FESHBACH'S TRANSFER THEOREM AND APPLICATIONS

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SUMMARY : *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group* G *over the field* $\mathbb{F}$. *Let* $\mathbb{F}[V]$ *be the ring of polynomial functions on* $V = \mathbb{F}^n$ *on which* G *acts via* ϱ, *and let* $\mathbb{F}[V]^G \subseteq \mathbb{F}[V]$ *be the ring of invariant polynomials. If the characteristic of* $\mathbb{F}$ *is* p *and* p *divides* $|G|$, *the order of* G, *then it is known that the transfer map*

$$\mathrm{Tr}^G : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^G$$

defined by $\mathrm{Tr}^G(f) = \sum_{g \in G} gf$ *has as image an ideal in* $\mathbb{F}[V]^G$ *of height strictly less than* n. *In* [1] *M. Feshbach described the radical of the image of the transfer. In this note we reprove this result and in addition describe the variety defined by the extended ideal* $\mathrm{Im}(\mathrm{Tr}^G))^e$ *in* $\mathbb{F}[V]$. *It turns out that this variety is nothing but the union of the fixed subspaces of the elements in* G *of order* p.

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Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group G over the field $\mathbb{F}$, and let $\mathbb{F}[V]$ be the algebra of polynomial functions on the vector space $V = \mathbb{F}^n$. The group G acts on V and hence also on $\mathbb{F}[V]$ via the representation ϱ and we denote by $\mathbb{F}[V]^G$ the subalgebra of G-invariant polynimials in $\mathbb{F}[V]$. (We refer to [5] for basic facts about the invariant theory of finite groups.) The **transfer homomorphism**

$$\mathrm{Tr}^G : \mathbb{F}[V] \longrightarrow \mathbb{F}[V]^G$$

is defined by

$$\mathrm{Tr}^G(f) = \sum_{g \in G} gf, \qquad \forall\ f \in \mathbb{F}[V].$$

If the characteristic of $\mathbb{F}$ is relatively prime to the order of G then the transfer map is surjective. By contrast, if the characteristic of $\mathbb{F}$ divides the order of G the image of the transfer, $\mathrm{Im}(\mathrm{Tr}^G)$, is an ideal of height strictly less than n [1], [3] [7]. We begin this note by reproving this result as well as M. Feshbach's unpublished description of the radical of $\mathrm{Im}(\mathrm{Tr}^G)$. We then go on to describe the variety defined by the extended ideal $(\mathrm{Im}(\mathrm{Tr}^G))^e \subset \mathbb{F}[V]$. It turns out that this variety has a particularly elegant description, namely it is the union of the fixed point sets of the elements of order p in G, where p is the characteristic of $\mathbb{F}$. As an application we show that the Dickson polynomial of least degree $\mathbf{d}_{n,n-1}$ is always a nonzero divisor in the quotient ring $\mathbb{F}[V]/\mathrm{Im}(\mathrm{Tr}^G)$, so this quotient ring has positive homological codimension (or depth).

§1. Feshbach's Transfer Theorem

Let $g \in \mathrm{GL}(n, \mathbb{F})$ and define the element $\partial_g = 1 - g \in \mathrm{Mat}_{n,n}(\mathbb{F})$, the algebra of $n \times n$ matrices over $\mathbb{F}$. The element ∂_g acts on the algebra $\mathbb{F}[V]$ as a linear **twisted differential**, i.e.

$$\partial_g(f + h) = \partial_g(f) + \partial_g(h)$$
$$\partial_g(f \cdot h) = \partial_g(f) \cdot h + (gf) \cdot \partial(h),$$

as the following computation shows:

$$\partial_g(f + h) = f + h - gf - gh = f - gf + h - gh = \partial_g(f) + \partial_g(h)$$
$$\begin{aligned}\partial_g(f \cdot h) &= f \cdot h - g(f \cdot h) = f \cdot h - (gf) \cdot h + (gf) \cdot h - (gf) \cdot (gh) \\ &= (f - gf) \cdot h + (gf)(h - gh) = \partial_g(f) \cdot h + (gf) \cdot \partial_g(h).\end{aligned}$$

Let $I_g \subseteq \mathbb{F}[V]$ denote the ideal generated by $\partial_g(V^*)$, where as usual V^* is the space of linear forms on $V = \mathbb{F}^n$.

Lemma 1.1: *Let $g \in \mathrm{GL}(n, \mathbb{F})$ and $\mathrm{Im}(\partial_g)$ the image of the action of ∂_g on $\mathbb{F}[V]$. Then $\mathrm{Im}(\partial_g) \subseteq I_g$.*

Proof: Choose a basis $z_1, \dots, z_n$ for V^* and identify $\mathbb{F}[V]$ with $\mathbb{F}[z_1, \dots, z_n]$ as usual. Both $\mathrm{Im}(\partial_g)$ and I_g are graded subsets of $\mathbb{F}[z_1, \dots, z_n]$ and they agree in grading 1 by definition. Therefore we may proceed inductively, and suppose that, $\mathrm{Im}(\partial_g)$ and I_g agree in degree $< d$. If $f \in \mathbb{F}[z_1, \dots, z_n]$ is a form, i.e., a homogeneous polynomial, of degree d then it is a sum of monomials $z^E = z_1^{e_1} \cdots z_n^{e_n}$, and so by linearity of ∂_g it is enough to show that

$\partial_g(z^E) \in I_g$. Without loss of generality we may suppose that $e_1 > 0$ so we may write $z^E = z_1 z^F$ where $F = (e_1 - 1, e_2, \ldots, e_n)$. Since the monomial z^F has degree $d-1$ we know by induction $\partial_g(z^F) \in I_g$. By the twisted derivation formula

$$\partial_g(z^E) = \partial_g(z_1) \cdot z^F + (g z_1) \cdot \partial_g(z^F).$$

By definition $\partial_g(z_1) \in I_g$ and since Ig is an ideal it follows that both terms on the right hand side of the preceeding equation are in I_g completing the induction step, and hence the proof. □

PROPOSITION 1.2 (M. Feshbach): *Let* $\mathbb{F}$ *be a field of characteristic* p, $u \in \mathrm{GL}(n, \mathbb{F})$ *an element of order* p, *and* $P < \mathrm{GL}(n, \mathbb{F})$ *the subgroup generated by* u. *Then* $\mathrm{Im}(\mathrm{Tr}^P) \subseteq I_u$ *and hence* $\mathrm{ht}(\sqrt{\mathrm{Im}(\mathrm{Tr}^P)}) \le n - \dim_{\mathbb{F}}(V^u)$.

PROOF: Note that in the group ring $\mathbb{F}(P)$ we have

$$\mathrm{Tr}^P = 1 + u + \cdots + u^{p-1} = (1-u)^{p-1} = \partial_u^{p-1}.$$

Hence $\mathrm{Im}(\mathrm{Tr}^P) = \mathrm{Im}(\partial_u^{p-1}) \subseteq \mathrm{Im}(\partial_u) \subset I_u$ by lemma 1.1. The ideal I_u is a prime ideal in $\mathbb{F}[V]$ since it is generated by linear forms. The height of I_u is equal to $\dim_{\mathbb{F}}(\partial_u(V^*)) = n - \dim_{\mathbb{F}}(\ker(\partial_u : V \longrightarrow V))$ and $\ker(\partial_u : V \longrightarrow V) = V^u$. Hence I_u has height $n - \dim_{\mathbb{F}}(V^u)$. Since $I_u \subset \mathbb{F}[V]$ is prime and $\mathbb{F}[V]^P \subset \mathbb{F}[V]$ is a finite extension the ideal $\mathbb{F}[V]^P \cap I_u$ is prime by the lying over theorem. By the going up theorem $\mathrm{ht}(\mathbb{F}[V]^P \cap I_u) = \mathrm{ht}(I_u)$ and since $\mathrm{Im}(\mathrm{Tr}^P) \subseteq \mathbb{F}[V]^P \cap I_u$ the result follows. □

LEMMA 1.3: *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group* G *over the field* $\mathbb{F}$ *and* $H \le G$ *a subgroup. Then* $\mathrm{Im}(\mathrm{Tr}^G) \subseteq \mathrm{Im}(\mathrm{Tr}^H)$.

PROOF: Choose a right transversal $g_1, \ldots, g_t$ for H in G. Then $G = Hg_1 \sqcup \cdots \sqcup Hg_t$ and hence for any $f \in \mathbb{F}[V]$

$$\mathrm{Tr}^G(f) = \sum_{g\in G} gf = \sum_{h\in H}\sum_{i=1}^{t} h g_i f = \sum_{h\in H} h(g_1 f + \cdots + g_t f) = \mathrm{Tr}^H(g_1 f + \cdots + g_t f)$$

and the result follows. □

THEOREM 1.4 (M. Feshbach): *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group over the field* $\mathbb{F}$. *Then*

(i) $\mathrm{Im}(\mathrm{Tr}^G) \subseteq \bigcap_{|g|=p} I_g$, *where the intersection runs over all the elements of order* p *in* G,

(ii) $\mathrm{ht}(\sqrt{\mathrm{Im}(\mathrm{Tr}^G)}) \le n - \max_{|g|=p} \{\dim_{\mathbb{F}}(V^g)\}$.

PROOF: For each element $g \in G$ of order p we have $\mathrm{Im}(\mathrm{Tr}^G) \subseteq \mathrm{Im}(Tr^{<g>})$ by 1.3, where $<g>$ denotes the subgroup of G generated by g, and the result follows from proposition 1.2. □

§2. The Transfer Variety

If $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ is a representation of a finite group and $\mathrm{Im}(\mathrm{Tr}^G) \subseteq \mathbb{F}[V]^G$ the transfer ideal, then we may extend $\mathrm{Im}(\mathrm{Tr}^G)$ to an ideal in $\mathbb{F}[V]$ in the usual way.[1] to an ideal in $\mathbb{F}[V]$ This extended ideal defines an algebraic set which we propose to study in this section. To be specific we introduce:

[1] Simply take the ideal in $\mathbb{F}[V]$ which is generated by the elements in $\mathrm{Im}(\mathrm{Tr}^G)$.

DEFINITION : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$. The* **transfer variety**, *denoted by $X_G \subseteq V$ is defined by*

$$X_G = \left\{ x \in V \mid \mathrm{Tr}^G(f)(x) = 0 \ \forall \ f \in \mathrm{Tot}(\mathbb{F}[V]) \right\}.$$

N.b. Since X_G is an ***affine*** variety, we must use all polynomial functions to define it, and not just homogeneous ones.

LEMMA 2.1 : *Let $g \in \mathrm{GL}(n, \mathbb{F})$ and $\mathcal{V}(I_g) \subseteq V$ the affine variety defined by the ideal $I_g \subset \mathrm{Tot}(\mathbb{F}[V])$. Then $\mathcal{V}(I_g) = V^g$, where $V^g \subseteq V$ is the fixed point set of the element g acting on v.*

PROOF : An element $x \in V$ belongs to $\mathcal{V}(I_g)$ if and only if $\partial_g \ell(x) = 0$ for every linear form ℓ. If $x \in V^g$ then

$$\partial_g \ell(x) = \ell(x) - \ell(g^{-1}x) = \ell(x) - \ell(x) = 0$$

so $x \in \mathcal{V}(I_g)$. Conversely, since

$$\partial_g \ell(x) = \ell(x) - \ell(g^{-1}x),$$

it follows that $x \in \mathcal{V}(I_g)$ if and only if $\ell(x) = \ell(g^{-1}x)$ for every linear form ℓ. Since the linear forms separate the points in V it follows $x = g^{-1}x$ and hence x belongs to V^g. □

The transfer variety is defined by the radical of the ideal in $\mathbb{F}[V]$ generated by $\mathrm{Im}(\mathrm{Tr}^G)$, which in turn is contained in the intersection of the ideals I_g, where g ranges over the elements of order p in G. Passing to varieties turns the inclusion around and the intersection into a union, so we get:

COROLLARY 2.2: *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$. Then*

$$\bigcup_{|g|=p} V^g \subseteq X_G,$$

where the union runs over all the elements of order p in G. □

We next describe the transfer variety.

LEMMA 2.3: *Let $B \subset V = \mathbb{F}^n$ be a finite subset and $b_0 \in B$. Then there exists a polynomial function $h \in \mathrm{Tot}(\mathbb{F}[V])$ such that*

$$h(b_0) = 1$$
$$h(b) = 0 \ \forall \ b \in B, b \neq b_0.$$

PROOF : It is enough to consider the case of a pair $x, y \in V$ of distinct points, and prove the existence of a polynomial function $h_{x,y} \in \mathrm{Tot}(\mathbb{F}[V])$ with the property

$$h_{x,y}(x) = 1$$
$$h_{x,y}(y) = 0.$$

For given this the general case is solved by setting

$$h = \prod_{\substack{b \in B \\ b \neq b_0}} h_{b_0,b}.$$

So suppose $x \neq y \in V = \mathbb{F}^n$. Write

$$x = (x_1, \dots, x_n)$$
$$y = (y_1, \dots, y_n)$$

and choose i between 1 and n with $x_i \neq y_i$. The function $h_{x,y} : V \longrightarrow \mathbb{F}$ defined by

$$\frac{h_{x,y}(a_1, \dots, a_n) = a_i - y_i}{x_i - y_i}$$

is linear, though not homogeneous, and has the desired property. □

LEMMA 2.4: *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group G over the field $\mathbb{F}$ and $x \in V$. Then for any $f \in \mathbb{F}[V]^{G_x}$ of positive degree we have*

$$\mathrm{Tr}^G(f)(x) = |G_x| \mathrm{Tr}^G_{G_x}(f)(x)$$

where $G_x \leq G$ is the isotropy group of x.

PROOF: Choose elements $g_1, \dots, g_m \in G$ that are simultaneouly a left and right transversal for G_x in G. This is always possible by König's Lemma [10] pp. 12–13. Then $g_1^{-1}, \dots, g_m^{-1}$ is also a left transversal for G_x in G and

$$\begin{aligned}
\mathrm{Tr}^G(f)(x) &= \sum_{g\in G} g f(x) = \sum_{i=0}^{m} \sum_{h\in G_x} h g_i f(x) \\
&= \sum_{i=1}^{m} \sum_{h\in G_x} f((h g_i)^{-1} x) = \sum_{i=1}^{m} \sum_{h\in G_x} f(g_i^{-1} h x) \\
&= \sum_{i=1}^{m} (\underbrace{f(g_i^{-1} x) + \dots + f(g_i^{-1} x)}_{|G_x|}) \\
&= \sum_{i=1}^{m} |G_x| f(g_i^{-1} x) = |G_x| \sum_{i=1}^{m} g_i f(x) \\
&= |G_x| \mathrm{Tr}^G_{G_x}(f)(x)
\end{aligned}$$

by the definition of the relative transfer. □

PROPOSITION 2.5: *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group G over the field $\mathbb{F}$ of characteristic p. Then a point $x \in V$ belongs to the transfer variety X_G if and only if p divides $|G_x|$.*

PROOF: The conditions of 2.4 are fullfilled, so, if p divides the order of the isotropy group G_x then by lemma 2.4 $\mathrm{Tr}^G(f)(x) = |G_x| \sum_{i=1}^{m} g_i f(x) = |G_x| \mathrm{Tr}^G_{G_x}(f)(x) = 0$ so $x \in X_G$.

On the other hand suppose p does not divide $|G_x|$. The orbit $B \subset V$ of x is a finite set, so by lemma 2.3 there is a polynomial function h on V such that

$$h(x) = 1$$
$$h(y) = 0 \ \forall \ y \in B, y \neq x.$$

Then by lemma 2.4 and the fact that $g_1(x), \ldots, g_t(x)$ are pairwise distinct, with $g_i(x) = x$ only if $g_i \in G_x$, we get

$$\begin{aligned} \mathrm{Tr}^G(h)(x) &= |G_x| \sum_{i=1}^{m} g_i h(x) = |G_x| \sum_{i=1}^{m} h(g_i^{-1} x) \\ &= |G_x| \mathrm{Tr}^G_{G_x}(h)(x) = |G_x| \left(\sum_{b \in B} h(b) \right) \\ &= |G_x| h(x) = |G_x| \neq 0 \in \mathbb{F} \end{aligned}$$

and hence $x \neq X_G$. □

COROLLARY 2.6: *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group over the field* $\mathbb{F}$ *of characteristic* p*. Then*

$$X_G = \bigcup_{\substack{g \in G \\ |g| = p}} V^g,$$

in other words, the transfer variety is the union of the fixed point sets of the elements in G *of order* p*.*

PROOF: By 2.5 any $x \in X_G$ is fixed by some element $g \in G$ with $|g| = p$ and hence $X_G \subseteq \bigcup_{|g|=p} V^g$. On the other hand, by 2.2 $\bigcup_{|g|=p} V^g \subseteq X_G$. □

By combining 2.5 with our previous work we arrive at Feshbach's main result for the modular transfer in invariant theory.

THEOREM 2.7 (M. Feshbach): *Let* $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ *be a representation of a finite group over the algebraically closed field* $\mathbb{F}$ *of characteristic* p*. Then*

$$\sqrt{\mathrm{Im}(\mathrm{Tr}^G)} = \left(\bigcap_{\substack{|g|=p \\ g \in G}} I_g \right) \cap \mathbb{F}[V]^G.$$

PROOF: By 2.6 $X_G = \bigcup_{\substack{|g|=p \\ g \in G}} V^g$. If the ground field is algebraically closed then passing back to ideals leads to the equality

$$\sqrt{(\mathrm{Im}(\mathrm{Tr}^G))} = \bigcap_{\substack{|g|=p \\ g \in G}} I_g$$

as ideals in $\mathbb{F}[V]$, where $(\mathrm{Im}(\mathrm{Tr}^G))$ is the ideal in $\mathbb{F}[V]$ generated by $\mathrm{Im}(\mathrm{Tr}^G)$. The ideals I_g are prime since they are generated by linear forms, hence their intersection with $\mathbb{F}[V]^G$ is also prime, and the result follows. □

PROBLEM: Is the representation

$$\sqrt{\mathrm{Im}(\mathrm{Tr}^G)} = \bigcap_{\substack{|g|=p \\ g \in G}} \mathfrak{p}_g,$$

where $\mathfrak{p}_g = I_g \cap \mathbb{F}[V]^G$, the primary decomposition of $\sqrt{\mathrm{Im}(\mathrm{Tr}^G)}$?

§3. The Depth of $\mathbb{F}[V]^G/\sqrt{\operatorname{Im}(\operatorname{Tr}^G)}$

It is known that $\sqrt{\operatorname{Im}(\operatorname{Tr}^G)}$ always contains the Dickson polynomial $\mathbf{d}_{n,0}$ of top degree for any $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$. As an application of Feshbach's transfer theorem we obtain the following complimentary result which provides a universal example of an invariant *not* in the image of the transfer.

PROPOSITION 3.1 : *Let $\varrho : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ be a representation of a finite group over the field $\mathbb{F}$ of characteristic p. If p divides $|G|$ then the Dickson polynomial $\mathbf{d}_{n,n-1}$ does not belong to $\sqrt{\operatorname{Im}(\operatorname{Tr}^G)}$.*

PROOF : The ideal $\sqrt{\operatorname{Im}(\operatorname{Tr}^G)} \subset \mathbb{F}[V]^G$ is invariant under the action of the Steenrod algebra P^* [6]. The action of the Steenrod reduced powers is given by the formulae [9]

$$P^{q^{i-1}}(\mathbf{d}_{n,i}) = \mathbf{d}_{n,i-1}$$

for $i = n-1, n-2 \cdots 1$. Therefore, if $\mathbf{d}_{n,n-1} \in \sqrt{\operatorname{Im}(\operatorname{Tr}^G)}$ then so are all the Dickson polynomials. Hence the ideal $\operatorname{Im}(\operatorname{Tr}^G)$ would have height n contrary to feshbach's transfer theorem. □

The quotient ring $\mathbb{F}[V]^G/\operatorname{Im}(\operatorname{Tr}^G)$ is of positive Krull dimension whenever the characteristic p of $\mathbb{F}$ divides the order $|G|$ of G. It therefore makes sense to ask for the existence of regular elements[2] in the quotient ring $\mathbb{F}[V]/\operatorname{Im}(\operatorname{Tr}^G)$. We conjecture that the Dickson polynomial $\mathbf{d}_{n,n-1}$ is such an element in all cases.

[2] An element is called **regular** if it is a nonzero divisor.

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