NEW RELATIONSHIPS AMONG LOOPSPACES, SYMMETRIC PRODUCTS, AND EILENBERG MACLANE SPACES

NICHOLAS J. KUHN

July 8, 1998

Abstract. Let $T(j)$ be the dual of the j^{th} stable summand of $\Omega^2 S^3$ (at the prime 2) with top class in dimension j. Then it is known that $T(j)$ is a retract of a suspension spectrum, and that the homotopy colimit of a certain sequence $T(j) \to T(2j) \to \dots$ is an infinite wedge of stable summands of $K(V,1)$'s, where V denotes an elementary abelian 2 group. In particular, when one starts with $T(1)$, one gets $K(Z/2,1) = RP^\infty$ as one of the summands.

I discuss a generalization of this picture using higher iterated loopspaces and Eilenberg MacLane spaces. I consider certain finite spectra $T(n,j)$ for $n,j \geq 0$ (with $T(1,j) = T(j)$), dual to summands of $\Omega^{n+1} S^N$, conjecture generalizations of the above, and prove that these conjectures are correct in cohomology. So, for example, $T(n,j)$ has unstable cohomology, and the cohomology of the hocolimit of a certain sequence $T(n,j) \to T(n,2j) \to \dots$ agrees with the cohomology of the wedge of stable summands of $K(V,n)$'s corresponding to the wedge occurring in the $n = 1$ case above.

One can also map the $T(n,j)$ to each other as n varies, and here the cohomological calculations imply a homotopical conclusion: the hocolimits that are nonzero, $T(\infty, 2^k)$, for $k \geq 0$, map to each other, giving rise to a fitration of $H\mathbf{Z}/2$ which is equivalent to the mod 2 symmetric powers of spheres filtration.

Our homotopical constructions use Hopf invariant methods and loopspace technology. These are quite general and should be of independent interest.

To study the action of the Steenrod operations on the cohomology of our spectra, we derive a Nishida formula for how $\chi(Sq^i)$ acts on Dyer–Lashof operations. This should be of use in other settings.

In an appendix, we explain connections with recent work by Greg Arone and Mark Mahowald on the Goodwillie tower of the identity.

Contents

1991 *Mathematics Subject Classification.* Primary 55P42; Secondary 55Q25,55S10,55S12.
Partially supported by the N.S.F. and the C.N.R.S.

1. Introduction

With all spaces and spectra localized at 2, let $T(j)$ be the $(2j)^{th}$ dual of the j^{th} stable summand of $\Omega^2 S^3$. These finite complexes were explored in the 1970's and 1980's in work by M.Mahowald, E.Brown, S.Gitler, F.Peterson, R.Cohen, G.Carlsson, H.Miller, J.Lannes, and P.Goerss, among others. (Entries into the extensive literature include [Mah, BC, Ca, Mi2, L2, GLM, HK].) They played an essential role in a number of the major achievements in homotopy theory during this time: Mahowald's construction [Mah] of an infinite family of 2-primary elements in $\pi_*^S(S^0)$ having Adams filtration 2; Goerss, Lannes, and F.Morel's work [GLM] on representing mod 2 homology by maps from (desuspensions of) the $T(j)$'s; and Miller's proof of the Sullivan conjecture [Mi2].

All of this work is a reflection of unexpected "unstable" properties of the $T(j)$'s. [Mah] is based on two facts: that as modules over the Steenrod algebra, the cohomology of the $T(j)$ are dual Brown–Gitler modules, and that one can construct maps $T(j) \to T(2j)$ realizing on cohomology certain canonical maps between these. [Ca, Mi2] are then based on the connection, just on the level of cohomology, between the classifying spaces BV of elementary abelian 2 groups V, and the homotopy colimits of the sequences

$$T(j) \to T(2j) \to T(4j) \to \cdots .$$

It is not hard to show that this cohomological connection can be realized homotopically: these hocolimits are always infinite wedges of stable wedge summands of BV's. In particular, if one starts with $T(1)$, one gets $B(\mathbf{Z}/2)$ as a summand. Finally, that $T(j)$ has unstable cohomology is explained by the fact that $T(j)$ is homotopic to a dual Brown–Gitler spectrum, which can be shown to be a wedge summand of suspension spectrum [L1, Goe, HK]. ([GLM] shows much more.)

In this paper, we first show that, at least on the level on cohomology, certain finite complexes $T(n,j)$ arising from $\Omega^{n+1}S^N$ appear to be unstable, and to be related to the Eilenberg-MacLane spaces $K(V,n)$ in the same way that the $T(j)$ are related to the spaces BV. Second, we let "n go to ∞", and obtain homotopical connections between these finite complexes and symmetric powers of spheres.

What I prove involves, first of all, some new observations about loopspace machinery and the Nishida relations which should be of independent interest. For Theorem 1.6 (which describes $T(n,j)$ as n goes to ∞), the author's old work on the Whitehead conjecture [K2, K3] is needed. The proof of Theorem 1.9 (which describes how the $T(n,j)$ are cohomologically related to $K(V,n)$'s) uses much of what the author knows about the relationship between the category of unstable modules over the Steenrod algebra and the "generic representation" category of [K6, K7, K8].

What I can't yet prove, but only conjecture, seems to suggest that there is a remarkable "naturally occurring" infinite loopspace (or perhaps E_∞–ring spectrum) waiting to be discovered.

To explain our main results, we need to introduce our cast of characters. Recall that [May], if X is path connected, there is a stable decomposition

$$\Sigma^\infty \Omega^n \Sigma^n X \simeq \bigvee_{j \geq 1} \Sigma^\infty D_{n,j} X,$$

where $D_{n,j}X = \mathcal{C}(n,j)_+ \wedge_{\Sigma_j} X^{[j]}$. Here $\mathcal{C}(n,j)$ is the configuration space of j tuples of distinct 'little cubes' in I^n, a space acted on freely by the j^{th} symmetric group Σ_j, and $X^{[j]}$ denotes the j-fold smash product of X with itself.

For a given n and j, there is a natural number d and a natural equivalence

$$D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X,$$

thus allowing $D_{n,j}X$ to be defined for a finite *spectrum*[1].

Definition 1.1. For $n \geq 0, j \geq 0$, let $T(n,j)$ be the S-dual of $D_{n+1,j}(S^{-n})$.

$T(n,j)$ is a finite spectrum with top cell in dimension nj, and with bottom mod 2 homology in dimension $n\alpha(j)$, where $\alpha(j)$ denotes the number of 1's in the 2-adic expansion of j. As examples, we note that, for all j and n, $T(0,j) = S^0 = T(n,0)$, $T(n,1) = S^n$, $T(1,j) = T(j)$ as above, and $T(n,2) =$ cofiber $\{\Sigma^n RP^{n-1}_+ \to S^n\}$.

This bigraded family of finite spectra has some extra structure we will need. The H-space structure on loopspaces induces copairings

$$\Psi : T(n,k) \to \bigvee_{i+j=k} T(n,i) \wedge T(n,j).$$

Evaluation on loopspaces induces maps

$$\delta : T(n,j) \to \Sigma^{-1} T(n+1,j).$$

Finally, looping Hopf invariants, together with the above periodicity, induces "Frobenious" maps

$$\Phi : T(n,j) \to T(n,2j).$$

These three families of maps will be shown to be compatible in the expected ways. In particular, δ and Φ commute up to homotopy.

Our first result is a description of $H^*(T(n,j); \mathbf{Z}/2)$ as a module over the mod 2 Steenrod algebra $\mathcal{A}$. Following the lead of others in the $n = 1$ case [Ca, Mi2, LZ1], we describe the bigraded object $H^*(T(n,*); \mathbf{Z}/2)$, with the extra structure afforded by Ψ^* and Φ^*. We need first to define variants on the category $\mathcal{U}$ of unstable $\mathcal{A}$ modules, and the category $\mathcal{K}$ of unstable $\mathcal{A}$ algebras.

Let $\mathcal{U}_\rho$ be the category whose objects are pairs (M, ρ): $M = M_{*,*}$ is an $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$ graded $\mathbf{Z}/2$ vector space[2] whose columns $M_{*,j}$ are unstable $\mathcal{A}$ modules, and ρ :

[1] There are more sophisticated ways to do this. See §2.

[2] Often $\mathbf{N} \times \mathbf{N}$ graded vector spaces will be considered $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$ graded by setting $M_{*,j} = \{0\}$ for $j \notin \mathbf{N}$.

$M \to M$ is a collection of $\mathcal{A}$ linear maps $\rho : M_{*,2j} \to M_{*,j}$. Morphisms in $\mathcal{U}_\rho$ are just maps $f : M \to N$ preserving all structure.

Let $\mathcal{K}_\rho$ be the category of "restricted algebras in $\mathcal{U}_\rho$", i.e. commutative, unital algebras K in $\mathcal{U}_\rho$ (a category with a tensor product) satisfying the "restriction axiom": $Sq^{|x|}x = (\rho(x))^2$ for all $x \in K$.

Let $U_\rho : \mathcal{U}_\rho \to \mathcal{K}_\rho$ be the free functor, left adjoint to the forgetful functor. Explicitly, $U_\rho(M, \rho) = S^*(M)/(Sq^{|x|}x - (\rho(x))^2)$.

If $I = (i_1, \dots, i_l)$, we set $Sq^I = Sq^{i_1} \dots Sq^{i_l}, l(I) = l$, and $e(I) = (i_1 - 2i_2) + \cdots + (i_{l-1} - 2i_l) + i_l$. I is called *admissible* if $i_s \geq 2i_{s+1}$ for all s. Define $E(n), L(k) \subset \mathcal{A}$ by

$$E(n) = \langle Sq^I \mid I \text{ is admissible and } e(I) > n \rangle$$
$$L(k) = \langle Sq^I \mid I \text{ is admissible and } l(I) > k \rangle.$$

Both of these are known to be left $\mathcal{A}$ modules [S, Prop.1.6.2], [Mi1] . Now let $F(n,k)$ be the unstable $\mathcal{A}$ module $\Sigma^n(A/(E(n) + L(k)))$, and then let $F_\rho(n) \in \mathcal{U}_\rho$ be the pair $(\bigoplus_{k \geq 0} F(n,k), \rho)$, where $F(n,k)$ has second grading 2^k, and $\rho : F(n, k+1) \to F(n,k)$ is the projection.

Theorem 1.2. *Let $n \geq 1$. With multiplication and restriction given by Ψ^* and Φ^*,*

$$H^*(T(n,*); \mathbf{Z}/2) \simeq U_\rho(F_\rho(n))$$

as objects in $\mathcal{K}_\rho$. In particular, $H^*(T(n,j); \mathbf{Z}/2)$ is an unstable $\mathcal{A}$ module.

This theorem suggests

Conjecture 1.3. $T(n,j)$ is a stable wedge summand of a suspension spectrum.

This is known to be true when $n = 1$ [L1, Goe, HK].

To discuss stablizing $T(n,j)$ with respect to δ, we make the following definition.

Definition 1.4. $T(\infty, j) = \text{hocolim } \{ \ T(0,j) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Sigma^{-n} T(n,j) \xrightarrow{\delta} \cdots \}$.

Theorem 1.5.
(1) $T(\infty, j) \simeq *$ *unless j is a power of 2.*
(2) $H^*((T(\infty, 2^k); \mathbf{Z}/2) \simeq \mathcal{A}/L(k)$ *as $\mathcal{A}$ modules.*

The $\mathcal{A}$ module $\mathcal{A}/L(k)$ is already known to arise as the cohomology of a spectrum: it is the cohomology of $SP^{2^k}_\Delta(S^0)$, the cofiber of the diagonal map $\Delta : SP^{2^{k-1}}(S^0) \to SP^{2^k}(S^0)$ between symmetric products of the sphere spectrum S^0 [MP].

Theorem 1.6. *The sequence* $T(\infty, 1) \to T(\infty, 2) \to T(\infty, 4) \to \dots$
is equivalent to the sequence $SP^1_\Delta(S^0) \to SP^2_\Delta(S^0) \to SP^4_\Delta(S^0) \to \dots$.
In particular, $T(\infty, 2^k) \simeq SP^{2^k}_\Delta(S^0)$.

Thus the maps $T(\infty, 2^k) \to T(\infty, 2^{k+1})$ have the striking properties proved in [K2], e.g. they induce the zero map in homotopy groups in positive degrees.

Corollary 1.7. $\underset{n,k \to \infty}{\text{hocolim}} \ \Sigma^{-n} T(n, 2^k) \simeq H\mathbf{Z}/2.$

We now turn our discussion to how $T(n,j)$ stablizes with respect to Φ.

Definition 1.8. $\Phi^{-1}T(n,j) = \operatorname{hocolim}\{\ T(n,j) \xrightarrow{\Phi} T(n,2j) \xrightarrow{\Phi} T(n,4j) \xrightarrow{\Phi} \cdots\}$

Our last theorem identifies $H^*(\Phi^{-1}T(n,j);\mathbf{Z}/2)$ as the cohomology of an infinite wedge of certain stable summands of the Eilenberg MacLane spaces $K(V,n)$, in a manner that is independent of n. In particular, just as $H^*(K(\mathbf{Z}/2,1);\mathbf{Z}/2)$ was shown in [Ca] to be an $\mathcal{A}$ module direct summand of $H^*(\Phi^{-1}T(1,1);\mathbf{Z}/2)$, so is $H^*(K(\mathbf{Z}/2,n);\mathbf{Z}/2)$ an $\mathcal{A}$ module summand of $H^*(\Phi^{-1}T(n,1);\mathbf{Z}/2)$.

To be more precise, we need yet more notation. As in [K6, K7, K8], let $\mathcal{F}$ be the category with objects the functors

$$F : \text{finite dimensional } \mathbf{Z}/2 \text{ vector spaces} \to\ \mathbf{Z}/2 \text{ vector spaces},$$

and with morphisms the natural transformations. For example, S^j and S_j, defined by $S^j(V) = V^{\otimes j}/\Sigma_j$ and $S_j(V) = (V^{\otimes j})^{\Sigma_j}$, are objects in $\mathcal{F}$.

Let Λ be an indexing set for the simple objects in this abelian category: algebraic group considerations suggest a number of Λ's, e.g. the set of 2-regular partitions [K7, Sections 5 and 6]. Given $\lambda \in \Lambda$, let $F_\lambda \in \mathcal{F}$ be the corresponding simple object, V_λ a vector space large enough so that $F_\lambda(V_\lambda) \neq 0$, $e_\lambda \in \mathbf{Z}_2[End(V_\lambda)]$ an idempotent chosen so that $\mathbf{Z}/2[End(V_\lambda)]e_\lambda$ is the projective cover of the $\mathbf{Z}/2[End(V_\lambda)]$ module $F_\lambda(V_\lambda)$, and $K(\lambda,n) = e_\lambda\Sigma^\infty K(V_\lambda,n)$ the corresponding stable summand of $K(V_\lambda,n)$. Finally, given $\lambda \in \Lambda$ and $j = 0,1,\dots,$ define $a(\lambda,j) \in \mathbf{N}$ by

$$a(\lambda,j) = \dim_{\mathbf{Z}/2} \operatorname{Hom}_{\mathcal{F}}(F_\lambda, S^{2^kj}),\ \text{for } k >> 0.$$

Theorem 1.9. $H^*(\Phi^{-1}T(n,j);\mathbf{Z}/2) \simeq H^*(\bigvee_{\lambda\in\Lambda} a(\lambda,j)K(\lambda,n);\mathbf{Z}/2)$ *as $\mathcal{A}$ modules.*

(Here $\bigvee_i b_iY_i$ means that each Y_i occurs in the wedge sum with multiplicity b_i.)

We remark that these large $\mathcal{A}$ modules are nevertheless of finite type.

Conjecture 1.10. $\Phi^{-1}T(n,j) \simeq \bigvee_{\lambda\in\Lambda} a(\lambda,j)K(\lambda,n)$.

Some form of the following has been known to the experts[3] since the late 1980's.

Proposition 1.11. *This conjecture is true when $n = 1$. In particular, $\Phi^{-1}T(1,1)$ has $B(\mathbf{Z}/2)$ as a stable summand.*

The organization of the rest of the paper is as follows.

§2,§3, and §4 are devoted to the geometric constructions used to define the three families of maps Ψ,Φ,δ on the $T(n,j)$. In hopes that these will be useful in other settings, we develop this material with perhaps more care than is traditional (at one point, proving a lemma using ideas from "Goodwillie calculus"). Theorem 2.4 summarizes our main geometric results. In §5, properties of these constructions are combined with standard formula [CLM] for the homology of iterated loopspaces to give descriptions of $H^*(T(n,j);\mathbf{Z}/2),\Psi^*,\Phi^*,$ and δ^* in terms of Dyer-Lashof-like operations. The standard Nishida relations then yield recursive formulae for how $\chi(Sq^i)$ acts on $H^*(T(n,j);\mathbf{Z}/2)$; we deduce more useful formulae for how Sq^i acts in

[3] By "experts" here I mean at least the authors of [LS], as well as myself.

§6. These should be of some independent interest. Theorem 1.2 and Theorem 1.5 are then deduced in §7. Using the author's proof of the Whitehead conjecture, Theorem 1.6 is quickly deduced from Theorem 1.5 in §8.

The proof of Theorem 1.9 is rather different. Recall [K6] that there are adjoint functors

$$\mathcal{U} \underset{r}{\overset{l}{\rightleftarrows}} \mathcal{F},$$

where $r(F) = \mathrm{Hom}_{\mathcal{F}}(S_*, F)$, with the Steenrod operations acting on the right of the S_j in the obvious way. Let $I_\lambda \in \mathcal{F}$ be the injective envelope of the simple functor F_λ, and let $\Phi^{-1}S^j \in \mathcal{F}$ be defined by

$$\Phi^{-1}S^j = \mathrm{colim}\left\{S^j \xrightarrow{\Phi} S^{2j} \xrightarrow{\Phi} S^{4j} \xrightarrow{\Phi} S^{8j} \cdots\right\},$$

where $\Phi : S^j \to S^{2j}$ is the squaring map.

The "Vanishing Theorem" of [K7] says that $\Phi^{-1}S^j$ is an injective object in the category $\mathcal{F}_\omega \subset \mathcal{F}$ of locally finite functors. It follows formally that there is a decomposition in $\mathcal{F}$

$$\Phi^{-1}S^j \simeq \bigoplus_{\lambda\in\Lambda} a(\lambda, j)I_\lambda.$$

Precomposing this with the functor S_n, and then applying the functor r, yields a decomposition in $\mathcal{U}$

$$\Phi^{-1}r(S^j \circ S_n) \simeq \bigoplus_{\lambda\in\Lambda} a(\lambda, j)r(I_\lambda \circ S_n).$$

The classical description of $H^*(K(V, n); \mathbf{Z}/2)$ reveals that $r(I_\lambda \circ S_n) = H^*(K(\lambda, n); \mathbf{Z}/2)$, so the righthand side of this last decomposition agrees with the righthand side of the the isomophism in Theorem 1.9. Meanwhile, the lefthand side of the isomorphism of Theorem 1.9 is known by Theorem 1.2; this is then shown to agree with $\Phi^{-1}r(S^j \circ S_n)$ by using a new result of ours [K9] that calculates $r(S^j \circ F)$ as a functor of $r(F)$.

§9 contains the details of this outline of the proof of Theorem 1.9. Finally in §10, we prove Proposition 1.11, as well as discussing approaches to the conjectures. In the appendix, we relate our spectra $T(\infty, 2^k)$ to work of Arone and Mahowald [AM]. Theorem 1.6 thus gives new information about their constructions.

We wish to give hearty thanks to Doug Ravenel. This project had its origins in a question that he was asking in late 1994: our Conjecture 1.10 amounts to a refinement and extension of this. Most of our results were presented in Gargnano, Italy in 1995 and Toronto, Canada in 1996. An earlier version of this preprint was circulated in the summer of 1996. In the two years that have followed, we have noticed that Theorem 1.6 follows from Theorem 1.5, and managed to connect our constructions to those of Arone and Mahowald. We hope the reader familiar with our older version will appreciate these improvements.

2. Geometric constructions

We begin by being a bit more specific about some notation introduced in the introduction. A point $\mathbf{c} \in \mathcal{C}(n,j)$ is a j tuple $\mathbf{c} = (c_1, \dots c_j)$ in which each $c_i : I^n \to I^n$ is a product of n linear embeddings from the unit interval I to itself, and the interiors of the images of the c_i are disjoint. Then the book of Gaunce Lewis, et. al. [LMMS] shows that the functor

$$D_{n,j}X = \mathcal{C}(n,j)_+ \wedge_{\Sigma_j} X^{[j]}$$

is well defined in the category of spectra.

Standard properties of equivariant homotopy then allow us to write

$$\begin{aligned} T(n,j) &= F(D_{n,j}S^{-n}, S^0) \\ &= F(\mathcal{C}(n,j)_+ \wedge_{\Sigma_j} S^{-nj}, S^0) \\ &= F(\mathcal{C}(n,j)_+, S^{nj})^{\Sigma_j}. \end{aligned}$$

This gives an interesting alternative (and technically simpler) definition of the spectra $T(n,j)$, reminiscent of some of the constructions recently occurring in the "Goodwillie Calculus" literature [AM]. (See the Appendix.)

Definition 2.1. Let $\tilde{D}_{n,j}X = F(\mathcal{C}(n,j)_+, X^{[j]})^{\Sigma_j}$.

With this definition, we have $T(n,j) = \tilde{D}_{n+1,j}S^n$, and, more generally, if X is a finite spectrum, then $\tilde{D}_{n,j}X =$ S-dual $(D_{n,j}(\text{S-dual }(X)))$.

In the usual way, the little cubes operad structure on the spaces $\mathcal{C}(n,j)$ induces natural maps

$$\mu : D_{n,i}X \wedge D_{n,j}X \to D_{n,i+j}X,$$

$$\Theta : D_{n,i}D_{n,j}X \to D_{n,ij}X,$$

and dually, natural maps

$$\Psi : \tilde{D}_{n,i+j}X \to \tilde{D}_{n,i}X \wedge \tilde{D}_{n,j}X,$$

and

$$\Gamma : \tilde{D}_{n,ij}X \to \tilde{D}_{n,i}\tilde{D}_{n,j}X.$$

In particular, we obtain maps

$$\Psi : T(n, i+j) \to T(n,i) \wedge T(n,j),$$

and

$$\Gamma : T(n, 2j) \to \tilde{D}_{n+1,2}T(n,j).$$

These two families of maps provide sufficient structure for the purposes of computing the mod 2 cohomology of the $T(n,j)$.

We turn our attention to constructing the maps

$$\delta : T(n,j) \to \Sigma^{-1}T(n+1,j).$$

In [K1] we noted that the evaluation map

$$\epsilon : \Sigma\Omega^{n+1}\Sigma^{n+1}X \to \Omega^n\Sigma^{n+1}X$$

induces maps

$$\epsilon : \Sigma D_{n+1,j} X \to D_{n,j} \Sigma X.$$

We note that the same geometric construction also yields natural maps

$$\delta : \tilde{D}_{n,j} X \to \Sigma^{-1} \tilde{D}_{n+1,j} \Sigma X.$$

Both of these families are induced by explicit Σ_j equivariant maps

$$\beta : \mathcal{C}(n+1,j)_+ \wedge S^1 \to \mathcal{C}(n,j)_+ \wedge S^j,$$

defined as follows.

Given a linear embedding $c : I \to I$, let $c^* : I \to I$ be the associated "Thom-Pontryagin collapse" map. Explicitly,

$$c^*(t) = \begin{cases} 0 & \text{if } t \leq Im(c) \\ s & \text{if } c(s) = t \\ 1 & \text{if } t \geq Im(c). \end{cases}$$

Note that $(c \circ d)^* = d^* \circ c^*$.

Given a little $n+1$ cube $c : I^{n+1} \to I^{n+1}$, we write $c = c' \times c''$, where $c' : I^n \to I^n$, and $c'' : I \to I$. Regarding S^1 as $I/\partial I$, and S^j as $(I/\partial I)^{[j]}$, we have the following definition.

Definition 2.2. (Compare with [May, page 47].)

$$\beta(c_1, \ldots, c_j, t) = (c_1', \ldots, c_j', c_1''^*(t), \ldots, c_j''^*(t)).$$

A straightforward check of definitions yields the next proposition, which shows how δ is related to the maps Ψ and Γ.

Proposition 2.3.
(1) *The composite* $\Sigma \tilde{D}_{n,i+j} X \xrightarrow{\delta} \tilde{D}_{n+1,i+j} \Sigma X \xrightarrow{\Psi} \tilde{D}_{n+1,i} \Sigma X \wedge \tilde{D}_{n+1,j} \Sigma X$ *is null if* $i > 0$ *and* $j > 0$.
(2) *There are commutative diagrams:*

$$\begin{array}{ccccc} \Sigma \tilde{D}_{n,ij} X & & \xrightarrow{\delta} & & \tilde{D}_{n+1,ij} X \\ \downarrow \Sigma\Gamma & & & & \downarrow \Gamma \\ \Sigma \tilde{D}_{n,i} \tilde{D}_{n,j} X & \xrightarrow{\delta} & \tilde{D}_{n+1,i} \Sigma \tilde{D}_{n,j} X & \xrightarrow{\tilde{D}_{n+1,i}\delta} & \tilde{D}_{n+1,i} \tilde{D}_{n+1,j} \Sigma X. \end{array}$$

Our last and most delicate construction is of the family

$$\Phi : T(n,j) \to T(n,2j).$$

The next theorem summarizes the properties we need to know.

Theorem 2.4. *There exist maps* $\Phi_{n,j} : T(n,j) \to T(n,2j)$ *such that the following five properties hold.*
(1) $\Phi_{0,j} : T(0,j) = S^0 \to T(0,2j) = S^0$ *is multiplication by* $(2j)!/j!2^j$.
(2) *There are commutative diagrams:*

$$\begin{array}{ccc} \Sigma T(n,j) & \xrightarrow{\Sigma\Phi_{n,j}} & \Sigma T(n,2j) \\ \big\downarrow{\scriptstyle \delta} & & \big\downarrow{\scriptstyle \delta} \\ T(n+1,j) & \xrightarrow{\Phi_{n+1,j}} & T(n+1,2j). \end{array}$$

(3) *For* $n \geq 1$, *there are commutative diagrams:*

$$\begin{array}{ccc} T(n,i+j) & \xrightarrow{\Phi_{n,i+j}} & T(n,2(i+j)) \\ \big\downarrow{\scriptstyle \Psi} & & \big\downarrow{\scriptstyle \Psi} \\ T(n,i) \wedge T(n,j) & \xrightarrow{\Phi_{n,i}\wedge\Phi_{n,j}} & T(n,2i) \wedge T(n,2j). \end{array}$$

(4) *If* $n \geq 1$, i *and* j *are odd, and* $i + j = 2k$, *the composite*

$$T(n,k) \xrightarrow{\Phi_{n,k}} T(n,2k) \xrightarrow{\Psi} T(n,i) \wedge T(n,j)$$

is null.
(5) *For* $n \geq 1$, *there are commutative diagrams:*

$$\begin{array}{ccc} T(n,2j) & \xrightarrow{\Phi_{n,2j}} & T(n,4j) \\ \big\downarrow{\scriptstyle \Gamma} & & \big\downarrow{\scriptstyle \Gamma} \\ \tilde{D}_{n,2}T(n,j) & \xrightarrow{\tilde{D}_{n,2}\Phi_{n,j}} & \tilde{D}_{n,2}T(n,2j). \end{array}$$

Proof. Fix $N \geq 0, J \geq 0$. Let $\mathcal{S}(N,J)$ be the collection of sets of maps $S = \{ \Phi_{n,j} \mid n \leq N, j \leq J \}$ such that properties (1) – (5) are true whenever the maps $\Phi_{n,j}$ appearing in those statements are chosen from S. (In other words, $S \in \mathcal{S}(N,J)$ makes true a finite number of the infinite lists of statements in (1) – (5).)

There are restriction maps $\mathcal{S}(N,J) \to \mathcal{S}(N-1,J)$ and $\mathcal{S}(N,J) \to \mathcal{S}(N,J-1)$. The theorem amounts to saying that the inverse limit, $\lim \mathcal{S}(N,J)$, taken over all N and J, is nonempty.

Since (1) and (2) determine $\Phi_{0,j}$ and $\Phi_{n,0}$, $\mathcal{S}(N,J)$ can be regarded as a subset of $\prod_{n=1}^{N}\prod_{j=1}^{J}\{T(n,j),T(n,2j)\}$, which is finite, as each $T(n,j)$ is a finite complex, and each $T(n,j)$ with $n \geq 1, j \geq 2$ is torsion. Since the inverse limit of nonempty finite sets is nonempty[4], the next theorem completes the proof of the theorem. □

Theorem 2.5. $\mathcal{S}(N,J)$ *is nonempty.*

[4] A standard application of the Tychonoff Theorem.

There are two ingredients in our construction of a set $\{\Phi_{n,j}\} \in \mathcal{S}(N,J)$. The first is the use of vector bundle trivializations to construct natural equivalences

$$\omega_{n,j} : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X,$$

for n and j in any *finite* range, compatible with the structure maps (ϵ, μ, Θ). The second is the use of Hopf invariants to construct maps, for $d > n$,

$$h^d_{n,j} : D_{n+1,2j} S^{d-n} \to D_{n+1,j} S^{2d-n}$$

with appropriate properties.

The next two theorems, whose proofs occupy the next two sections, more precisely describe what we need.

Theorem 2.6. *Fix N and J. Then there exists $d > 0$, and natural equivalences*

$$\omega_{n,j} : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j} X,$$

defined for $1 \le n \le N, 1 \le j \le J$, such that the following diagrams commute:
(1) *for all $1 \le n \le N-1, 1 \le j \le J$,*

$$\begin{array}{ccc} \Sigma D_{n+1,j}(\Sigma^d X) & \xrightarrow{\omega_{n+1,j}} & \Sigma^{1+dj} D_{n+1,j}(X) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle (-1)^{d(j-1)}\epsilon} \\ D_{n,j}(\Sigma^{d+1} X) & \xrightarrow{\omega_{n,j}} & \Sigma^{dj} D_{n,j}(\Sigma X), \end{array}$$

(2) *for all $1 \le n \le N, i+j \le J$,*

$$\begin{array}{ccc} D_{n,i}(\Sigma^d X) \wedge D_{n,j}(\Sigma^d X) & \xrightarrow{\omega_{n,i} \wedge \omega_{n,j}} & \Sigma^{di} D_{n,i}(X) \wedge \Sigma^{dj} D_{n,j}(X) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ D_{n,i+j}(\Sigma^d X) & \xrightarrow{\omega_{n,i+j}} & \Sigma^{d(i+j)} D_{n,i+j}(X), \end{array}$$

(3) *for all $1 \le n \le N, ij \le J$,*

$$\begin{array}{ccccc} D_{n,i} D_{n,j}(\Sigma^d X) & \xrightarrow{D_{n,i}\omega_{n,j}} & D_{n,i} \Sigma^{dj} D_{n,j}(X) & \xrightarrow{\omega^j_{n,i}} & \Sigma^{dij} D_{n,i} D_{n,j}(X) \\ \downarrow{\scriptstyle \Theta} & & & & \downarrow{\scriptstyle \Theta} \\ D_{n,ij}(\Sigma^d X) & & \xrightarrow{\omega_{n,ij}} & & \Sigma^{dij} D_{n,ij}(X). \end{array}$$

Theorem 2.7. *For all $0 \le n < d$ and for all j, there exist maps*

$$h^d_{n,j} : D_{n+1,2j} S^{d-n} \to D_{n+1,j} S^{2d-n}$$

with the following properties.
(1) *If d is even, $h^d_{0,j} : D_{1,2j} S^d = S^{2jd} \to D_{1,j} S^{2d} = S^{2jd}$ is multiplication by*

$(2j)!/j!2^j$.

(2) *There are commutative diagrams:*

$$\begin{array}{ccc} \Sigma D_{n+1,2j}S^{d-n} & \xrightarrow{\Sigma h^d_{n,j}} & \Sigma D_{n+1,j}S^{2d-n} \\ \downarrow \epsilon & & \downarrow \epsilon \\ D_{n,2j}S^{d-n+1} & \xrightarrow{h^d_{n-1,j}} & D_{n,j}S^{2d-n+1}. \end{array}$$

(3) *There are commutative diagrams:*

$$\begin{array}{ccc} D_{n+1,2i}S^{d-n} \wedge D_{n+1,2j}S^{d-n} & \xrightarrow{h^d_{n,i}\wedge h^d_{n,j}} & D_{n+1,i}S^{2d-n} \wedge D_{n+1,j}S^{2d-n} \\ \downarrow \mu & & \downarrow \mu \\ D_{n+1,2(i+j)}S^{d-n} & \xrightarrow{h^d_{n,i+j}} & D_{n+1,i+j}S^{2d-n}. \end{array}$$

(4) *If i and j are odd, and $i+j=2k$, the composite*

$$D_{n+1,i}S^{d-n} \wedge D_{n+1,j}S^{d-n} \xrightarrow{\mu} D_{n+1,2k}S^{d-n} \xrightarrow{h^d_{n,k}} D_{n+1,k}S^{2d-n}$$

is null.

(5) *There are commutative diagrams:*

$$\begin{array}{ccc} D_{n,2}D_{n+1,2j}S^{d-n} & \xrightarrow{D_{n,2}h^d_{n,j}} & D_{n,2}D_{n+1,j}S^{2d-n} \\ \downarrow \Theta & & \downarrow \Theta \\ D_{n+1,4j}S^{d-n} & \xrightarrow{h^d_{n,2j}} & D_{n+1,2j}S^{2d-n}. \end{array}$$

Assuming these two theorems, we note that Theorem 2.5 follows easily. First choose d as in Theorem 2.6 (but with J replaced by $2J$). We can also assume d is even. Then, with $h^d_{n,j}$ as in Theorem 2.7, we define $\Phi_{n,j}: T(n,j) \to T(n,2j)$ to be the S-dual of the composite

$$D_{n+1,2j}S^{-n} \xrightarrow{\omega^{-1}_{n,2j}} \Sigma^{-2dj}D_{n+1,2j}S^{d-n} \xrightarrow{h^d_{n,j}} \Sigma^{-2dj}D_{n+1,j}S^{2d-n} \xrightarrow{\omega^2_{n,j}} D_{n+1,j}S^{-n}.$$

Courtesy of Theorem 2.6, each statement in Theorem 2.7 translates immediately into the corresponding statement in Theorem 2.4, proving Theorem 2.5.

3. Quasiperiodicity of the sphere spectrum

In this section we prove Theorem 2.6, which asserts that given N and J, there exists $d > 0$ and natural equivalences

$$\omega_{n,j}: D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j}X,$$

defined for $1 \le n \le N, 1 \le j \le J$ which are appropriately compatible with the three families of structure maps

$$\epsilon: \Sigma D_{n+1,j}X \to D_{n,j}\Sigma X,$$

$$\mu: D_{n,i}X \wedge D_{n,j}X \to D_{n,i+j}X, \text{ and}$$

$$\Theta : D_{n,i}D_{n,j}X \to D_{n,ij}X.$$

To put this theorem in context, recall that as an aid to constructing power operations and studying Thom isomorphisms, the authors of [BMMS] defined the notion of an H^d_∞–ring spectrum. For the sphere spectrum S^0 to admit an H^d_∞ structure would be roughly equivalent to natural equivalences $\omega_{n,j}$ as in the theorem for all $n < \infty, j < \infty$. Though it is easy to see that this cannot be done, our theorem says that it partially *can* be. If one defines the notion of an H^d_n structure in the obvious way, we know of no reason why the following conjecture might not be true.

Conjecture 3.1. Localized at a prime p, for each n, S^0 admits the structure of an H^d_n–ring spectrum for some $d > 0$.

The origin of the natural equivalences is as follows.

Suppose ξ and ζ are two r dimensional vector bundle over a space B, respectively classified by maps $f_\xi, f_\zeta : B \to BO$. Then a homotopy $H : B \times I \to BO$ between f_ξ and f_ζ induces an bundle isomorphism $\omega_H : \xi \to \zeta$ and thus a homeomorphism $\omega_H : M(\xi) \to M(\zeta)$ of Thom spaces. In particular, given a map $i : B \to C$ to a contractible space C, and an extension $F : C \to BO$ of f_ξ, there is an induced homeomorphism of spaces

$$\omega_F : M(\xi) \to \Sigma^r(B_+).$$

Furthermore, given a second extension $F' : C' \to BO$, ω_F and $\omega_{F'}$ will be homotopic if the map

$$F \cup_{f_\xi} F' : C \cup_B C' \to BO$$

is null. This last map can be regarded an obstruction $o(F, F') : \Sigma B \to BO$.

We apply these general remarks to the case of interest. Let $\xi_{n,j}$ be the vector bundle

$$\mathcal{C}(n,j) \times_{\Sigma_j} \mathbf{R}^j \to B(n,j) = \mathcal{C}(n,j)/\Sigma_j,$$

with classifying map $f_{n,j} : B(n,j) \to BO$. This is easily seen to be a bundle of finite order, and an extension $F : CB(n,j) \to BO$ of $df_{n,j}$ to the cone on $B(n,j)$ induces a homeomorphism

$$\omega_F : \mathcal{C}(n,j)_+ \wedge_{\Sigma_j} S^{dj} \to \Sigma^{ds}(B(n,j)_+),$$

and thus a Σ_j–equivariant homeomorphism

$$\omega_F : \mathcal{C}(n,j)_+ \wedge S^{dj} \to \Sigma^{ds}(\mathcal{C}(n,j)_+),$$

and finally a natural equivalence

$$\omega_F : D_{n,j}(\Sigma^d X) \simeq \Sigma^{dj} D_{n,j}X.$$

A straightforward check of definitions shows

Lemma 3.2. *In this situation, if $F : CB(n,j) \to BO$ is the restriction of a map $F' : CB(n+1,j) \to BO$ extending $df_{n+1,j}$ then the following diagram commutes:*

$$\begin{array}{ccc} \Sigma D_{n+1,j}(\Sigma^d X) & \xrightarrow{\omega_{F'}} & \Sigma^{1+dj} D_{n+1,j}(X) \\ \downarrow \epsilon & & \downarrow (-1)^{d(j-1)}\epsilon \\ D_{n,j}(\Sigma^{d+1}X) & \xrightarrow{\omega_F} & \Sigma^{dj} D_{n,j}(\Sigma X). \end{array}$$

Now fix N and J as in Theorem 2.6. Let $d > 0$ and let $\mathcal{F} = \{F_j : CB(N,j) \to BO \mid j = 1, \ldots, J\}$ be a collection of extensions of the maps $df_{N,j}$. We define the obstruction set $o(\mathcal{F})$ to be the following set of maps:

$$o^{\mu}_{i,j}(\mathcal{F}) : \Sigma(B(N,i) \times B(N,j)) \to BO,$$

for $i + j = J$, and

$$o^{\Theta}_{i,j}(\mathcal{F}) : \Sigma(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i) \to BO,$$

for $ij = J$, where these maps are defined as follows.

For $o^{\mu}_{i,j}(\mathcal{F})$, we regard $\Sigma(B(N,i) \times B(N,j))$ as

$$C(B(N,i) \times B(N,j)) \cup_{B(N,i)\times B(N,j)} CB(N,i) \times CB(N,j),$$

and we let

$$o^{\mu}_{i,j}(\mathcal{F}) = \begin{cases} F_{i+j} \circ \mu & \text{on } C(B(N,i) \times B(N,j)), \\ \mu_{BO} \circ (F_i \times F_j) & \text{on } CB(N,i) \times CB(N,j). \end{cases}$$

Here $\mu_{BO} : BO \times BO \to BO$ is the H-space structure map.

For $o^{\Theta}_{i,j}(\mathcal{F})$, we regard $\Sigma(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i)$ as

$$C(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i) \cup_{\mathcal{C}(N,i)\times_{\Sigma_i} B(N,j)^i} \mathcal{C}(N,i) \times_{\Sigma_i} CB(N,j)^i,$$

and we let

$$o^{\Theta}_{i,j}(\mathcal{F}) = \begin{cases} F_{ij} \circ \Theta & \text{on } C(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i), \\ \Theta_{BO} \circ (Id \times_{\Sigma_i} (F_j)^i) & \text{on } \mathcal{C}(N,i) \times_{\Sigma_i} CB(N,j)^i. \end{cases}$$

Here $\Theta_{BO} : \mathcal{C}(n,i) \times_{\Sigma_i} BO^i \to BO$ is the infinite loopspace structure map.

Theorem 2.6 will follow if we can show that there is a choice of d and $\mathcal{F}$ for which $o(\mathcal{F})$ is a set of null maps. Firstly, we note that there *do* exist collections $\mathcal{F}$ as above: we just need to choose d equal to a common multiple of the orders of the bundles $\xi_{N,1}, \ldots, \xi_{N,J}$. By making d possibly bigger, we can even ensure that $\mathcal{F}$ is the restriction of a similar family $\tilde{\mathcal{F}}$ defined for the pair $(N+1, J)$, and the obstruction set $o(\mathcal{F})$ is the restriction of $o(\tilde{\mathcal{F}})$.

Given a family $\mathcal{F}$, let $r\mathcal{F}$ be the family with j^{th} function equal to rF_j. Note that if F_j extends $df_{N,j}$, then rF_j extends $(rd)f_{N,j}$. It is easy to check

Lemma 3.3.
(1) $o^{\mu}_{i,j}(r\mathcal{F}) = ro^{\mu}_{i,j}(\mathcal{F}) \in K^1(B(N,i) \times B(N,j))$.
(2) $o^{\Theta}_{i,j}(r\mathcal{F}) = ro^{\Theta}_{i,j}(\mathcal{F}) \in K^1(\mathcal{C}(N,i) \times_{\Sigma_i} B(N,j)^i)$.

Proposition 3.4. *Let $X(N)$ be one of the spaces $B(N,j)$, $B(N,i) \times B(N,j)$, or $\mathcal{C}(n,i) \times_{\Sigma_i} B(N,j)^i$. If $x \in K^*(X(N))$ is in the image of the restriction from $K^*(X(N+1))$, then x is torsion.*

Postponing the proof of this proposition for the moment, we show that there is a choice of d and $\mathcal{F}$ for which $o(\mathcal{F})$ is a set of null maps. Start with any family $\mathcal{F}$ (and associated d) as above. Let r be a common multiple of the orders of the obstructions $o^{\mu}_{i,j}(\mathcal{F})$ and $o^{\Theta}_{i,j}(\mathcal{F})$. (Proposition 3.4 tells us that these elements *do* have finite order.) Then the family $r\mathcal{F}$ has an obstruction set consisting only of null maps, as needed.

It remains to prove Proposition 3.4. This will follow from three lemmas.

Lemma 3.5. *Let $f : X \to Y$ be a map between finite complexes. If $H_*(f;\mathbf{Q}) = 0$, then $Im\{E^*(f) : E^*(Y) \to E^*(X)\}$ is torsion for all generalized cohomology theories E^*.*

Proof. For finite complexes Z, $E^*(Z_{\mathbf{Q}}) \simeq E^*(Z) \otimes \mathbf{Q}$. $H_*(f;\mathbf{Q}) = 0$ implies that $f_{\mathbf{Q}} \simeq *$, and thus that $E^*(f) \otimes \mathbf{Q} = 0$. □

Lemma 3.6. *If $X(N)$ is as in Proposition 3.4, $X(N)$ has the homotopy type of a finite complex.*

Proof. There are many ways to see this. The author's favorite is to note that the explicit cell decomposition for $B(2,j)$ given by Fox and Neuwirth in [FN] generalizes to $B(n,j)$: $B(n,j)$ has the homotopy type of an $(n-1)(j-1)$ dimensional cell complex with exactly n^{j-1} cells. □

Lemma 3.7. *With $X(N)$ as in Proposition 3.4,*

$$H_*(X(N);\mathbf{Q}) \to H_*(X(N+1);\mathbf{Q})$$

is 0.

Proof. This follows from standard homology calculations [CLM]. □

4. Hopf invariants

In this section we use Hopf invariants to define maps

$$h^d_{n,j} : D_{n+1,2j}S^{d-n} \to D_{n+1,j}S^{2d-n},$$

for $0 \leq n < d$, and then show that they have the properties listed in Theorem 2.7.

The maps are not hard to define. Let

$$H_Y : \Omega\Sigma Y \to \Omega\Sigma(Y \wedge Y)$$

be the classic Hopf invariant. Replacing Y by $\Sigma^n X$, and looping n times, defines an unstable natural map

$$\Omega^n H_{\Sigma^n X} : \Omega^{n+1}\Sigma^{n+1}X \to \Omega^{n+1}\Sigma^{n+1}(\Sigma^n X \wedge X).$$

Now let $D_n X$ denote $\bigvee_{j=1}^{\infty} D_{n,j}X$, and, for connected X, let

$$s_n : D_n X \simeq \Omega^n \Sigma^n X$$

be the natural *stable* Snaith equivalence as studied in [LMMS, Chapter VII]. Finally,

$$H_n(X) : D_{n+1}X \to D_{n+1}(\Sigma^n X \wedge X)$$

will be the stable map given by the composite $s^{-1}_{n+1} \circ (\Omega^n H_{\Sigma^n X}) \circ s_{n+1}$.

Definition 4.1. For all $0 \leq n < d$, and for all j,

$$h^d_{n,j} : D_{n+1,2j} S^{d-n} \to D_{n+1,j} S^{2d-n}$$

is defined to be the $(2j,j)^{th}$ component of $H_n(S^{d-n})$.

The first of the properties in Theorem 2.7 is easily checked. If d is even, $h^d_{0,j} : S^{2jd} \to S^{2jd}$ is multiplication by $(2j)!/j!2^j$, as cup product considerations easily show that $H : \Omega S^{d+1} \to S^{2d+1}$ induces multiplication by this number in cohomology in dimension $2dj$ [H, p.294].

Property (2) of Theorem 2.7, the compatibility of $h^d_{n,j}$ with the maps ϵ, follows from the main result of [K1]: under the Snaith equivalence, the evaluation

$$\epsilon : \Sigma\Omega^{n+1}\Sigma^{n+1}X \to \Omega^n\Sigma^{n+1}X$$

is carried to

$$\bigvee_{j=1}^{\infty} \epsilon : \bigvee_{j=1}^{\infty} \Sigma D_{n+1,j} X \to \bigvee_{j=1}^{\infty} D_{n,j}\Sigma X.$$

The remaining three properties follow from the next two propositions.

Proposition 4.2. *There is a commutative diagram:*

$$\begin{array}{ccc} D_n D_{n+1} X & \xrightarrow{D_n H_n(X)} & D_n D_{n+1}(\Sigma^n X \wedge X) \\ \downarrow \Theta & & \downarrow \Theta \\ D_{n+1} X & \xrightarrow{H_n(X)} & D_{n+1}(\Sigma^n X \wedge X). \end{array}$$

Here $\Theta : D_n D_{n+1} X \to D_{n+1} X$ is the restriction of the structure map $\Theta : D_{n+1} D_{n+1} X \to D_{n+1} X$.

Proposition 4.3. *The $(i,j)^{th}$ component of $H_n(X)$ is null unless $i \leq 2j$.*

This tells us that $H_n(X)$ can be regarded as an "upper triangular matrix" of maps. With this information fed into Proposition 4.2, the three last properties of Theorem 2.7 can be read off immediately.

Proof of Proposition 4.3. The $(i,j)^{th}$ component of $H_n(X)$ is a natural transformation

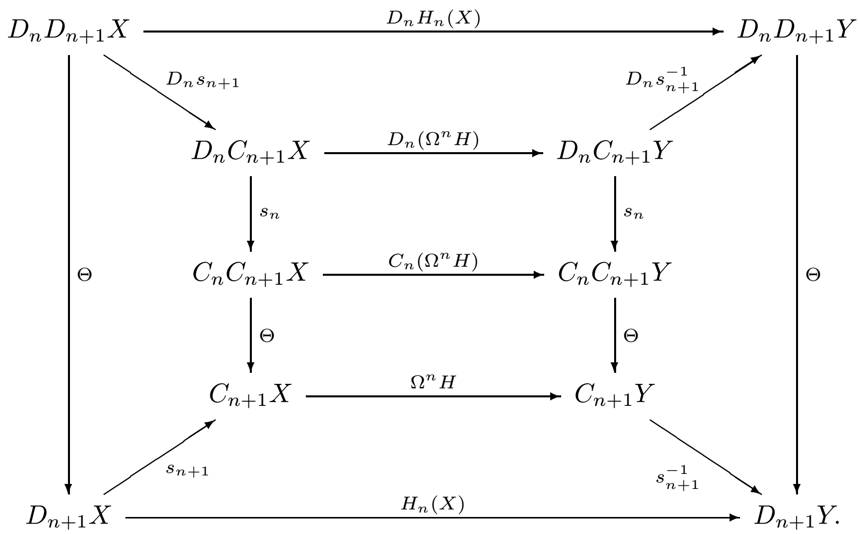
$$D_{n+1,i} X \to D_{n+1,j}(\Sigma^n X \wedge X).$$

In the terminology of [Goo], the domain is a homogeneous functor of degree i, while the range is a functor of degree $2j$. Thus there are *no* nontrivial natural transformations from the former to the latter if $i > 2j$. $\square$

Remark 4.4. This proposition presumably has a direct proof, along the lines of the proofs of similar results in [K5].

Proof of Proposition 4.2. This is a consequence of the fact that $H_n(X)$ corresponds to an n fold loop map. Let C_nX denote the usual approximation to $\Omega^n\Sigma^nX$, with monad structure map $\Theta : C_nC_n \to C_n$, and let Y denote $\Sigma^nX \wedge X$.

With this notation, we assert that there is a commutative digram:



The lower central square commutes since Ω^nH is a C_n–map. The upper square commutes by naturality. Finally the argument in [K4, §4] shows that the two side trapezoids commute. □

5. Cohomology calculations

We use the following notational conventions in the next three sections. $H_*(X)$ and $H^*(X)$ will denote homology and cohomology with $\mathbf{Z}/2$ coefficients. The binomial coefficient $\binom{b}{a}$ is defined, for *all* integers a and b, as the a^{th} Taylor coefficient of $(x+1)^b$ if $a \geq 0$, and 0 otherwise. We will use, without further comment, that $\binom{b}{a} = \binom{a-b-1}{a}$.

In this section we describe $H^*(T(n,*))$, and the maps Ψ^*, Φ^*, and δ^*, in terms of "dual" Dyer-Lashof operations. We begin by remarking that since $T(n,j)$ is the S–dual of $D_{n+1,j}S^{-n}$, and $H_*(D_{n+1,j}S^{-n})$ embeds in $H_*(D_{\infty,j}S^{-n})$, we will not need to confront the Browder operations, and the "top" Dyer-Lashof operation will be additive (as are the others).

As part of the general theory [CLM], the product maps μ induce a bigraded product on $H_*(D_{n+1,*}S^{-n})$, and associated to the structure maps Θ, there are Dyer-Lashof operations

$$Q^s : H_q(D_{n+1,j}S^{-n}) \to H_{q+s}(D_{n+1,2j}S^{-n}).$$

These are defined for $s \leq q+n$, and are 0 for $s < q$. Furthermore, these satisfy the Cartan formula, Adem relations, and restriction axiom: $Q^{|x|}x = x^2$. $H_*(D_{n+1,*}S^{-n})$ is the free object with all this structure, generated by a class in degree $-n$.

There is a canonical isomorphism $H^q(T(n,j)) = H_{-q}(D_{n+1,j}S^{-n})$. Under this isomorphism, Ψ^* will correspond to μ_*, and will induce a bigraded product (occasionally denoted " $*$ ") on $H^*(T(n,*))$. We define operations

$$\tilde{Q}^s : H^q(T(n,j)) \to H^{q+s}(T(n,2j))$$

to correspond to

$$Q^{-s} : H_{-q}(D_{n+1,j}S^{-n}) \to H_{-q-s}(D_{n+1,2j}S^{-n}).$$

These are defined for $s \geq q-n$, and are 0 for $s > q$. These satisfy the Cartan formula,

$$\tilde{Q}^t(x * y) = \sum_{r+s=t} \tilde{Q}^r x * \tilde{Q}^s y,$$

Adem relations,

$$\tilde{Q}^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} \tilde{Q}^i x,$$

and restriction axiom,

$$\tilde{Q}^{|x|} x = x^2.$$

(We note that in the Adem relations, whenever the iterated operation on the left is defined, so are those appearing with nonzero coefficient on the right, though not conversely[5].)

Theorem 5.1. $H^*(T(n,*))$ *is the free object with all this structure, generated by a class x_n in degree n. Explicitly, if*

$$\tilde{R}_n = \langle \tilde{Q}^I x_n \mid I \text{ is admissible } \rangle / \langle \tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) > n \rangle,$$

$H^*(T(n,*)) = S^*(\tilde{R}_n)/(\tilde{Q}^{|x|}x - x^2)$. *Thus, as a bigraded algebra, $H^*(T(n,*))$ is a polynomial algebra on the set $\{\tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) < n\}$, with $\tilde{Q}^I x_n \in H^*(T(n, 2^{l(I)}))$.*

Here, if $I = (i_1, \dots, i_l)$, $\tilde{Q}^I = \tilde{Q}^{i_1} \dots \tilde{Q}^{i_l}$, and $e(I)$, $l(I)$, and *admissible* mean what they did in §1. There is a little wrinkle here however: as $\tilde{Q}^0$ is not the identity, an admissible sequence can end with 0's.

The geometric results of §2 allow us to quickly deduce the behavior of δ^* and Φ^*.

Proposition 5.2. $\delta^* : H^{*+1}(T(n+1,j)) \to H^*(T(n,*))$ *is determined by*
(1) $\delta^*(\tilde{Q}^I x_{n+1}) = \tilde{Q}^I x_n$, *and*
(2) δ^* *is 0 on decomposables.*

Proof. This follows from Proposition 2.3, and the fact that Dyer–Lashof operations commute with the evaluation [CLM, p.6, p.218]. $\square$

[5]The relation $\tilde{Q}^1\tilde{Q}^2 = \tilde{Q}^3\tilde{Q}^0$ illustrates this.

Proposition 5.3. *$\Phi^*: H^*(T(n,*)) \to H^*(T(n,*))$ is determined by*
(1) When $n=0$, $\Phi^*(x_0^{2j}) = x_0^j$.
(2) $\Phi^*(\tilde{Q}^s x) = \tilde{Q}^s(\Phi^* x)$ if $s > |x| - n$.
(3) Whenever the iterated operation $\tilde{Q}^I x_n$ is defined, $\Phi^*(\tilde{Q}^I x_n) = \tilde{Q}^{I'} x_n$ if $I = (I', 0)$, and is 0 otherwise.
(4) When $n \geq 1$, Φ^* is an algebra map (with the second grading in the domain of Φ^* doubled).

Proof. This follows from Theorem 2.4 and the last proposition. As $(2j)!/j!2^j$ is always odd, statement (1) of Theorem 2.4 implies that statement (1) here is true. Statement (2) here is implied by statement (5) of Theorem 2.4. To see that statement (3) is true, we first prove this in the special case when I consists only of 0's. Note that (1) includes the $n=0$ subcase of this special case, and then the statement for general n follows by combining the last proposition with statement (2) of Theorem 2.4 (which implies that Φ^* and δ^* commute). Now use (2) to deduce (3) for general I from the special case already established. Finally, (4) follows from statements (3) and (4) of Theorem 2.4. □

Note that as a corollary of Proposition 5.2, we have partially proved Theorem 1.5.

Corollary 5.4.
(1) $T(\infty, j) \simeq *$ unless j is a power of 2.
(2) $H^*((T(\infty, 2^k)) = \tilde{R}[k]$, where $\tilde{R}[k] = \langle \tilde{Q}^I x_0 \mid I$ is admissible and $l(I) = k\rangle$.

6. New Nishida relations

In the last section, we determined $H^*(T(n,*))$ in terms of dual Dyer–Lashof operations. Here we describe the Steenrod algebra action.

The standard Nishida relations [CLM, p.6, p.214] tell us how $(Sq^r)_*$ commutes with Q^s in $H_*(D_{n+1,*}S^{-n})$. Since $\chi(Sq^r)$[6] acting on $H^*(T(n,*))$ corresponds to $(Sq^r)_*$ acting on $H_{-*}(D_{n+1,*}S^{-n})$, we immediately have the following formula.

Lemma 6.1.

$$\chi(Sq^r)\tilde{Q}^s x = \sum_i \binom{-r-s}{r-2i} \tilde{Q}^{r+s-i}\chi(Sq^i)x.$$

Though this does completely specify the $\mathcal{A}$ module structure on $H^*(T(n,*))$, it is in a form completely unsuitable for proving theorems like those in the introduction. The point of this section is to prove

Theorem 6.2.

$$Sq^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} Sq^i x.$$

[6] χ is the antiautomorphism of the connected Hopf algebra $\mathcal{A}$.

The reader may find it amusing to compare this formula to the Adem relation of the last section,

$$\tilde{Q}^r \tilde{Q}^s x = \sum_i \binom{s-i-1}{r-2i} \tilde{Q}^{r+s-i} \tilde{Q}^i x,$$

the Adem relations in $\mathcal{A}$,

$$Sq^r Sq^s x = \sum_i \binom{s-i-1}{r-2i} Sq^{r+s-i} Sq^i x,$$

and the formula defining the "Singer construction" [Si]

$$Sq^r (t^{s-1} \otimes x) = \sum_i \binom{s-i-1}{r-2i} t^{r+s-i-1} \otimes Sq^i x.$$

Proof of Theorem 6.2. With Sq denoting the total square $1 + Sq^1 + Sq^2 + \dots$, to verify the formula, it suffices to check that it is consistent with the identity $Sq(\chi(Sq)) = 1$ and Lemma 6.1 above. Fixing n and s, we compute

$$\begin{aligned}
\sum_r Sq^{n-r} \chi(Sq^r) \tilde{Q}^s x &= \sum_r Sq^{n-r} \left[\sum_i \binom{-r-s}{r-2i} \tilde{Q}^{r+s-i} \chi(Sq^i) x \right] \\
&= \sum_{i,j} \left[\sum_r \binom{r+s-i-j-1}{n-r-2j} \binom{-r-s}{r-2i} \right] \tilde{Q}^{n+s-i-j} Sq^j \chi(Sq^i) x \\
&= \sum_{i,j} \left[\sum_p \binom{i+s-j-1+p}{n-2i-2j-p} \binom{-2i-s-p}{p} \right] \tilde{Q}^{n+s-i-j} Sq^j \chi(Sq^i) x
\end{aligned}$$

(letting $p = r - 2i$)

$$= \sum_{i,j} \binom{-(i+j)}{n-2(i+j)} \tilde{Q}^{n+s-(i+j)} Sq^j \chi(Sq^i) x$$

(using J. Adem's formula [A, (25.3)]: $\sum_p \binom{b+p}{c-p}\binom{a-p}{p} \equiv \binom{a+b+1}{c} \mod 2$)

$$\begin{aligned}
&= \sum_k \binom{-k}{n-2k} \tilde{Q}^{n+s-k} \left[\sum_i Sq^{k-i} \chi(Sq^i) x \right] \\
&= \binom{0}{n} \tilde{Q}^{n+s} x = \begin{cases} \tilde{Q}^s x & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

Remark 6.3. Our method of proof also shows that the analogues of the formula in Lemma 6.1,

$$\chi(Sq^r) Sq^s x = \sum_i \binom{-r-s}{r-2i} Sq^{r+s-i} \chi(Sq^i) x,$$

and

$$\chi(Sq^r)(t^{s-1} \otimes x) = \sum_i \binom{-r-s}{r-2i} t^{r+s-i-1} \otimes Sq^i x,$$

respectively hold in the Steenrod algebra and Singer construction. The formula in $\mathcal{A}$ already appears in the literature as [BaMi, (4.4)], where it is given a proof in the style of Bullett and MacDonald [BuMacD].

7. The proofs of Theorem 1.2 and Theorem 1.5

To prove Theorem 1.2, first recall the description of $H^*(T(n,*))$ given in Theorem 5.1:

$$H^*(T(n,*)) = S^*(\tilde{R}_n)/(\tilde{Q}^{|x|}x - x^2),$$

where

$$\tilde{R}_n = \langle \tilde{Q}^I x_n \mid I \text{ is admissible } \rangle / \langle \tilde{Q}^I x_n \mid I \text{ is admissible and } e(I) > n \rangle.$$

Note that $\tilde{R}_n$ is closed under both the action of $\mathcal{A}$ and Φ^*, thanks to our Nishida relations and Proposition 5.3, i.e. $(\tilde{R}_n, \Phi^*)$ is an object in $\mathcal{U}_\rho$. Thus Theorem 1.2 will follow from the next two proposition.

Proposition 7.1. $(\tilde{R}_n, \Phi^*) \simeq F_\rho(n)$ *as objects in* $\mathcal{U}_\rho$.

Proposition 7.2. *Let* $n \geq 1$. *In* $S^*(\tilde{R}_n)$*, the ideal generated by elements of the form* $\tilde{Q}^{|x|}x - x^2$ *equals the ideal generated by elements of the form* $Sq^{|y|}y - (\Phi^* y)^2$.

Both propositions will follow from the next result.

Theorem 7.3. $Sq^I \tilde{Q}^J x_n = (\Phi^*)^{l(I)}(\tilde{Q}^I \tilde{Q}^J x_n)$, *whenever the iterated operation* $\tilde{Q}^I \tilde{Q}^J x_n$ *is defined.*

Proposition 7.1 then follows from

Corollary 7.4. *If* I *is admissible,* $Sq^I(\tilde{Q}^0)^k x_n = \begin{cases} \tilde{Q}^I(\tilde{Q}^0)^{k-l(I)} & \text{if } l(I) \leq k, \\ 0 & \text{if } l(I) > k. \end{cases}$

This same corollary, together with Corollary 5.4 proves Theorem 1.5.

Proof of Proposition 7.2. Let $F(x) = \tilde{Q}^{|x|}x - x^2$ and $G(x) = Sq^{|x|}x - (\Phi^* x)^2$. Using the fact that $\tilde{R}_n$ is unstable, it is easy to deduce that the two ideals in question are generated by elements of the form $F(x)$ and $G(x)$ respectively, where $x \in \tilde{R}_n$. We claim that the sets of such elements are the same; more precisely, $F(\tilde{Q}^I x_n) = G(\tilde{Q}^I \tilde{Q}^0 x_n)$ and $G(\tilde{Q}^I x_n) = F(\Phi^*(\tilde{Q}^I x_n))$.

To see that these hold, we let $d = |I| + n$ and compute:

$$\begin{aligned} F(\tilde{Q}^I x_n) &= \tilde{Q}^d \tilde{Q}^I x_n - (\tilde{Q}^I x_n)^2 \\ &= Sq^d \tilde{Q}^I \tilde{Q}^0 x_n - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2, \end{aligned}$$

using Theorem 7.3 and Proposition 5.3,

$$= G(\tilde{Q}^I \tilde{Q}^0 x_n).$$

Similarly,

$$\begin{aligned} G(\tilde{Q}^I x_n) &= Sq^d \tilde{Q}^I x_n - (\Phi^*(\tilde{Q}^I x_n))^2 \\ &= \Phi^*(\tilde{Q}^d \tilde{Q}^I x_n) - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2, \end{aligned}$$

using Theorem 7.3 and Proposition 5.3,

$$= \tilde{Q}^d \Phi^*(\tilde{Q}^I x_n) - (\Phi^*(\tilde{Q}^I \tilde{Q}^0 x_n))^2,$$

using part (2) of Proposition 5.3 (since $n \geq 1$),

$$= F(\Phi^*(\tilde{Q}^I x_n)).$$

□

It remains to prove Theorem 7.3. This will follow from a couple of lemmas.

Lemma 7.5. *$Sq^r \tilde{Q}^J \tilde{Q}^0 x_n = \tilde{Q}^r \tilde{Q}^J x_n$, whenever the iterated operation $\tilde{Q}^r \tilde{Q}^J x_n$ is defined.*

Proof. This is proved by induction on $l(J)$. The induction is started by using the Nishida relations to verify that $Sq^r \tilde{Q}^0 x_n = \tilde{Q}^r x_n$.

For the inductive step, suppose $J = (j, J')$. Then

$$\begin{aligned} Sq^r \tilde{Q}^J \tilde{Q}^0 x_n &= Sq^r \tilde{Q}^j \tilde{Q}^{J'} \tilde{Q}^0 x_n \\ &= \sum_i \binom{r-j-1}{r-2i} \tilde{Q}^{r+j-i} Sq^i \tilde{Q}^{J'} \tilde{Q}^0 x_n \quad \text{(using the Nishida relations)} \\ &= \sum_i \binom{r-j-1}{r-2i} \tilde{Q}^{r+j-i} \tilde{Q}^i \tilde{Q}^{J'} x_n \quad \text{(by induction)} \\ &= \tilde{Q}^r \tilde{Q}^j \tilde{Q}^{J'} x_n \quad \text{(using the Adem relations)} \\ &= \tilde{Q}^r \tilde{Q}^J x_n. \end{aligned}$$

□

Lemma 7.6. *$Sq^I \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n = \tilde{Q}^I \tilde{Q}^J x_n$, whenever the iterated operation $\tilde{Q}^I \tilde{Q}^J x_n$ is defined.*

Proof. This is proved by induction on $l(I)$, and the last lemma is the case $l(I) = 1$. Let $I = (I', i)$. Then

$$\begin{aligned} Sq^I \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n &= Sq^{I'} Sq^i \tilde{Q}^J (\tilde{Q}^0)^{l(I)} x_n \\ &= Sq^{I'} \tilde{Q}^i \tilde{Q}^J (\tilde{Q}^0)^{l(I)-1} x_n \quad \text{(by the case } l(I) = 1) \\ &= \tilde{Q}^{I'} \tilde{Q}^i \tilde{Q}^J x_n \quad \text{(by induction)} \\ &= \tilde{Q}^I \tilde{Q}^J x_n. \end{aligned}$$

□

Proof of Theorem 7.3. Applying $(\Phi^*)^{l(I)}$ to the formula in the previous lemma yields

$$(\Phi^*)^{l(I)}(Sq^I\tilde{Q}^J(\tilde{Q}^0)^{l(I)}x_n) = (\Phi^*)^{l(I)}(\tilde{Q}^I\tilde{Q}^J x_n).$$

As it has a topological origin, $(\Phi^*)^{l(I)}$ commutes with Steenrod operations. By Proposition 5.3, $(\Phi^*)^{l(I)}(\tilde{Q}^J(\tilde{Q}^0)^{l(I)}x_n) = \tilde{Q}^J x_n$. The theorem follows. □

8. The Whitehead conjecture resolution and Theorem 1.6

In this section, we note that the homotopical equivalence of Theorem 1.6 can be deduced from the homological isomorphism of Theorem 1.5, using work of Lannes and Zarati [LZ2] to improve previous work of the author [K2, K3].

Letting $Z_k = T(\infty, 2^k)$ in the next theorem, Theorem 1.6 follows from Theorem 1.5.

Theorem 8.1. *Any sequence of 2 complete, connective spectra*

$$Z_0 \xrightarrow{\Phi} Z_1 \xrightarrow{\Phi} Z_2 \xrightarrow{\Phi} \dots$$

that realizes the length filtration of $\mathcal{A}$ in cohomology is equivalent to the sequence

$$SP^1_\Delta(S^0) \to SP^2_\Delta(S^0) \to SP^4_\Delta(S^0) \to \dots.$$

This is proved in [K2, K3], assuming the extra geometric condition:

$$\Sigma^{-k}(Z_k/Z_{k-1}) \text{ is a wedge summand of a suspension spectrum.}$$

We note that this geometric condition is *automatically* satisfied! Under our cohomological hypothesis, $H^*(\Sigma^{-k}(Z_k/Z_{k-1}))$ is isomorphic to $H^*(M(k))$, where $M(k)$ is the stable wedge summand of $B(\mathbf{Z}/2)^k$ associated to the Steinberg module. Now consider the Adams spectral sequence for computing maps from $M(k)$ to $\Sigma^{-k}(Z_k/Z_{k-1})$. An $\mathcal{A}$–module isomorphism $H^*(\Sigma^{-k}(Z_k/Z_{k-1})) \simeq H^*(M(k))$ can be regarded as an element in $E_2^{0,0}$. The following proposition implies that such an element is a permanent cycle, i.e. one can topologically realize this isomorphism.

Proposition 8.2. [LZ2, Proposition 5.4.7.1] *If M is an unstable $\mathcal{A}$–module, and N is a summand of $H^*(B(\mathbf{Z}/2)^k)$, then* $\operatorname{Ext}^{s,t}_{\mathcal{A}}(M,N) = 0$ *for all $t - s < 0$.*

Lannes and Zarati prove this using ideas of W.Singer. As explained in [HK], this proposition can also be deduced from [BC, Lemma 2.3(i)] (slightly modified) in the spirit of Carlsson's work [Ca].

9. The proof of Theorem 1.9

This sections contains the details of the proof of Theorem 1.9, which was outlined at the end of §1.

As in [K6], $F \in \mathcal{F}$ is said to be *finite* if it has a finite length composition series with simple subquotients, and is said to be *locally finite* (written $F \in \mathcal{F}_\omega$) if it is the union of its finite subfunctors. Recall that $I_\lambda \in \mathcal{F}$ is the injective envelope of the simple functor F_λ. The I_λ are locally finite [K6]. Then the general theory of locally Noetherian abelian categories [S, p.92] [P, Theorem 5.8.11] implies that, if $J \in \mathcal{F}_\omega$ is any injective, then there is a decomposition in $\mathcal{F}$

$$J \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, J) I_\lambda,$$

where $a(\lambda, J) = \dim_{\mathbf{Z}/2} \operatorname{Hom}_{\mathcal{F}}(F_\lambda, J)$.

Applying this to the case $J = \Phi^{-1} S^j$, and noting [KK] that

$$\dim_{\mathbf{Z}/2} \operatorname{Hom}_{\mathcal{F}}(F_\lambda, \Phi^{-1} S^j) = \dim_{\mathbf{Z}/2} \operatorname{Hom}_{\mathcal{F}}(F_\lambda, S^{2^k j}), \text{ for } k >> 0,$$

we deduce that

$$\Phi^{-1} S^j \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) I_\lambda,$$

with $a(\lambda, j)$ as in the introduction.

Recall that $r : \mathcal{F} \to \mathcal{U}$ is defined by letting $r(F)_j = \operatorname{Hom}_{\mathcal{F}}(S_j, F)$. The fact that S_j is finite implies that r will commute with filtered direct limits. In particular, we can deduce the decomposition in $\mathcal{U}$

$$\Phi^{-1} r(S^j \circ S_n) \simeq \bigoplus_{\lambda \in \Lambda} a(\lambda, j) r(I_\lambda \circ S_n).$$

Proposition 9.1. *$r(I_\lambda \circ S_n) \simeq H^*(K(\lambda, n); \mathbf{Z}/2)$ as $\mathcal{A}$ modules.*

Momentarily postponing the proof of this, to prove Theorem 1.9, we need to show

$$H^*(\Phi^{-1} T(n, j); \mathbf{Z}/2) \simeq \Phi^{-1} r(S^j \circ S_n) \text{ as } \mathcal{A} \text{ modules.}$$

Note that this asserts that a certain inverse limit of finite dimensional modules is isomorphic to a certain direct limit of *nilclosed* modules (i.e. modules of the form $r(F)$).

To show this, observe that $\Phi^{-1} r(S^* \circ S_n)$ is $\mathbf{N} \times \mathbf{N}[\frac{1}{2}]$ graded. It is even an object in $\mathcal{K}_\rho$, using $\Phi^{-1} : \Phi^{-1} r(S^{2j} \circ S_n) \to \Phi^{-1} r(S^j \circ S_n)$ as the restriction.

Theorem 9.2. *$H^*(\Phi^{-1} T(n, *); \mathbf{Z}/2) \simeq \Phi^{-1} r(S^* \circ S_n)$ as objects in $\mathcal{K}_\rho$.*

Returning to the proof of Proposition 9.1, we first note that $H^*(K(\lambda, n); \mathbf{Z}/2) = H^*(K(V_\lambda, n); \mathbf{Z}/2) e_\lambda$ and $r(I_\lambda \circ S_n) = r(I_{V_\lambda} \circ S_n) e_\lambda$, where $I_W \in \mathcal{F}$ is the injective defined by $I_W(V) = (\mathbf{Z}/2)^{\operatorname{Hom}(V, W)}$. Thus we need just show that

$$r(I_W \circ S_n) = H^*(K(W, n); \mathbf{Z}/2).$$

Now one has the classic calculation [S, p.184] $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2) = U(F(n))$, where $F(n) = \mathcal{A}/E(n)$ is the free unstable module on an n dimensional class, and where $U : \mathcal{U} \to \mathcal{K}$ is the free functor, left adjoint to the forgetful functor. Explicitly, $U(M) = S^*(M)/(Sq^{|x|} x - x^2)$. Similarly, $H^*(K(W, n); \mathbf{Z}/2) = U_W(F(n))$ where $U_W : \mathcal{U} \to \mathcal{K}$ is given by $U_W(M) = U(M \otimes W^*)$.

A simple calculation reveals that $F(n) = r(S_n)$[7] (see e.g. [K8, Prop.8.1]), so the proof of Proposition 9.1 is completed with

Lemma 9.3. [K9] *There are natural isomorphisms $U_W(r(F)) \simeq r(I_W \circ F)$, for all $F \in \mathcal{F}_\omega$.*

[7] This is false at odd primes: $F(n)$ is not nilclosed in the odd prime case.

Sketch proof. It is easy to reduce to the case when $W = \mathbf{Z}/2$. Let $I = I_{\mathbf{Z}/2}$. By filtering $U(M)$ one then verifies that if M is nilclosed, so is $U(M)$. Thus to identify $U(r(F))$ with $r(I \circ F)$, it suffices to check that $l(U(r(F))) = I \circ F$, where $l : \mathcal{U} \to \mathcal{F}$ is left adjoint to r. The functor l is exact, preserves tensor products, and can be regarded as localization away from nilpotent modules [HLS, K6]. Thus it carries

$$S^*(r(F))/(Sq^{|x|}x - x^2)$$

to the functor that sends V to

$$S^*(l(r(F))(V))/(x - x^2).$$

Since $l(r(F)) = F$, and $I(V) = S^*(V)/(x - x^2)$ [K6], this functor is just $I \circ F$. □

To prove Theorem 9.2, we need to use the main result of [K9].

As in [K8], let $\mathcal{U}^2$ be the category of $\mathbf{N} \times \mathbf{N}$ graded modules over the bigraded algebra $A \otimes A$, unstable in each grading. For $M \in \mathcal{U}^2$, there are natural maps $\Phi_1 : M_{m,*} \to M_{2m,*}$ and $\Phi_2 : M_{*,n} \to M_{*,2n}$[8], and we let $\mathcal{K}^2$ denote the category of commutative algebras M in $\mathcal{U}^2$ satisfying the "restriction" axiom: for all $x \in M$, $(\Phi_1 \otimes \Phi_2)(x) = x^2$. Let $U_2 : \mathcal{U}^2 \to \mathcal{K}^2$ be left adjoint to the forgetful functor: explicitly, $U_2(M) = S^*(M)/((\Phi_1 \otimes \Phi_2)(x) - x^2)$.

Given $M \in \mathcal{U}$, $M \otimes F(1)$ is an object in $\mathcal{U}^2$. $F(1)$ can be regarded as the module $\langle x_1, \dots, x^{2^k}, \dots \rangle$, with x^{2^k} having bidegree $(1, 2^k)$. Now define

$$\mathrm{Hom}_{\mathcal{F}}(S_*, F) \otimes F(1) \to \mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F)$$

by sending $(S_i \xrightarrow{\alpha} F) \otimes x^{2^k}$ to the composite $S_i \xrightarrow{\alpha} F \to S^{2^k} \circ F$. Since $\mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F)$ is easily checked to be in $\mathcal{K}^2$, this map extends to a natural map in $\mathcal{K}^2$:

$$\Theta_F : U_2(\mathrm{Hom}_{\mathcal{F}}(S_*, F) \otimes F(1)) \to \mathrm{Hom}_{\mathcal{F}}(S_*, S^* \circ F).$$

Theorem 9.4. [K9] *For all $F \in \mathcal{F}_\omega$, Θ_F is an isomorphism.*

This is proved in a manner similar to the way Lemma 9.3 is proved.

Corollary 9.5. *$r(S^* \circ S_n) \simeq U_2(F(n) \otimes F(1))$, as objects in $\mathcal{K}^2$.*

Corollary 9.6. *$\Phi^{-1} r(S^* \circ S_n) \simeq U_\rho(F(n) \otimes \Phi^{-1}F(1))$, as objects in $\mathcal{K}_\rho$.*

Here $\Phi^{-1}F(1) = \langle x^{2^k} \mid k \in \mathbf{Z} \rangle$, with the restriction map (part of the $\mathcal{K}_\rho$ structure), taking x^{2^k} to $x^{2^{k-1}}$.

By Theorem 1.2, $H^*(\Phi^{-1}T(n,*); \mathbf{Z}/2) \simeq U_\rho(F_\rho(n)_{\hat{\Phi}})$ as objects in $\mathcal{K}_\rho$, where $F_\rho(n)_{\hat{\Phi}}$ denotes the inverse limit

$$F_\rho(n) \xleftarrow{\rho} F_\rho(n) \xleftarrow{\rho} F_\rho(n) \xleftarrow{\rho} \dots.$$

The following observation completes the proof of Theorem 9.2, and thus the proof of Theorem 1.9.

Lemma 9.7. *$F_\rho(n)_{\hat{\Phi}} = F(n) \otimes \Phi^{-1}F(1)$, as objects in $\mathcal{U}_\rho$.*

[8]These are the Steenrod squares in the right degree

10. Towards the conjectures

In this section we outline some possible approaches to the conjectures of the introduction.

We start with a rigorous proof of Proposition 1.11.

Proof of Proposition 1.11. Let $X(j) = \bigvee_{\lambda \in \Lambda} a(\lambda, j)K(\lambda, 1)$, and recall that we wish to topologically realize an $\mathcal{A}$ module isomorphism:

$$H^*(\Phi^{-1}T(1,j); \mathbf{Z}/2) \simeq H^*(X(j); \mathbf{Z}/2).$$

But Proposition 8.2 tells us that any such $\mathcal{A}$–module map *can* be realized: in the Adams spectral sequence for computing maps from $X(j)$ to $\Phi^{-1}T(1,j)$, $E_2^{s,t} = 0$ for $t - s < 0$. □

Thus far, we have been unable to find any way in which this proof, or the related proofs of Conjecture 1.3 [L1, Goe, HK] in the $n = 1$ case, generalize to prove the $n > 1$ cases of the conjectures. These Adams spectral sequence based proofs rely on magical properties of the spectra $T(j)$ and $K(V, 1)$, which, in turn, are (partly) due to the fact that $H^*(T(j); \mathbf{Z}/2)$ and $H^*(K(V, 1))$ are injective in $\mathcal{U}$. A search for similar proofs of the conjectures leads to the following questions.

Question 10.1. For $n > 1$, do $H^*(T(n,j); \mathbf{Z}/2)$ and $H^*(K(\mathbf{Z}/2, n); \mathbf{Z}/2)$ have any sort of injectivity properties in some well chosen subcategory of $\mathcal{U}$?

Question 10.2. Is $\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(K(W,n)), H^*(K(V,n))) = 0$ if $t - s < 0$?

Question 10.3. Is $\mathbf{Z}/2[\mathrm{Hom}(V,W)] \to \mathrm{Hom}_{\mathcal{A}}(H^*(K(W,n)), H^*(K(V,n)))$ an isomorphism?

Note that an affirmative answer to the second question would allow us to prove Conjecture 1.10 along the lines of the above proof of Proposition 1.11. It is not hard to show that, if the last question has an affirmative answer, then Conjecture 1.10 would follow if one could just construct a family of stable maps

$$T(n, 2^k) \to K(\mathbf{Z}/2, n) \tag{10.1}$$

nonzero in cohomology in dimension n.

Related to the [HK] proof of Conjecture 1.3 in the $n = 1$ case, we note that, by [HK, Proposition 1.6], if Conjecture 1.10 were true, then one could conclude that $\epsilon : \Sigma^\infty \Omega^\infty T(n,j) \to T(n,j)$ is onto in mod 2 homology, which is a weak form of Conjecture 1.3.

Now we discuss a rather intriguing "conceptual" approach to Conjectures 1.3 and 1.10. The idea would be to start with the $n = 0$ case (!) of the conjectures, using the concept of S–algebras (a.k.a. E_∞–ring spectra).

Question 10.4. Let Λ denote a divided power algebra over $\mathbf{Z}_2$. Does there exist an $\mathbf{N}$–graded commutative augmented S–algebra structure on $T = \bigvee_{j \geq 0} T(0,j) = \bigvee_{j \geq 0} S^0$ such that

(1) $\pi_0(T) = \Lambda$,
(2) $\Phi : T \to T$ is a map of S–algebras, and
(3) T admits a nonzero S-algebra map to $\Sigma^\infty(\mathbf{Z}/2)_+$?

An affirmative answer to parts (1) and (3) would presumably yield a construction of maps as in (10.1) upon applying the "bar construction" n times to map (3).

We can refine this question, motivated by work in [K8]. The key is to rearrange the untidy right side of the isomorphism

$$H^*(\Phi^{-1}T(n,j);\mathbf{Z}/2) \simeq H^*(\bigvee_{\lambda\in\Lambda} a(\lambda,j)K(\lambda,n);\mathbf{Z}/2).$$

We know that this module corresponds to the functor $(\Phi^{-1}S^j)\circ S_n \in \mathcal{F}$. The proof in [K8] that $\Phi^{-1}S^j$ is injective in $\mathcal{F}_\omega$ reveals that

$$\Phi^{-1}S^j \simeq \lim_{s\longrightarrow\infty} I_{(\mathbf{F}_{2^s})^*}[j],$$

where $(\mathbf{F}_{2^s})^*$ is the $\mathbf{F}_2$ linear dual of the finite field $\mathbf{F}_{2^s}$, and $I_{(\mathbf{F}_{2^s})^*}[j]$ is the j^{th} eigenspace of $I_{(\mathbf{F}_{2^s})^*}$ under the action of $\mathbf{F}_{2^s}^\times$. Furthermore, if we extend the scalars to the algebraic closure $\bar{\mathbf{F}}_2$, this isomorphism is well behaved with respect to pairings (between various j's).

It follows that

$$H^*(\Phi^{-1}T(n,*);\bar{\mathbf{F}}_2) \simeq H^*_{\text{cont}}(K(\bar{\mathbf{F}}_2,n);\bar{\mathbf{F}}_2)[*]$$

as $\mathbf{N}[\frac{1}{2}]$ graded algebras in $\mathcal{U}$, where we write

$$H^*_{\text{cont}}(K(\bar{\mathbf{F}}_2,n);\bar{\mathbf{F}}_2) = \lim_{s\longrightarrow\infty} H^*(K(\mathbf{F}_{2^s},n);\bar{\mathbf{F}}_2).$$

Just as one can discuss S–algebras, one can discuss $SW(\bar{\mathbf{F}}_2)$–algebras, where $W(\bar{\mathbf{F}}_2)$ are the Witt vectors of $\bar{\mathbf{F}}_2$.

Question 10.5. With T the S-algebra as in Question 10.4 above, does there exist an equivalence of $\mathbf{N}[\frac{1}{2}]$ graded $SW(\bar{\mathbf{F}}_2)$–algebras

$$\Phi^{-1}T \wedge_S SW(\bar{\mathbf{F}}_2) \simeq \Sigma^\infty((\bar{\mathbf{F}}_2)^*)_+ \wedge_S SW(\bar{\mathbf{F}}_2)?$$

As before, an affirmative answer to this formidable question would presumably yield a proof of Conjecture 1.10 upon applying the bar construction to the equivalence n times.

We end with a question about the most straightforward way to try to get at these sorts of things.

Question 10.6. Does there exist a "naturally occurring" spectrum E, with a group action, such that the group action can be used to establish a splitting

$$\Sigma E_n \simeq \bigvee_j T_1(n,j),$$

where E_n is the n^{th} infinite loop space of the spectrum E, and $T_1(n,j)$ is a desuspension of $\Sigma T(n,j)$?

When $n=1$, this would be consistent with [GLM]. However, anyone searching for such a spectrum should make sure their search is compatible with results in [McG].

Appendix A. Connections with work of Arone and Mahowald

In this appendix, we explain how our constructions are related to those appearing in [AM] in their work on the Goodwillie tower of the identity. (Our arguments are a bit sketchy as we plan to elaborate on these ideas elsewhere.)

Recall our definition: $\tilde{D}_{n,j}(X) = F(\mathcal{C}(n,j)_+, X^{[j]})^{\Sigma_j}$. We begin by rewriting this in a useful way.

Let $\Delta(n,j) \subset S^{nj}$ be the singular part of the Σ_j–space S^{nj}. Then $\mathcal{C}(n,j)$ is equivariantly homotopy equivalent to $S^{nj} - \Delta(n,j)$ (the configuration space). Thus, by equivariant Alexander duality [LMMS, Theorem III.4.1],

$$F(\mathcal{C}(n,j)_+, (\Sigma^n X)^{[j]}) \simeq S^{nj}/\Delta(n,j) \wedge X^{[j]}$$

as Σ_j spectra. Now note that this latter spectrum is clearly Σ_j–free, as $S^{nj}/\Delta(n,j)$ is, thus its fixed point spectrum is naturally equivalent to its orbit spectrum [LMMS, Theorem II.7.1]. We have proved

Proposition A.1. *$\tilde{D}_{n,j}(\Sigma^n X)$ is naturally equivalent to $((S^{nj}/\Delta(n,j)) \wedge X^{[j]})_{\Sigma_j}$.*

Checking definitions reveals

Lemma A.2. *$\beta : \mathcal{C}(n+1,j)_+ \wedge S^1 \to \mathcal{C}(n,j)_+ \wedge S^j$ is equivariantly S–dual to the evident diagonal map $S^1 \wedge (S^{nj}/\Delta(n,j)) \to S^{(n+1)j}/\Delta(n+1,j)$.*

Definition A.3. Let $\tilde{D}_j(X) = \operatorname*{hocolim}_n \Sigma^{-n}\tilde{D}_{n,j}(\Sigma^n X)$, with the colimit induced by either of the maps in the last lemma.

Note that, with this notation, $T(\infty,j) = \Sigma\tilde{D}_j(S^{-1})$.

Now let K_j be the Σ_j–space introduced in [AM]: K_j is the unreduced suspension of $\tilde{K}_j$, the classifying space of the poset of the nontrivial partitions of a set with cardinality j. (By nontrivial, we mean to exclude the partitions (j) and $(1,1,\dots,1)$.)

Proposition A.4. [AM, early versions] *and* [AD, §6] *There is a Σ_j equivariant map*

$$\operatorname*{hocolim}_n \Sigma^{-n}(S^{nj}/\Delta(n,j)) \to \Sigma K_j$$

that is a nonequivariant equivalence.

Corollary A.5. *$\tilde{D}_j(X) = (\Sigma K_j \wedge X^{[j]})_{h\Sigma_j}$.*

Combining this corollary with Theorem 1.6 yields

Theorem A.6. *$(K_{2^k} \wedge S^{-2^k})_{h\Sigma_{2^k}} \simeq \Sigma^{-2} SP^{2^k}_{\Delta}(S^0)$.*

In work in progress, we have established the following.

Proposition A.7. *Localized at 2, there are cofibration sequences*

$$\Sigma\tilde{D}_{2^{k-1}}(S^{2n-1}) \to \Sigma\tilde{D}_{2^k}(S^{n-1}) \to \tilde{D}_{2^k}(S^n)$$

which are short exact in cohomology.

The first map here is constructed with Hopf invariant techniques, and is the generalization of $\Phi : T(\infty, 2^{k-1}) \to T(\infty, 2^k)$.

Using these sequences when $n=0$ and $n=1$, one can deduce

Corollary A.8. *Localized at 2, there are equivalences*
(1) $(K_{2^k})_{h\Sigma_{2^k}} \simeq \Sigma^{-1} SP^{2^k}_\Delta(S^0)/SP^{2^{k-1}}_\Delta(S^0)$,
(2) $(K_{2^k} \wedge S^{2^k})_{h\Sigma_{2^k}} \simeq SP^{2^k}(S^0)/SP^{2^{k-1}}(S^0)$.

Part (2) of this corollary is due to Arone and Mahowald who sketch the following elegant and direct short proof in their early versions of [AM]. (See also [AD].)

Lemma A.9. *The space S^{nj}/Σ_j is homeomorphic to $SP^j(S^n)/SP^{j-1}(S^n)$.*

Lemma A.10. $(\Delta(n,j) \wedge S^j)_{\Sigma_j}$ *is contractible.*

Sketch proof. The partition filtration of $\Delta(n,j)$ induces a filtration of $(\Delta(n,j) \wedge S^j)_{\Sigma_j}$ in which each subquotient has the form $SP^i(S^1)/SP^{i-1}(S^1)$, and so is contractible. □

Corollary A.11. *There are homotopy equivalences of spaces*

$$SP^j(S^{n+1})/SP^{j-1}(S^{n+1}) \simeq (S^{nj}/\Delta(n,j) \wedge S^j)_{\Sigma_j}.$$

Proof. $SP^j(S^{n+1})/SP^{j-1}(S^{n+1}) \simeq (S^{nj} \wedge S^j)_{\Sigma_j} \simeq (S^{nj}/\Delta(n,j) \wedge S^j)_{\Sigma_j}$. □

Now Corollary A.8(2) follows by letting n go to infinity, and using Proposition A.4.

We finish with one last observation. Let $\mathcal{D}_j(X) = F(\Sigma K_j, X^{[j]})_{h\Sigma_j}$. Arone and Mahowald [AM] show that $\Omega^\infty \mathcal{D}_j(X)$ is the j^{th} fiber of the Goodwillie tower of the identity applied to a space X. Arone and Dwyer [AD] show that, if X is an odd dimensional sphere, then $\Sigma^{2k}\mathcal{D}_{2^k}(X) \simeq (\Sigma K_{2^k} \wedge X^{[2^k]})_{h\Sigma_{2^k}}$. Thus we have

Corollary A.12. *If X is an odd dimensional sphere, then $\mathcal{D}_{2^k}(X) \simeq \Sigma^{-2k}\tilde{D}_{2^k}(X)$.*

Corollary A.13. $\mathcal{D}_{2^k}(S^{-1}) \simeq \Sigma^{-(2k+1)} SP^{2^k}_\Delta(S^0)$.

References

[A] J. Adem, *The relations on Steenrod powers of cohomology classes*, Algebraic Geometry and Topology (R. H. Fox, et. al., editors), Princeton University Press, 1957, 191–242.

[AD] G.Z. Arone and W.G. Dwyer, *Partition complexes, Tits buildings, and symmetric product spectra*, preliminary preprint, 1997.

[AM] G. Arone and M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, preprint, 1996.

[BaMi] M. G. Barratt and H. R. Miller, *On the antiautomorphism of the Steenrod algebra*, A. M. S. Cont. Math. **12**(1982), 47–52.

[BC] E.H.Brown and R.L.Cohen, *The Adams spectral sequence of $\Omega^2 S^3$ and Brown-Gitler spectra,* Annals of Math.Study **113**(1987), 101–125.

[BMMS] R. R. Bruner, J. P. May, J. McClure, and M. Steinberger H_∞ *Ring Spectra and their Applications*, Springer L. N. Math. **1176**, 1986.

[BuMacD] S. R. Bullett and I. G. MacDonald, *On the Adem relations*, Topology **21**(1982), 329–332.

[Ca] G.Carlsson, *G.B.Segal's Burnside ring conjecture for* $(\mathbf{Z}/2)^k$, Topology **22**(1983), 83–103.

[CLM] F. R. Cohen, T. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Springer L. N. Math. **533**, 1976.

[FN] R. Fox and L. Neuwirth, *The braid groups*, Math. Scand. **10**(1962), 119–126.

[Goe] P.Goerss, *A direct construction for the duals of Brown-Gitler spectra*, Ind. Math. J. **34**(1985), 733–751.

[GLM] P.Goerss, J. Lannes, and F.Morel, *Vecteurs de Witt non-commutatifs et representabilité de l'homologie modulo p*, Inv.Math. **108**(1992), 163–227.

[Goo] T. G. Goodwillie, *Calculus III: the Taylor series of a homotopy functor*, preprint, 1994?.

[HLS] H.-W. Henn, J. Lannes, and L. Schwartz *The categories of unstable modules and unstable algebras modulo nilpotent objects*, Amer. J. Math. **115**(1993), 1053–1106.

[HK] D.J.Hunter and N.J.Kuhn, *Characterizations of spectra with $\mathcal{U}$–injective cohomology which satisfy the Brown–Gitler property*, to appear in Trans.A.M.S.

[H] D. Husemoller, *Fibre Bundles*, Springer Graduate Texts in Math. **20**, 1966.

[KK] P. Krason and N. J. Kuhn, *On embedding polynomial functors in symmetric powers*, J. Algebra. **99**(1994), 281–294.

[K1] N. J. Kuhn, *The geometry of the James Hopf maps*, Pac. J. Math. **102**(1982), 397–412.

[K2] N. J. Kuhn, *A Kahn-Priddy sequence and a conjecture of G. W. Whitehead*, Math. Proc. Camb. Phil. Soc. **92**(1982), 467–483.

[K3] N. J. Kuhn, *Spacelike resolutions of spectra*, A.M.S. Cont. Math. Series 19(1983), 153–165.

[K4] N. J. Kuhn, *Suspension spectra and homology equivalences*, Trans. A. M. S. **283**(1984), 303–313.

[K5] N. J. Kuhn, *The transfer and the James-Hopf invariants*, Math. Zeit. **196**(1987), 391–405.

[K6] N. J. Kuhn, *Generic representation theory of the finite general linear groups and the Steenrod algebra: I*, Amer. J. Math. **116**(1994), 327–360.

[K7] N. J. Kuhn, *Generic representation theory of the finite general linear groups and the Steenrod algebra: II*, K-Theory J. **8**(1994), 395–428.

[K8] N. J. Kuhn, *Generic representation theory of the finite general linear groups and the Steenrod algebra: III*, K-Theory J. **9**(1995), 273–303.

[K9] N. J. Kuhn, *Computations in generic representation theory: maps from symmetric powers to composite functors* , Trans. A.M.S., to appear.

[L1] J.Lannes, *Sur le n dual du n-ème spectre de Brown-Gitler,* Math. Zeit. **159**(1988), 29–42.

[L2] J.Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire,* Pub. I.H.E.S. **75**(1992), 135–244.

[LS] J.Lannes and L.Schwartz, *Sur la structure des A-modules instables injectifs,* Topology **28**(1989), 153–169.

[LZ1] J.Lannes and S.Zarati, *Sur les U-injectifs,* Ann.Scient.Ec.Norm.Sup **19**(1986), 1–31.

[LZ2] J.Lannes and S.Zarati, *Sur les foncteurs dérivés de la déstablisation,* Math.Zeit. **194**(1987), 25–59.

[LMMS] L. G. Lewis, J. P. May, J. McClure, and M. Steinberger *Equivariant Stable Homotopy Theory*, Springer L. N. Math. **1213**, 1986.

[Mah] M. Mahowald, *A new infinite family in ${}_2\pi_*^S$,* Topology **16**(1977), 249–256.

[May] J. P. May, *The Geometry of Iterated Loop Spaces*, Springer L. N. Math. **271**, 1972.

[McG] C.A.McGibbon, *Infinite loop spaces and Neisendorfer localization*, Proc.A.M.S. **125**(1997), 309–313.

[Mi1] H. R. Miller, *An algebraic analogue of a conjecture of G.W.Whitehead,* Proc. A.M.S. **84**(1982), 131–137.

[Mi2] H. R. Miller, *The Sullivan conjecture on maps from classifying spaces,* Ann. Math. **120**(1984), 39–87.

[MP] S.A.Mitchell and S.B.Priddy, *Stable splittings derived from the Steinberg module,* Topology **22**(1983), 285–298.

[P] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, London, 1973.

[S] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point conjecture,* Chicago Lectures in Math., Unversity of Chicago Press, 1994.

[Si] W. M. Singer, *A new chain complex for the homology of the Steenrod algebra*, Math. Proc. Camb. Phil. Soc. **90**(1981), 279–292.

Department of Mathematics, University of Virginia, Charlottesville, VA 22903