Brown-Peterson cohomology of $\Omega^\infty\Sigma^\infty S^{2n}$

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Abstract

In this paper we compute $BP^*(QS^{2n})$.

1 Introduction

Throughout the paper, $H^*(-)$ denotes the ordinary cohomology with Z/p-coefficient. In this paper, we compute $BP^*(QS^{2n})$. Our first result concernes with its BP^*-module genrators. Namely, we prove,

Theorem 1.1 *$BP^*(QS^{2n})\hat{\otimes}_{BP^*}Z/p$ injects to $H^*(QS^{2n})$ and its image coincides with the image of the map $H^*(\underline{BP}_{2n}) \to H^*(QS^{2n})$. In terms of the Dyer-Lashof operations, it is dual to the subalgebra of $H^*(QS^{2n})$ generated by the elements $Q^I\iota_{2n}$ where I is an admissible sequence without Bockstein (To be more accurate, all ϵ's in I are 0. See §3 for the details) if $n > 0$. If $n = 0$, we just add the generator $[-1]$ (see §3 for the definition). Furthermore $BP^*(\underline{BP}_{2n})$ surjects to $BP^*(QS^{2n})$. Here $\underline{E}_{2n}$ denotes the $2n$-th infinite loop space associated to the Ω-spectrum for E.*

A concrete description of the relations among those generators seems to be difficult to obtain. However, we will show

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Theorem 1.2 *$BP^*(QS^{2n})$ is the quotient of $BP^*(\underline{BP}_{2n})$ by the ideal generated by the image of the augmentation ideal under the map $\Pi_i BP^*(r_{p^i})$: $\Pi BP^*(\underline{BP}_{2n+p^i})$, where r_{p^i} is the Landweber-Novikov operation corresponding to the sequence $(p^i, 0, \ldots 0, \ldots)$ (e.g., [1][part II])*

Remark 1.3 *The result of [23] shows that $BP^*(r_{p^i})$ can be expressed completely algebraically (for details, see [30]). Thus this corollary determines $BP^*(QS^{2n})$.*

We also obtain a characterization of $BP^*(QS^{2n})$ as BP-unstable algebra (Theorem 7.4).

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2 Outline of the proof

In this section we present a brief sketch of the proof. First we compute the image of the Thom homomorphism $BP^*(QS^{2n}) \to H^*(QS^{2n})$. Since in general we know little about the BP-cohomology (that is why we are trying to study it via the Thom homomorphism) or the Thom homomorphism (what can we say about the map whose domain is unknown?), we are naturally led to compare it with other spaces for which both the BP-cohomology and the Thom homomorphism is known. First we establish a lower bound by compairing QS^{2n} with $\underline{BP}_{2n}$. We use

Theorem 2.1 ([30]) *(i) $H^*(\underline{BP}_{2n}, Z_{(p)})$ is torsion free.*

(ii) The kernel of the map $H_*(QS^{2n}) \to H_*(\underline{BP}_{2n})$ is precisely the ideal generated by $Q^I(\iota_{2n})$ where Q^I is an admissible operation with at least one Bockstein in it. (To be more accurate, I is an admissible sequence with at least one $\epsilon = 1$. See §3.)

Next we go on to establish an upper bound. In view of the Snaith splitting

Theorem 2.2 ([26]) *If $n > 0$ QS^{2n} is stably homotopy equivalent to $\vee_{0 \le k \le \infty} D_k(S^{2n})$, where $D_k(S^{2n}) = (E\Sigma_k)_+ \wedge_{\Sigma_k} (S^{2n})^{\wedge k}$, where $(-)_+$ denotes a space with a disjoint base point added.*

the Thom isomorphism (note that $D_k(S^{2n})$ is just the Thom complex of the $U(nk)$-bundle $E\Sigma_k \times_{\Sigma_k} (\mathbf{C}^n)^k$ over $B\Sigma_k$),

Theorem 2.3 (c.f.[14]) *Let E be a complex oriented spectrum. Then we have $\widetilde{E}^{*+2nk}(D_k S^{2n}) \cong E^*(B\Sigma_k)$.*

and the approximation theorem

Theorem 2.4 ([20]) *QS^0 has the homotopy type of the group completion of CS^0, where CS^0 is the space $\coprod_q B\Sigma_q$, together with a monoid structure induced from the map $B\Sigma_q \times B\Sigma_r \to B\Sigma_{q+r}$.*

everything is reduced to the case for CS^0. Again, we use the naturality argument. This time we need spaces that map into $B\Sigma_q$'s. According to Quillen,

Theorem 2.5 ([21]) *Elements of mod p cohomology of $B\Sigma_q$ are detected by elementary abelian p-subgroups.*

Since we know from a work of Landweber,

Theorem 2.6 ([11]) $BP^*(B(Z/p)^l) \cong (BP^*(B(Z/p))^{\hat{\otimes} l}$

and the image of the Thom homomorphism $BP^*(BZ/p) \to H^*(BZ/p)$, we can establish an upper bound this way. Of course, one has to know how the elements of $B\Sigma_q$ restricts to elementary abelian p-subgroups. For the purpose, we use May's formula

Theorem 2.7 ([13], also [12, 15, 29]) *Denote by $\circ$ the product induced by the multiplicative pairing $\Sigma_q \times \Sigma_r \to \Sigma_{qr}$ in $H_*(CS^0)$. Then one has $Q^i[1] \circ x = \Sigma_{j \ge 0} Q^{i+j}(P^j_*(x))$.*

These two bounds happen to coincide.

Next we study the BP^*-module generators. Again Thom isomorphism and Snaith splitting reduces the problem to the case for $B\Sigma_q$. However, since the p-Sylow subgroup of Σ_q is just a product of iterated wreath products $Z/p \wr \cdots \wr Z/p$, $B\Sigma_q$ is p-locally a stable summand of a product of the spaces $BZ/p \wr \cdots \wr Z/p$. Now the BP^*-module generators for $BP^*(Z/p \wr \cdots \wr Z/p)$ can be determined using

Theorem 2.8 ([7, 24]) *Let G be good (see Definition 3.10). Then*

(i) so is $Z/p \wr G$,

(ii) and $BP^*(BG)$ is generated by transfer image of Euler classes of subgroups of G.

(iii) Furthermore, Let $BP^*(BG)\hat{\otimes}_{BP^*}Z/p \cong Z/p\{b_\lambda\}$ with b_λ's such tranfer image element. Then

$$BP^*(B(Z/p\wr G))\hat{\otimes}_{BP^*}Z/p \cong Z/p\{\varrho(\lambda)y^s, y^{s+1}, \sigma(\lambda_1, \ldots \lambda_p)|s \geq 0, \exists(i,j)s.t.\lambda_i \neq \lambda_j\}$$

where $\sigma(\lambda_1, \ldots \lambda_p) = Tr(b_{\lambda_1}\otimes\cdots\otimes b_{\lambda_p}), Tr : BP^*((BG)^p) \to BP^*(B(Z/p\wr G))$, $y = \pi^*(\tilde{y})$ with $\pi : Z/p \wr G \to Z/p$, $\tilde{y}$ is the Euler class of a non-trivial one dimensional complex representation of Z/p, and $\varrho(\lambda) = Tr(e(\hat{\rho}))$ if $b_\lambda = Tr(e(\rho))$ for some representation ρ of $H \subset G$, and $\hat{\rho}$ is the representation of $Z/p \wr H$ which restricts to $\rho \otimes \cdots \otimes \rho$ in H^p.

Now anyone who is familiar with the mod p cohomology of the wreath products must have noticed a similarity with the following classical theorem

Theorem 2.9 ([19]) *$H^*(B(Z/p \wr G); Z/p) \cong H^*(Z/p; (H^*(BG))^{\otimes p})$ where Z/p acts on $(H^*(BG))^{\otimes p}$ by permutation of factors.*

By comparing them we deduce that $BP^*(B(Z/p \wr \cdots \wr Z/p))\hat{\otimes}_{BP^*}Z/p \hookrightarrow H^*(B(Z/p \wr \cdots \wr Z/p))$, and thus $BP^*(QS^{2n})\hat{\otimes}_{BP^*}Z/p \hookrightarrow H^*(QS^{2n})$. Since this image is seen to agree with the image of $BP^*(\underline{BP}_{2n})$ we see that $BP^*(\underline{BP}_{2n})$ surjects onto $BP^*(QS^{2n})$. Next we will see that this surjection implies surjection in Morava K-cohomologies, which then shows that Marava K-homologies of QS^{2n}'s are polynomial algebras (Corollaries 6.4,6.5,6.7). This implies a decomposition result for Marava K-homologies of some infinite loop spaces (Proposition 6.10). Finally we apply it to the Novikov-Adams resolution to determine the structure of $BP^*(QS^{2n})$.

3 Preliminaries

Throughout the main text of the paper, p will be an odd prime. However, most of our results also hold for $p = 2$, and the necessary modifications are indicated in the Appendix.

Definition 3.1 *Let* $I = (\epsilon_1, s_1, \dots \epsilon_k, s_k)$ *such that* $s_j \geq \epsilon_j$ *and* $\epsilon_j = 0$ *or* 1*. Define the degree (d), the excess (e), the length (l), and the presence of Bockstein at the end (b) of I by*

$$\begin{aligned} d(I) &= \Sigma_{j=1}^{k}[2(s_j(p-1)-\epsilon_j] \\ e(I) &= 2s_1 - \epsilon_1 - \Sigma_{j=2}^{k}[2(s_j(p-1)-\epsilon_j] \\ l(I) &= k \\ b(I) &= \epsilon_1 (\text{if } p > 2). \end{aligned}$$

I is said to be admissible if $ps_j - \epsilon_j \geq s_{j-1}$ *for* $2 \leq j \leq k$*.*

For any such sequence I (not necessarily admissible), we have corresponding homology operation on E_∞ spaces $Q^I = \beta^{\epsilon_1} Q^{s_1} \dots\dots \beta^{\epsilon_k} Q^{s_k}$, that raises the degree of elements by $d(I)$ and vanishes on elements of degree greater than $e(I)$.

Theorem 3.2 ([5]) $H_*(QS^{2n})$ $(n > 0)$ *is a free commutative algebra on generators* $Q^I(\iota_{2n})$*, where I is admissible,* $e(I) + b(I) > 2n$ *if* $n > 0$*.* $H_*(CS^0)$ *is a free commutative algebra on generators* $Q^I([1])$*, where I is admissible,* $e(I) > 0$*, and* $[i]$ *denotes the image of the Hurewicz homomorphism* $\pi_0(X) \to H_0(X)$ *for any space X. Finally,* $H_*(QS^0)$ *is an algebra generated by* $H_*(CS^0)$*, and* $[-1]$*, subject to the relation* $[1] \cdot [-1] = 1$*.*

Remark 3.3 The upper index notation is related to the lower index notation by the formula $Q^s(x) = \nu(q) Q_{(2s-q)(p-1)}(x)$ where q is the degree of x, and $\nu(q)$ is a certain multiplicative unit of Z/p. In terms of lower indices, $H_*(QS^{2n})$ $(n > 0)$ ($H_*(CS^0)$ resp.) are free commutative algebras on $Q_I(\iota_{2n})$ ($Q_I([1])$, resp.) where $I = (\epsilon_1, s_1, \dots \epsilon_k, s_k)$ satisfies following conditions.

(i) each s_j is a multiple of $p-1$.

(ii) $s_{j-1} \leq s_j$

(iii) $s_j/(p-1) \equiv s_{j-1}/(p-1) + \epsilon_{j-1} \bmod 2$

(iv) $s_k/(p-1) \equiv 0 \bmod 2$

Here $Q_I = \beta^{\epsilon_1} Q_{s_1} \dots\dots \beta^{\epsilon_k} Q_{s_k}$. We call such an sequence I strongly allowable. The word "strongly" is added to emphasize we are not considering the operations corresponding to p-th powers.

For the sake of counting arguments, we need to know the following.

Theorem 3.4 ([13], Theorem 4.1. See also [14], §2.**)** *$H_*(D_i(S^{2n}))$ has a basis that corresponds to the monomials of weight i in the admissible operations on the fundamental class via the stable homotopy equivalence of Theorem 2.2, where the weight is defined by $weight(Q^I\iota_{2n}) = p^{length(I)}$, for admissible sequences I and extended by the formula $weight(xy) = weight(x)weight(y)$.*

Since the sphere spectrum is a ring spectrum, its multiplication induces a pairing $QS^i \times QS^j \to QS^{i+j}$. When both $i = j = 0$, this pairing agrees with the map induced by $CS^0 \times CS^0 \to CS^0$ whose components are given by the maps induced by $\Sigma_a \times \Sigma_b \to \Sigma_{ab}$. It induces a pairing in homology, denoted by $\circ : H_*(CS^0) \otimes H_*(CS^0) \to H_*(CS^0)$. Furthermore, the usual product of QS^0 coming from its loop space structure $QS^0 = \Omega QS^1$ coincides with the product induced by the map $CS^0 \times CS^0 \to CS^0$ whose components are given by the map induced by $\Sigma_a \times \Sigma_b \to \Sigma_{a+b}$. The map it induces in mod p homology is just the usual pontrjagin product of $H_*(CS^0)$, which will be denoted by $\star$ or just by juxtaposition. We will need the following.

Theorem 3.5 (c.f. [13])

- $Q^s[1] \circ x = \Sigma_{t\geq 0} Q^{s+t} P^t_* x$
- $Q^s(x \star y) = \Sigma_{t+u=s} Q^t(x) \star Q^u(y)$
- $P^r_*(Q^s(x)) = \Sigma_i (-1)^{r+i} \binom{(p-1)(s-r)}{r-pi}$
- $Q^r Q^s = \Sigma_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i$ *if $r > ps$*

Here P^r_* denotes the dual Steenrod reduced power operation.

Now Theorem 2.5 (if one considers its proof) can be reformulated as follows.

Theorem 3.6 ([13], p.81) *$QH_*(CS^0)$ is generated under $\circ$ product by $[1]$, $Q^s[1]$'s and $\beta Q^s[1]$'s. Here Q denotes the module of indecomposables with respect to $\star$ product.*

Next we need to know $H^*(B\Sigma_p)$.

Proposition 3.7 (e.g. [25]) *$H^*(B\Sigma_p) \hookrightarrow H^*(BZ/p) = \Lambda(x) \otimes Z/p[y]$, the image is the subalgebra generated by y^{p-1} and xy^{p-2}.*

We denote by $e_{2i(p-1)}$ the element dual to $y^{i(p-1)}$. Then by definition[2, 5], in $H_*(B\Sigma_p) \subset H_*(CS^0)$, $Q^i[1] = e_{2i(p-1)}$. From the theory of the Brown-Peterson cohomology, we need to know the following.

Proposition 3.8 *$BP^*(BZ/p)\hat{\otimes}_{BP^*}Z/p \hookrightarrow H^*(BZ/p)$, and the image is $Z/p[y]$.*

We conclude this section with a brief review of the theory of characteristic classes.

Proposition 3.9 (e. g. [1]p.49) *Let E be a complex oriented cohomology theory. Then $E^*(BU(n)) = E^*[[c_1, \ldots c_n]]$.*

The generators c_i's are called (Conner-Floyd-)Chern classes. By convention $c_0 = 1$. If G is any group then a unitary representation $\phi : G \to U(n)$ gives rise to a homotopy class of a map $B\phi : BG \to BU(n)$. We denote by $c_i(\phi)$ $(0 \leq i \leq n)$), we denote by $c_i(\phi)$ the element $B\phi^*(c_i) \in E^*(BG)$. The element $c_n(\phi)$ plays a distinguished role and we call $c_n(\phi) = e(\phi)$, the Euler class of ϕ. Associated to an inclusion $H \subset G$, there is a stable map from BG_+ to BH_+, denoted by Tr^G_H of just by Tr. By abuse of notation, we denote by Tr the map it induces in any homology or cohomology theory.

Definition 3.10 *A finite group G is called good if $K(n)^*(BG)$ is spanned by elements of the form $Tr^G_H(e(\phi))$ with $\phi : H \to U(m)$.*

Now in the statement of Theorem 2.8, note that $Tr(b_{\lambda_1} \otimes \cdots \otimes b_{\lambda_p})$ restricts to $\Sigma_{i \in Z/p} b_{\lambda_{i+1}} \otimes \cdots \otimes b_{\lambda_{i+p}}$ and $\varrho(\lambda)$ to $b_\lambda \otimes \cdots \otimes b_\lambda$ (this was how the Theorem was proved) here, the subscripts on λ are considered as elements in Z/p.

4 The image of Thom map

Denote by K_n the image of $BP^*(QS^{2n}) \to H^*(QS^{2n})$, and by K'_0 the image of $BP^*(CS^0) \to H^*(CS^0)$. In this section we determine K_n's and K'_0. First we find a lower bound. Let $A_n \subset H_*(QS^{2n})$ be the subalgebra of $H_*(QS^{2n})$ generated by the elements $Q^I(\iota_{2n})$ with $I = (\epsilon_1, s_1, \ldots \epsilon_k, s_k)$, $\epsilon_1 = \cdots \epsilon_k = 0$, i.e., Q^I has no Bockstein in it, and I is admissible. if $n = 0$, A'_0 is as above with n replaced by 0, and A_0 is the subalgebra generated by A'_0 and $[-1]$.

Lemma 4.1 *The map $H^*(QS^n) \to \mathrm{Hom}(H_*(QS^n), Z/p))$ induces an epimorphism $K_n \to A_n^*$, $K_0' \to A_0'^*$, where * denotes the Z/p- dual.*

Proof. By dualizing Theorem 2.1, we see that the image of $H^*(\underline{BP}_{2n}) \to H^*(QS^{2n})$ maps isomorphically to A_n^*. Now, consider the following diagram.

$$\begin{array}{ccc} BP^*(\underline{BP}_{2n}) & \longrightarrow & BP^*(QS^{2n}) \\ \downarrow & & \downarrow \\ H^*(\underline{BP}_{2n}) & \longrightarrow & H^*(QS^{2n}) \end{array}$$

Since $H^*(\underline{BP}_{2n}, Z_{(p)})$ is torsion-free by 2.1 , the left column is surjective. Therefore, K_n contains the image of $H^*(\underline{BP}_{2n}) \to H^*(QS^{2n})$. Similarly for K_0'. ∎

Next we find an upper bound. We start from CS^0. We refine Theorem 2.5 using Theorem 3.6.

Lemma 4.2 *There exists a family of groups $G_{l,i} \cong (\Sigma_p)^l$ and a homomorphism $f_{l,i} : (\Sigma_p)^l \to \Sigma_{\phi(l,i)}$ such that*

(i)

$$\coprod_{l,i} Bf_{l,i} : \coprod (B\Sigma_p)^l \to \coprod_{k=0}^{\infty} B\Sigma_k = CS^0$$

induces a monomorphism in mod p cohomology and,

(ii) $Bf_{l,i\,*}(e_{2j_1(p-1)} \otimes \cdots \otimes e_{2j_l(p-1)}) \in A_0'$.

Proof. Theorem 3.6 implies that to satisfy i), it is enough to take the family

$$\{\overbrace{(\Sigma_p \times \cdots \times \Sigma_p)}^{m_1 \text{ factors}} \times \cdots \times \overbrace{(\Sigma_p \times \cdots \times \Sigma_p)}^{m_k \text{ factors}} \to \Sigma_{p^{m_1}} \times \cdots \times \Sigma_{p^{m_k}} \to \Sigma_{p^{m_1} + \cdots + p^{m_k}}\}$$

where the first map is induced by the multiplication map $\Sigma_a \times \Sigma_b \to \Sigma_{ab}$ and the second by the addition $\Sigma_a \times \Sigma_b \to \Sigma_{a+b}$. (See also [16, 17, 18].) Now by definitions the element

$$(e_{2j_1(p-1)} \otimes \cdots \otimes e_{2j_{m_1}(p-1)}) \otimes \cdots \otimes (e_{2j_{m_1+\cdots+m_{k-1}+1}(p-1)} \otimes \cdots \otimes e_{2j_{m_1+\cdots+m_{k-1}+m_k}(p-1)})$$

is mapped to the element

$$(Q^{j_1}[1] \circ \cdots \circ Q^{j_{m_1}}[1]) \star \cdots \star (Q^{j_{m_1+\cdots+m_{k-1}+1}}[1] \circ \cdots \circ Q^{j_{m_1+\cdots+m_{k-1}+m_k}}[1]).$$

Using the formulas 3.5, one can rewrite each factor $Q^{j_1}[1] \circ \cdots \circ Q^{j_{m_1}}[1]$ etc. by induction on length as a linear combination of the elements of the form $(Q^I[1])^{p^k}$ where I is an admissible sequence, and of the form $I = (\epsilon_1, s_1, \dots \epsilon_k, s_k)$, with all ϵ's equal to 0. Thus the condition ii) is satisfied. ■

Proposition 4.3 *K'_0 is isomorphic to A'^*_0.*

Proof. From Lemma 4.2 , we have $H^*(CS^0) \hookrightarrow \oplus H^*(BG_{l,i})$. Now note that we have the following commutative diagram.

$$\begin{array}{ccccc} BP^*(CS^0) & \longrightarrow & \oplus BP^*(BG_{l,i}) & \longrightarrow & \oplus BP^*(BA_{l,i}) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(CS^0) & \longrightarrow & \oplus H^*(BG_{l,i}) & \longrightarrow & \oplus H^*(BA_{l,i}) \end{array}$$

Here $A_{l,i} = (Z/p)^l \subset \Sigma_p^l$. Note that the maps in the bottom row are injective. Using 2.6 and 3.8 one sees that the image of $BP^*(BA_{l,i})$ in $H^*(A_{l,i}) \cong \Lambda(x_1, \dots, x_l) \otimes Z/p[y_1, \dots, y_l]$ is just $Z/p[y_1, \dots, y_l]$. Therefore the image of $BP^*(BG_{l,i})$ in $H^*(BG_{l,i})$ is contained in $Z/p[y_1^{p-1}, \dots, y_l^{p-1}]$, which is nothing but the dual of the vector subspace of $H^*(BG_{l,i})$ generated by $e_{2j_1(p-1)} \otimes \cdots \otimes e_{2j_l(p-1)}$, which we will denote by $H_{l,i}$. Thus we have

$$K \hookrightarrow \oplus_{l,i}(\mathrm{Im}(BP^*(BG_{l,i}) \to H^*(BG_{l,i})) \hookrightarrow \oplus H^*_{l,i}.$$

That is, $\oplus H_{l,i}$ surjects to K'^*_0. But by ii) of Lemma 4.2, the image of $H_{l,i}$ in $H_*(CS^0)$ is contained in A'_0. Denote by B this image. Then we have seen that B surjects to K'^*_0, and that B injects to A'_0. By dualizing, we see that A'^*_0 surjects to B^* and that K'_0 injects to B^*. However, we have previously shown that K'_0 surjects to A'^*_0 (Lemma 4.1). Since each component of all of the objects we are considering are of finite dimension at each degree, even without tracing the definitions of maps, we can conclude that $K'_0 \cong A'^*_0 \cong B^*$. ■

Corollary 4.4 $K_n \cong A_n^*$.

Proof. The case for $n = 0$ is an immediate consequence of the Proposition, and Theorem 2.4. When $n > 0$, by Theorem 2.3 we have the following commutative diagram in which the horizontal arrows are isomorphisms.

$$\begin{array}{ccc} \widetilde{BP}^{*+2nk}(D_k S^{2n}) & \longrightarrow & BP^*(B\Sigma_k) \\ \downarrow & & \downarrow \\ \tilde{H}^{*+2nk}(D_k S^{2n}) & \longrightarrow & H^*(B\Sigma_k) \end{array}$$

Note that

$$\deg(Q_I(\iota_{2n})) = \deg Q_I[1] + 2n \cdot p^{l(I)}.$$

Thus it is easy to see that the number of monomials of weight w, degree d in $Q_I([1])$'s with strongly allowable I's coincides with the number of monomials of weight w, degree $d + 2nw$ in $Q_I([1])$'s with strongly allowable I's. (Of course this has to be true in view of Theorem 3.4 and the mod p cohomology version of Theorem 2.3.) Furthermore, we see that the number of monomials of weight w, degree d in $Q_I([1])$'s with strongly allowable I's with all ϵ's equal to 0 coincides with the number of monomials of weight w, degree $d+2nw$ in $Q_I([1])$'s with strongly allowable I's with all ϵ's equal to 0. Since $Q_I(\iota_{2n})$ for strongly allowable I is up to invertible constant multiple $Q^{I'}(\iota_{2n})$ for unique admissible I' with $e(I') > 2n$, and all ϵ's in I are zero if and only if all ϵ's in I' are zero, the fact that the lower bound obtained from Lemma 4.1 and the upperbound obtained from Proposition 4.2 coincide for CS^0 forces them to coincide for QS^{2n} as well. ■

5 BP^*-module generators for CS^0 (and QS^0)

In this section we determine the BP^*-module generators for $BP^*(QS^{2n})$. First we need

Proposition 5.1 *Let G be a good group such that $BP^*(BG)\hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(BG)$. Then $BP^*(B(Z/p\wr\cdots\wr Z/p\wr G))\hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(B(Z/p\wr\cdots\wr Z/p\wr G))$.*

Proof. Let M be a Z/p-vector space, and let Z/p act on $M^{\otimes p}$ by permutation of factors. Then it is well-known that as a $Z/p[Z/p]$-module, $M^{\otimes p} = F \oplus T$, where Z/p acts trivially on T and by permutation of a basis on F. Furthermore, if $\{a_\lambda : \lambda \in \Lambda\}$ forms a basis for M, then bases for F and T are given by $\{\Sigma_{i \in Z/p} a_{\lambda_i} \otimes \cdots \otimes a_{\lambda_{i+p-1}}$: (here the subscripts on λ are considered as an element of Z/p.) not all λ_j's are equall, $\lambda_j \in \Lambda\}$ and $\{a_\lambda \otimes \cdots \otimes a_\lambda : \lambda \in \Lambda\}$. Thus $H^*(Z/p; H^*(BG)^{\otimes p}) = H^0(Z/p, F) \oplus H^*(BZ/p) \otimes T$ [19]. Now choose a basis for $H^*(BG)$ such that the image of the elements b_λ ($\lambda \in \Lambda'$) becomes its subset, where b_λ's are BP^*-module generators of $BP^*(BG)$ consisting of transfer image of Euler classes of subgroups of G. One can reindex the set of indices in such a way that b_λ maps to a_λ. Then by the remark at the end of §3, we see that the elements $y^s \varrho(\lambda)$ reduces to $y^s \otimes a_\lambda \otimes \cdots \otimes a_\lambda \in H^*(BZ/p) \otimes T$ and that the elements $Tr(b_{\lambda_1} \otimes \cdots \otimes b_{\lambda_p})$ reduces to $\Sigma_{i \in Z/p} a_{\lambda_{i+1}} \otimes \cdots \otimes a_{\lambda_{i+p}}$ (here again the subscripts on λ is an element of Z/p). Thus by Theorem 2.8, a basis for $BP^*(B(Z/p \wr G)) \hat{\otimes}_{BP^*} Z/p$ is mapped to a subset of a basis for $H^*(B(Z/p \wr G))$. By iterating, one obtains the desired result. ■

Since the trivial group is good, and $B\Sigma_k$ is p-locally a stable summand of a product of $B(Z/p \wr \cdots \wr Z/p)$, we obtain

Corollary 5.2 *$BP^*(B\Sigma_k) \hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(B\Sigma_k)$, and $BP^*(CS^0) \hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(CS^0)$.*

By passing to limit, we have

Corollary 5.3 *$BP^*(QS^0) \hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(QS^0)$*

Furthermore, using Theorems 2.3 and 2.2, one obtains immediately,

Corollary 5.4 *$BP^*(QS^{2n}) \hat{\otimes}_{BP^*} Z/p \hookrightarrow H^*(QS^{2n})$.*

Now Theorem 1.1 of the introduction follows from Corollaries 5.3, 5.4, Lemma 4.1, Proposition 4.2 and Corollary 4.4.

6 Morava K-theories

In this section we show that the Morava K-homology of QS^{2n}'s are polynomial algebras (except for $n = 0$ when it is a polynomial algebra tensored with a Laurent polynomial algebra in one variable), and deduce a decomposition result for Morava K-homology of some infinite loop spaces. Our result applies to infinite loop spaces associated to such spectra as $T(n)$, BP, and MU. First we note the following.

Proposition 6.1 $K(n)^*(B\Sigma_k) \cong BP^*(B\Sigma_k) \otimes_{BP^*} K(n)^*$.

Proof. Since $K(n)^{odd}(B\Sigma_k) = 0$ ([8, 7]), this follows from [24]. ∎

Thus, we have

Corollary 6.2 $K(n)^*(CS^0) \cong BP^*(CS^0) \otimes_{BP^*} K(n)^*$.

Since $K(n)$-cohomology is just dual to $K(n)$-homology, this fact together with Theorem 1.1 imply

Proposition 6.3 $K(n)_*(CS^0) \hookrightarrow K(n)_*(\underline{BP}_0)$, $K(n)_*(CS^0) \hookrightarrow K(n)_*(\underline{MU}_0)$.

These inclusions are inclusions of Hopf algebras. In next section we deduce from it the following.

Corollary 6.4 *$K(n)_*(CS^0)$ is a polynomial algebra.*

Corollary 6.5 *For $m > 0$, $K(n)_*(QS^{2m})$ is a polynomial algebra.*

Proof. By the naturality and the multiplicativity of the Thom isomorphism, we have the following commutative diagram in which vertical arrows are isomorphisms.

$$\begin{array}{ccc}
\widetilde{K(n)}_{*+2im}(D_i(S^{2m})) \otimes \widetilde{K(n)}_{*+2jm}(D_j(S^{2m})) & \longrightarrow & \widetilde{K(n)}_{*+2(i+j)m}(D_{i+j}(S^{2m})) \\
\downarrow & & \downarrow \\
K(n)_*(B\Sigma_i) \otimes K(n)_*(B\Sigma_j) & \longrightarrow & K(n)_*(B\Sigma_{i+j})
\end{array}$$

Thus $\widetilde{K(n)}_*(\vee_i D_i(S^{2m}))$ is a polynomial algebra. Since $\widetilde{K(n)}_*(\vee_i D_i(S^{2m}))$ is an associated graded object of $K(n)_*(QS^{2m})$ by Theorem 2.2, $K(n)_*(QS^{2m})$ itself must be a polynomial algebra. ■

Remark 6.6 *The case for $n = 1$ was known by [14]. The case fon $n = 2$ was proved in [10].*

When $n = 0$, by Theorem 2.4 we get the following.

Corollary 6.7 *$K(n)_*(QS^0)$ is a polynomial algebra tensored with a Laurent polynomail algebra in one variable.*

Remark 6.8 The case for $n = 1$ was known by [6].

As a consequence, we have,

Corollary 6.9 *For $m \geq 0$, $K(n)_*(QS^{2m+1})$ is an exterior algebra generated by odd degree elements.*

Consider the bar spectral sequence associated to the fibration $QS^{2m} \to * \to QS^{2m+1}$, $E_2 = \mathrm{Tor}^{K(n)_*(QS^{2m})}(K(n)_*, K(n)_*) \Rightarrow K(n)_*(QS^{2m+1})$. Since $K(n)_*(QS^{2m})$ is a polynomial if $m > 0$, and polynomial tensored with a Laurent polynomial in one variable if $m = 0$, we have, $E_2 \cong \Lambda(\sigma(QK(n)_*(QS^{2m})))$. Thus E_2 term is generated by the elements in $E_2^{1,*}$, so that there can be no differentials. Since these generators are in odd degree, there is no algebra extension problem, and we obtain the result. ■

Proposition 6.10 *For any spectrum X that admits a stable cell decomposition consisting only of non-negative even dimensional cells, $K(n)_*(\underline{X}_0)$ decomposes as a tensor product of $K(n)_*(QS^{2m})$'s corresponding to the stable cell decomposition of X. Furthermore, this decomposition is an isomorphism of algebras.*

Proof. First assume that X has finitely many cells. Thus there exist spectra

$$\begin{aligned} &X^0 = S^{2d}, X^1, \ldots, X^l = X \text{ such that} \\ &\quad \exists \text{ cofibrations } S^{2d_i - 1} \to X^{i-1} \to X^i. \end{aligned}$$

This cofibration gives rise to a fibration of infinite loop spaces $QS^{2d_i-1} \to \underline{X^{i-1}}_0 \to \underline{X^i}_0$. By the induction hypothesis, $K(n)_*(\underline{X^{i-1}}_0)$ decomposes, and thus is concentrated in even degree. Therefore, the map $K(n)_*(QS^{2d_i-1}) \to K(n)_*(\underline{X^{i-1}}_0)$ is trivial since the former is generated by odd degree elements. Thus

$$\begin{aligned} E_2 &= \mathrm{Tor}^{K(n)_*(QS^{2d_i-1})}(K(n)_*(\underline{X^{i-1}}_0), K(n)_*) \\ &\cong \mathrm{Tor}^{K(n)_*(QS^{2d_i-1})}(K(n)_*, K(n)_*) \otimes K(n)_*(\underline{X^{i-1}}_0) \\ &\cong \Gamma[\sigma Q K(n)_*(QS^{2d_i-1})], \end{aligned}$$

where Γ denotes the divided power algebra. Thus E_2 term is concentrated in even degrees, and the spectral sequence collapses. To solve the algebra extension problem, use the map that "pinches" X_i to QS^{2d_i}. One obtains the result for the general case by passing to the colimit. ■

Remark 6.11 Since $\underline{X}_{2m} = \underline{\Sigma^{2m} X}_0$, we can obtain a similar decomposition result for $\underline{X}_{2m}$, $m > 0$. Moreover, using the bar spectral sequence for the fibration $\underline{X}_{2n} \to * \to \underline{X}_{2n+1}$, we can obtain similar decomposition for odd spaces as well.

7 BP-unstable algebra structure

Definition 7.1 *Let X be a spectrum that has stable cells only in even non-negative dimensions. A sequence of maps of spectra $X = X_{-1} \to X_0 \to \cdots \to X_n \stackrel{f_{n+1}}{\to} X_{n+1} \cdots$ is called Novikov-Adams resolution if it satisfies following properties.*

(i) X_n's ($n \geq 0$) are BP-module spectra

(ii) $\pi_*(X_n)$'s ($n \geq 0$) are free BP_*-module

(iii) The composition $X_{n-1} \to X_n \to X_{n+1}$ is null homotopic

(iv) If we denote by C_n the cofiber of f_n, then $BP^*(X_{n+1})$ surjects to $BP^*(C_n)$,

(v) The cofiber of the map $C_n \to X_{n+1}$ has only positive even dimensional cells.

Remark 7.2 *This last condition is usually not required. However, it is easy to see that when X satisfies the hypothesis, one can obtain a resolution which satisfies this extra condition. As a matter of fact, the method of [1][part III] produces such a resolution.*

Now we are ready to state our main result of the section.

Theorem 7.3 *Let $X = X_{-1} \to X_0 \to \cdots \to X_n \overset{f_{n+1}}{\to} X_{n+1} \cdots$ be as above. Then $BP^*(\underline{X_{-1}}_0) \leftarrow BP^*(\underline{X_0}_0) \leftarrow \cdots BP^*(\underline{X_n}_0) \leftarrow \cdots$ is a coexact sequence in the category of BP^*-algebras. Furthermore, $BP^*(\underline{X_n}_0)$'s are projective objects in the category of BP-unstable algebras.*

As an immediate consequence, we have,

Corollary 7.4 *Let D denote the functor that is left adjoint to the forgetful functor from the category of BP-unstable algebra ([3]) to the category of $BP^*(BP)$-module. Then under the hypotheses of the Theorem 7.3, the canonical map $DBP^*(X) \to BP^*(\underline{X}_0)$ is an isomorphism.*

Proof of the Theorem 1.2. This follows from the Theorem 7.3 directly using the fact that $BP^*(BP)$ is generated by r_{p^i}'s [32].
Proof of the Theorem 7.3. Notice that we have a cofibration sequence $\Sigma^{-1}C_n \to X_{n-1} \to X_n$, where $\Sigma^{-1}C_n$ has only positive odd dimensional stable cells. Thus the Proposition 6.10 implies that we have an isomorphisme of algebras $K(m)_*(\underline{X_n}_0) \cong K(m)_*(\underline{X_{n-1}}_0) \otimes K(m)_*(\underline{C_n}_0)$. Now, this can be shown by comparing the bar spectral sequences converging to $K(m)_*(\underline{X_n}_0)$ and to $K(m)_*(\underline{C_n}_0)$ as in the proof of Proposition 6.10 instead of adding one cell at a time. Therefore, taking into acount the fact that $K(m)$-cohomology of these two spaces are complete with respect to the bar spectral sequence filtration, and the fact that the bar spectral sequence is a spectral sequence of Hopf algebras, one can conclude that the sequence $K(m)_*(\underline{X_{n-1}}_0) \to K(m)_*(\underline{X_n}_0) \to K(m)_*(\underline{C_n}_0)$ is an exact sequence of Hopf algebras. Therefore, according to [24], we have a short coexact sequence of algebras $BP^*(\underline{X_{n-1}}_0) \leftarrow BP^*(\underline{X_n}_0) \leftarrow BP^*(\underline{C_n}_0)$. Similarly, we can show that $BP^*(\underline{X_{n+1}}_0)$ surjects to $BP^*(\underline{C_n}_0)$. By patching them together one gets the desired coexact sequence. The second assertion follows from the fact that X_n's are wedge of suspensions of BP, which in turn follows from the requirements i) and ii) in the definition of the Novikov-Adams resolution. ■

8 Almost coconnected polynomial Hopf algebra

In this section, we deduce Corollary 6.4 from Proposition 6.3.

Definition 8.1 *Let A be a Hopf algebra over $K(n)_*$. A is called almost coconnected if it satisfies the following property.*

(i) $\exists$ Hopf subalgebra B, such that $A \cong B \otimes K(n)_*[Z^+]$ where $K(n)_*[Z^+]$ is the semi-group ring of $[Z^+]$, the additive semi-group of non-negative integers, over $K(n)_*$.

(ii) $\exists$ a filtration F on B $K(n)_* = F_0(B) \subset \cdots \subset F_i(B) \subset \cdots$, $\cup_i F_i(B) = B$ such that $F_i(B) \cdot F_j(b) \subset F_{i+j}(B)$, $\Delta(F_i(B)) \subset \Sigma_{j=0}^{i} F_j(B) \otimes F_{i-j}(B)$.

Example 8.2 (i) $K(n)_*(BU \times Z^+) = K(n)_*[b_1, \ldots] \otimes K(n)_*[Z^+]$. Let $B = K(n)_*(BU \times 0) \subset K(n)_*(BU \times Z^+)$. Define the *weight* of monomial $b_1^{j_1} \cdots b_m^{j_m}$ by $j_1 + \ldots + m j_m$. Then define $F_i(B)$ to be the vector subspace of B spanned by monomials of weight less than or equal to i.

(ii) $K(n)_*(\underline{MU}'_0 \times Z^+)$, where $\underline{MU}'_0$ denotes the connected component of $\underline{MU}_0$ that corresponds to $0 \in \pi_*(MU)$. Let $B = K(n)_*(\underline{MU}'_0 \times \{0\})$. Define a filtration F on $K(n)_*(CP^\infty \times \cdots \times CP^\infty)$ by giving weight $i_i + \cdots + i_k$ to the basis element $\beta_{i_1} \otimes \cdots \otimes \beta_{i_k}$, where β_i is the class dual to x^i, where x is the orientation class, and by defining $F_i(K(n)_*(CP^\infty \times \cdots \times CP^\infty))$ to be the vector subspace spanned by elements of weight not exceeding i. Define F on B by

$$z \in F_i(B)$$
$$\Leftrightarrow \exists \text{ a collection of maps } f_j : CP^\infty \times \cdots \times CP^\infty \to \underline{MU}'_0 \text{ with } z \in \Sigma_j \, \mathrm{Im} f_{j*}(F_i)$$

Then the result of [9] shows that this filtration has the desired property. (This filtration can be defined using Ravenel-Wilson description of the Hopf ring[23], too.)

Proposition 8.3 *Let A be an almost coconected Hopf algebra which is, as an algebra, is a polynomial algebra. Let C be its Hopf subalgebra with the following properties.*

(i) $C \cap K(n)_*\{0\} \otimes B = K(n)_*$*,*

(ii) C *contains* $K(n)_*[Z^+]$*,*

(iii) C *is generated by elements in* $C \cap K(n)_*\{d\} \otimes B$*,* $d \in Z^+$*,*

(iv) $C \cap K(n)_*\{d\} \otimes B$ *is finite dimensional over* $K(n)_*$ *for each* d*.*

Then C *is a polynomial algebra.*

Remark 8.4 The conditions i) and iv) are not really necessary. However, this is good enough for our purpose.

Proof. Define a filtration F on A by $F_i(A) = F_i(B) \otimes K(n)_*[Z^+]$. Then it satisfies all the properties the F on B satisfies except for $F_0(A) = K(n)_*[Z^+]$, instead of $K(n)_*$. We prove by induction on i and j the following statement $P_{i,j}$.

$P_{i,j}$:$\exists D_{i,j}$, a Hopf subalgebra of A that is a polynomial algebra containing $C_{i,j} = C \cap (F_i(A) + F_{i+1}(A) \cap K(n)_*\{0, 1, \ldots, k\} \otimes B)$. and generated by elements in $C_{i,j}$

Note that the condition ii) implies that $P_{0,0}$ holds. Suppose that $P_{i,j}$ holds. Let D be any polynomial Hopf algebra containing $C_{i,j}$ and generated by elements in $C_{i,j+1}$. Let $x \in F_{i+1}(A) \cap K(n)_*\{j+1\} \otimes B$ such that $x \notin D$. If x is algebraically dependent on D, Let $f(X) = a_0 X^d + \cdots + a_d$ be a polynomial with $f(x) = 0$ such that $a_i \in D$, and such that d and the filtration of a_0 is minimum. Then by applying the diagonal, one sees that f must be of the form $f(X) = X^{p^e} + a$. Thus in any case the subalgebra generated by D and x is a polynomial algebra. Since $C \cap F_i(A) \subset D$, the algebra generated by x and D is a Hopf subalgebra since A is almost coconnected. We can repeat this process to obtain $D_{i,j+1}$ starting from $D_{i,j}$. This process terminates thanks to the condition v), and thus $D_{i,j+1}$ is still a polynomial algebra. Now let $D_{i+1,0} = \cup_j D_{i,j}$. The only thing that has to be proved is that it is still a polynomial algebra. But this is true since the only possibile failure could be caused by the replacement of a generator by its p^f-th root, which would create an infinite sequence of elements of the form $x_1, x_2, \ldots$ such that $x_1 = x_2^{p_1^e}, x_2 = x_3^{p_2^e}, \ldots$, but noting that $x_i = x_{i+1}^q$, $x_i \in K(n)_*\{k\} \otimes B$ implies $x_{i+1} \in K(n)_*\{k/p\} \otimes B$, our hypothesis i) excludes this possibility. ■

Corollary 8.5 *(i) Let q be a power of l where l is a prime different from p. Then $K(n)_*(\coprod_m BGL(m;F_q))$ is a polynomial algebra.*

(ii) $K(n)_*(\coprod_m B\Sigma_m)$ is a polynomial algebra.

Proof. According to [27] , $K(n)_*(BGL(m,F_q)) \hookrightarrow K(n)_*(BU(m))$. and this induces an inclusion of Hopf algebras. This proves i). ii) follows from Proposition 6.3. ∎

9 Appendix:Modifications for the case $p = 2$

In this section, we indicate how to modify our arguments when $p = 2$. We have to start from modifying definitions.

Definition 9.1 *$I = (s_1, \ldots, s_k)$ is called admissible if $s_j \leq 2s_{j+1}$, the excess, the egree, and the length of I are defined by $d(I) = \Sigma_{j=1}^k s_j$, $l(I) = k$ and $e(I) = s_1 - \Sigma_{j=2}^k s_j$.*

Now, 3.1, 3.2, 3.4, and 3.6 hold after one modifies the statements accordingly ([2]). In 3.3, the condition becomes $0 < s_1 \leq \cdots \leq s_k$. In 3.5, P^t has to be replaced by Sq^t. In 2.1, the condition becomes "I is an admissible sequence with at least one s_j odd". The modification required for 3.7 and 3.8 are left to the reader. Now the problem is with Lemma 4.2. The condition ii) is not satisfied because we have

$$Q_{4i+2}[1] \circ [2] = Q_{4i+2}[2] = (Q_{2i+1}[1])^2.$$

So we need a more detailed analysis. We will show

Lemma 9.2 *Let J denote the ideal of $H^*(CS^0)$ generated by $Q^I[1]$'s with $I = (s_1, \ldots s_k)$ with at least one s_t is odd. Then J is spanned by the elements of the form $f_{i,l*}(e_{j_1} \otimes \cdots \otimes e_{j_l})$ with at least one of j_m's is odd. (Odd prime analogue of this is still true, but we wanted to present the shortest proof.)*

Now using this Lemma, the proof of 4.2 goes as follows. Let $\phi \in \mathrm{Im}(BP^*(CS^0) \to H^*(CS^0))$. Let $\langle , \rangle$ denote the pairing between homology and cohomology. $\langle \phi, J \rangle = 0$, since $\langle f_{i,l}^*(\phi), (e_{j_1} \otimes \cdots \otimes e_{j_l}) \rangle = 0$ if one of j_m's is odd. Thus $\phi \in \mathrm{Im}(BP^*(\underline{BP}_0) \to BP^*(CS^0))$. Now we prove the Lemma. Let R denote the Dyer-Lashof algebra, i.e., the algebra generated by symbols Q^i's subject to the Adem relation (3.5 iv)). We identify R with the vector subspace of $H_*(CS^0)$ by the correspondance $Q^I \mapsto Q^I[1]$. Let E be the vector subspace of R spanned by the elements Q^I's ($I = s_1, \dots, s_k$) with at least one s_m odd. We need

(i) E is closed under the action of the Steenrod algebra.

(ii) $E \cdot R \subset E$ and $R \cdot E \subset E$ (i.e. E is an ideal).

To prove ii) it suffices to note that if $2i+1 > 2(2j+1)$, in the right hand side of the formula

$$Q^{2i+1}Q^{2j+1} = \Sigma_k \binom{k-(2j+1)-1}{2k-(2i+1)} Q^{2i+2j+2-k}Q^k,$$

the binomial coefficient vanishes if k is even. We prove i) by induction on $l(I)$. First note that $Sq_*^{2i+1}Q^{2j+1}(x) = 0$ by Nishida relation 3.5 iii). Thus when $l(I) = 1$, $Sq_*^i(Q^{2j+1}[1]) \in E$, since if i is odd this is 0, and if i is even, then $Sq_*^i(Q^{2j+1}[1])$ has odd degree. Now suppose $l(I) = 1$, $I = (i_1, \dots i_l)$. Denote by I' the sequence $(i_2, \dots i_l)$. Consider $Sq_*^j Q^{i_1} Q^{I'}[1]$. If both j and i_1 are odd, $\deg Sq_*^j Q^{i_1} Q^{I'}[1]$ has the same parity as $\deg Q^{i_1} Q^{I'}[1]$. Thus if $\deg(I)$ is odd, then by degree reason, it is in E. If $\deg(I)$ is even, then $\deg(I')$ is odd, so by Nishida relation and ii), we can use the inductive hypothesis to get the conclusion. If i_1 is even, then we can use the nishida relation and apply inductive hypothesis to I'. this finishes the proof of i).

Finally we are ready to prove the Lemma. We show that for any $I = (i_1, \dots, i_l)$, at least one of i_m's is odd (not necessarily admissible) can be expressed as a linear combination of $Q_{j_1}[1] \circ \cdots \circ Q_{j_l}[1]$'s with at least one of j_m's is odd. This is shown induction both on the length of I and degree of I'. There is nothing to prove when $l(I) = 1$. By May's formula

$$Q^I[1] = Q^{i_1}[1] \circ Q^{I'}[1] + \Sigma_{j>0} Q^{i+j} Sq_*^j Q^{I'}[1].$$

If $Q^{I'}[1] \notin E$, then i_1 has to be odd and so is $\deg(I)$, thus the statement is true for degree reason. Therefore we can assume that $Q^{I'}[1] \in E$. Then we can apply the induction hypothesis on the length to show that the first term of the right hand side can be written in a desired form. As to the terms in the summation , noting that $\deg(Sq_*^j Q^{I'}[1]) < \deg(Q^{I'}[1])$, using i) above one can apply induction hypothesis. ■

There is another place where a modification is necessary. Namely, the bar spectral sequence arguments in §6. First, In Corollary 6.9, one has to solve algebra extension problems. However, for degree reasons, the only possible extensions have to have the form $a^2 = c$ or $ab - ba = c$ where a, b are exterior algebra generators in E_2 term, and c is an element in $K(n)_*$. In any case, the elements in the left hand sides are in the kernel of homology suspension, where as those it the right hand sides are not. This solves the algebra extension problem. In Proposition 6.10, the decomposition is just as $K(n)_*$-vector spaces, since without commutativity, polynomial algebras are not free algebras.

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