(X_1) \otimes \ldots \otimes C^*(X_n)] \nabla [\Omega(\Delta_*) \otimes \ldots \otimes \Omega(\Delta_*)]$. Up to a quasi-isomorphism, we may replace $\Omega(\Delta_*)$ by $\hat{\Omega}(\Delta_*)$. The “bisimplicial symmetric kernel” $\hat{\Omega}(\Delta_r) \overline{\otimes} \hat{\Omega}(\Delta_s)$ is then defined (as in 2.5) to be the k-submodule of $\hat{\Omega}(\Delta_r) \otimes \hat{\Omega}(\Delta_s)$ of elements which restrictions to $\hat{\Omega}(\Delta_t) \otimes \hat{\Omega}(\Delta_t)$ belong to $\hat{\Omega}(\Delta_t)^{\otimes 2}$ (with t = Inf(r, s)). If we set $A = \hat{\Omega}(X)$ and A_2 = the reduced product $[C^*(X) \otimes C^*(X)] \nabla [\hat{\Omega}(\Delta_*) \overline{\otimes} \hat{\Omega}(\Delta_*)]$, we can check easily that $\hat{\Omega}(X)$ is also a neo-algebra.

3.4. If A and B are neo-algebras over k = **Z** or a field, it is not difficult to see that $A \otimes B$ is also a neo-algebra (with the usual sign conventions for the tensor product of differential graded k-modules).

3.5. Finally, a morphism between two neo-algebras A and B is defined as a morphism of differential graded k-modules $f : A \longrightarrow B$ such that

1. $(f \otimes f)(A_2) \subset B_2$
2. The following diagram commutes

$$\begin{array}{ccc} A_2 & \longrightarrow & B_2 \\ \downarrow\mu & & \downarrow\mu \\ A & \longrightarrow & B \end{array}$$

3.6. THEOREM. *Let us consider two connected nilpotent p-complete simplicial sets*[13] X *and* Y *of finite type*. *We assume that there is a zigzag sequence of quasi-isomorphisms of neo-algebras* (*with* $k = F_p$)

$$\hat{\Omega}(X) \longrightarrow A \longleftarrow B \longrightarrow \ldots. \longleftarrow \hat{\Omega}(Y)$$

Then X *and* Y *have the same homotopy type*.

Sketch of the proof. Let Z be a simplicial set. According to section II, there is a natural quasi-isomorphism between the differential graded k-modules $\hat{\Omega}^*(Z)$ and $C^*(Z)$. On the other hand, we can associate to a neo-algebra an E_∞-algebra using the method in the book of I. Kriz and P. May [6]. Let $\mathcal{P}$ (resp. $\mathcal{E}$, resp $\mathcal{E}\mathcal{S}\mathcal{P}$) denote the category of partial algebras (resp. E_∞-algebras, resp. E_∞-simplicial partial algebras). In [6] one describes a diagram of categories and functors which is commutative up to isomorphism (φ and ψ being quasi-isomorphisms of underlying differential graded modules)

$$\begin{tikzcd} & \mathcal{P} \\ \mathcal{P} \arrow[ur, "Id"] \arrow[r, "V"] \arrow[dr, "W"'] & \mathcal{E}\mathcal{S}\mathcal{P} \arrow[u, "\varphi"'] \arrow[d, "\psi"] \\ & \mathcal{E} \end{tikzcd}$$

The quasi-isomorphisms in the hypothesis of the theorem

$$\hat{\Omega}^*(X) \longrightarrow A \longleftarrow B \longrightarrow \ldots. \longleftarrow \hat{\Omega}^*(Y)$$

imply therefore a sequence of quasi-isomorphisms between the associated E_∞-algebras via the functor W.

On the other hand, according to a recent result of M.-A. Mandell [9], for any finite simplicial set Z, the E_∞-algebras $\hat{\Omega}^*(Z)$ and $C^*(Z)$ are also related by a sequence of E_∞-algebras quasi-isomorphisms. From the previous conclusion, we deduce that $C^*(X)$ and $C^*(Y)$ are also related by a sequence of E_∞-algebras quasi-isomorphisms.

Since X and Y are nilpotent, a second key result of M.-A. Mandell [8] implies that X and Y have the same homotopy type, which concludes the proof of our theorem.

A weaker version of the theorem is the following :

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[13] This means that its Postnikov tower can be choosen such that each fiber is of type $K(\mathbf{Z}/p, n)$ or $K(\hat{\mathbf{Z}}p, n)$, where $\hat{\mathbf{Z}}_p$ denotes the ring of p-adic integers [8].

3.7. THEOREM. *Let us consider two connected* <u>*finite*</u> *simplicial sets* X *and* Y *such that their homotopy groups are finite* p*-groups*.*We assume that there is a zigzag sequence of quasi-isomorphisms of neo-algebras* (*with* $k = F_p$)

$$\hat{\Omega}^*(X) \longrightarrow A \longleftarrow B \longrightarrow \dots \longleftarrow \hat{\Omega}^*(Y)$$

Then X *and* Y *have the same homotopy type*.

3.8. As a matter of fact, if the homotopy groups of X are finite p-groups, there is a procedure to compute algebraically the homotopy groups of X via a suitable "iteration" of the bar construction [2], starting from the neo-algebra $A = \hat{\Omega}^*(X)$. More precisely, the correspondence $(r_1,\dots, r_n) \mapsto A_{r_1\dots r_n}$ defines a n-simplicial graded module (one has to use the base point to define some face maps). The associated total cohomology complex $\mathrm{Tot}\,(A_{(-r_1)\dots(-r_n)})$, located in the second quadrant[14] , has the cohomology of the n^{th} iterated loop space of X, denoted here $\mathfrak{L}^n(X)$. The coalgebra structure on the total complex determines the group structure on $\pi_n(X)$ (note that $H^0(\mathfrak{L}^n(X)) = \mathrm{Hom}_{sets}(\pi_n(X), \mathbf{Z}/p)$).

IV. Braided differential graded coalgebras[15] and q-homology. Neo-coalgebras.

4.1. Let A denote the fundamental example of braided DGA defined in 1.2. Its k-dual $\mathrm{Hom}(A, k) = k[[x]] \oplus k[[x]]\, dx$ is NOT a coalgebra (the dual of a tensor product is not a tensor product). However, we are going to define a coalgebra of "quantum algebraic currents" $\mathfrak{D}(x)$ (in duality with $\Omega(t)$) as a suitable k-submodule of Hom(A, k), which will be a <u>covariant</u> algebraic analog of the unit interval [0, 1]. Its definition makes use of the quantum exponential[16] $e_q(x)$, considered as an element of Hom(A, k) :

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!_q}$$

More precisely, let us define the following graded k-submodule $\mathfrak{D}(x) = \mathfrak{D}_0(x) \oplus \mathfrak{D}_1(x)$ of Hom(A, k) : $\mathfrak{D}_0(x)$ consists of formal power series $f(x) = U(x)e_q(x)$, where U is a polynomial ; the elements of $\mathfrak{D}_1(x)$ may be written formally as $T.dx$, where $T \in \mathfrak{D}_0(x)$.

[14] which is obtained by considering the <u>sum</u> of elements in the diagonals.

[15] Braided DGC in short

[16] We denote by $n!_q$ the product $1_q \times 2_q \times \dots \times n_q$, where m_q represents the "quantum integer" $\frac{q^m - 1}{q - 1}$. Note that if q = 0, $n!_q = 1$.

The duality $\Omega(t) \times \mathfrak{D}(x) \longrightarrow k$ is induced by a "continuous" scalar product on polynomials in t and power series in x, given by the following formulas :

$$< t^n, x^m> = < t^n dt, x^m dx> = 0 \text{ if } n \neq m \text{ and } < t^n, x^n> = < t^n dt, x^n dx> = n!_q$$

Moreover, we assume that elements of different degrees are orthogonal to each other. Another way of looking the situation is to take as a basis of $\Omega(t)$ the "q-divided powers" $\dfrac{t^n}{n!_q}$ and $\dfrac{t^n}{n!_q}\,dt$.

The x^n and $x^n\,dx$ are then in duality with this basis.

4.2. In order to define the comultiplication of $\mathfrak{D}(x)$ in a convenient way, we use a continuous "twisted product" on Hom(A, k) $\otimes$ Hom(A, k) : we assume that

$$(1 \otimes x^n).(x^m \otimes 1) = q^{nm} (x^m \otimes 1).(1 \otimes x^n) = q^{nm}(x^m \otimes x^n),$$
$$(1 \otimes x^n dx).(x^m \otimes 1) = q^{(n+1)m} (x^m \otimes 1).(1 \otimes x^n dx) = q^{(n+1)m}(x^m \otimes x^n dx)$$
$$(1 \otimes x^n).(x^m dx \otimes 1) = q^{n(m+1)}(x^m dx \otimes 1).(1 \otimes x^n) = q^{n(m+1)}(x^m dx \otimes x^n)$$
$$(1 \otimes x^n dx).(x^m dx \otimes 1) = - q^{(n+1)(m+1)}(x^m dx \otimes 1).(1 \otimes x^n dx)$$
$$= - q^{(n+1)(m+1)}(x^m dx \otimes x^n dx)$$

The comultiplication Δ is then deduced from the following formulas

$$\Delta(x^m) = (x \otimes 1 + 1 \otimes x)^m = \sum_{n=0}^{m} \frac{m!_q}{n!_q (m-n)!_q} \quad x^n \otimes x^{m-n}$$
$$\Delta(e_q(x)) = e_q(x) \otimes e_q(x)$$
$$\Delta(x^m dx) = (x \otimes 1 + 1 \otimes x)^m \ (dx \otimes 1 + 1 \otimes dx)$$

$$\Delta(e_q(x)dx) = [e_q(x) \otimes e_q(x)]\,(dx \otimes 1 + 1 \otimes dx) = e_q(x)dx \otimes e_q(qx) + e_q(x) \otimes e_q(x)dx$$

and in general $\Delta(u.v) = \Delta(u).\Delta(v)$ each time the <u>product</u> u.v makes sense in Hom(A, k) [note that $\bar{e}_q(x) = e_q(qx)$ is equal to $((q-1)x + 1)).e_q(x)$].

4.3. Finally, the (co)differential $\bar{d} : \mathfrak{D}_1(x) \longrightarrow \mathfrak{D}_0(x)$ is defined by

$$\bar{d}[(U(x)e_q(x))dx] = [U(x)e_q(x)].x$$

In order to see that $\bar{d}$ is a differential of coalgebra, it is convenient to introduce formally the variables $X = x \otimes 1$ and $Y = 1 \otimes x$ (with $YX = q\,XY$). With obvious notations, we then have

$$\bar{d}[\Delta(f(x).dx)] = \bar{d}[f(X + Y).d(X + Y)] = f(X + Y)\,(X + Y) = \Delta(f(x)x) = \Delta[\bar{d}(f(x).dx)]$$

By definition, $\mathfrak{D}(x)$ is the elementary DGC of <u>quantum algebraic currents on the unit interval</u> [0, 1]. The structure morphisms of $\Omega(t)$ and $\mathfrak{D}(x)$ are in duality one to another.

4.4. If $1 - q$ is invertible, it is important to notice that $\mathcal{D}(x)$ has two remarkable "group like" elements g (i.e. such that $\Delta(g) = g \otimes g$)). They are $g_1 = e_q(x)$ and $g_0 = e_q(qx) = [(q - 1)x + 1].e_q(x)$. The two coalgebras-morphisms $\alpha_0 : k \longrightarrow \mathcal{D}(x)$ and $\alpha_1 : k \longrightarrow \mathcal{D}(x)$ corresponding to these group-like elements show that $\mathcal{D}(x)$ is the <u>covariant</u> algebraic analog of the unit interval, with its two end points (whereas $\Omega(t)$ is the <u>contravariant</u> analog).

4.5. On the other hand, <u>braided</u> coalgebras are defined dually to braided algebras. For instance, the β axiom in 1.1 for algebras can be translated into the following formula for coalgebras :

$$\mu_{23} R = R_{12}R_{23}\, \mu_{12} \quad \text{and} \quad \mu_{12} R = R_{23}R_{12}\, \mu_{23}$$

where μ is the comultiplication and R is the braiding.

With these definitions, taking into account the scalar product defined above, the braiding on $\mathcal{D}(x)$ may be defined by explicit formulas. For $f \in \mathcal{D}_0(x)$, let us put $\bar{f}(x) = f(qx)$. Then we have

$$R(u \otimes v) = v \otimes u$$
$$R(udx \otimes v) = v \otimes u.dx$$
$$R(u \otimes v.dx) = v.dx \otimes \bar{u} + (1 - q)vx \otimes du = v.dx \otimes \bar{u} + v(x - \bar{x}) \otimes du$$
$$R(u.dx \otimes v.dx) = -vdx \otimes \bar{u}.d\bar{x}$$

The proof of the following theorem is obvious :

4.6. THEOREM (Poincaré's lemma for $\mathcal{D}(x)$). *If* $1 - q$ *is invertible in* k, *the* <u>*homology*</u> *of the complex*

$$\mathcal{D}_1(x) \xrightarrow{\bar{d}} \mathcal{D}_0(x)$$

is trivial, except in degree 0 *where it is isomorphic to* k.

4.7. If $x_0, \dots, x_n$ are indeterminates, we define a <u>simplicial</u> DGC as follows

$$S_{(r)} = \prod_{i_0 < \dots < i_r} \mathcal{D}(x_0, \dots, \hat{x}_{i_0}, \dots, \hat{x}_{i_r}, \dots, x_n)$$

In particular, the two face operators

$$S_{(1)} = \prod_{i<j} \mathcal{D}(x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \rightrightarrows S_{(0)} = \prod_i \mathcal{D}(x_0, \dots, \hat{x}_i, \dots, x_n)$$

are the obvious inclusions obtained by tensoring the identity with the coalgebra morphism $\alpha_0 : k \longrightarrow \mathcal{D}(x)$ defined above. The coequalizer of these two morphisms defines a commutative braided DGC which we denote by $\mathcal{D}(\Delta_n)$ or $\mathcal{D}_*(\Delta_n)$. It is easy to see that $\mathcal{D}(\Delta_n)$ is the k-submodule of $\mathcal{D}(x_0, \dots, x_n) = \mathcal{D}(x_0) \otimes \dots \otimes \mathcal{D}(x_n)$, which consists of sums of tensors of the type $\mathcal{D}(x_0, \dots, \hat{x}_i, \dots, x_n).e_q(qx_i)$ for various i's. If $q = 0$, $\mathcal{D}(\Delta_n)$ is just the sum of all the $\mathcal{D}(x_0, \dots, \hat{x}_i, \dots, x_n)$'s inside $\mathcal{D}(x_0, \dots, x_n)$.

4.8. THEOREM. *Let us assume* q = 0. *Then the correspondence* $n \mapsto \mathcal{D}(\Delta_n)$ *defines a* *<u>cosimplicial</u> braided* DGC. *In particular, the coface operators are defined by replacing the missing variable* x_i *by the multiplication with the quantum exponential* $e_q(x_i) = \sum_{t=0}^{\infty} (x_i)^t$. *Therefore, for any non decreasing set map* : $[s] \longrightarrow [r]$, *the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D}(\Delta_s) \times \mathcal{D}(\Delta_s) & \xrightarrow{R} & \mathcal{D}(\Delta_s) \times \mathcal{D}(\Delta_s) \\ \downarrow & & \downarrow \\ \mathcal{D}(\Delta_r) \times \mathcal{D}(\Delta_r) & \xrightarrow{R} & \mathcal{D}(\Delta_r) \times \mathcal{D}(\Delta_r) \end{array}$$

Proof. We use the duality between $\mathcal{D}(\Delta_n)$ and $\Omega(\Delta_n)$ made explicit in 4.1. In particular, the algebra morphism $\Omega(t) \longrightarrow k$ defined by putting t = 1 is the transpose of the coalgebra morphism $k \longrightarrow \mathcal{D}(x)$ which associates to the number 1 the quantum exponential $e_q(x) = \sum_{t=0}^{\infty} x^t$ (since q = 0). In the same way, the degeneracy operators are induced by the k-module map of two variables $\phi : \mathcal{D}(x) \otimes \mathcal{D}(y) \longrightarrow \mathrm{Hom}(A, k) = k[[z]] \oplus k[[z]].dz$ given by the following formula : if $u = \sum a_n x^n$, $v = \sum b_n y^n$, we have $\phi(u, v) = \sum n!_q\, a_n b_n\, z^n$, $\phi(u, vdy) = \sum (n+1)!_q\, a_{n+1} b_n z^n dz$, $\phi(udx, v) = \sum (n+1)!_q\, a_n b_{n+1} z^n dz$ and, finally, $\phi(udx, vdy) = 0$. For q = 0, these formulas are reduced to $\phi(u, v) = \sum a_n b_n\, z^n$, $\phi(u, vdy) = \sum a_{n+1} b_n z^n dz$, $\phi(udx, v) = a_n b_{n+1} z^n dz$ and finally $\phi(udx, vdy) = 0$. We should notice that by choosing u and v functions of the type $x^r e_q(x)$ or $y^s e_q(y)$, the image of ϕ is really included in $\mathcal{D}(z)$.

4.9. The $\mathcal{D}_m(\Delta_*)$ define a <u>cosimplicial</u> k-module which is acyclic[17]. This follows from the same argument as in 1.8, where we showed that the homotopy groups of $\Omega^s(\Delta_*)$ are equal to 0. Let know S be a <u>simplicial</u> k-module. The reduced product $\mathcal{D}_m(\Delta_*) \nabla S_*$ (cf. 1.12), denoted simply $\mathcal{D}_m(S)$, is by definition the k-module of quantum algebraic currents of degree m associated to the simplicial module S. According to the general considerations of 1.12, $\mathcal{D}_m(S)$ is also isomorphic to $\overline{\mathcal{D}}_m(\Delta_*) \nabla \overline{S}_*$. The differential $\mathcal{D}_m(\Delta_*) \longrightarrow \mathcal{D}_{m-1}(\Delta_*)$ induces a differential $\mathcal{D}_m(S) \to \mathcal{D}_{m-1}(S)$. An interesting case is when S is the simplicial chain functor S(X) associated to a simplicial set X : in this case, we note simply $\mathcal{D}_m(X)$ instead of $\mathcal{D}_m(S)$. This notation is coherent with the previous one for $\mathcal{D}_m(\Delta_n)$; as a matter of fact, if X is a finite simplicial set (i.e. with a finite number of non degenerate simplices) and if we put S = S(X) as before, we have $\overline{\mathcal{D}}_m(\Delta_*) \nabla \overline{S}_* \cong \mathrm{Mor}(\overline{S}_*{}^*, \overline{\mathcal{D}}_m(\Delta_*)) \cong \mathrm{Mor}_\Delta((S)^*, \mathcal{D}_m(\Delta_*))$, where *

[17] Following 1.10, we may also replace $\mathcal{D}_m(\Delta_*)$ by its "stabilization" $\hat{\mathcal{D}}_m(\Delta_*) = \mathrm{colim}_r\, \mathcal{D}_m(\Delta_*)^{\otimes r}$, without changing the homology.

denotes the dual. When $X = \Delta_n$, we recover $\mathfrak{D}_m(\Delta_n)$ as expected.

4.10. The functors $S \mapsto \mathfrak{D}_m(S)$ and $X \mapsto \mathfrak{D}_m(X)$ satisfy the same formal properties as for the functor $\Omega(X)$ defined in 1.12 and 2.6 In particular, from an exact sequence of <u>flat</u> simplicial modules

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0$$

we deduce another exact sequence with currents :

$$0 \longrightarrow \mathfrak{D}_m(S') \longrightarrow \mathfrak{D}_m(S) \longrightarrow \mathfrak{D}_m(S'') \longrightarrow 0$$

4.11. By induction on m, starting with m = 0, we are going to define elements I_m of $\mathfrak{D}_m(\Delta_m)$ such that the classes J_m in $\mathfrak{D}_m(\Delta_m/\partial\Delta_m)$ generate the reduced homology of the spheres[18]. For this purpose, we look at the following exact sequence of singular complexes

$$0 \longrightarrow S(\partial\Delta_m) \longrightarrow S(\Delta_m) \oplus S(*) \longrightarrow S(\Delta_m/\partial\Delta_m) \longrightarrow 0$$

From the previous considerations, we deduce an exact sequence of corresponding reduced modules of algebraic currents

$$0 \longrightarrow \mathfrak{D}_r(\partial\Delta_m) \longrightarrow \mathfrak{D}_r(\Delta_m) \oplus \mathfrak{D}_r(*) \longrightarrow \mathfrak{D}_r(\Delta_m/\partial\Delta_m) \longrightarrow 0$$

The connecting homomorphism associated to this exact sequence enables us to identify the homology of $\Delta_m/\partial\Delta_m$ with the shifted homology of $\partial\Delta_m$ (in positive degrees). On the other hand, if we write $\partial\Delta_m$ as the union of a cone Λ_{m-1} and a face Δ_{m-1}, we have a Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathfrak{D}_r(\partial\Delta_{m-1}) \longrightarrow \mathfrak{D}_r(\Lambda_{m-1}) \oplus \mathfrak{D}_r(\Delta_{m-1}) \longrightarrow \mathfrak{D}_r(\partial\Delta_m) \longrightarrow 0$$

Therefore, we can also identify the reduced homology of $\partial\Delta_m$ with the shifted homology of $\partial\Delta_{m-1}$: that's the way the simplicial homology of spheres may be computed. Once the J_m are choosen (in such a way that the cohomology classes are linked by the connecting homomorphisms deduced from the previous exact sequences), lift them in the currents I_m in $\mathfrak{D}_r(\Delta_m)$ and write $\bar{I}_m$ its class in $\overline{\mathfrak{D}}_m(\Delta_m)$ and put $\bar{u}_m = \partial_m(\bar{I}_m)$ where $\partial_m : \overline{\mathfrak{D}}_m(\Delta_m) \longrightarrow \overline{\mathfrak{D}}_m(\Delta_{m+1})$

[18] This method has been shown to me by M. Zisman.

4.12. THEOREM. *Let* S_* *be a flat simplicial complex and* $\phi : \overline{S}_m \longrightarrow \mathfrak{D}_m(S)$ *defined by associating to a normalized chain* c *of degree* m *the sum* $\bar{\iota}_m \otimes c + (-1)^m \bar{u}_m \otimes dc$ *in* $\overline{\mathfrak{D}}_m(\Delta_m) \otimes \overline{S}_m + \overline{\mathfrak{D}}_m(\Delta_{m+1}) \otimes \overline{S}_{m-1}$.*Then* ϕ *induces a quasi-isomorphism between the normalized chain complex* $\overline{S}_*$ *and the complex of quantum algebraic currents* $\mathfrak{D}_m(S) = \mathfrak{D}_m(\Delta_*) \nabla S_*$

Proof . As we have seen before, an exact sequence of flat simplicial modules

$$0 \longrightarrow S'_* \longrightarrow S_* \longrightarrow S''_* \longrightarrow 0$$

induces an exact sequence of complexes

$$0 \longrightarrow \mathfrak{D}(S'_*) \longrightarrow \mathfrak{D}(S_*) \longrightarrow \mathfrak{D}(S''_*) \longrightarrow 0$$

On the other hand, in order to prove surjectivity, as well as injectivity, we may assume that the normalized complex $\overline{S}$ is bounded. Moreover, by taking inductive limits and reducing the size of the complex $\overline{S}_*$ by induction, we may assume that $\overline{S}_*$ is concentrated in a single degree, say m. The complex $\mathfrak{D}(S_*)$ is then isomorphic to $\mathfrak{D}_*(\Delta_m/\partial\Delta_m) \otimes \overline{S}_m$. In that case, the theorem becomes clear, since $H_i(\mathfrak{D}(S_*)) \cong \overline{S}_m \otimes \tilde{H}_i(\Sigma^m)$, where Σ^m is the sphere of dimension m (one has to use the flatness of S_* again).

4.13. *Remark*. If we put $C^* = \mathrm{Hom}(S_*, k)$ and $\Omega^m(S) = C^* \nabla \Omega^m(\Delta_*)$, there is a pairing between $\Omega^m(C)$ and $\mathfrak{D}_m(S) = \mathfrak{D}_m(\Delta_*) \nabla S_*$. This is induced by the composition

$$[C^* \nabla \Omega^m(\Delta_*)] \times [\mathfrak{D}_m(\Delta_*) \nabla S_*] \longrightarrow C^* \otimes (\Omega^m(\Delta_*) \otimes \mathfrak{D}_m(\Delta_*)) \otimes S_* \longrightarrow C^* \otimes S_* \longrightarrow k.$$

4.14. If S_* is a simplicial coalgebra, we can provide $\mathfrak{D}_*(S)$ with a coalgebra structure. The comultiplication follows from the composition of the following maps

$$\mathfrak{D}(\Delta_*) \nabla S_* \longrightarrow [\mathfrak{D}(\Delta_*) \otimes \mathfrak{D}(\Delta_*)] \nabla [S_* \otimes S_*)]$$
$$\cong [\mathfrak{D}(\Delta_*) \nabla S_*] \otimes [\mathfrak{D}(\Delta_*) \nabla S_*]$$

4.15. If S is a commutative braided coalgebra, the symmetric cokernel of $S^{\otimes n}$ (denoted by $S^{\overline{\otimes} n}$) is the quotient of $S^{\otimes n}$ by the equivalence relation which identifies $\sigma_{u,v}(\omega)$ and $R_{u,v}(\omega)$ for all couples (u, v), with the notations of 2.1. This quotient is stable by the action of the symmetric group. In the case where S is the coalgebra $\mathfrak{D}(x)$ defined in 4.1, $C^{\otimes n}$ may be identified with $\mathcal{D}(x_1,.., x_n)$. These currents are linear combinations of elements of the type

$$e_q(x_1)...e_q(x_n).\omega(x_1, ..., x_n), \text{ with } \omega(x_1, ..., x_n) \in \Omega(x_1,\ldots, x_n)$$

The following theorem may be deduced by duality from the analogous theorem in 2.2.

4.16. THEOREM. *Let* $S = \mathcal{D}(x)$ *be the elementary coalgebra of quantum algebraic currents on the unit interval and let us assume that* $q = 0$. *Then, all the elements of* $S^{\overline{\otimes} n}$ *are of degree* 0 *or* 1. *In degree* 0, *we obtained all the elements of degree* 0 *in* $S^{\otimes n}$. *The elements of degree 1 in* $S^{\overline{\otimes} n}$ *are the classes of elements of degree* 1 *in* $S^{\otimes n}$ *for the equivalence relation which identifies* $f(x_1, .., x_n).dx_i$ *and* $g(x_1, ..., x_n).dx_j$ *if the products* $f(x_1, ..., x_n).x_i$ *and* $g(x_1,..., x_n).x_j$ *coincide. In particular, the quotient map* $S^{\otimes n} \longrightarrow S^{\overline{\otimes} n}$ *is a quasi-isomorphism*.

4.17. Let $\hat{\mathcal{D}}(\Delta_r)$ be the coalgebra of stabilized currents (cf. the footnote 18 p. 17). Since $\hat{\mathcal{D}}(\Delta_r)$ is a braided coalgebra, the "symmetric cokernel" $\hat{\mathcal{D}}(\Delta_r) \overline{\otimes} \hat{\mathcal{D}}(\Delta_s)$ may be identified with the push-out of $\hat{\mathcal{D}}(\Delta_r) \otimes \hat{\mathcal{D}}(\Delta_s)$ in the following diagram (where t = Inf (r, s))

$$\begin{array}{ccc}
\hat{\mathcal{D}}(\Delta_t) \otimes \hat{\mathcal{D}}(\Delta_t) & \longrightarrow & \hat{\mathcal{D}}(\Delta_r) \otimes \hat{\mathcal{D}}(\Delta_s) \\
\downarrow & & \downarrow \\
\hat{\mathcal{D}}(\Delta_t) \overline{\otimes} \hat{\mathcal{D}}(\Delta_t) & \longrightarrow & \hat{\mathcal{D}}(\Delta_r) \overline{\otimes} \hat{\mathcal{D}}(\Delta_s)
\end{array}$$

The symmetric cokernel $\hat{\mathcal{D}}(\Delta_{r_1}) \overline{\otimes} \ldots \overline{\otimes} \hat{\mathcal{D}}(\Delta_{r_n})$ is analogously defined as the quotient of $\hat{\mathcal{D}}(\Delta_{r_1}) \otimes \ldots \otimes \hat{\mathcal{D}}(\Delta_{r_n})$ by the sum of all the kernels of the morphisms of $\hat{\mathcal{D}}(\Delta_{r_i}) \otimes \hat{\mathcal{D}}(\Delta_{r_j})$ into the various symmetric cokernels $\hat{\mathcal{D}}(\Delta_{r_i}) \overline{\otimes} \hat{\mathcal{D}}(\Delta_{r_j})$. With these definitions, we can give a n-simplicial meaning to the symmetric cokernel $(r_1,\ldots, r_n) \mapsto \hat{\mathcal{D}}(\Delta_{r_1}) \overline{\otimes} \ldots \overline{\otimes} \hat{\mathcal{D}}(\Delta_{r_n})$, as explained in 2.5 for the dual situation.

4.18. Let us now consider the coalgebra $\hat{\mathcal{D}}(X)$ of quantum stabilized algebraic currents on the simplicial set X. The "symmetric cokernel" $\hat{\mathcal{D}}(X)^{\overline{\otimes} n}$ may be defined as the reduced product $(S(X)^{\otimes n}) \nabla (\hat{\mathcal{D}}(\Delta_*) \overline{\otimes} \ldots \overline{\otimes} \hat{\mathcal{D}}(\Delta_*))$. The quotient morphism $\hat{\mathcal{D}}(X)^{\otimes n} \longrightarrow \hat{\mathcal{D}}(X)^{\overline{\otimes} n}$ is then a quasi-isomorphism.

4.19. With all these definitions, we can say at last what should be a "neo-coalgebra". The axioms are simply dual to the ones for neo-algebras. What is given essentially is a differential graded quotient module C_2 of C^2 and a "partial comultiplication"

$$\mu : C \longrightarrow C_2$$

with properties dual to α, β, γ and δ in 3.1. In particular, C_n is a quotient module of $C_{n_1} \otimes \ldots \otimes C_{n_p}$ for $n = n_1 + \ldots + n_p$. To any simplicial set X, we can associate the neo-coalgebra $C = \hat{\mathcal{D}}(X)$ defined above with its symmetric cokernel C_2 defined in 4.20.

4.20. *Remark.* The dual of a coalgebra (resp. a neo-coalgebra) is an algebra (resp. a neo-algebra). Therefore, the dual of $\hat{\mathcal{D}}(X)$ is a neo-algebra : it is quasi-isomorphic to the neo-algebra $\hat{\Omega}(X)$ defined in 1.18.

4.21. *Remark.* If the homotopy groups of X are finite p-groups and if C is the neo-coalgebra $\hat{\mathcal{D}}(X)$, the homology complex[19] Tot $(C_{(-r_1)\ldots(-r_n)})$ associated to the n-<u>cosimplicial</u> graded differential module $(r_1,\ldots, r_n) \mapsto C_{r_1\ldots r_n}$ has an homology which is isomorphic to the homology of the n^{th}-iterated loop space of X.

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[19] In the Tot homology complex, we should consider the <u>product</u> of elements located on the diagonals, in order to get a situation in duality with the cohomological one.