STABLE SPLITTINGS AND THE DIAGONAL

NICHOLAS J. KUHN

October 14, 1999. Revised January 6, 2000.

Abstract. Many approximations to function spaces admit natural stable splittings, with a typical example being the stable splitting of a space C_dX approximating $\Omega^d\Sigma^dX$. With an eye towards understanding cup products in the cohomology of such function spaces, we describe how the diagonal interacts with the stable splitting. The description involves group theoretic transfers.

In an appendix independent of the rest of the paper, we use ideas from Goodwillie calculus to show that such natural stable splittings are unique, and discuss three different constructions showing their existence.

1. Introduction

Many mapping space functors of a topological space X admit homotopy equivalent combinatorial models CX constructed in the following way. Let Λ be the category with objects the finite sets $\mathbf{0} = \emptyset$ and $\mathbf{n} = \{1, 2, \ldots, n\}$, $n \geq 1$, and morphisms the injective functions. Following the terminology of [CMT], a *coefficient system* is a contravariant functor

$$\mathcal{C} : \Lambda^{op} \to \text{topological spaces}.$$

Meanwhile, a based space X defines a covariant functor

$$X^{\times} : \Lambda \to \text{topological spaces}$$

by the formula $X^{\times}(n) = X^n$. Then CX is defined to be the coend of $\mathcal{C}$ and $X^{\times}$:

$$CX = (\coprod_{n=0}^{\infty} \mathcal{C}(n) \times X^n)/(\sim)$$

where $(\alpha^*(c), x) \sim (c, \alpha_*(x))$ for all $c \in \mathcal{C}(n), x \in X^m$, and $\alpha : \mathbf{m} \to \mathbf{n}$.

An important family of examples is the case when $\mathcal{C}(n) = \mathcal{C}_d(n)$, the space of '$n$ disjoint little d–cubes in a big d–cube', i.e. the n^{th} space in the Boardman–Vogt little d-cubes operad $\mathcal{C}_d$. Here the associated space CX maps naturally to the iterated loopspace $\Omega^d\Sigma^dX$, and this is a homotopy equivalence if X is path connected.

The coefficient systems $\mathcal{C}$ arising in this family are Σ*–free*: for all n, the action of the symmetric group Σ_n on the space $\mathcal{C}(n)$ is free. This is typical of coefficient

1991 *Mathematics Subject Classification.* Primary 55P35; Secondary 55P42.
Partially supported by the N.S.F.

systems arising in both of the most common ways: from Σ–free (unital) operads, and from configuration spaces. For the rest of the paper we will assume that all our coefficient systems are Σ–free.

The space CX has a natural increasing filtration, with

$$F_nCX = (\coprod_{m=0}^{n} \mathcal{C}(m) \times X^m)/(\sim).$$

Letting $D_nX = F_nCX/F_{n-1}CX = \mathcal{C}(n)_+ \wedge_{\Sigma_n} X^{\wedge n}$, and $DX = \bigvee_{n=0}^{\infty} D_nX$, there is a natural *stable* equivalence[1] [CMT] that splits the filtration

$$s : (CX)_+ \underset{\sim}{\longrightarrow} DX.$$

In the author's experience, many applications of the combinatorial models CX arise from an analysis of how the stable splitting interacts with unstable structure. By and large, such interactions were analyzed by the early 1980's (see e.g. [K1, K2, K3, K4, LMMS]), but the most fundamental unstable structure, the diagonal on CX, seems to have been not previously studied from this point of view. It is the point of this note to remedy this.

Our immediate interest in this question comes from the fact that recent projects by D. Tamaki [T], N. Strickland [S1], and the author (joint with S. Ahearn), have all involved studying the product structure on $h^*(CX)$ for various $\mathcal{C}$ as above and for various multiplicative generalized cohomology theories h^*. Understanding such cup product stucture amounts to understanding the homomorphisms induced by the maps Δ^l in h^*–theory, where we let $\Delta^l : CX \to (CX)^l$ be the l–fold diagonal map. For this, it suffices to study Δ^l stably.

Definitions 1.1. Let $\Psi^l : DX \to (DX)^{\wedge l}$ be the stable composite

$$DX \underset{\sim}{\xrightarrow{s^{-1}}} (CX)_+ \xrightarrow{\Delta^l_+} (CX)^l_+ \underset{\sim}{\xrightarrow{s^{\wedge l}}} (DX)^{\wedge l}.$$

Given n and $I = (i_1, \dots, i_l)$, let $D_IX = D_{i_1}X \wedge \cdots \wedge D_{i_l}X$, and let $\Psi^n_I : D_nX \to D_IX$ be the $(n, I)^{th}$ component of Ψ^l.

We will often identify D_IX with the homeomorphic space $\mathcal{C}(I)_+ \wedge_{\Sigma_I} X^{\wedge |I|}$, where $\mathcal{C}(I) = \mathcal{C}(i_1) \times \cdots \times \mathcal{C}(i_l)$, $\Sigma_I = \Sigma_{i_1} \times \cdots \times \Sigma_{i_l}$, and $|I| = i_1 + \cdots + i_l$.

Our main theorem, Theorem 2.4, is a description of each Ψ^n_I as a sum of more familiar maps arising from transfers and diagonals, with the sum indexed by equivalence classes of covers of $\mathbf{n}$ by $\mathbf{i_1}, \mathbf{i_2}, \dots, \mathbf{i_l}$. An immediate corollary is easy to state:

Theorem 1.2. *Suppose $I = (i_1, \dots, i_l)$.*
(1) *$\Psi^n_I \simeq *$ unless $|I| \geq n$ and $i_t \leq n$, for all t.*
(2) *If $|I| = n$, then $\Psi^n_I : D_nX \to D_IX$ is the composite*

$$\mathcal{C}(n)_+ \wedge_{\Sigma_n} X^{\wedge n} \to \mathcal{C}(n)_+ \wedge_{\Sigma_I} X^{\wedge n} \to \mathcal{C}(I)_+ \wedge_{\Sigma_I} X^{\wedge n}$$

[1] We will, in general, not distinguish a space Z from its suspension spectrum $\Sigma^\infty Z$, but will try to alert readers as to when maps are defined only stably.

where the first map is the (stable) transfer induced by the inclusion $\Sigma_I \subseteq \Sigma_n$ and the second map is induced by the Σ_I–equivariant map $\mathcal{C}(n) \to \mathcal{C}(I)$ arising from the coefficient system structure.
(3) *If X is a coH–space, then furthermore $\Psi_I^n \simeq *$ if $|I| > n$.*

Remarks 1.3.
(1) If $l = 2$ and X is a coH–space, the theorem says that the composite

$$DX \xrightarrow[\sim]{s^{-1}} (CX)_+ \xrightarrow{\Delta_+^l} (CX \times CX)_+ \xrightarrow[\sim]{s \wedge s} DX \wedge DX$$

is the wedge over all $i + j = n$ of the maps $D_nX \to D_iX \wedge D_jX$ which are the composites

$$\mathcal{C}(n)_+ \wedge_{\Sigma_n} X^{\wedge n} \xrightarrow{\text{Tr}} \mathcal{C}(n)_+ \wedge_{\Sigma_i \times \Sigma_j} X^{\wedge n} \to (\mathcal{C}(i) \times \mathcal{C}(j))_+ \wedge_{\Sigma_i \times \Sigma_j} X^{\wedge n}.$$

Here Tr is the transfer associated to $\Sigma_i \times \Sigma_j \subseteq \Sigma_n$.

When X is a suspension and $\mathcal{C} = \mathcal{C}_d$, so that $CX \simeq \Omega^d\Sigma^d X$, a closely related statement was given by D. Tamaki [T, Prop. 3.5]. More recently, N. Strickland has also noted and used a version of the $\mathcal{C} = \mathcal{C}_\infty$ case [S1, S2] (with $X = S^2$). In this last situation, the second map in the composite can be viewed as the identity in homotopy. The methods used by both of these authors heavily use that their X is a coHspace, and we hope that our paper offers some perspective on their arguments.
(2) In [A], G. Arone gave an explicit model for the Goodwillie tower associated to the functor

$$X \qquad \Sigma^\infty Map_*(K, X),$$

where K is a fixed finite complex. Working with this, the author, together with S. Ahearn, have been studing multiplicative properties of the associated spectral sequences computing $h^*(Map_*(K, X))$, where h^* is a multiplicative cohomology theory. This joint project overlaps with the present paper in that we can recover the first two parts of Theorem 1.2 in the many classical situations (see [B]) for which there is a coefficient system $\mathcal{C}$ and natural number m, both depending on K, and a natural map

$$CX \to Map_*(K, \Sigma^m X)$$

which is an equivalence for all connected X. It was partially a desire to 'check our work' that led to the present note.
(3) Related to (2), we observe that the first statement of Theorem 1.2 is predicted by Goodwillie calculus.

The organization of the paper is as follows. The main theorem is stated in section 2, after the maps arising in its statement have been carefully defined. This is then proved in section 3, following the strategy followed in [K4], where we analyzed the James-Hopf maps in the same way that we here analyze diagonals. In the short Appendix A, we discuss statement (3) of Theorem 1.2 and similar simplifications of the results in [K4] which arise when X is a coH–space.

In proving Theorem 2.4, we use a simple characterization from [K4] of the stable equivalence of [CMT]. A reader might wonder if other people's constructions of stable splittings yield the same maps. Appendix B, which can be read independently of the rest of the paper, addresses this question. Using ideas from Goodwillie calculus, we show that there is a *unique* equivalence that 'splits the filtration' and is appropriately 'natural', and discuss three different constructions of such splittings. The author would like to thank the referee for a careful critique of an earlier version of this section.

2. The main theorem

We begin with various definitions of a combinatorial nature.

Definitions 2.1. Fix n and $I = (i_1, \ldots, i_l)$. A *partial* I *cover of* $\mathbf{n}$ is an l–tuple $S = (S_1, \ldots, S_l)$ of subsets of $\mathbf{n}$ so that S_t has cardinality i_t for all t. Such an S is a *cover* if $\mathbf{n}$ is the union of $S_1, \ldots, S_l$.

We let $\mathcal{S}(n, I)$ be the set of all partial I covers of $\mathbf{n}$. The tautological action of Σ_n on $\mathbf{n}$ induces an action on $\mathcal{S}(n, I)$ in the obvious way, and two partial I covers of $\mathbf{n}$ are said to be *equivalent* if they are in the same Σ_n–orbit.

Definitions 2.2. Fix $S = (S_1, \ldots, S_l) \in \mathcal{S}(n, I)$.
(1) For $1 \leq t \leq l$, let $\alpha_{S,t} : \mathbf{i}_t \to \mathbf{n}$ be the unique monic order preserving map with image S_t.
(2) Let $\alpha_S : |\mathbf{I}| \to \mathbf{n}$ be the map defined by the maps $\alpha_{S,t}$, viewing $|\mathbf{I}|$ as the disjoint union $\mathbf{i}_1 + \cdots + \mathbf{i}_l$.
(3) Let $\Sigma_S \subseteq \Sigma_n$ be the stablizer subgroup.
(4) Let $\rho_S : \Sigma_S \to \Sigma_I$ be the group homomorphism with t^{th} component $\rho_t : \Sigma_S \to \Sigma_{i_t}$ defined by $\rho_t(\sigma)(x) = \alpha_{S,t}^{-1}(\sigma \alpha_{S,t}(x))$.

Given a partial cover $S \in \mathcal{S}(n, I)$, we now proceed to construct stable maps

$$\Psi_S : (\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+.$$

natural in both X and Σ–free coefficient systems $\mathcal{C}$.

Firstly, the maps $\alpha_{S,t} : \mathbf{i}_t \to \mathbf{n}$ induce

$$p_S : \mathcal{C}(n) \longrightarrow \mathcal{C}(I) = \mathcal{C}(i_1) \times \cdots \times \mathcal{C}(i_l).$$

Then we define

$$\Delta_S : X^n \longrightarrow X^{|I|},$$

by letting, the t^{th} component of $\Delta_S(x)$ be $x_{\alpha_S(t)}$, if $x = (x_1, \ldots, x_n)$. Both these maps are equivariant with respect to $\rho_S : \Sigma_S \to \Sigma_I$. Thus they assemble into an unstable map

$$f_S : (\mathcal{C}(n) \times_{\Sigma_S} X^n)_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+.$$

The stable map Ψ_S is then the stablization of this map precomposed with the stable transfer

$$\mathrm{Tr}_{\Sigma_S}^{\Sigma_n} : (\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \to (\mathcal{C}(n) \times_{\Sigma_S} X^n)_+.$$

If the partial cover S is a cover, then $\alpha_S : |I| \to \mathbf{n}$ is onto. Then Δ_S prolongs to a natural map

$$\Delta_S : X^{\wedge n} \longrightarrow X^{\wedge |I|},$$

and thus Ψ_S prolongs to a natural stable map

$$\Psi_S : D_n X \longrightarrow D_I X.$$

The proof of [K4, Lemma 2.6], based on the naturality of the transfer and the behavior of extended power constructions under conjugation, generalizes to the setting here to prove the next lemma.

Lemma 2.3. *If S is equivalent to S' then Ψ_S is homotopic to $\Psi_{S'}$.*

Our main theorem now goes as follows.

Theorem 2.4. $\Psi_I^n \simeq \sum_S \Psi_S : D_n X \to D_I X$, *with the sum ranging over equivalence classes of I covers of $\mathbf{n}$.*

Note that the first two statements of Theorem 1.2 follow. There are *no* I covers of $\mathbf{n}$ unless $|I| \geq n$ and $i_t \leq n$, for all t. If $|I| = n$, there is a single equivalence class of I covers of $\mathbf{n}$ represented by the disjoint union decomposition $\mathbf{n} = \mathbf{i}_1 + \cdots + \mathbf{i}_l$, and for this S, Ψ_S is the composite of Theorem 1.2(2).

Let S be an I cover of $\mathbf{n}$. In the appendix we check that if $n < |I|$ and X is a coH–space, then the diagonal $\Delta_S : X^{\wedge n} \to X^{\wedge |I|}$ is Σ_S–equivariantly null. Thus Ψ_S will be null, and so Theorem 1.2(3) also follows from Theorem 2.4.

3. Proof of the main theorem

We begin this section by describing a characterization of $s : (CX)_+ \longrightarrow DX$ which we noted in [K4].

To explain this, we first note that, since $Df : D(Y) \to D(Z)$ is defined for all stable maps $f : Y \to Z$, the fact that $X_+ \to X$ is stably split epic implies the next lemma.

Lemma 3.1. *For all spaces X, the projection $D(X_+) \to D(X)$ is stably split epic, up to homotopy.*

Thus, to understand $s : (CX)_+ \longrightarrow DX$, one need just study $s : (C(X_+))_+ \longrightarrow D(X_+)$, and to do this, we make the following observation. There is a natural (unstable) homeomorphism of spaces

$$s' : C(X_+)_+ = (\coprod_{n=1}^{\infty} \mathcal{C}(n) \times_{\Sigma_n} X^n)_+ = \bigvee_{n=1}^{\infty} (\mathcal{C}(n)_+ \wedge_{\Sigma_n} (X_+)^{\wedge n}) = D(X_+).$$

As in [K4], we have the next definitions.

Definitions 3.2. Let $\Theta : D(X_+) \to D(X_+)$ be the composite of stable equivalences

$$D(X_+) \xrightarrow[\sim]{(s')^{-1}} C(X_+)_+ \xrightarrow[\sim]{s} D(X_+).$$

Given n and m, let $\Theta_{n,m} : D_n(X_+) \to D_m(X_+)$ be the $(n,m)^{th}$ component of Θ.

Thanks to the last lemma, the next proposition, describing the maps $\Theta_{n,m}$ in terms of more fundamental maps, can be regarded as a characterization of s.

Proposition 3.3. *$\Theta_{n,m} \simeq *$ if $n < m$, and for $n \geq m$, $\Theta_{n,m}$ is the composite*

$$D_n(X_+) \to (\mathcal{C}(n) \times_{\Sigma_m \times \Sigma_{n-m}} X^n)_+ \to D_m(X_+) \wedge D_{n-m}(X_+) \to D_m(X_+)$$

where the first map is the stable transfer associated to $\Sigma_m \times \Sigma_{n-m} \subseteq \Sigma_n$, the second map is induced by the map $\mathcal{C}(n) \to \mathcal{C}(m) \times \mathcal{C}(n-m)$ associated to the coefficient system, and the last map is the projection.

In the case when $\mathcal{C} = \mathcal{C}_\infty$ this was shown in [K4]. In fact, this was shown twice. For s constructed as the adjoint to the James–Hopf maps of [CMT], this is [K4, Proposition 4.5]. For s constructed using R. Cohen's 'stable proofs of stable splittings' [C], this is [K4, Theorem A.2]. In both cases, the proofs given there generalize easily to general coefficient systems $\mathcal{C}$. (This was already hinted at in [K4]: in that paper, see the last paragraph of section 5.)

We now turn to proving Theorem 2.4, following the strategy used in [K4, §4].

Recall that we are trying to identify Ψ^n_I, the $(n,I)^{th}$ component of the composite

$$\Psi^l : DX \underset{\sim}{\xrightarrow{s^{-1}}} (CX)_+ \xrightarrow{\Delta^l_+} (CX)^l_+ \underset{\sim}{\xrightarrow{s^{\wedge l}}} (DX)^{\wedge l}.$$

The observations above show that it suffices to identify this component with X replaced by X_+. In this case, Ψ^n_I is the $(n,I)^{th}$ component of the composite

$$\Psi^l : D(X_+) \underset{\sim}{\xrightarrow{\Theta^{-1}}} D(X_+) \xrightarrow{\Delta^l_+} D(X_+)^{\wedge l} \underset{\sim}{\xrightarrow{\Theta^{\wedge l}}} D(X_+)^{\wedge l},$$

where Θ is the stable map described by Proposition 3.3.

Let $\bar{\Psi}^n_I$ be the $(n,I)^{th}$ component of the composite

$$D(X_+) \xrightarrow{\Delta^l_+} D(X_+)^{\wedge l} \underset{\sim}{\xrightarrow{\Theta^{\wedge l}}} D(X_+)^{\wedge l}.$$

Proposition 3.4. $\bar{\Psi}^n_I \simeq \sum_S \Psi_S : (\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+$, *with the sum ranging over equivalence classes of partial I covers of $\mathbf{n}$.*

Assuming this key result for the moment, we finish the proof of Theorem 2.4.

If S is a partial I cover of $\mathbf{n}$, we let $S + \mathbf{k}$ denote S viewed as a partial cover of $\mathbf{n} + \mathbf{k}$, under the inclusion $\mathbf{n} \to \mathbf{n} + \mathbf{k}$. Note that every partial I cover of $\mathbf{n}$ is equivalent to one of the form $S + \mathbf{n} - \mathbf{m}$ with S an I cover of $\mathbf{m}$, and S is unique up to equivalence. (The number $n - m$ will be the number of 'uncovered' points of the original partial cover.)

Lemma 3.5. *If $n \geq m$ and S is an I cover of $\mathbf{m}$, then*

$$\Psi_{S+\mathbf{n}-\mathbf{m}} \simeq \Psi_S \circ \Theta_{n,m} : (\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+.$$

This follows from standard properties of the transfer as in [K4, Proof of 4.13].

Theorem 2.4 now follows. Let $\tilde{\Psi}^l : D(X_+) \to D(X_+)^{\wedge l}$ have $(n,I)^{th}$ component given by $\sum_S \Psi_S : (\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+$, with the sum ranging over equivalence classes of I covers of $\mathbf{n}$. Proposition 3.4 and Lemma 3.5 combine to say

$$\Theta^{\wedge l} \circ \Delta^l_+ \simeq \tilde{\Psi}^l \circ \Theta.$$

Precomposing with Θ^{-1} yields the main theorem: $\Psi^l \simeq \tilde{\Psi}^l$.

It remains to prove Proposition 3.4. If $I = (i_1, \ldots, i_l)$, let $I' = (n-i_1, \ldots, n-i_l)$. One can view $\Sigma_I \times \Sigma_{I'}$ as a subgroup of Σ_n^l in the obvious way, and $|I| + |I'| = nl$. Then $\bar{\Psi}_I^n$ will be the composite of the stable map

$$(\mathcal{C}(n) \times_{\Sigma_n} X^n)_+ \xrightarrow{\Delta} (\mathcal{C}(n)^l \times_{\Sigma_n^l} X^{nl})_+ \xrightarrow{\mathrm{Tr}} (\mathcal{C}(n)^l \times_{\Sigma_I \times \Sigma_{I'}} X^{nl})_+$$

and the stablization of the unstable map

$$(\mathcal{C}(n)^l \times_{\Sigma_I \times \Sigma_{I'}} X^{nl})_+ \to ((\mathcal{C}(I) \times \mathcal{C}(I')) \times_{\Sigma_I \times \Sigma_{I'}} X^{nl})_+ \to (\mathcal{C}(I) \times_{\Sigma_I} X^{|I|})_+.$$

The first of these maps, $\mathrm{Tr} \circ \Delta$, can be analyzed using the naturality of the transfer with respect to pullbacks of covering spaces. To describe our pullback, we need yet more notation. If $S = (S_1, \ldots, S_l)$ is a partial I cover of $\mathbf{n}$, let $S' = (\mathbf{n} - S_1, \ldots, \mathbf{n} - S_l)$ denote the complementary partial I' cover of $\mathbf{n}$. Note that $\Sigma_{S'} = \Sigma_S$ and that $(S')' = S$.

Lemma 3.6. *There is a pullback diagram*

$$\begin{array}{ccc} \coprod_S \mathcal{C}(n) \times_{\Sigma_S} X^n & \longrightarrow & \mathcal{C}(n)^l \times_{\Sigma_I \times \Sigma_{I'}} X^{nl} \\ \downarrow & & \downarrow \\ \mathcal{C}(n) \times_{\Sigma_n} X^n & \xrightarrow{\Delta} & \mathcal{C}(n)^l \times_{\Sigma_n^l} X^{nl} \end{array}$$

where S runs through the partial I covers of $\mathbf{n}$, and the component maps

$$\mathcal{C}(n) \times_{\Sigma_S} X^n \to \mathcal{C}(n)^l \times_{\Sigma_I \times \Sigma_{I'}} X^{nl}$$

are induced by $\rho_S \times \rho_{S'} : \Sigma_S \to \Sigma_I \times \Sigma_{I'}$ and $\Delta_S \times \Delta_{S'} : X^n \to X^{nl}$.

Note that Proposition 3.4 follows from this.

By naturality, to verify the lemma, it suffices to check that there is a pullback diagram

$$\begin{array}{ccc} \coprod_S B\Sigma_S & \xrightarrow{\coprod_S \rho_S \times \rho_{S'}} & B(\Sigma_I \times \Sigma_{I'}) \\ \downarrow & & \downarrow \\ B\Sigma_n & \xrightarrow{\Delta} & B\Sigma_n^l \end{array}$$

where S runs through the partial I covers of $\mathbf{n}$.

This we check using the double coset formula, i.e. we check that there is an isomorphism of Σ_n–sets

$$\coprod_S \Sigma_n / \Sigma_S \simeq \Sigma_n^l / (\Sigma_I \times \Sigma_{I'}),$$

where Σ_n acts diagonally on Σ_n^l. But this is easy to see. By definition, there is an isomorphism of Σ_n–sets

$$\coprod_S \Sigma_n / \Sigma_S \simeq \mathcal{S}(n, I).$$

Now note that $\mathcal{S}(n, I) = \mathcal{S}(n, i_1) \times \cdots \times \mathcal{S}(n, i_l)$ is acted on transitively by Σ_n^l with isotropy group precisely $\Sigma_I \times \Sigma_{I'}$.

Appendix A. Simplifications when X is a coHspace.

In this appendix, we verify the intuitively reasonable fact stated at the end of §2: if S is an I cover of n and $|I| > n$, so that the diagonal is involved in an interesting way in the construction of $\Psi_S : D_n X \to D_I X$, then Ψ_S is null if X is a coH–space. We verify this by proving a more general lemma, which applies also to the constructions of [K4].

If A is a finite set, and X is a pointed space, we let $W_A(X) \subset X^A$ denote the fat wedge, and $X^{\wedge A} = X^A / W_A(X)$. Then $X^{\wedge A}$ is a covariant functor of X, and a contravariant functor of A if we restrict maps in the category of finite sets to just the epimorphisms. Given an epimorphism of finite sets $\alpha : B \to A$, we write the induced map as $\Delta_\alpha : X^{\wedge A} \to X^{\wedge B}$ to emphasize that it is a generalized diagonal map.

It is easy to check that there are natural homeomorphisms $X^{\wedge(A \amalg B)} \simeq X^{\wedge A} \wedge X^{\wedge B}$ and $X^{\wedge(A\times B)} \simeq (X^{\wedge A})^{\wedge B}$.

If a group G acts on a finite set A, then $X^{\wedge A}$ will naturally be a G–space. Given an epimorphism of G–sets $\alpha : B \to A$, and $a \in A$, let $G_a \subseteq G$ be the stabilizer subgroup, and let $B_a = \alpha^{-1}(a) \subseteq B$. Then B_a is a G_a–set.

Lemma A.1. *Let $\alpha : B \to A$ be an epimorphism of G–sets. Suppose that there exists an element $a \in A$ such that G_a acts trivially on B_a, and B_a contains more than one element. Then $\Delta_\alpha : X^{\wedge A} \to X^{\wedge B}$ will be G–equivariantly null if X is a coH–space.*

To see this, we first note that there is a decomposition of $\alpha : B \to A$ as a G–map:

$$\coprod_{Ga \in G\backslash A} G \times_{G_a} B_a \to \coprod_{Ga \in G\backslash A} G/G_a.$$

Thus Δ_α is the smash product, over the G–orbits in A, of G–maps

$$\Delta_{\alpha_a} : X^{\wedge G/G_a} \to X^{\wedge(G\times_{G_a} B_a)},$$

where $\alpha_a : G \times_{G_a} B_a \to G/G_a$ is the evident epimorphism.

Choosing $a \in A$ as in the hypotheses, we will have $G \times_{G_a} B_a = G/G_a \times B_a$, so that Δ_{α_a} can be identified with

$$\Delta^{\wedge G/G_a} : X^{\wedge G/G_a} \to (X^{\wedge B_a})^{\wedge G/G_a},$$

where $\Delta : X \to X^{\wedge B_a}$ is a diagonal map into a smash product of more than one copy of X. If X is a coH–space, then Δ will be (nonequivariantly) null, so that $\Delta^{\wedge G/G_a}$ will be G–equivariantly null, establishing the lemma.

For the purposes of verifying the statement at the end of §2, let $A = \mathbf{n}$, $B = |\mathbf{I}|$, $\alpha = \alpha_S : |\mathbf{I}| \to \mathbf{n}$, and $G = \Sigma_S$, where S is an I cover of $\mathbf{n}$. Then, for all $a \in A$, G_a acts trivially on B_a, so the lemma applies.

As a second application, in [K4] we defined an (r, q)–set S with $|S| = n$ to be a cover of $\mathbf{n}$ by r distinct subsets each of cardinality q. As discussed there, if $\Sigma_S \subset \Sigma_n$ is the stabilizer of this configuration, one gets a homomorphism $\Sigma_S \to \Sigma_r \wr \Sigma_q$, and an epimorphism $\alpha_S : \mathbf{rq} \to \mathbf{n}$ which is Σ_S–equivariant. Letting $A = \mathbf{n}$, $B = \mathbf{rq}$, $G = \Sigma_S$, and $\alpha = \alpha_S$, we again have that for all $a \in A$, G_a acts trivially on B_a, so the lemma applies.

In this case, the topological consequence, a simplification of the main result of [K4] in the coH–space case, goes as follows. Let C_∞ and D_∞ denote the C and D constructions associated to the E_∞ operad $\mathcal{C}_\infty$. Given an arbitrary coefficient system $\mathcal{C}$, let $j_q : CX \to C_\infty D_q X$ be the q^{th} James–Hopf invariant, i.e. the unstable map adjoint to the stable splitting projection $s_q : CX \to D_q X$. Then let $J^n_{r,q} : D_n X \to D_{\infty,r} D_q X$ be the $(n,r)^{th}$ component of the stable map

$$DX \underset{\sim}{\xrightarrow{s^{-1}}} (CX)_+ \xrightarrow{j_{q+}} (C_\infty D_q X)_+ \underset{\sim}{\xrightarrow{s}} D_\infty D_q X.$$

In [K4], we described $J^n_{r,q}$ as a sum of maps associated to the equivalence classes of (r,q)–sets S with $|S| = n$. If X is a coH–space, and $rq > n$, our lemma applies, and we have the following corollary of [K4, Theorem 2.3].

Theorem A.2. *If X is a coH–space, then $J^n_{r,q} \simeq *$ unless $rq = n$, and $J^{rq}_{r,q}$ is the composite*

$$\mathcal{C}(rq)_+ \wedge_{\Sigma_{rq}} X^{\wedge rq} \to \mathcal{C}(rq)_+ \wedge_{\Sigma_r \wr \Sigma_q} X^{\wedge rq} \to (\mathcal{C}_\infty \times \mathcal{C}(q)^r)_+ \wedge_{\Sigma_r \wr \Sigma_q} X^{\wedge rq}.$$

Here the first map is the (stable) transfer induced by the inclusion $\Sigma_r \wr \Sigma_q \subset \Sigma_{rq}$, and the second map is induced by maps $\mathcal{C}(rq) \to \mathcal{C}(q)^r$ arising from the coefficient system structure, together with any Σ_r–equivariant map $\mathcal{C}(rq) \to \mathcal{C}_\infty(r)$. (As $\mathcal{C}_\infty(r)$ is contractible, this will exist and be unique up to equivariant homotopy.)

Appendix B. Characterizations of stable splittings

Given a stable map $s : (CX)_+ \to DX$, we let $s_n : (CX)_+ \to D_n X$ denote the composite $(CX)_+ \xrightarrow{s} DX \to D_n X$, where the second map is projection onto the n^{th} wedge summand.

Implicit in the statement 'the stable map $s : (CX)_+ \to DX$ splits the filtration of CX' is that s satisfies the following property:

(B.1) For all $n \geq 0$, the composite $(F_n CX)_+ \overset{i_n}{\hookrightarrow} (CX)_+ \xrightarrow{s_n} D_n X$

is homotopic to the stablization of the projection

$$\pi_n : (F_n CX)_+ \to F_n CX / F_{n-1} CX = D_n X.$$

We will also insist that s preserve the increasing filtrations on $(CX)_+$ and DX:

(B.2) For all $n \geq 0$, the composite $(F_n CX)_+ \hookrightarrow (CX)_+ \xrightarrow{s} DX$

factors, up to homotopy, through $\bigvee_{m=0}^n D_m X \hookrightarrow DX$.

Equivalently, this last property says that the composite

$$(F_n CX)_+ \hookrightarrow (CX)_+ \xrightarrow{s} DX \to \bigvee_{m=n+1}^{\infty} D_m X$$

is null homotopic.

Lemma B.1. *If s satisfies (*B.1*) and (*B.2*), then s is an equivalence. Furthermore, such an s is uniquely determined by the maps s_n.*

Proof. Let $\tilde{s}_n : (F_nCX)_+ \to \bigvee_{m=0}^{n} D_mX$ have 'm^{th} component' $s_m \circ i_n$.

By assumption, there are homotopy commutative diagrams

$$\begin{array}{ccccc} (F_{n-1}CX)_+ & \longrightarrow & (F_nCX)_+ & \longrightarrow & D_nX \\ \downarrow \tilde{s}_{n-1} & & \downarrow \tilde{s}_n & & \downarrow Id \\ \bigvee_{m=0}^{n-1} D_mX & \longrightarrow & \bigvee_{m=0}^{n} D_mX & \longrightarrow & D_mX, \end{array}$$

and

$$\begin{array}{ccc} (F_nCX)_+ & \longrightarrow & (CX)_+ \\ \downarrow \tilde{s}_n & & \downarrow s \\ \bigvee_{m=0}^{n} D_mX & \longrightarrow & DX. \end{array}$$

By induction on n, the first diagram implies that each $\tilde{s}_n$ is a homotopy equivalence. One consequence of this is that $\{(CX)_+, DX\} = \lim_n \{(F_nCX)_+, DX\}$, i.e. there are no nonzero $\lim^1$ terms, as the Mittag–Leffler condition clearly holds in the inverse system $\{\Sigma(\bigvee_{m=0}^{n} D_mX), DX\}$. From the second diagram, we thus conclude that s can be viewed as $\operatorname{hocolim}_n \tilde{s}_n$. The map s is thus determined by the maps s_n, and will be a homotopy equivalence. □

For a fixed space X, there can be many maps s satisfying properties (B.1) and (B.2). Thus it appears ambiguous to refer to *the* stable splitting of CX. However, as we will see below, this ambiguity disappears if one assumes that s is appropriately *natural* in X. Since our proof of this evokes T.Goodwillie's only partially published theory of the 'calculus' of homotopy functors, categories of spaces and spectra that we work with need to have enough structure to support his proofs.[2]

Given two functors F and G from pointed spaces to spectra, it is useful to generalize the notion of a natural transformation from F to G in the following way.

Definition B.2. A *weak* natural transformation $h : F \to G$ will be a triple (H, f, g), with H a functor from spaces to spectra, $g : H \to G$ a natural transformation, and $f : H \to F$ a natural transformation such that $f(X) : H(X) \to F(X)$ is a homotopy equivalence for all X.

Note that, if F and G are homotopy functors, then a weak natural transformation $h : F \to G$ induces a well defined natural transformation in the homotopy category: $h(X) = g(X) \circ f(X)^{-1} \in \{F(X), G(X)\}$. Furthermore, using homotopy pullbacks, one can define the composition of weak natural transformations.[3]

In particular, viewing CX and DX as functors from the category of pointed spaces to spectra, it makes sense to say that a weak natural transformation $s : (CX)_+ \to DX$ satisfies properties (B.1) and (B.2).

Our uniqueness result goes as follows.

[2] Since the details of [Goo] have not been forthcoming, it is a bit hard to know exactly what this structure is. (But see the 'short review of Goodwillie calculus' in the preprint [McC].) However, it seems clear that the category of coordinate free spectra of [LMMS], or the category of symmetric spectra of [HSS] each support more than enough structure.

[3] To blithely declare that the weak natural transformations lead to an actual category of fractions raises set theoretic questions that we would rather not discuss.

Proposition B.3. *Let s and t be two weak natural transformations from $(CX)_+$ to DX satisfying properties (*B.1*) and (*B.2*). Then s and t agree up to homotopy: for all X,*

$$s(X) \simeq t(X) : CX \to DX.$$

Sketch proof. We will show that, up to homotopy, there is a formula for the components $s_n : (CX)_+ \to D_nX$ that is independent of s.

Let $p_n : (CX)_+ \to \mathcal{P}_n((CX)_+)$ be projection onto the n^{th} degree polynomial functor approximation to $(CX)_+$ in the Goodwillie sense. His general theory says that $D_n(X)$ is homogeneous of polynomial degree n and that there is a weak natural transformation

$$\mathcal{P}_n((CX)_+) \xrightarrow{\bar{s}_n} D_nX,$$

unique up to homotopy, such that in the homotopy category

$$s_n : (CX)_+ \to D_nX$$

factors as the composite

$$(CX)_+ \xrightarrow{p_n} \mathcal{P}_n((CX)_+) \xrightarrow{\bar{s}_n} D_nX.$$

Furthermore, using that $\mathcal{P}_n$ preserves fibrations up to homotopy, one deduces that $(F_nCX)_+$ has polynomial degree at most n and that the composite

$$(F_nCX)_+ \xrightarrow{i_n} (CX)_+ \xrightarrow{p_n} \mathcal{P}_n((CX)_+)$$

is a homotopy equivalence. Thus

$$\begin{aligned} s_n &\simeq \bar{s}_n \circ p_n \\ &\simeq \bar{s}_n \circ (p_n \circ i_n) \circ (p_n \circ i_n)^{-1} \circ p_n \\ &\simeq (\bar{s}_n \circ p_n) \circ i_n \circ (p_n \circ i_n)^{-1} \circ p_n \\ &\simeq s_n \circ i_n \circ (p_n \circ i_n)^{-1} \circ p_n \\ &\simeq \pi_n \circ (p_n \circ i_n)^{-1} \circ p_n, \end{aligned}$$

a formula that is evidently independent of s. □

We now discuss the existence of stable splittings.

One construction comes from the proof of the last proposition: with notation as there, one defines a weak natural transformation $s_n : (CX)_+ \to D_nX$ as the composite of $p_n : (CX)_+ \to \mathcal{P}_n((CX)_+)$ with $\mathcal{P}_n((CX)_+) \underset{\sim}{\xleftarrow{p_n \circ i_n}} (F_nCX)_+ \xrightarrow{\pi_n} D_nX$.

A second construction is to use the natural transformations arising as adjoints of the James–Hopf maps of [CMT]. These are combinatorially defined, unstable maps $j_n : CX \to \Omega^\infty\Sigma^\infty D_nX$ which generalize James' original construction of global Hopf invariants. These are natural in X, and are quite explicit, except for embeddings of certain configuration spaces in $\mathbf{R}^\infty$.

Finally, a third construction uses the idea of R. Cohen [C]. (See also [LMMS, VII.5].) We briefly review this, tweaked slightly so that the splittings are natural.

One starts by observing that, working in a category of spectra (e.g. [EKMM]) with an associative, commutative smash product with unit $S = \Sigma^\infty S^0$, a map of spectra $\eta : S \to E$ defines a covariant functor

$$E^\wedge : \Lambda \to Spectra$$

by the formula $E^\wedge(n) = E^{\wedge n}$. Thus, given a coefficient system $\mathcal{C}$, we can define a filtered spectrum CE to be the coend of $\Sigma^\infty \mathcal{C}$ and $E^\wedge$: CE is the coequalizer of the two apparent maps

$$\bigvee_{\alpha:\mathbf{m}\to\mathbf{n}} \mathcal{C}(n)_+ \wedge E^{\wedge m} \ \underset{\longrightarrow}{\overset{\longrightarrow}{}} \ \bigvee_n \mathcal{C}(n)_+ \wedge E^{\wedge n}.$$

This is related to the construction of the introduction by

$$\Sigma^\infty (CX)_+ = C(\Sigma^\infty (X_+)).$$

Furthermore, there is an isomorphism of spectra

$$C(Z \vee S) = D(Z),$$

natural in spectra Z.

The construction CE is functorial with respect to maps under S.

The construction of a natural splitting s now goes as follows. Let X' denote the homotopy fiber of the projection $X_+ \to S$. Note that the composite $X' \to X_+ \to X$ is a natural homotopy equivalence. A weak natural transformation $s : (CX)_+ \to DX$ satisfying properties (B.1) and (B.2) is then obtained by applying the construction C to the natural maps under (and over) S

$$X_+ \longleftarrow X' \vee S \longrightarrow X \vee S.$$

Remark B.4. These days, some readers might find our first, Goodwillie calculus based, construction of the stable splittings the preferred one. However, the author knows a proof of Proposition 3.3, the characterization of the stable splittings needed in the body of the note, only using one of the other two constructions. Of the three constructions given here, it is our last one that seems best suited to use when studying further structure of the stable splittings.

References

[A] G. Arone, *A generalization of Snaith–type filtration* , Trans.A.M.S. **351**(1999), 1123–1250.

[B] C.–F. Bödigheimer *Stable splitting of mapping spaces*, in Algebraic Topology (Seattle, Wash., 1985), 174–187, Springer L. N. Math. **1286**, 1987.

[CMT] F. R. Cohen, J. P. May, and L. R. Taylor, *Splitting certain spaces $C\mathbf{X}$*, Math.Proc.Camb.Phil.Soc. **84**(1978), 465–496.

[C] R. Cohen, *Stable proofs of stable splittings*, Math. Proc. Camb. Phil. Soc. **88**(1980), 149–151.

[EKMM] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, Modules, and Algebras in Stable Homotopy Theory*, A. M. S. Math. Surveys and Monographs **47**, 1997.

[Goo] T. G. Goodwillie, *Calculus III: the Taylor series of a homotopy functor*, partial preprint from before 1994.

[HSS] M. Hovey, B. Shipley, and Jeff Smith, *Symmetric spectra*,

[K1] N. J. Kuhn, *The geometry of the James-Hopf maps*, Pac. J. Math. **102**(1982), 397–412.

[K2] N. J. Kuhn, *Suspension spectra and homology equivalences*, Trans. A. M. S. **283**(1984), 303–313.

[K3] N. J. Kuhn, *Extended powers of spectra and a generalized Kahn–Priddy Theorem*, Topology **23**(1985), 473–480–405.

[K4] N. J. Kuhn, *The transfer and the James-Hopf invariants*, Math. Zeit. **196**(1987), 391–405.
[LMMS] L. G. Lewis, J. P. May, J. McClure, and M. Steinberger *Equivariant Stable Homotopy Theory*, Springer L. N. Math. **1213**, 1986.
[McC] R. McCarthy, *On a universal tower for approximating functors by those of finite degree, and its relationship to Goodwillie calculus*, preprint, University of Illinois.
[S1] N. P. Strickland, *Morava E–theory of symmetric groups*, Topology **37**(1998), 759–779.
[S2] N. P. Strickland, *Correction to: "Morava E–theory of symmetric groups"*, Topology **38**(1999), 931.
[T] D. Tamaki, *A dual Rothenberg–Steenrod spectral sequence*, Topology **33**(1994), 631–662.

Department of Mathematics, University of Virginia, Charlottesville, VA 22903