BP infinite loop algebras

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Abstract

In this paper we define the notion of BP infinite loop algebras, a sort of BP-analogue of A-R-allowable Hopf algebras, and show that under some conditions, BP-cohomology of infinite loop spaces has such a structure. Furthermore we show that the BP infinite loop algebra structure gives a serious restriction on underlying unstable BP-algebra structure.

Key words: Brown-Peterson cohomology, Infinite loop spaces, Generalized cohomology operations, Unstable Algebras, Dyer-Lashof operations, Nishida relations, Landweber-Novikov algebra, Steenrod algebra, Hopf ring
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1. Introduction

It is well-known that the mod p ordinary homology of an infinite loop space has the structure of a so called A-R-allowable Hopf algebra [13], i.e., an Hopf algebra on which both the Steenrod algebra and the Dyer-Lashof algebra act satisfying certain compatibility conditions. What about generalized (co)homology? In the case of the complex K-theory, the question was more or less solved by McClure in [14]. The rather complicated answer, however, is due to the presence of torsion elements, and the situation appears to simplify when dealing with spaces whose integral (p-complete) K-theory has no torsion elements (c. f. [5, 8]. See also [7] on the relationship between the results of [5] and [14].) In an unpublished work, N. Strickland deals with E_n-homology of QX when E_n-homology of X is free. What about connective theories? In [10] the author computed BP-cohomology of QX when X satisfies certain conditions. In this paper we use this computation to give a definition of a sort of BP- analogue of the category of A-R allowable Hopf algebras, and we will discuss some properties.

Another purpose of this paper is to exhibit that the unstable BP-cohomology operations are rather accessible, contrary to a common belief. For this purpose we rely rather heavily on known results on Steenrod operations and use the relationship between ordinary cohomology and BP-cohomology.

The paper is organised as follows. In section 2, we define the category of BP infinite loop algebras and show that the BP-cohomology of an infinite loop space becomes naturally a BP infinite loop algebra under some conditions. In section 3 we describe explicitly BP infinite loop algebra structure of BP-cohomology of some infinite loop spaces. In section 4, we show that the structure of BP infinite loop algebra imposes a serious restriction on the structure of the underlying unstable BP-algebra, and show that there is no nontrivial BP infinite loop algebra with a spherical class in dimension greater than 2 and that is finitely generated as BP^*-module. In appendix A we deal with some issues concerning the topology of BP-cohomology of spaces, and in appendix B we recall some standard facts on $BP_*(\underline{BP}_*)$ for readers who are not familiar with.

The following convention will be used throughout the paper. BP will denote the p-completed version of the Brown-Peterson spectrum unless otherwise specified, where p will be a fixed odd prime. For spaces or spectra X, $BP^*(X)$ will be equipped with the BP-skeletal topology as will be defined in appendix A unless otherwise specified. However, the use of BP-skeletal topology instead of the usual skeletal topology is not crucial. In fact it is only used in Theorem 2.11 ii), and in case of interests the two topologies agree anyway. We denote $H^*(-)$ and $H_*(-)$ the mod p ordinary cohomology and homology.

2. The definition of BP infinite loop algebras

2.1. $BP^*(BP)$-modules, unstable BP- algebras, and the destabilization

Let X be a spectrum, E a generalized cohomology theory with product. It is well-known (c. f. [1]) that $E^*(X)$ is a module over the ring of stable cohomology operations E^*E. Less well-known is the fact that, if X is a space, under certain conditions on E, $E^*(X)$ has a richer set of structures called unstable E-algebra (c. f. [3]). BP is known to verify the required condition (loc. cit.) by results in [25, 19]. In view of recent works by Andrews and Whitehouse [24], it is not clear if the definition given there is the best one. In any event, we need following properties of the category of unstable BP-algebras (which we will denote $\mathcal{K}_{BP}$) :

i Let X be a space. Then $BP^*(X)$ is naturally an unstable BP-algebra.

ii The following diagram of functors is commutative

$$
\begin{array}{ccc}
Pointed\ spaces & \xrightarrow{\ \Sigma^\infty\ } & Spectra \\
\downarrow{\scriptstyle BP^*(-)} & & \downarrow{\scriptstyle BP^*(-)} \\
\mathcal{K}_{BP} & \xrightarrow{\ I\ } & \mathcal{M}_{BP^*BP}
\end{array}
$$

where I denotes the augmentation ideal functor, and $\mathcal{M}_{BP^*BP}$ the category of BP^*BP-modules.

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iii There is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{K}_{BP}}(BP^*(\underline{BP}_n), A) \cong A^n$$

for unstable BP-algebras A.

iv In $\mathcal{K}_{BP}$ the categorical sum is represented by a completed tensor product over BP^*.

v Cokernels exist in $\mathcal{K}_{BP}$.

All of the properties follow easily from the definition of unstable BP-algebras in [3]. These properties allow us to define

Definition 2.1. *The destabilization functor $\mathcal{M}_{BP^*BP} \to \mathcal{K}_{BP}$ is the left adjoint to the augmentation ideal functor $I : \mathcal{K}_{BP} \to \mathcal{M}_{BP^*BP}$.*

(here we use the augmentation ideal and not the forgetful functor because we are using the unreduced theory for spaces).

Note that such a functor transforms the direct sum into the tensor product, is right exact, and sends $\Sigma^n BP^*(BP)$ to $BP^*(\underline{BP}_n)$. These properties characterize the destabilization functor. As we know completely algebraically $BP^*(\underline{BP}_n)$'s and the induced map among them, our functor is determined in a completely algebraic way (including the topology/filtration).

2.2. BP infinite loop algebras

In this section we define the category of BP infinite loop algebras, which can be considered as a some sort of BP counterpart of the category of A-R-allowable Hopf algebras, and show that indeed when X is a nice infinite loop space, its BP-cohomology has a structure of a BP infinite loop algebra.

First we recall from [10] the following

Theorem 2.2. *Let X be a space satisfying the following conditions.*

(H1) $BP^*(X)\hat{\otimes}_{BP^*}Z/p \subset H^*(X)$.

(H2) $BP^*(X)$ *is Landweber-flat, namely the sequence $(p, v_1, v_2, \cdots)$ is regular on $BP^*(X)$.*

Then the natural map $\mathcal{D}\tilde{BP}^*(X) \to BP^*(QX)$ is an isomorphism.

Now suppose X itself is an infinite loop space, i.e., $X = \Omega^\infty Y$ for a spectrum Y. Then by adjunction we get a map of infinite loop spaces $QX \to X$. Thus we get a map of unstable BP-algebras $BP^*(X) \to BP^*(QX)$. One can see that this map satisfies several compatibility conditions coming from the formal properties of adjunction, notably it is a section of the obvious map $BP^*(QX) \to BP^*(X)$. This motivates the following definition :

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Definition 2.3. *An BP infinite loop algebrais a coalgebra over the comonad ([12]) associated to the adjunction pair $(\mathcal{D}, I)$, that is, an unstable BP-algebra A equipped with a map of unstable BP-algebras $\xi_A : A \to \mathcal{D}I(A)$ making the following diagrams commutative*

$$
\begin{array}{ccc}
A & \xrightarrow{\;\xi_A\;} & \mathcal{D}I(A) \\
{\scriptstyle \xi_A}\downarrow & & \downarrow{\scriptstyle \mathcal{D}(\phi_{IA})} \\
\mathcal{D}I(A) & \xrightarrow{\;\mathcal{D}I(\xi_A)\;} & \mathcal{D}I\mathcal{D}I(A)
\end{array}
\qquad
\begin{array}{ccc}
A & & \\
{\scriptstyle \xi_A}\downarrow & \searrow{\scriptstyle id} & \\
\mathcal{D}I(A) & \xrightarrow{\;\varphi_A\;} & A
\end{array}
$$

where $\phi : id \to I\mathcal{D}$ and $\varphi : \mathcal{D}I \to id$ are the adjunction maps. Often by abuse of language we say simply that A is a BP infinite loop algebra.

Thus the above discussion leads to :

Theorem 2.4. *Let X be an infinite loop space satisfying the properties (H1) and (H2). Then $BP^*(X)$ is a BP infinite loop algebra.*

Before we study the structure of BP infinite loop algebras, we need to know more on the structure of unstable BP-algebras obtained by the destabilization. The first results are more or less formal :

Proposition 2.5. *Let M be a $BP^*(BP)$-module. Then*

  (i) *$\mathcal{D}(M)$ is naturally a completed Hopf algebra.*
  (ii) *There is a composition copairing*

$$\mathcal{D}(M) \to \mathcal{D}(M)\hat{\otimes}_{BP^*}\mathcal{D}(BP^*(S^0))$$

   which agrees with the map induced by the composition pairing map

Proof. The diagonal map $M \to M \oplus M$ induces a map $\mathcal{D}(M) \to \mathcal{D}(M)\hat{\otimes}_{BP^*}\mathcal{D}(M)$ which makes $\mathcal{D}(M)$ a completed coalgebra. One easily sees that the coproduct is compatible with product. This proves i). The assertion ii) is a special case of the following :

Lemma 2.6. *Let L, M, and N be $BP^*(BP)$-modules with a map $L \to M\hat{\otimes}_{BP^*}N$. Then we have a natural map $\mathcal{D}(L) \to \mathcal{D}(M)\hat{\otimes}_{BP^*}\mathcal{D}(N)$.*

Proof. By adjunction we have natural maps $M \to I\mathcal{D}(M)$ and $N \to I\mathcal{D}(N)$, which give rise to a map $M\hat{\otimes}_{BP^*}N \to I\mathcal{D}(M)\hat{\otimes}_{BP^*}I\mathcal{D}(N) \cong I(\mathcal{D}(M)\hat{\otimes}_{BP^*}\mathcal{D}(N))$. By composing with the map $L \to M\hat{\otimes}_{BP^*}N$ we get a map $L \to I(\mathcal{D}(M)\hat{\otimes}_{BP^*}\mathcal{D}(N))$. By adjunction we get the desired map. $\qquad\square$

We can also relate them to ordinary homology, we start with :

Proposition 2.7. *Let A be an object in $\mathcal{K}_{BP}$. Then $A\hat{\otimes}_{BP^*}Z/p$ has a natural structure of unstable algebra over $\mathcal{A}_p$, the mod p Steenrod algebra with trivial action of the Bockstein.*

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Proof. Let P^i be the Steenrod power operation. It is well-known that there is a stable BP-operation Θ^i which covers P^i ([29]). Let M be an object in $\mathcal{M}_{BP^*BP}$. Since the ideal $I_{(\infty)} = (p, v_1, \cdots, v_n)$ is invariant under the action of BP^*BP ([11]) one sees that one can let the subalgebra of $\mathcal{A}_p$ generated by P^i's on $M\hat{\otimes}_{BP^*}Z/p$ via the formula $P^i(\overline{x}) = \overline{\Theta^i(x)}$. Now let A be an unstable BP-algebra. We have already seen that $\mathcal{A}_p$ acts on $A\hat{\otimes}_{BP^*}Z/p$ (with trivial action of Bockstein). Let x be an element of degree k in $A\hat{\otimes}_{BP^*}Z/p$. We will show that if $k < 2i$ then $P^i(x) = 0$. Consider $\Theta^i(\iota_k)$ where $\iota_k \in BP^k(\underline{BP_k})$ is the class corresponding to the identity map. As $P^i(\overline{\iota_k}) = 0$ in $HZ/p^*(\underline{BP_k})\overline{\otimes}BP^*(\underline{BP_k})\hat{\otimes}_{BP^*}Z/p$, and since $BP^*(\underline{BP_k})$ is generated by ι_k as an unstable BP-algebra, we see that there is a series of element $\Theta^i_j \in BP^*(\underline{BP_k})$ such that

$$\Theta^i(\iota_k) = p\Theta^i_0(\iota_k) + \cdots + v_n\Theta^i_n(\iota_k)\cdots .$$

By the universality of ι_k we get

$$\Theta^i(x) = p\Theta^i_0(x) + \cdots + v_n\Theta^i_n(x)\cdots$$

for any x in degree k part of an unstable BP-algebra. Therefore we have

$$P^i(\overline{x}) = \overline{\Theta^i(x)} = 0.$$

Other conditions can be verified in a similar way. $\square$

As a corollary, we can recover the following result originally due to Quillen ([18]).

Corollary 2.8. *Let A be an unstable BP-algebra. Then $A\hat{\otimes}_{BP^*}Z/p$ is trivial in negative degrees.*

Proof. It suffices to note that $id = P^0 = 0$ in negative degrees for an unstable $\mathcal{A}_p$-algebra. $\square$

Now we can prove

Proposition 2.9. *Let M be a $BP^*(BP)$-module. Then $\mathcal{D}(M)_{\#} \cong \mathrm{Hom}_{Z/p}(\mathcal{D}(M)\hat{\otimes}_{BP^*}Z/p, Z/p)$ has a natural structure of A-R-allowable Hopf algebra.*

Proof. All of the structure has been already shown to exist except the action of the Dyer-Lashof algebra ([13]). To see this, let $M \leftarrow P_0 \leftarrow P_1$ be a $BP^*(BP)$-module presentation of M. Thus we have a coexact sequence of algebras (which also is an exact sequence of completed Hopf algebras in view of results above) $BP^* \leftarrow \mathcal{D}(M) \leftarrow \mathcal{D}(P_0) \leftarrow \mathcal{D}(P_1)$. As the tensor product is right exact we see that $\mathcal{D}(M)\hat{\otimes}_{BP^*}Z/p$ is the cokernel (as algebras/Hopf algebras) of the map $\mathcal{D}(P_0)\hat{\otimes}_{BP^*}Z/p \leftarrow \mathcal{D}(P_1)\hat{\otimes}_{BP^*}Z/p$. By dualizing we see that $\mathcal{D}(M)_{\#}$ is the Hopf kernel of the map $\mathcal{D}(P_0)_{\#} \to \mathcal{D}(P_1)_{\#}$. Since $\mathcal{D}(P_i)_{\#}$'s are just mod p homology of the infinite loop spaces associated to a free BP-module spectra, and the maps between them are induced by a spectra map, we see that the kernel has the strucutre of an *A-R*-allowable Hopf algebra. $\square$

Remark 2.10. *Note that one can identify $N_{\#}$ with the set of continuous BP^*-linear maps from N to Z/p.*

With these preparations we are now ready to study properties of BP infinite loop algebras. As a matter of fact we get all the properties of $\mathcal{D}(M)$'s that we have seen, namely

Theorem 2.11. *Let A be an BP infinite loop algebra.*

(i) *A is a completed Hopf algebra, and the structure map ξ_A is a map of completed Hopf algebras.*

(ii) *The category of BP infinite loop algebras is abelian.*

(iii) *There is a natural "composition copairing"*

$$A \to A \hat{\otimes}_{BP^*} \mathcal{D}(BP^*(S^0))$$

and ξ_A commutes with the copairing.

(iv) *$A \hat{\otimes}_{BP^*} Z/p$ is an A-R-allowable Hopf algebra.*

Proof. Basic idea is to use the fact that these properties hold for $\mathcal{D} \circ I(A)$, and the fact that A embeds naturally into $\mathcal{D} \circ I(A)$. One can use the composition

$$A \to \mathcal{D} \circ I(A) \to \mathcal{D} \circ I(A) \hat{\otimes}_{BP^*} \mathcal{D} \circ I(A) \to A \hat{\otimes}_{BP^*} A$$

to define the coproduct. Unfortunately with such a definition of the coproduct, it is not obvious that the structure map commutes with the coproduct. To see this, first consider the case when $A = \mathcal{D}(M)$. We show that the coproduct defined earlier agrees with the new one, which proves the compatibility of the coproduct with the structure map in the general case by naturality. To prove that the two coproducts agree, we need to show the commutativity of the following square.

$$
\begin{array}{ccccccc}
\mathcal{D}(M) & \longrightarrow & \mathcal{D}I\mathcal{D}(M) & \longrightarrow & \mathcal{D}(I\mathcal{D}(M) \oplus I\mathcal{D}(M)) & \to & \mathcal{D}I\mathcal{D}(M) \hat{\otimes}_{BP^*} \mathcal{D}I\mathcal{D}(M) \\
\downarrow & & & & & & \downarrow \\
\mathcal{D}(M \oplus M) & & & \longrightarrow & & & \mathcal{D}(M) \hat{\otimes}_{BP^*} \mathcal{D}(M)
\end{array}
$$

The top left triangle commutes by naturality and the bottom pentagon commutes by generalities of adjunctions. The part ii) follows immediately in view of the appendix A. Note that without using the properties of the BP-skeletal topology, we still get all of the properties of the abelian category except the equality between the image and the coimage. The part iii) can be proved by a method similar to the part i). To prove the part iv), notice that the functor $\hat{\otimes}_{BP^*} Z/p$ transforms our composition copairing into the composition pairing $H_*(QS^0) \otimes A \hat{\otimes}_{BP^*} Z/p \to A \hat{\otimes}_{BP^*} Z/p$. As one can recover the action of the Dyer-Lashof algebra from the composition pairing and the action of the Steenrod algebra, we can make the Dyer-Lashof algebra act on $A \hat{\otimes}_{BP^*} Z/p$. All the compatibility conditions are satisfied because they are satisfied on $\mathcal{D}I(A) \hat{\otimes}_{BP^*} Z/p$ which surjects to $A \hat{\otimes}_{BP^*} Z/p$. $\qquad \square$

3. Examples of BP infinite loop algebras

3.1. The adjunction maps

In this section we will exhibit explicitly the structure of BP infinite loop algebra for some known BP-cohomology of infinite loop spaces. We start with the simplest case when the structure maps can be obtained from the adjunction maps, especially for $BP^*(\underline{BP}_n) \cong \mathcal{D}(\Sigma^n BP^*(BP))$ or $BP^*(QX) \cong \mathcal{D}I(BP^*(X))$ with $BP^*(X)$ satisfying the conditions (H1) and (H2) of Theorem 2.2. So we explain how to describe algebraically the adjunction map. Let's first consider the adjunction $id_{\mathcal{M}_{BP^*BP}} \to I \circ \mathcal{D}$. By the construction of $\mathcal{D}$, the general cases can be reduced to the case when M is free and monogenic, i.e., M is of the form $M = \Sigma^n BP^*(BP)$, in which case the adjunction map $\Sigma^n BP^*(BP) \cong BP^*(\Sigma^n BP) \to BP^*(\underline{BP}_n)$ is obtained by considering a stable operation as an unstable operation, in other words by evaluating a stable operation on the class $\iota_n \in BP^*(\underline{BP}_n)$.

Next let's consider the adjunction $\mathcal{D} \circ I \to id_{\mathcal{K}_{BP}}$. For the ease of the description we only deal with the case of BP-cohomology of a space, the general case being left to the reader.

Consider a free BP^*BP-resolution of $IBP^*(X)$ realized by maps $X \to P_0 \to P_1 \to \cdots$ where P_i's are wedge of suspensions of BP. Thus $BP^*(X)$ is a quotient of $BP^*(P_0)$ and $\mathcal{D}I(BP^*(X))$ that of $\mathcal{D}(BP^*(P_0)) \cong BP^*(\Omega^\infty P_0)$. As we are supposed to know $BP^*(X)$ as an object of $\mathcal{K}_{BP}$, we know how BP-unstable operations act on it, and thus we can extend the map $BP^*(P_0) \to IBP^*(X)$ to $\mathcal{D}(BP^*(P_0)) \to BP^*(X)$. One verifies easily that by passing to the quotient, we get the desired map $\mathcal{D}I(BP^*(X)) \to BP^*(X)$.

3.2. Wilson spaces

Let's examine now the case of so-called Wilson spaces, namely the spaces of the form $\underline{BP\langle i\rangle}_j$ with $j \leq 2(1 + p + \cdots p^i)$, which splits off $\underline{BP}_j$ ([26, 4, 3]), that is we have a map of spaces $\theta : .\underline{BP\langle i\rangle}_j \to \underline{BP}_j$ such that the composition with the usual map $\underline{\rho_{i}}_j : \underline{BP}_j \to \underline{BP\langle i\rangle}_j$ is homotopic to the identity. We shall prove

Proposition 3.1. *Let $j \leq 2(1 + p + \cdots p^i)$. Then the structure map*

$$\xi_{BP^*(\underline{BP\langle i\rangle}_j)} : BP^*(\underline{BP\langle i\rangle}_j) \to \mathcal{D}I(BP^*(\underline{BP\langle i\rangle}_j))$$

is given by the composition

$$BP^*(\underline{BP\langle i\rangle}_j) \overset{BP^*(\theta)}{\to} BP^*(\underline{BP}_j) \overset{\xi_{BP^*(\underline{BP}_j)}}{\to} \mathcal{D}I(BP^*(\underline{BP}_j)) \overset{\mathcal{D}I(BP^*(\rho_i_j))}{\to} \mathcal{D}I(BP^*(\underline{BP\langle i\rangle}_j)).$$

Proof. It suffices to take the adjoint of the commutative diagram

$$BP^*(\underline{BP\langle i\rangle}_j) \overset{id}{\longleftarrow} BP^*(\underline{BP\langle i\rangle}_j)$$

$$BP^*(\underline{BP}_j)$$

Note that each of theses arrows can be described completely algebraically (c. f. [3, 4] for $BP^*(\theta)$), which means that we have a completely algebraic description of $\xi_{BP^*(\underline{BP\langle i\rangle}_j)}$.

3.3. Eilenberg-Maclane spaces

The Brown-Peterson cohomology of the Eilenberg-Maclane spaces were calculated in [20]. For concreteness' sake, we will deal with the case of $K(Z/p, n)$'s, but the other cases are similar. According to [20], there are maps of infinite loop spaces

$$K(Z/p, n) \to K(Z, n+1) \to \underline{BP\langle 1 \rangle}_{n+2p} \to \cdots \to \underline{BP\langle n-1 \rangle}_{2(1+p+\cdots+p^{n-1})}$$

which induces the surjection $BP^*(\underline{BP\langle n-1 \rangle}_{2(1+p+\cdots+p^{n-1})}) \to BP^*(K(Z/p, n))$. As this is a map of BP infinite loop algebra, we deduce

Proposition 3.2. $\xi_{BP^*(K(Z/p,n)}$ *is determined by the commutative diagram*

$$
\begin{array}{ccc}
BP^*(\underline{BP\langle n-1 \rangle}_{2(1+p+\cdots+p^{n-1})}) & \longrightarrow & BP^*(K(Z/p, n)) \\
\downarrow {\scriptstyle \xi_{BP^*(\underline{BP\langle n-1 \rangle}_{2(1+p+\cdots+p^{n-1})})}} & & \downarrow {\scriptstyle \xi_{BP^*(K(Z/p,n))}} \\
\mathcal{DI}(BP^*(\underline{BP\langle n-1 \rangle}_{2(1+p+\cdots+p^{n-1})})) & \longrightarrow & \mathcal{DI}(BP^*(K(Z/p, n)))
\end{array}
$$

4. Finitely generated BP infinite loop algebras

In this section we will see how the structure of BP infinite loop algebra gives restrictions on the structure of the underlying unstable BP-algebra structure. Let's first see what happens if X is an infinite loop space with torsion-free cohomology. (It happens that by a work of Slack [21] we know all of them, but let's forget if for the time being.) Suppose also X is $n-1$ connected, but not n-connected. Thus we have the following commutative diagram.

$$
\begin{array}{ccc}
S^n & \longrightarrow & QS^n \\
\downarrow & & \downarrow \\
X \longrightarrow & QX \longrightarrow X & \longrightarrow K(Z/p, n) \\
& \downarrow & \nearrow \\
& \underline{BP}_n &
\end{array}
$$

Here the composition of maps all the way from S^n to $K(Z/p, n)$ gives a generator of Z/p. Such a space X satisfies clearly the conditions (H1) and (H2), thus by taking BP-cohomology

we get the following diagram

$$BP^*(S^n) \longleftarrow \mathcal{D}I(BP^*(S^n))$$

$$\Big\uparrow \qquad\qquad\qquad \Big\uparrow$$

$$BP^*(X) \longleftarrow \mathcal{D}I(BP^*(X)) \xleftarrow[\xi_{BP^*(X)}]{} BP^*(X)$$

where the composition from $BP^*(X)$ to $BP^*(S^n)$ is surjective. This motivates the following definition.

Definition 4.1. *Let A be an unstable BP-algebra. We say that A has a spherical class in degree n if There is a surjective map of unstable BP-algebras from A to $BP^*(S^n)$.*

Remark 4.2. *By the above, we see that if X is an infinite loop space with torsion-free cohomology then $BP^*(X)$ has a spherical bottom in degree n.*

Now we are ready to state our main result.

Theorem 4.3. *Let X be an unstable BP-algebra with a spherical class in dimension n. If $n \geq 3$ then $A \hat{\otimes}_{BP^*} Z/p$ is not bounded above. In particular A is not finitely generated as BP^*-module.*

Proof. The key point is the following

Proposition 4.4. *Let A be as above. Then $\exists a \in Hom_{Z/p}(A\hat{\otimes}_{BP^*}Z/p, Z/p)_n$ such that*

$$\begin{cases} Q_1(a) \neq 0 & \text{if } n \text{ is odd} \\ Q_2(a) \neq 0 & \text{if } n \text{ is even} \end{cases}$$

Granted Proposition, we can conclude the proof of Theorem 4.3 using the following

Lemma 4.5. *Let M be an A-R-allowable module ([13], Definition 2.8), and $x \in M$ be an element such that $P^{i'}(x) = 0$ for $\forall i > 0$, $\exists j > 0$ such that $Q^j(x) \neq 0$. Then we have $Q^{pj}(x) \neq 0$. We use the notation $P^{i'}$ rather than the more usual P^{i*} to keep the notation θ^* for the maps induced by θ.*

Proof. By Nishida relation ([13, 16]), we have $P^{(p-1)j'}Q^{pj}(x) = Q_j(x)$. $\qquad\square$

Now we go back to the proof of Proposition. First we need to know the action of Q_1 and Q_2 on the bottom class of $H_*(\underline{BP}_n)$. Denote ι_n the element of $H_n(\underline{BP}_n)$ which is the image of the unit map in $\pi_n(\underline{BP}_n)$ by the Hurewicz map.

Lemma 4.6. (i) *Let $n \geq 0$. In $H_*(\underline{BP}_{2n+1})$ we have*

$$Q_1(\iota_{2n+1}) = -e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}.$$

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(ii) *Let $n \geq 1$. In $H_*(\underline{BP}_{2n})$ we have*

$$Q_2(\iota_{2n}) = -[v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}.$$

Proof. Basically in [19], Theorem 6.1, the action of Q_i's on ι_n was determined, and their result was extended by Turner ([23]) for the action of Q_i's on any element of $H_*(\underline{BP}_*)$. However here we present a simpler proof in the spirit of [25]. First of all we have

$$
\begin{aligned}
-e_1 \circ (e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}) &= -[v_1] \circ b_1^{\circ p} \circ b_p^{\circ n} \text{ (as } e_1 \circ e_1 = b_1) \\
&= (b_1^{\star p}) \circ b_p^{\circ n} \text{ (by main relation, see appendix B)} \\
&= (b_1 \circ b_1^{\circ n})^{\star p} \text{ (the distributivity, see B)} \\
&= \iota_{2n+1}^{\star p} \\
&= e_1 \circ Q_1(\iota_{2n+1}) \text{ (property of Dyer-Lashof operations)}
\end{aligned}
$$

We also know that the map $e_1 \circ : H_*(\underline{BP}_{2n+1}) \to H_{*+1}(\underline{BP}_{2n+2})$ factors through $QH_*(\underline{BP}_{2n+1})$, and by [25] we have $PH_*(\underline{BP}_{2n+1}) \cong QH_*(\underline{BP}_{2n+1}) \hookrightarrow H_{*+1}(\underline{BP}_{2n+2})$. Thus we get i) since both $Q_1(\iota_{2n+1})$ and $-e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}$ map to $\iota_{2n+1}^{\star p}$. To prove ii), note that the map $e_1 \circ : QH_*(\underline{BP}_{2n+1}) \to PH_{*+1}(\underline{BP}_{2n+2})$ is also injective ([25]). Furthermore, the kernel of the map $PH_*(\underline{BP}_{2n}) \to QH_*(\underline{BP}_{2n})$ consists of the p-th powers. Since $QH_*(\underline{BP}_{2n})$ is trivial if $* \leq 2n + 2(p-1)$ unless $* = 0$ or $2n$, we see that in degree $|Q_2(\iota_{2n})| = 2np + 4(p-1)$, the map $PH_*(\underline{BP}_{2n}) \to QH_*(\underline{BP}_{2n})$ is bijective. Thus we derive ii) from i). □

These elements have an interesting property. Let θ be an element of $BP^{2n+\epsilon}(\underline{BP}_{2n+\epsilon})$, $\epsilon = 1, 2$. If θ comes from a stable map, then we clearly have

$$\theta_*(Q_\epsilon \iota_{2n+\epsilon}) = Q_\epsilon(\theta_* \iota_{2n+\epsilon}) = a Q_\epsilon \iota_{2n+\epsilon}$$

for some $a \in Z/p$. It turns out that this still is the case with unstable maps θ, namely

Proposition 4.7. *Let $\theta \in BP^{2n+\epsilon}(\underline{BP}_{2n+\epsilon})$, $\epsilon = 1, 2$, $2n+\epsilon \geq 1$. Then in $H_{p(2n+\epsilon)-\epsilon}(\underline{BP}_{2n+\epsilon})$ we have*

$$\exists a \in Z/p, \text{ such that } \theta_*(Q_\epsilon \iota_{2n+\epsilon}) = a Q_\epsilon \iota_{2n+\epsilon} \text{ modulo decomposables.}$$

Proof. We only deal with the case $\epsilon = 1$, the other case being similar. Denote by s^1 and σ_1 the fundamental classes in $BP^1(S^1)$ and $BP_1(S^1)$ respectively. We then have

$$
\begin{aligned}
\theta_*(-Q_1 \iota_{2n+1}) &= \theta_*(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}) \\
&= \theta_*((v_1 s^1 x_1 \cdots x_{p-1} x_p \cdots x_{p+n-1})_*(\sigma_1 \otimes \overbrace{\beta_1 \otimes \cdots \otimes \beta_1}^{p-1 \text{ factors}} \otimes \overbrace{\beta_p \otimes \cdots \otimes \beta_p}^{n \text{ factors}})) \\
&= (\theta(v_1 s^1 x_1 \cdots x_{p-1} x_p \cdots x_{p+n-1}))_*(\sigma_1 \otimes \overbrace{\beta_1 \otimes \cdots \otimes \beta_1}^{p-1 \text{ factors}} \otimes \overbrace{\beta_p \otimes \cdots \otimes \beta_p}^{n \text{ factors}})
\end{aligned}
$$

So we can conclude that $\theta_*(Q_1 \iota_{2n+1})$ lies in the coalgebraic subring of $H_*(\underline{BP}_*)$ generated by e_1, b_1 and b_p over $Z/p[BP^*]$. Denote B this coalgebraic subring. Let $y \in PH_{p(2n+1)-1}(\underline{BP}_*) \cap B$. Then for degree reasons y is a linear combination of elements of the form

$$[\alpha] \circ e_1 \circ b_1^{\circ p-1+jp} \circ b_p^{\circ n-j} \text{ with } [\alpha] \in BP^{2(1+j)(p-1)}.$$

Consider $e_1 \circ y = [\alpha] \circ b_1^{\circ p + jp} \circ b_p^{\circ n - j}$. Suppose v_i divides α. Then $1 + j \geq 1 + p + \cdots p^{i-1}$, so by Lemma B.4, $e_1 \circ y$ is decomposable. Since α has to be divided by some v_i, we see that $e_1 \circ y$ is decomposable. As in the proof of Lemma 4.6, we see that y is a multiple of $Q_1 \iota_{2n+1}$. Combining everything together, we get the desired result. $\qquad\square$

We also need to know some particularity of unstable BP-operations that are "dual" to these Dyer-Lashof operations. Denote ρ the obvious map $BP \to HZ/p$. Note that ρ can also be considered as an element of $H^n(\underline{BP}_n)$ that is dual to ι_n. Then we have

Proposition 4.8. (i) *Let $n \geq 2$, $\theta_1 \in BP^{2n}(\underline{BP}_{2n})$, $\theta_2 \in BP^{(2n+2)p-2}(\underline{BP}_{2n})$ such that*

$$
\begin{aligned}
< \rho(\theta_2), \quad Q_2(\iota_{2n}) > &= 1 \\
< \rho(\theta_1), \qquad \iota_{2n} > &= 1.
\end{aligned}
$$

Then we have $< \rho(\theta_2 \circ \theta_1), Q_2(\iota_{2n}) >= 1$.
(ii) *Let $n \geq 1$, $\theta_1 \in BP^{2n+1}(\underline{BP}_{2n+1})$, $\theta_2 \in BP^{(2n+2)p-1}(\underline{BP}_{2n+1})$ such that*

$$
\begin{aligned}
< \rho(\theta_2), \quad Q_1(\iota_{2n+1}) > &= 1 \\
< \rho(\theta_1), \qquad \iota_{2n+1} > &= 1.
\end{aligned}
$$

Then we have $< \rho(\theta_2 \circ \theta_1), Q_1(\iota_{2n+1}) >= 1$.

Remark 4.9. (i) *An analogous fact holds for any stable operations, since $BP^*(BP)$ is generated as a topological algebra by operations that cover Steenrod reduced powers.*
(ii) *Let $\theta_1 \in BP^1(\underline{BP}_1)$ be the splitting map $\underline{BP}_1 \to S^1 \to \underline{BP}_1$, $\theta_2 \in BP^{2p-1}(\underline{BP}_1)$ be any element with $< \rho(\theta_2), \iota_1 >= 1$. Then we have $< \rho(\theta_1), \iota_1 >= 1$, but $\theta_2 \circ \theta_1 = 0$ as $BP^{2p-1}(S^1) = 0$, so an analogous statement doesn't hold for $\underline{BP}_1$.*

Proof. We have

$$
< \rho(\theta), \iota_k >=< \theta^*(\rho), \iota_k >=< \rho, \theta_*(\iota_k) >
$$

which implies that

$$
< \rho(\theta), \iota_k >= 1 \iff \theta_*(\iota_k) = \iota_k.
$$

Similarly we have

$$
< \rho(\theta), Q_\epsilon(\iota_{2n+\epsilon}) >= 1 \iff \theta_*(Q_\epsilon(\iota_{2n+\epsilon})) = (\iota_{(2n+2)p-\epsilon})
$$

so it suffices to show that $\theta_*(\iota_{2n+\epsilon}) = \iota_{2n+\epsilon}$ implies $\theta_*(Q_\epsilon(\iota_{2n+\epsilon})) = Q_\epsilon(\iota_{2n+\epsilon})$ for $\epsilon = 1, 2$. To prove this, we consider the induced map in BP-homology. As it is a map of $BP^*(BP)$-modules, we can use the action of $BP^*(BP)$. Recall $HZ/p_*(HZ/p) \cong \Lambda(q_1, q_2, \cdots) \otimes P[\xi_1, \xi_2, \cdots]$, and denote $\mathcal{P}_{n\Delta_1}$ the element of dual polynomial basis (the Milnor basis [15]) dual to ξ_1^n. Let $r_{n\Delta_1} \in BP^*(BP)$ be as in [29]. We know that its coproduct is given by $\Delta(r_{n\Delta_1}) = \Sigma_k r_{k\Delta_1} \otimes r_{(n-k)\Delta_1}$ and that the following diagram commutes

$$
\begin{array}{ccc}
BP & \xrightarrow{r_{n\Delta_1}} & BP \\
\downarrow{\rho} & & \downarrow{\rho} \\
HZ/p & \xrightarrow{\mathcal{P}_{n\Delta_1}} & HZ/p
\end{array}
$$

(According to [29], Proof of Lemma 3.7, we get $\chi(\mathcal{P}_{n\Delta_1})$ instead of $\mathcal{P}_{n\Delta_1}$, but as $\chi(\xi_1) = \xi_1$, χ being an algebra map we have $\chi(\xi_1^n) = \xi_1^n$ so $\chi(\mathcal{P}_{n\Delta_1}) = \mathcal{P}_{n\Delta_1}$.) The operation $\mathcal{P}_{n\Delta_1}$ is known to coincide with the Steenrod reduced power P^n, so for spaces X, we have $\mathcal{P}'_{n\Delta_1}(x) = 0$ if $x \in H_*(X)$ with $* \le pn$ and $\mathcal{P}'_{n\Delta_1}(x) = V(x)$ where V is the Verschiebung (and $'$ denotes the action in homology). Now we have

Lemma 4.10.   (i) *Let $n \ge 1$. In $BP_*(\underline{BP}_{2n+1})/p$ we have* $r'_{n\Delta_1}(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}) = v_1 \iota_{2n+1}$.

(ii) *Let $n \ge 2$. In $BP_*(\underline{BP}_{2n})/p$ we have* $r'_{n\Delta_1}([v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}) = v_1 \iota_{2n+1}$.

Proof. We see easily that $r'_{n\Delta_1}(e_1 \circ [v_1] \circ b_1^{\circ p-1}) = 0$ so by the Cartan formula we get

$$r'_{n\Delta_1}(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n}) = e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ r'_{n\Delta_1}(b_p^{\circ n}).$$

We also see that $r'_{1\Delta_1}(b_p) = b_1$ modulo $[p]$, p, and $\star$-decomposables. Since $[p] \circ e_1 = pe_1$, and $e_1 \circ (-)$ kills the decomposables, we get the equality i) using the lemma B.5. The proof of ii) is similar and omitted. $\qquad\qquad\square$

Now we can finish the proof of Proposition 4.8. Note that by the naturality of all structures of coalgebraic rings, all named elements in $H_*(\underline{BP}_*)$ is the image of the elements with same name in $BP_*(\underline{BP}_*)$. Let θ be an element of $BP^{2n+1}(\underline{BP}_{2n+1})$ such that $H_*(\theta)(\iota_{2n+1}) = \iota_{2n+1}$ (from now on, we use notations $H_*(\theta)$ and $BP_*(\theta)$ instead of θ_* to avoid confusion). Then we have also $BP_*(\theta)(\iota_{2n+1}) = \iota_{2n+1}$ modulo p. Note that as in the proof of Proposition 4.7 we see that $BP_*(\theta)(\iota_{2n+1})$ lies in the coalgebraic subring generated by e_1, b_1 and b_p over $BP_*[BP^*]$. With a little more careful analysis, we see also that modulo decomposables it has to lie in the image of the circle multiplication with $e_1 \circ [v_1] \circ b_1^{p-1}$. Denote by C the sets of BP_*-linear combinations of the elements of the form $e_1 \circ [v_1] \circ b_1^{p-1} \circ [\alpha] \circ b_1^{\circ j_0} \circ b_p^{\circ j_1}$ in $QBP_{(2n+1)p-1}(\underline{BP}_{2n+1})$. For degree reasons $|\alpha| \ne 0|$ unless $j_0 = p-1$ and $j_1 = n$. The proof of Proposition 4.7 shows us that the elements of $[I^2] \circ BP_*(\underline{BP}_*) \cap C$ is in the kernel of ρ. Thus we have

$$\begin{aligned}
Ker(\rho_{|C}) &= C \cap ([I^2] \circ BP_*(\underline{BP}_*) + p\langle e_1 \circ [v_1] \circ b_1^{p-1} \circ b_p^{\circ n}\rangle) \\
&\subset C \cap ([I^2] \circ BP_*(\underline{BP}_*) + I[I] \circ BP_*(\underline{BP}_*) + I^2 BP_*(\underline{BP}_*))
\end{aligned}$$

where $\langle a \rangle$ denotes the submodule generated by a. However, $[I^2] \circ BP_*(\underline{BP}_*) + I[I] \circ BP_*(\underline{BP}_*) + I^2 BP_*(\underline{BP}_*)$ is invariant under $r'_{n\Delta_1}$, so from

$$\begin{aligned}
r'_{n\Delta_1}(BP_*(\theta)(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n})) &= BP_*(\theta)(r'_{n\Delta_1}(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n})) \\
&= BP_*(\theta)(v_1 \iota_{2n+1}) \\
&= v_1 \iota_{2n+1}
\end{aligned}$$

we deduce that

$$r'_{n\Delta_1}(BP_*(\theta)(e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n})) = e_1 \circ [v_1] \circ b_1^{\circ p-1} \circ b_p^{\circ n} \text{ modulo } Ker\rho.$$

Thus we have the desired result. $\qquad\qquad\square$

Remark 4.11. *The case of $BP^2(\underline{BP}_2)$ deserves a little discussion. Lemma 4.10 ii) almost holds, with an extra term $\iota_2^{\star p}$ on the right hand side. As θ doesn't have to be additive, $BP_*(\theta)$ doesn't commute with the $\star$ product, so $BP_*(\theta)(\iota_2^{\star p})$ is not necessarily a decomposable element, and as a matter of fact it can be equal to $v_1\iota_2$.*

The proof of Proposition 4.4 can be concluded as follows. We treat the case when n is odd, the case n is even is left to the reader. Take an element $\alpha \in A$ that maps to a generator $BP^n(S^n)$. Consider the element $\mathcal{D}(u)(\xi_A(\alpha)) \in \mathcal{D}(BP^*(S^n)) \cong BP^*(QS^n)$. By the hypothesis on α it maps to a generator in $BP^n(S^n)$, so it also maps to a generator in $H^n(QS^n)$. Therefore, $\exists \theta_1 \in BP^n(\underline{BP}_n)$ such that

(i) $\mathcal{D}(u)(\xi_A(\alpha)) = \tau^*(\theta_1) = \theta_1(\tau)$ where $\tau : QS^n \to \underline{BP}_n$ is the unit map.
(ii) $\rho(\theta_1) \in H^n(\underline{BP}_n)$ is a generator.

We can suppose that $< \rho(\theta_1), \iota_n >= 1$, multiplying α with a unit if necessary. Now choose an element $\theta_2 \in BP^*(\underline{BP}_n)$ as in Proposition 4.8. Given an unstable BP-algebra B, denote by $<,>$ the pairing $B \times Hom_{Z/p}(B\hat{\otimes}_{BP^*}Z/p; Z/p) \to Z/p$. Then we have

$$
\begin{aligned}
< \theta_2\theta_1(\tau), Q_1(\sigma_n) > \; &= \; < \tau^*(\theta_2\theta_1), Q_1(\sigma_n) > \\
&= \; < (\theta_2\theta_1), \tau_* Q_1(\sigma_n) > \\
&= \; < (\theta_2\theta_1), Q_1(\iota_n) > \quad \text{as } \tau \text{ is a map of infinite loop space} \\
&= \; 1 \text{ by Proposition 4.8}
\end{aligned}
$$

But we also have

$$
\begin{aligned}
< \theta_2\theta_1(\tau), Q_1(\sigma_n) > \; &= \; < \theta_2(\mathcal{D}(u)(\xi_A(\alpha))), Q_1(\sigma_n) > \\
&= \; < \theta_2(\alpha), \xi_{A\#}\mathcal{D}(u)_\#(Q_1(\sigma_n)) > \\
&= \; < \theta_2(\alpha), Q_1(\xi_{A\#}\mathcal{D}(u)_\#(\sigma_n)) >
\end{aligned}
$$

as ξ and $\mathcal{D}(u)$ are maps of BP infinite loop algebras. This concludes the proof of Proposition 4.4. □

Remark 4.12. *Here we see some of the difficulties of dealing with the generalized cohomology theories. First of all unlike the case of BP-homology (which is connective), the existence of spherical class is not guaranteed at all in BP-cohomology of a space. Secondly, it is tempting to consider our arguments above as some sort of refinement of Nishida relations, and to state an equality like $r'_{n\Delta_1}Q_1 = -v_1 id$. Unfortunately it is not clear at all on what kind of algebraic structure this equality makes sense.*

To conclude, we show that one can use Proposition 4.4 to get some restrictions on the homotopy type of infinite loop spaces. For example,

Corollary 4.13. *Let X be a $(n-1)$-connected space with $H_*(X; Z_{(p)})$ is free over $Z_{(p)}$, $H_n(X; Z_{(p)}) \neq 0$ and $H_{p(n+\epsilon)-\epsilon}(X; Z_{(p)}) \cong 0$ where $\epsilon = 1$ if n is odd and 2 if n is even. Let G_* be a torsion abelian graded group of finite type such that $G_m \cong 0$ unless $2(1+p+\cdots p^m) \geq (p-1)(n+\epsilon)$. Then the product space $X \times K(G_*, *)$ doesn't have the homotopy type of an infinite loop space.*

Proof. By [20] $K(G_*, *)$ satisfies the conditions (H1) and (H2), and we have $BP^*(X \times K(G_*, *)) \cong BP^*(X) \hat{\otimes}_{BP^*} BP^*(K(G_*, *))$, thus $X \times K(G_*, *)$ also satisfies the conditions (H1) and (H2). On the other hand according to [20], $BP^*(K(G_*, *)) \hat{\otimes}_{BP^*} Z/p \cong 0$ in degrees less than $2(1 + p + \cdots p^{m-1})$. Thus $BP^*(X \times K(G_*, *)) \hat{\otimes}_{BP^*} Z/p$ is trivial in the degree $p(n + \epsilon) - \epsilon$. Thus by Proposition 4.4 it can't have a structure of a BP infinite loop algebra. $\square$

A. Topologies on $BP^*(X)$

One of the technical difficulties concerning the BP-cohomology is the issue of its topology. That is, quite often while dealing with the BP-cohomology of an infinite dimensional complex, one would like to consider infinite sums, which means that we need a topology. The traditional solution is tu use the "classical" skeletal topology, that is the topology associated to the filtration given by $F^s(BP^*(X)) \cong Ker(BP^*(X) \to BP^*(sk_{s-1}X))$. It turns out that this topology is nice enough so that it has become the default topology to work with. Unfortunately it also has several draw-backs, notably the lack of the rigidity. That is, for example, if $f : BP^*(X) \to BP^*(Y)$ is a continuous homomorphism of BP^*-modules, then it is not clear whether the topology on $Im(f)$ induced by that of the topology of $BP^*(Y)$ agrees with the quotient topology. In [10] one approach to settle this was attempted, unfortunately it requires the ordinary cohomology as a part of initial data, and we certainly don't want to use such an approach to deal with general problems involving unstable BP-algebras, even though in practice we are only interested in BP-cohomology of spaces or spectra whose ordinary cohomology is known.

Tamanoi, on the other hand, used another natural topology called BP-topology in [22], and showed that it has some nice properties. Unfortunately his topology is too fine for our purpose. For example, a sum of the form $\Sigma_i(v_1)^i x^{(p-1)i}$ with $x \in BP^2(CP^\infty)$ doesn't converge in this topology. Consequently $BP^*(CP^\infty) \hat{\otimes}_{BP^*} Z/p$ where the completed tensor product is taken with respect to the BP-topology doesn't inject to $H^*(CP^\infty; Z/p)$.

There also are several other natural topology on $BP^*(X)$, arising from its algebraic structure. In [28], Yamaguchi mentions the "skeletal topology", which we will refer to as "algebraic skeletal topology" to distinguish from the classical skeletal topology. With this topology $\Sigma_i(v_1)^i x^{(p-1)i}$ converges. However, a homogeneous sum of the form $\Sigma_i v_i y_i$ doesn't converge. Now we notice that the problems of convergence we have with the BP-topology and with the algebraic skeletal topology are complementary. That is, with the BP-topology, the non-convergence comes from the high powers of the ideal $(v_1, \cdots, v_n)$ whereas with the algebraic skeletal topology the problem comes from the presence of v_n's with infinitely many n's. This motivates us to consider the intersection of the two topologies, which we will call the BP-skeletal topology. It turns out that it agrees with the classical skeletal topology in many cases of interest, and it also has a good rigidity. We will discuss the details in the rest of this appendix.

We start with some definitions.

Definition A.1. *Let X be a space or spectrum.*

(i) *The BP-topology on $BP^*(X)$ is the topology defined by the decreasing filtration*

$$BP^k(X) = F'^{-1}(X) \supset F'^0(X) \supset \cdots \supset F'^n(X) = Ker(BP^k(X) \to BP\langle n\rangle^k(X)) \supset \cdots.$$

Note that we complete BP at p, so we will do the same with $BP\langle n\rangle$.

(ii) *The algebraic skeletal topology on $BP^*(X)$ is defined by the decreasing filtration*

$$F''^n(BP^k(X)) \text{ is the submodule generated by } \cup_{i \geq n} BP^{k+i}(X).$$

(iii) *The classical skeletal topology on $BP^*(X)$ is defined by the filtration*

$$F^s(BP^*(X)) \cong Ker(BP^*(X) \to BP^*(sk_{s-1}X)).$$

(iv) *The BP-skeletal topology on $BP^*(X)$ is the intersection of the BP-topology and the algebraic skeletal topology, in other words it is the topology defined by the fundamental system of neighbourhood of 0 $\{F'^n(BP^*(X)) + F''^m(BP^*(X))\}$.*

Now, according to [22] Proposition 2.8, the BP-topology is finer than the classical skeletal topology. It is clear that the algebraic skeletal topology is finer than the classical skeletal topology. Thus the BP-skeletal topology is finer than the classical skeletal topology, too. We prove a partial inverse, namely

Theorem A.2. *Let X be a space satisfying the conditions (H1) and (H2) of Theorem 2.2. Then the BP-skeletal topology on $BP^*(X)$ agrees with the classical skeletal topology.*

Proof. We start with the simplest case, when $BP^*(X)$ is topologically free. In this case we don't need X to be a space, the proof will be valid when X is a spectrum as well. Let $\{x_i\}$ be a topological basis of $BP^*(X)$ with respect to the classical skeletal topology. Thus all elements of $BP^*(X)$ can be written uniquely as

$$x = \sum_i \alpha_i x_i, \text{ with } |\alpha_i| + |x_i| = |x|, \ x_i \in F^{|x_i|}(BP^*(X)).$$

Now fix n. Note that $BP\langle n\rangle^* \cong \hat{Z}_p[v_1, \cdots v_n]$ is a Noetherian ring, so the ideal $\{f \in BP\langle n\rangle^*, |f| \leq l\}$ is finitely generated. Call the generators $f_1, \cdots, f_m$. Now, one can rewrite the sum as

$$x = \sum_i \alpha_i' x_i + \sum_i \alpha_i'' x_i \text{ where } \alpha_i' \in BP\langle n\rangle^*, \alpha_i'' \in Ker(BP^* \to BP\langle n\rangle^*).$$

Suppose $x \in F^{|x|+l}(BP^*(X))$. Then we have $|\alpha_i'| \geq l$, so each α_i' can be rewritten as linear combination of $f_1, \cdots, f_m$. Thus the first sum is contained in $F'''^l(BP^*(X))$. Obviously the second sum is in $F'^n(BP^*(X))$, so we get $F^{d+l}(BP^d(X)) \subset F'^n(BP^d(X)) + F'''^l(BP^d(X))$. Thus the BP-skeletal topology is coarser than the classical skeletal topology as desired.

Now we will deal with the general case. According to the proof of Theorem 1.20 of [20], the minimum set of generators of $BP^*(X)$ also generates the E_∞-term of the Atiyah-Hirzebruch spectral sequence $H^*(X, BP^*) \to BP^*(X)$. Thus any element of $BP^*(X)$ can

be represented in the E_∞-term of the Atiyah-Hirzebruch spectral sequence by an element of the form $x = \sum_i \alpha_i x_i$, with $|\alpha_i| + |x_i| = |x|$, and x_i's are in $E_\infty^{*,0}$. The rest of the argument is similar.

The prototype case of the second situation is as follows. Consider the Atiyah-Hirzebruch spectral sequence $H^*(BZ/p, BP^*) \to BP^*(BZ/p)$. We have $E_2 \cong E_\infty \cong BP^*/p[[x]]$, and $BP^*(BZ/p) \cong BP^*[[x]]/([p](x))$ where $[p]x = px + v_1 x + \cdots$. In $BP^*(BZ/p)$, the element x has filtration 2, however $px = -v_1 x + \cdots$ has filtration $2p$. In the Atiya-Hirzebruch spectral sequence, $px = 0$ and it is represented by $-v_1 x + \cdots$, which, indeed, has the correct filtration. □

Remark A.3. *It is not clear if the condition (H2) is really necessary here, as we know from [20] that the Atiyah-Hirzebruch spectral sequence for $BP^*(X)$ behaves more or less reasonably.*

Of course, the interest of defining a new topology is not that it agrees with an old one, but that it has something new to offer, in our case the rigidity. Note that the algebraic skeletal topology is completely algebraic. We show that for a space X, if we take into account unstable operations, the BP-topology on $BP^*(X)$ is determined by its algebraic structure. Fix d. Let n be a positive integer such that $2(1 + \cdots p^n) > d$. Then using the H-space splitting $\underline{BP\langle n\rangle}_d \to \underline{BP}_d$ ([25, 4]), we get an operation $\theta_{d,n} : \underline{BP}_d \to \underline{BP}_d$ such that

$$x \in Ker(BP^d(X) \to BP\langle n\rangle^d(X)) \text{ if and only if } \theta_{d,n}(x) = x.$$

Thus the the BP-topology on $BP^*(X)$ is determined by the action of $\theta_{d,n}$'s and the abelian group structure. As the algebraic skeletal topology on $BP^*(X)$ is determined by the (discrete) BP^*-module structure, we see that the BP-skeletal topology on $BP^*(X)$ is determined by its underlying (discrete) algebraic structure. Furthermore our argument apply to any unstable BP-algebra. Putting them altogether, we have proven :

Theorem A.4. *An unstable BP-algebra admits a natural inherent topology $\mathcal{T}_{BP}$ such that*

(i) *$\mathcal{T}_{BP}$ is finer than the "classical skeletal topology".*
(ii) *$\mathcal{T}_{BP}$ agrees with the classical skeletal topology on $BP^*(X)$ where X is a space satisfying the conditions (H1) and (H2) in Theorem 2.2.*
(iii) *$\mathcal{T}_{BP}$ on an unstable BP-algebra A depends only on the "algebraic structures" on A. More precisely, let A and A' be unstable BP-algebras, and $f : A \to A'$ a homomorphism of BP^*-modules that commutes with all unstable operations. Then f is a homeomorphism (with respect to $\mathcal{T}_{BP}$).*

Thus, for example, we have

Corollary A.5. *The category of completed unstable BP-Hopf algebras equipped with $\mathcal{T}_{BP}$ (instead of the default topology) is abelian. Here a completed unstable BP-Hopf algebra means an unstable BP-algebra A equipped with the diagonal $A \to A \hat{\otimes}_{BP^*} A$ which is a map of unstable BP-algebras.*

Proof. It is easy to see that the standard proof of the fact that the category of Hopf algebras is abelian applies. The only issue would be the uniqueness of the topology on the image, but in view of iii) above, we see that this doesn't cause a problem. □

B. The Coalgebraic ring $BP_*(\underline{BP}_*)$

In this appendix we gather some facts on $E_*(\underline{BP}_*)$ where $E = H$ or BP. Most of the material presented here is taken from [19], [27] and [9]. Another good reference on the subject is [3]. The results of this section hold for the usual (p-local, without completion) BP-theory.

The first computation of $H_*(\underline{BP}_*)$ was done in [25]. It was shown

Theorem B.1 ([25]). *$H_*(\underline{BP}_i)$ is a polynomial algebra concentrated in even degrees for i even, and an exterior algebra generated by odd degree elements for i odd. Furthermore, $dim(QH_j(\underline{BP}_i)) = rank(BP_{j-i})$.*

This is somewhat surpassed by later works that we shall describe. However, for low degree computations, this dimension formula is quite handy. Besides we used it implicitly in the proof of Lemma 4.6.

Note that by the space $\underline{BP}_i$ represent the degree i part of the BP-cohomology. Thus the ring structure of the BP-cohomology is represented by maps

$$\begin{aligned} \mu_+ : & \quad \underline{BP}_i \times \underline{BP}_i & \to & \quad \underline{BP}_i \\ \mu_x : & \quad \underline{BP}_i \times \underline{BP}_j & \to & \quad \underline{BP}_{i+j} \end{aligned}$$

which induce in homology the following maps.

$$\begin{aligned} \star = H_*(\mu_+) : & \quad H_*(\underline{BP}_i) \otimes H_*(\underline{BP}_i) & \to & \quad H_*(\underline{BP}_i) \\ \circ = H_*(\mu_x) : & \quad H_*(\underline{BP}_i) \otimes H_*(\underline{BP}_j) & \to & \quad H_*(\underline{BP}_{i+j}) \end{aligned}$$

Since $\underline{BP}_i$'s are spaces, they have the diagonal : $\underline{BP}_i \to \underline{BP}_i \times \underline{BP}_i$ which makes $H_*(\underline{BP}_i)$ a coalgebra. Clearly the two products $\star$ and $\circ$ are maps of coalgebras, and they are related to each other via the "distributivity law". More precisely, there is a relation of the form

$$a \circ (b \star c) = \Sigma(a' \circ b) \star (a'' \circ c) \text{ where } \Delta(a) = \Sigma a' \otimes a''$$

"up to sign". That is according to the bidegrees of elements concerned, there are the multiplication by -1 and/or the conjugation that appear. However we only need to use the distributivity law for elements in $H_{2*}(\underline{BP}_{2i})$'s so the reader can forget about the signs. All these make $H_*(\underline{BP}_*)$ a ring object in the category of coalgebras, which were called Hopf rings in [19]. However we follow [6] and call them coalgebraic ring.

The theory of coalgebraic rings have its own interest, especially in connection with that of coalgebraic modules (c. f. [6]). Here, we are interested in them because thanks to these products, we can produce lots of elements starting from a few elements, which we are going to define now. First of all for $a \in BP^* \cong [S^0, \underline{BP}_{-*}]$ we have $[a] = a_*(1) \in H_0(\underline{BP}_{-*})$. Note that we have $\Delta([a]) = [a] \otimes [a], [a] \star [a'] = [a + a']$ and $[a] \circ [a'] = [aa']$. Second, let $x \in BP^2(CP^\infty)$ be the orientation class, $\beta_i \in BP_{2i}(CP^\infty)$ to be dual to x^i. Then we have $b_i = x_*(\beta_i) \in H_{2i}(\underline{BP}_2)$. Note that we have $\Delta(\beta_i) = \Sigma_{j+k=i}(\beta_j \otimes \beta_k)$. So far all elements we have defined live in even degrees, and to remedy this we define $e_1 \in H_1(\underline{BP}_1)$ to be

the image under the suspension map of $1 \in H_0(\underline{BP}_0)$. It turns out that these elements "generate" $H_*(\underline{BP}_*)$ under our two products, that is , every element of $H_*(\underline{BP}_*)$ can be written as a linear combination of $\star$ products of $\circ$ products of theses elements. However we don't have the uniqueness, i.e., there are some relations that we describe now.

Consider the composition $CP^\infty \times CP^\infty \to CP^\infty \xrightarrow{x} \underline{BP}_2$. This is just the formal sum $x_1 +_{BP} x_2$ where $+_{BP}$ denotes the universal p-typical formal group law ([17]). Thus in homology it induces the map

$$\beta(x_1) +_{[BP]} \beta(x_2) \in Hom(H_*(CP^\infty \times CP^\infty), H_*(\underline{BP}_2)) \cong H_*(\underline{BP}_2)[[x_1, x_2]]$$

where we denote $\beta(X) = \Sigma \beta_i X^i$, and $+_{[BP]}$ means the formal sum with $\times$ and $+$ replaced with $\circ$ and $\star$. However, the map $CP^\infty \times CP^\infty \to CP^\infty$ induces in cohomology the ring map that sends x to $x_1 + x_2$ so we see that the above map is equal to $\beta(x_1 + x_2)$. Thus we have

$$\beta(x_1) +_{[BP]} \beta(x_2) = \beta(x_1 + x_2) \text{ (The main relation) .}$$

Then the main result of [19] is

Theorem B.2 ([19], Theorem 4.2). $H_*(\underline{BP}_*)$ *is the quotient of the free coalgebraic ring generated by the elements $[a]$'s for $a \in BP^*$, b_i's and e_1 by the main relation and the relation $e_1 \otimes e_1 = b_1$.*

Among other things, this implies that everything comes from a product of S^1's and CP^∞'s. Denote $\mathcal{C}PS$ the full subcategory of the homotopy category of spaces whose objects are finite products of S^1's and CP^∞'s, and $\mathcal{C}PS/\underline{BP}_*$ the category whose objects are maps from an object of $\mathcal{C}PS$ to $\underline{BP}_*$ and whose morphisms are commutative triangles. Then we have

Theorem B.3 ([9]). *The natural map $colim_{\mathcal{C}PS/\underline{BP}_*} H_*(source(-)) \to H_*(\underline{BP}_*)$ is an isomorphism.*

Now, although the main relation gives a complete set of relations in purely algebraic way, it is not quite practical to work with. Fortunately there are simpler versions. Consider the maps $CP^\infty \xrightarrow{p} CP^\infty \xrightarrow{x} \underline{BP}_2$ and the induced maps in homology. Then as in above, we get

$$b(px) = [p_{[BP]}](b(x))$$

where $[p_{[BP]}](X)$ is the p-series for BP ($[p_{BP}](X)$)with the sum and product replaced by the star and circle products. It turns out that b_i's with i equal to a power of p is necessary to generate $H_*(\underline{BP}_*)$, and if one uses only these b's instead of all of them, then only the simplified form of the main relation is necessary ([19]). Now, let's take a look at these relations. Modulo $([v_1], [v_2] \cdots [v_n] \cdots) \circ ([p], [v_1], [v_2] \cdots [v_n] \cdots)$ we get

$$[v_1] \circ b_1^{\circ p} + b_1^{\star p} = 0$$
$$[v_1] \circ b_p^{\circ p} + [v_2] \circ b_1^{\circ p^2} + b_p^{\star p} = 0$$
$$\vdots$$
$$[v_1] \circ b_{p^n}^{\circ p} + [v_2] \circ b_{p^n-1}^{\circ p^2} + \cdots [v_n] \circ b_1^{\circ p^n} + b_{p^n}^{\star p} = 0$$

from which we derive

Lemma B.4 ([2]). *We have* $[v_n] \circ b_1^{\circ p^n + \cdots + p} = (-1)^n (b_1^{\circ})^{\star p}$

As we know that $BP_*(\underline{BP}_*)$ is free, the above arguments apply to $BP_*(\underline{BP}_*)$ with a slight modification (especially concerning the main relation). For example, the simplified form of the main relation becomes

$$b([p_{BP}]x) = [p_{[BP]}](b(x))$$

and we get

Lemma B.5. *In* $BP_*(\underline{BP}_*)/p$ *we have*

$$[v_1] \circ b_1^{\circ p} + b_1^{\star p} = v_1 b_1.$$

Note also that the freeness implies that $BP^*(\underline{BP}_*)$ is just its dual, which should mean that it is enough to know $BP_*(\underline{BP}_*)$ to understand $BP^*(\underline{BP}_*)$. However, as $BP^j(\underline{BP}_i) \cong [\underline{BP}_i, \underline{BP}_j]$, there is a composition $BP^j(\underline{BP}_i) \times BP^k(\underline{BP}_j) \to BP^k(\underline{BP}_i)$, and there is no structure in $BP_*(\underline{BP}_*)$ which is dual to this. Recent works in [24] suggest a new approach to deal with this problem, but here we will stick to the traditional approach (c. f. [27]). Given $\theta \in BP^*(\underline{BP}_*) \cong [\underline{BP}_*, \underline{BP}_*]$, consider $\theta_* \in Hom_{BP_*}(BP_*(\underline{BP}_*), BP_*(\underline{BP}_*))$. Of course we know the most concretely $Hom_{BP_*}(BP_*(\underline{BP}_*), BP_*)$. So if denote by $\hat{\theta}$ the element corresponding to θ in $Hom_{BP_*}(BP_*(\underline{BP}_*), BP_*)$ what is really interesting to know is the formula expressing θ_* in terms of $\hat{\theta}$. Although such a formula exists ([27], [3]) as we don't need it in the current version of our paper, we content ourselves to express θ_* in terms of θ, and we just consider the induced map in ordinary homology (although the case for BP-homology is just as simple). This is extremely simple using the colimit model, and we have

Lemma B.6. *Let* $\theta \in BP^j(\underline{BP}_i) \cong [\underline{BP}_i, \underline{BP}_j]$. *In* $H_*(\underline{BP}_j)$ *we have* $\theta_*(f_*(\beta)) = (\theta(f))_*(\beta)$ *where* $f \in BP^i(X)$, $\beta \in H_*(X)$ *with* X *a finite product of* S^1's *and* CP^{∞}'s.

Proof. Juste note that $\theta(f) = \theta \circ f$. □

The last formula we need is the action of $BP^*(BP)$ on $BP_*(\underline{BP}_*)$. Again, the colimit model makes things easy. Let $r \in BP^*(BP)$ and denote by r' its right action on BP-homology. Then we have

Lemma B.7. *In* $BP_*(\underline{BP}_*)$ *we have* $r'(f_*(\beta)) = f_*(r'(\beta))$ *where* $f \in BP^i(X)$, $\beta \in BP_*(X)$ *with* X *a finite product of* S^1's *and* CP^{∞}'s.

Proof. By the definition of the homology operations f_* commutes with r'. □

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