MULTIPLICATIVE PROPERTIES OF QUINN SPECTRA

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Abstract. We give a simple sufficient condition for Quinn's "bordism-type spectra" to be weakly equivalent to strictly associative ring spectra. We also show that Poincaré bordism and symmetric L-theory are naturally weakly equivalent to monoidal functors. Part of the proof of these statements involves showing that Quinn's functor from bordism-type theories to spectra lifts to the category of symmetric spectra. We also give a new account of the foundations.

1. Introduction

Our main goal in this paper and its sequel is to give a systematic account of multiplicative properties of Quinn's "bordism-type spectra." The present paper deals with associativity and the sequel with commutativity.

We also give a new account of the foundations, and we have made our paper mostly self-contained in the hope that it can serve as an introduction to [Ran92], [WW89], [WW] and other work in this area.

1.1. **Quinn's bordism-type spectra.** The Sullivan-Wall manifold structure sequence is one of the central results of surgery theory. In his thesis ([Qui70a], also see [Qui70b] and [Nic82]) Frank Quinn showed how to interpret this as part of the long exact homotopy sequence of a fiber sequence of spectra. In particular, for each group G he constructed a spectrum $\mathbf{L}(G)$ (which is now called the quadratic L-spectrum of G) whose homotopy groups are Wall's groups $L_*(G)$.

More generally, Quinn gave a machine for constructing spectra from "bordism-type theories" (see [Qui95]). As motivation for his construction, consider the Thom space of the universal $\mathbb{R}^k$-bundle with structure group Top_k; we denote this by $T(\mathrm{Top}_k)$.[1] The usual simplicial model $S_\bullet T(\mathrm{Top}_k)$ has as n-simplices the continuous maps

$$f : \Delta^n \to T(\mathrm{Top}_k).$$

Let us consider the subobject of $S_\bullet^\pitchfork T(\mathrm{Top}_k)$ consisting of maps whose restrictions to each face of Δ^n are transverse to the zero section $B(\mathrm{Top}_k) \subset T(\mathrm{Top}_k)$. Note that $S_\bullet^\pitchfork T(\mathrm{Top}_k)$ is closed under face maps but not under degeneracy maps; that is, it is a *semisimplicial set*.[2] There is a concept of homotopy in the category of

Date: July 9, 2009.

2000 *Mathematics Subject Classification.* Primary 55P43; Secondary 57R67, 57P10.

The second author was partially supported by NSF grants. He thanks the Lord for making his work possible.

[1] The smooth case is technically more difficult because it requires careful attention to manifolds with corners; the second author plans to pursue this in a future paper.

[2] In the literature these are often called Δ-sets, but that terminology seems infelicitous since the category that governs simplicial sets is called Δ.

semisimplicial sets ([RS71, Section 6]), and a transversality argument using [FQ90, Section 9.6] shows that $S^{\pitchfork}_{\bullet}T(\mathrm{Top}_k)$ is a deformation retract of $S_{\bullet}T(\mathrm{Top}_k)$.

Next observe that for each simplex $f : \Delta^n \to T(\mathrm{Top}_k)$ in $S^{\pitchfork}_{\bullet}T(\mathrm{Top}_k)$, the intersections of $f^{-1}(B(\mathrm{Top}_k))$ with the faces of Δ^n form a manifold n-ad;[3] that is, a collection of topological manifolds X_σ, indexed by the faces of Δ^n, with monomorphisms $X_\tau \hookrightarrow \partial X_\sigma$ for $\tau \subsetneq \sigma$ such that

$$\partial X_\sigma = \operatorname*{colim}_{\tau \subsetneq \sigma} X_\tau,$$

where the colimit is taken in the category of topological spaces (the simplest example of a manifold n-ad is the collection of faces of Δ^n itself). The n-ads obtained in this way are of degree k (that is, $\dim X_\sigma = \dim \sigma - k$).

Quinn observed that something interesting happens if one considers the semisimplicial set of *all* n-ads of degree k; we denote this semisimplicial set by P_k and its realization by Q_k. Each P_k is a Kan complex and its homotopy groups are the topological bordism groups (shifted in dimension by k). There are suspension maps $\Sigma Q_k \to Q_{k+1}$ which make the sequence $\mathbf{Q} = \{Q_k\}$ an Ω spectrum. (See Section 13 below for proofs of all of these statements.) The natural map

$$S^{\pitchfork}_{\bullet}T(\mathrm{Top}_k) \to Q_k$$

is an isomorphism on π_{i+k} for $2i^2 + 7i + 4 \leq k$ (by [Hir66, Theorem 1(ab)]) and thus $\mathbf{Q}$ is weakly equivalent to $M\mathrm{Top}$.

An important advantage of this construction is that it depends only on the *category* of topological manifolds, not on the bundle theory. Quinn gave an axiomatization of the structures to which one can apply this construction, which he called bordism-type theories [Qui95, Section 3.2]. One example of a bordism-type theory arises from Poincaré n-ads; in this situation transversality does not hold but one obtains a bordism spectrum from Quinn's construction (cf. Section 7 below). Other important examples are Ranicki's quadratic and symmetric algebraic Poincaré n-ads, which lead to a purely algebraic description of quadratic and symmetric L-spectra ([Ran92]; also see Sections 8 and 9 below).

1.2. **Previous work on multiplicative structures.** In [Ran80a] and [Ran80b], Ranicki used product structures on the L-groups to give product formulas for the surgery obstruction and the symmetric signature. In [Ran92, Appendix B] he observed that these products come from pairings (in the sense of [Whi62]) at the spectrum level, and he used one of these pairings to give a new construction of the assembly map in quadratic L-theory. He also suggested that the pairings could be obtained from a bisemisimplicial construction. This idea, which was developed further in [WW00], is a key ingredient in our work.

1.3. **Smash products in the category of spectra.** Given spectra E, F and G, a pairing in the sense of [Whi62] is a family of maps

$$E_i \wedge F_j \to G_{i+j}$$

satisfying certain conditions. That is, a pairing relates the spaces of the spectra rather than the spectra themselves. Starting in the early 1960's topologists realized that the kind of information given by pairings of spectra could be captured more effectively by using smash products of spectra. The earliest constructions were in

[3] In the literature these are often called $(n + 2)$-ads.

the stable category (that is, the homotopy category of spectra). A smash product that was defined at the spectrum level and not just up to homotopy was given in [LMSM86]; however, this satisfied associativity and commutativity only up to higher homotopies, which was a source of considerable inconvenience. In the early 1990's there were two independent constructions of categories of spectra in which the smash product was associative and commutative up to coherent natural isomorphism. These were the categories of symmetric spectra (eventually published as [HSS00]) and the category of S-modules [EKMM97]. In these categories it is possible to speak of strictly associative and commutative ring spectra (these are equivalent to the A_∞ and E_∞ ring spectra of [May77]).

A later paper [MMSS01] gave a version of the category of symmetric spectra which was based on topological spaces rather than simplicial sets, and this is the version that we will use. (Our reason for using symmetric spectra rather than S-modules is that the former have a combinatorial flavor that makes them well-suited to constructions using n-ads.)

1.4. **Our work.** Our goal is to relate Quinn's theory of bordism-type spectra to the theory of symmetric spectra. As far as we can tell, Quinn's original axioms are not strong enough to do this. We give a stronger set of axioms for a structure that we call an *ad theory* and we show that our axioms are satisfied by all of the standard examples.

Next we show that there is a functor from ad theories to symmetric spectra which is weakly equivalent to Quinn's spectrum construction. We also give a sufficient condition (analogous to the existence of Cartesian products in the category of topological manifolds) for the symmetric spectrum arising from an ad theory to be a strictly associative ring spectrum. Finally, we show that Poincaré bordism is naturally weakly equivalent to a monoidal (that is, coherently multiplicative) functor from a category $\mathcal{T}$ (Definition 11.1) to symmetric spectra and that symmetric L-theory is naturally weakly equivalent to a monoidal functor from the category of rings with involution to symmetric spectra.

In the sequel we will give a sufficient condition for the symmetric spectrum arising from an ad theory to be a strictly commutative ring spectrum. We will also show that Poincaré bordism and symmetric L-theory are naturally weakly equivalent to symmetric monoidal functors. Finally, we will show that the symmetric signature from Poincaré bordism to symmetric L-theory can be realized as a monoidal natural transformation.

1.5. **Outline of the paper.** As we have mentioned, an n-ad is indexed by the faces of Δ^n. For our purposes we need a more general point of view, in which an ad is indexed by the cells of a ball complex (i.e., a regular CW complex with a compatible PL structure). In Section 2 we collect some relevant terminology from [BRS76, pages 4–5].

In Section 3 we give the axioms for an ad theory, together with a simple example (the cellular cocyles on a ball complex).

In Section 4 we define the bordism sets of an ad theory and show that they are abelian groups.

In Sections 5–10 we consider the standard examples of bordism-type theories and show that they are ad theories. Section 5 gives some preliminary terminology. Oriented topological bordism is treated in Section 6, geometric Poincaré bordism in Section 7, symmetric and quadratic Poincaré bordism (following [WW89]) in Sections 8 and 9. Section 10 gives a gluing result which is needed for Sections 7–9 and may be of independent interest.

It is our hope that new families of ad theories will be discovered (your ad here).

In Section 11 we use an idea of Blumberg and Mandell to show that the various kinds of Poincaré bordism are functorial—this question seems not to have been considered in the literature.

In Sections 12–14 we consider the cohomology theory associated to an ad theory; this is needed in later sections and is important in its own right. There is a functor (which we denote by T^*) that takes a ball complex K to the graded abelian group of K-ads modulo a certain natural bordism relation. Ranicki [Ran92, Proposition 13.7] stated that (for symmetric and quadratic Poincaré bordism, and assuming that K is a simplicial complex) T^* is the cohomology theory represented by the Quinn spectrum $\mathbf{Q}$ (Quinn stated a similar result [Qui95, Section 4.7] but seems to have had a different equivalence relation in mind). The proof of this fact in [Ran92] is not correct (see Remark 14.2 below). We give a different proof (for general ad theories, and general K). First, in Section 12 we use ideas from [BRS76] to show that T^* is a cohomology theory. In Section 13 we review the construction of the Quinn spectrum $\mathbf{Q}$. Then in Section 14 we show that T^* is naturally isomorphic to the cohomology theory represented by $\mathbf{Q}$ by giving a morphism of cohomology theories which is an isomorphism on coefficients.

In Section 15 we review the definition of symmetric spectrum and show that the functor $\mathbf{Q}$ from ad theories to spectra lifts (up to weak equivalence) to a functor $\mathbf{M}$ from ad theories to symmetric spectra. In Section 16 we consider multiplicative ad theories and show that for such a theory the symmetric spectrum $\mathbf{M}$ is a strictly associative ring spectrum. In Section 17 we show that the functors $\mathbf{M}$ given by the geometric and symmetric Poincaré bordism ad theories are monoidal functors.

In an appendix we review some simple facts from PL topology that are needed in the body of the paper.

Acknowledgments. The authors benefited from a workshop on forms of homotopy theory held at the Fields Institute. They would like to thank Matthias Kreck for suggesting the problem to the first author and also Carl-Friedrich Bödigheimer, Jim Davis, Steve Ferry, Mike Mandell, Frank Quinn, Andrew Ranicki, John Rognes, Stefan Schwede, Michael Weiss and Bruce Williams for useful hints and helpful discussions. The first author is grateful to the Max Planck Institute in Bonn for its hospitality.

2. Ball complexes

Definition 2.1. (i) Let K be a finite collection of PL balls in some $\mathbb{R}^n$, and write $|K|$ for the union $\cup_{\sigma\in K}\,\sigma$. We say that K is a *ball complex* if the interiors of the balls of K are disjoint and the boundary of each ball of K is a union of balls of K (thus the interiors of the balls of K give $|K|$ the structure of a regular CW complex). The balls of K will also be called *closed cells* of K.

(ii) A *subcomplex* of a ball complex K is a subset of K which is a ball complex.

(iii) A *morphism* of ball complexes is the composite of an isomorphism with an inclusion of a subcomplex.

Definition 2.2. A *subdivision* of a ball complex K is a ball complex K' with two properties:
(a) $|K'| = |K|$, and
(b) each cell of K' is contained in a cell of K.
A subcomplex of K which is also a subcomplex of K' is called *residual*.

Notation 2.3. Let I denote the unit interval with its standard structure as a ball complex (two 0 cells and one 1 cell).

3. Axioms

Definition 3.1. A category with involution is a category together with an endofunctor i which satisfies $i^2 = 1$.

Example 3.2. The set of integers $\mathbb{Z}$ is a poset and therefore a category. We give it the trivial involution.

Definition 3.3. A $\mathbb{Z}$*-graded category* is a category $\mathcal{A}$ with involution together with involution-preserving functors $d : \mathcal{A} \longrightarrow \mathbb{Z}$ (called the *dimension function*) and $\emptyset : \mathbb{Z} \longrightarrow \mathcal{A}$ such that $d\,\emptyset$ is equal to the identity functor. A k*-morphism* between $\mathbb{Z}$-graded categories is a functor which decreases the dimensions of objects by k and strictly commutes with $\emptyset$ and i.

We will write $\emptyset_n$ for $\emptyset(n)$.

Note that the existence of d implies that when $d(A) > d(B)$ there are no morphisms $A \to B$.

Example 3.4. Given a chain complex C, let $\mathcal{A}_C$ be the $\mathbb{Z}$-graded category whose objects in dimension n are the elements of C_n, with a unique morphism from every object of $\mathcal{A}$ to every object of higher dimension. i is multiplication by -1 and the object $\emptyset_n$ is the 0 element in C_n. The boundary map is a 1-morphism.

Example 3.5. Let $\mathcal{A}_{\mathrm{STop}}$ be the category whose objects in dimension n are the n-dimensional oriented topological manifolds with boundary (with an empty manifold of dimension n for every n); the morphisms which preserve dimension are the orientation-preserving inclusions and the morphisms which increase dimension are the inclusions with image in the boundary. The involution i reverses the orientation, and $\emptyset_n$ is the empty manifold of dimension n. Again, the boundary map is a 1-morphism.

For examples related to geometric and algebraic Poincare bordism see Definitions 7.2, 8.4 and 9.2 below.

Example 3.6. Let K be a ball complex and L a subcomplex. Define $\mathcal{C}ell(K, L)$ to be the $\mathbb{Z}$-graded category whose objects in dimension n are the oriented closed n-cells (σ, o) which are not in L, together with an object $\emptyset_n$ (the empty cell of dimension n). The morphisms which preserve dimension are identity maps, and the morphisms which increase dimension are the inclusions of cells (with no requirements on the orientations), together with a morphism from $\emptyset_n$ to each object of higher dimension. The involution i reverses the orientation.

We will write $\mathcal{C}ell(K)$ instead of $\mathcal{C}ell(K, \emptyset)$.

It will be important for us to consider abstract isomorphisms between categories of the form $\mathcal{C}ell(K_1, L_1)$, $\mathcal{C}ell(K_2, L_2)$ (not necessarily induced by maps of pairs).

The motivation for the first part of the following definition is the fact that, if f is a chain map which lowers degrees by k, then $f \circ \partial = (-1)^k \partial \circ f$.

Definition 3.7. Let $\theta : \mathcal{C}ell(K_1, L_1) \to \mathcal{C}ell(K_2, L_2)$ be a k-morphism.

(i) θ is *incidence-compatible* if it takes incidence numbers in $\mathcal{C}ell(K_1, L_1)$ (see [Whi78, page 82]) to $(-1)^k$ times the corresponding incidence numbers in $\mathcal{C}ell(K_2, L_2)$.

(ii) If $\mathcal{A}$ is a $\mathbb{Z}$-graded category and $F : \mathcal{C}ell(K_2, L_2) \to \mathcal{A}$ is an l-morphism then

$$\theta^* F : \mathcal{C}ell(K_1, L_1) \to \mathcal{A}$$

is the composite $i^{kl} \circ F \circ \theta$.

Now we fix a $\mathbb{Z}$-graded category $\mathcal{A}$.

Definition 3.8. Let K be a ball complex and L a subcomplex. A *pre* (K, L)*-ad* of degree k is a k-morphism $\mathcal{C}ell(K, L) \to \mathcal{A}$.

Let $\mathrm{pre}^k(K)$ be the set of pre K-ads of degree k and let $\mathrm{pre}^k(K, L)$ be the set of pre (K, L)-ads of degree k.

Note that pre^k is a functor from ball complexes (resp., pairs of ball complexes) to sets.

Definition 3.9. An *ad theory* consists of

(i) a $\mathbb{Z}$-graded category $\mathcal{A}$, and

(ii) for each k, and each ball complex pair (K, L), a subset $\mathrm{ad}^k(K, L)$ of $\mathrm{pre}^k(K, L)$ (called the set of (K, L)*-ads of degree* k)

such that the following hold.

(a) ad^k is a subfunctor of pre^k, and $\mathrm{ad}^k(K, L) = \mathrm{pre}^k(K, L) \cap \mathrm{ad}^k(K)$.

(b) The element of $\mathrm{pre}^k(K)$ which takes every object of $\mathcal{C}ell(K)$ to $\emptyset$ is a K-ad, called the *trivial* K*-ad* of degree k.

(c) i takes K-ads to K-ads.

(d) Any pre K-ad which is isomorphic to a K-ad is a K-ad.

(e) A pre K-ad is a K-ad if it restricts to a σ-ad for each closed cell σ of K.

(f) (Reindexing.) Suppose

$$\theta : \mathcal{C}ell(K_1, L_1) \to \mathcal{C}ell(K_2, L_2)$$

is an incidence-compatible k-isomorphism of $\mathbb{Z}$-graded categories. Then the induced bijection

$$\theta^* : \mathrm{pre}^l(K_2, L_2) \to \mathrm{pre}^{l+k}(K_1, L_1)$$

restricts to a bijection

$$\theta^* : \mathrm{ad}^l(K_1, L_1) \to \mathrm{ad}^{l+k}(K, L).$$

(g) (Gluing.) For each subdivision K' of K and each K'-ad F there is a K-ad which agrees with F on each residual subcomplex.

(h) (Cylinder.) There is a natural transformation $J : \mathrm{ad}^k(K) \to \mathrm{ad}^k(K \times I)$ (where $K \times I$ has its canonical ball complex structure [BRS76, page 5]) such that the restrictions of $J(F)$ to $K \times 0$ and to $K \times 1$ are both equal to F. J takes trivial ads to trivial ads.

We call $\mathcal{A}$ the *target category* of the ad theory. A *morphism* (resp., *equivalence*) of ad theories is a functor (resp., equivalence) of target categories which takes ads to ads.

Remark 3.10. This definition is based in part on [Qui95, Section 3.2] and [BRS76, Theorem I.7.2].

Example 3.11. Let C be a chain complex and let $\mathcal{A}_C$ be the $\mathbb{Z}$-graded category of Example 3.4. We define an ad-theory (denoted by ad_C) as follows. Let $\mathrm{cl}(K)$ denote the cellular chain complex of K; specifically, $\mathrm{cl}_n(K)$ is generated by the symbols $\langle\sigma, o\rangle$ with σ n-dimensional, subject to the relation $\langle\sigma, -o\rangle = -\langle\sigma, o\rangle$. A pre K-ad F gives a map of graded abelian groups from $\mathrm{cl}(K)$ to C, and F is a K-ad if this is a chain map (thus a K-ad is the same thing as a cycle on K with values in C). We define a (K, L)-ad to be a K-ad which is zero on $\mathcal{C}ell(L)$ (this is forced by part (a) of Definition 3.9). Gluing is addition and $J(F)$ is 0 on all the objects of $K \times I$ which are not contained in $K \times 0$ or $K \times 1$.

4. The bordism groups of an ad theory

Fix an ad theory. Let $*$ denote the one-point space.

Definition 4.1. Two elements of $\mathrm{ad}^k(*)$ are *bordant* if there is an I-ad which restricts to the given ads at the ends.

This is an equivalence relation: reflexivity follows from part (h) of Definition 3.9, symmetry from part (f), and transitivity from part (g).

Definition 4.2. Let Ω_k be the set of bordism classes in $\mathrm{ad}^{-k}(*)$.

Example 4.3. Let C be a chain complex and let ad_C be the ad theory defined in Example 3.11. Then a $*$-ad is a cycle of C and there is a bijection between Ω_k and H_kC. We will return to this example at the end of the section.

Our main goal in this section is to show that Ω_k has an abelian group structure (cf. [Qui95, Section 3.3]). For this we need some notation.

Let M' be the pushout of ball complexes

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & I \times I \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ I \times I & \xrightarrow{\delta} & M' \end{array}$$

where α takes t to $(1, t)$ and β takes t to $(0, t)$; see Figure 1.

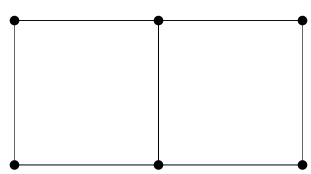

Figure 1

Let M be the ball complex with the same total space as M' whose (closed) cells are: the union of the two 2-cells of M', the 1-cells $\gamma(I \times 0)$, $\delta(I \times 0)$, $\gamma(0 \times I)$, $\delta(1 \times I)$ and $\gamma(I \times 1) \cup \delta(I \times 1)$, and the vertices of these 1-cells; see Figure 2.

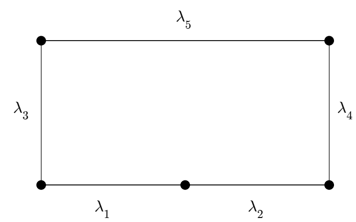

FIGURE 2

We will need explicit parametrizations of the 1-cells of M: for $t \in I$ define

$$\begin{aligned}\lambda_1(t) &= \gamma(t,0)\\ \lambda_2(t) &= \delta(t,0)\\ \lambda_3(t) &= \gamma(0,t)\\ \lambda_4(t) &= \delta(1,t)\\ \lambda_5(t) &= \begin{cases}\gamma(2t,1) & \text{if } t \in [0,1/2],\\ \delta(2t-1,1) & \text{if } t \in [1/2,1]\end{cases}\end{aligned}$$

Let us write κ for the isomorphism of categories

$$\mathcal{C}ell(I,\{0,1\}) \to \mathcal{C}ell(*)$$

which takes I with its standard orientation to $*$ with its standard orientation. The map

$$\kappa^* : \mathrm{ad}^k(*) \to \mathrm{ad}^{k+1}(I,\{0,1\})$$

is a bijection by part (f) of Definition 3.9.

Lemma 4.4. *For $F, G \in \mathrm{ad}^k(*)$, there is an $H \in \mathrm{ad}^{k+1}(M)$ such that $\lambda_1^* H = \kappa^* F$, $\lambda_2^* H = \kappa^* G$, and $\lambda_3^* H$ and $\lambda_4^* H$ are trivial.*

Proof. By part (e) of Definition 3.9, there is an M'-ad which restricts to the cylinder $J(\kappa^* F)$ on the image of γ and to the cylinder $J(\kappa^* G)$ on the image of δ. The result now follows by part (g) of Definition 3.9. □

We will write $[F]$ for the bordism class of a $*$-ad F.

Definition 4.5. Given $F, G \in \mathrm{ad}^k(*)$, let H be an M-ad as in Lemma 4.4 and define $[F] + [G]$ to be

$$[(\kappa^{-1})^* \lambda_5^* H].$$

We need to show that this is well-defined. Let F_1 and G_1 be bordant to F and G, and let H_1 be an M-ad for which $\lambda_1^* H_1 = \kappa^* F_1$, $\lambda_2^* H_1 = \kappa^* G_1$, and $\lambda_3^* H_1$ and $\lambda_4^* H_1$ are trivial. Figure 3, together with part (f) of Definition 3.9, gives a bordism from $[(\kappa^{-1})^* \lambda_5^* H]$ to $[(\kappa^{-1})^* \lambda_5^* H_1]$.

Remark 4.6. Our definition of addition agrees with that in [Qui95, Section 3.3] because the $\mathbb{Z}$-graded category $\mathcal{C}ell(M, \lambda_3(I) \cup \lambda_4(I))$ is isomorphic to $\mathcal{C}ell(\Delta^2)$.

Proposition 4.7. *The operation $+$ makes Ω_k an abelian group.*

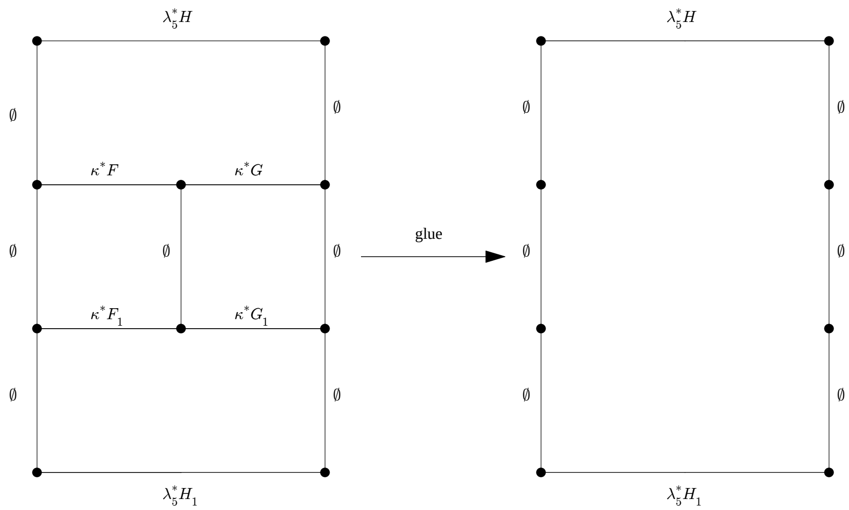

FIGURE 3

Proof. Let 0 denote the bordism class of the trivial $*$-ad. The cylinder $J(F)$, together with part (f) of Definition 3.9, shows both that 0 is an identity element and that $[iF]$ is the inverse of $[F]$. Figure 4, together with part (f) of Definition 3.9, gives the proof of associativity.

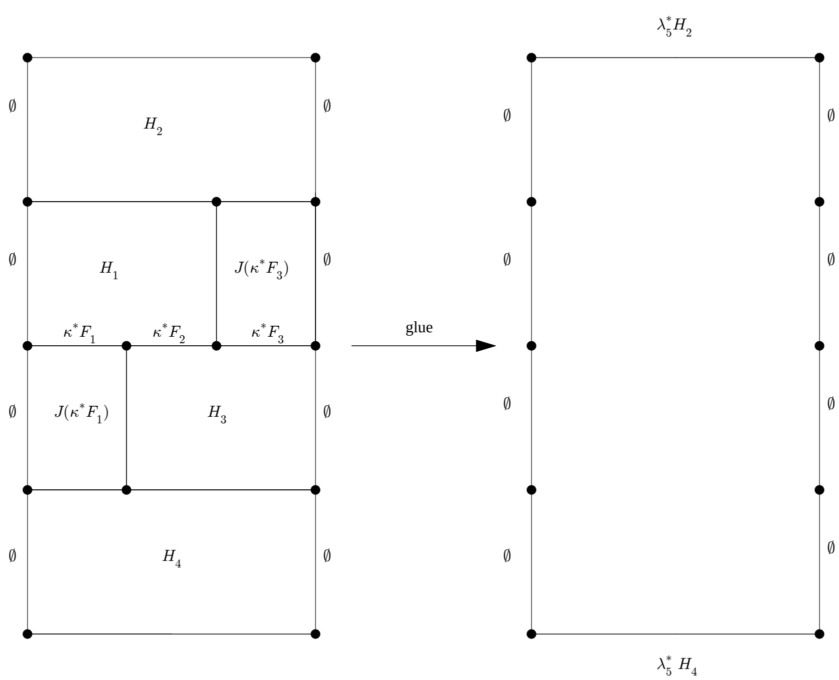

FIGURE 4

To see commutativity, let F, G and H be as in Lemma 4.4. Then iH is an M-ad and Definition 4.5 gives

$$[iF] + [iG] = [(\kappa^{-1})^* \lambda_5^*(iH)].$$

The left-hand side of this equation is equal to $-[F] + (-[G])$, and the right-hand side is $-([F] + [G])$; this implies that $+$ is commutative. □

Remark 4.8. An equivalence of ad theories (see Definition 3.9) induces an isomorphism of bordism groups.

Remark 4.9. In Example 4.3, the addition in Ω_k is induced by addition in C, as one can see from the proof of Lemma 4.4 and the fact that gluing in ad_C is given by addition. Thus Ω_k is isomorphic to $H_k C$ as an abelian group.

5. Balanced categories and functors

For the examples in Sections 6–9, it will be convenient to have some additional terminology.

Let $\mathcal{A}(A, B)$ denote the set of morphisms in $\mathcal{A}$ from A to B.

Definition 5.1. A *balanced category* is a $\mathbb{Z}$-graded category $\mathcal{A}$ together with a natural bijection

$$\eta : \mathcal{A}(A, B) \to \mathcal{A}(A, i(B))$$

for objects A, B with $\dim A < \dim B$, such that

(a) $\eta \circ i = i \circ \eta : \mathcal{A}(A, B) \to \mathcal{A}(i(A), B)$, and

(b) $\eta \circ \eta$ is the identity.

If $\mathcal{A}$ and $\mathcal{A}'$ are balanced categories then a *balanced functor* $F : \mathcal{A} \to \mathcal{A}'$ is a morphism of $\mathbb{Z}$-graded categories for which

$$F \circ \eta = \eta \circ F : \mathcal{A}(A, B) \to \mathcal{A}'(F(A), i(F(B))).$$

All of the $\mathbb{Z}$-graded categories in the previous section are balanced. In particular $\mathcal{C}ell(K, L)$ is balanced.

Definition 5.2. Let $\mathcal{A}$ be a balanced category. A *balanced pre* (K, L)*-ad* with values in $\mathcal{A}$ is a pre (K, L)-ad F which is a balanced functor.

6. Example: Oriented Topological Bordism

In this section we construct an ad theory with values in the category $\mathcal{A}_{\mathrm{STop}}$ of Example 3.5.

Let $\mathcal{B}$ denote the category of compact orientable topological manifolds; the morphisms which preserve dimension are the orientable inclusions (that is, the inclusions which preserve some choice of the orientations) and the morphisms which increase dimension are the inclusions with image in the boundary.

Definition 6.1. For a ball complex K, let $\mathcal{C}ell^\flat(K)$ denote the category whose objects are the cells of K (including an empty cell in each dimension) and whose morphisms are the inclusions of cells.

A balanced pre K-ad F with values in $\mathcal{A}_{\mathrm{STop}}$ induces a functor

$$F^\flat : \mathcal{C}ell^\flat(K) \to \mathcal{B}.$$

Given cells $\sigma' \subsetneq \sigma$ of K, let $i_{(\sigma',o'),(\sigma,o)}$ denote the map in $\mathcal{C}ell(K)$ from (σ', o') to (σ, o) and let $j_{\sigma',\sigma}$ denote the map in $\mathcal{C}ell^\flat(K)$ from σ' to σ.

Definition 6.2. Let K be a ball complex. A K-ad with values in $\mathcal{A}_{\mathrm{STop}}$ is a balanced pre K-ad F with the following properties.

(a) If (σ', o') and (σ, o) are oriented cells with $\dim \sigma' = \dim \sigma - 1$ and if the incidence number $[o, o']$ is equal to $(-1)^k$ (where k is the degree of F) then the map

$$F(i_{(\sigma',o'),(\sigma,o)}) : F(\sigma', o') \to \partial F(\sigma, o)$$

is orientation preserving.

(b) For each σ, $\partial F^\flat(\sigma)$ is the colimit in Top of $F^\flat|_{\mathcal{C}ell^\flat(\partial\sigma)}$.

Remark 6.3. The sign in part (a) of this definition is needed in order for part (f) of Definition 3.9 to hold.

Example 6.4. The functor $\mathcal{C}ell(\Delta^n) \to \mathcal{A}_{\mathrm{STop}}$ which takes each oriented simplex of Δ^n to itself (considered as an oriented topological manifold) is a Δ^n-ad of degree 0.

We write $\mathrm{ad}_{\mathrm{STop}}(K)$ for the set of K-ads with values in $\mathcal{A}_{\mathrm{STop}}$.

Theorem 6.5. $\mathrm{ad}_{\mathrm{STop}}$ *is an ad theory.*

The rest of this section is devoted to the proof of Theorem 6.5. The only parts of Definition 3.9 which are not obvious are (g) and (h).

For part (h), we define $J(F)$ on oriented cells of the form $(\sigma \times I, o \times o')$ to be $F(\sigma, o) \times (I, o')$, where (I, o') denotes the topological manifold I with orientation o'.

For part (g), let K be a ball complex and K' a subdivision of K. The proof is by induction on the lowest dimensional cell of K which is not a cell of K'. For the inductive step, we may assume that $|K|$ is a PL n-ball, that K has exactly one n cell, and that K' is a subdivision of K which agrees with K on the boundary of $|K|$. Let F be a K'-ad. It suffices to show that the colimit of $F^\flat$ over the cells of K' is a topological manifold with boundary and that its boundary is the colimit of $F^\flat$ over the cells of the boundary of $|K|$.

We will prove something more general:

Proposition 6.6. *Let* (L, L_0) *be a ball complex pair such that* $|L|$ *is a PL manifold with boundary* $|L_0|$. *Let* F *be an* L*-ad. Then* $\mathrm{colim}_{\sigma\in L} F^\flat(\sigma)$ *is a topological manifold with boundary* $\mathrm{colim}_{\sigma\in L_0} F^\flat(\sigma)$.

Proof. (The proof is essentially the same as the proof of Lemma II.1.2 in [BRS76].)

Using the notation of the Appendix, let us write $D^\circ(\sigma)$ for $D(\sigma) - \dot{D}(\sigma)$. If σ is not in L_0 then, by Proposition A.4(i), $D^\circ(\sigma)$ is topologically homeomorphic to $\mathbb{R}^{n-m}$. If σ is in L_0 then, by Proposition A.4(ii) and [RS82, Theorem 3.34], there is a homeomorphism from $D^\circ(\sigma)$ to the half space $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-m-1}$ which takes $\hat{\sigma}$ to a point on the boundary.

There is another way to describe $D^\circ(\sigma)$. Given a (possibly empty) sequence $T = (\sigma_1, \ldots, \sigma_l)$ with $\sigma \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_l$, let us write $[0, 1)^T$ for $[0, 1)^l$ and $T[i]$ for the sequence obtained by deleting σ_i. Given $u \in [0, 1)^l$ let us write $u[i]$ for the element of $[0,1)^{l-1}$ obtained by deleting the i-th coordinate of u. Let $E(\sigma)$ be the quotient of

$$\coprod_T [0,1)^T$$

in which a point u in $[0,1)^T$ with i-th coordinate 0 is identified with the point $u[i]$ in $[0,1)^{T[i]}$. Let $\mathbf{0}$ denote the equivalence class of $(0,\ldots,0) \in [0,1)^T$ (which is independent of T). Then there is a homeomorphism $D^\circ(\sigma) \to E(\sigma)$ which takes $\hat{\sigma}$ to $\mathbf{0}$.

Now consider the space $X = \operatorname{colim}_{\sigma\in L} F^\flat(\sigma)$. Let $x \in X$. There is a unique σ for which x is in the interior of $F^\flat(\sigma)$. Let m be the dimension of σ, and k the degree of F. Let U be an $(m-k)$-dimensional Euclidean neighborhood of x in $F^\flat(\sigma)$. An easy inductive argument, using the collaring theorem for topological manifolds, gives an imbedding

$$h : U \times E(\sigma) \to X$$

such that $h(x, \mathbf{0}) = x$ and $h(U\times E(\sigma))$ contains a neighborhood of x in X. If σ is not a cell of L_0 this shows that x has an $(n-k)$-dimensional Euclidean neighborhood in X. If σ is a cell of L_0 we obtain a homeomorphism from a neighborhood of x in X to the half space of dimension $n-k$ which takes x to a boundary point. □

Remark 6.7. The description of gluing in the proof of Theorem 6.5, together with the proof of Lemma 4.4, shows that addition in the bordism groups of $\mathrm{ad}_{\mathrm{STop}}$ is induced by disjoint union. Thus the bordism groups are the usual oriented topological bordism groups.

7. Example: Geometric Poincaré ad Theories

Fix a group π and a properly discontinuous left action of π on a simply connected space Z; then Z is a universal cover of Z/π.

Fix a homomorphism $w : \pi \to \{\pm 1\}$.

Ranicki [Ran80b, page 243] defines the bordism groups $\Omega_*^P(Z/\pi, w)$ of geometric Poincaré complexes over $(Z/\pi, w)$; our goal in this section is to define an ad theory whose bordism groups are a slightly modified version of Ranicki's (see the end of this section for a precise comparison).

Let $\mathbb{Z}^w$ denote the right π action on $\mathbb{Z}$ determined by w.

Definition 7.1. Given a map $f : X \to Z/\pi$, define $S_*(X, \mathbb{Z}^f)$ to be $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} S_*(\tilde{X})$, where $\tilde{X}$ is the pullback of Z to X and $S_*(\tilde{X})$ denotes the singular chain complex of $\tilde{X}$.

Definition 7.2. We define a category $\mathcal{A}_{\pi,Z,w}$ as follows. An object of $\mathcal{A}_{\pi,Z,w}$ is a triple

$$(X,\ f : X \to Z/\pi,\ \xi \in S_*(X, \mathbb{Z}^f)),$$

where X is homotopy equivalent to a finite CW complex. If $\dim\xi < \dim\xi'$ then a morphism from (X, f, ξ) to (X', f', ξ') is a map $g : X \to X'$ such that $f' \circ g = f$. If $\dim\xi = \dim\xi'$ we require in addition that $g_*(\xi) = \xi'$. If $\dim\xi > \dim\xi'$ there are no morphisms.

$\mathcal{A}_{\pi,Z,w}$ is a balanced $\mathbb{Z}$-graded category, where the dimension of (X, f, ξ) is $\dim\xi$, i takes (X, f, ξ) to $(X, f, -\xi)$, and $\emptyset_n$ is the n-dimensional object with $X = \emptyset$.

Next we must say what the K-ads with values in $\mathcal{A}_{\pi,Z,w}$ are. We will build this up gradually by considering several properties that a pre K-ad can have.

For a balanced pre K-ad F we will use the notation

$$F(\sigma, o) = (X_\sigma, f_\sigma, \xi_{\sigma,o});$$

note that X_σ and f_σ do not depend on o.

Recall Definition 6.1.

Definition 7.3. (cf. [WW89, page 50]) A functor X from $\mathcal{C}ell^\flat(K)$ to topological spaces is *well-behaved* if the following conditions hold:

(a) For each inclusion $\tau \subset \sigma$, the map $X_\tau \to X_\sigma$ is a cofibration.

(b) For each cell σ of K, the map

$$\operatorname*{colim}_{\tau \subsetneq \sigma} X_\tau \to X_\sigma$$

is a cofibration.

If F is a balanced pre K-ad for which X is well-behaved, let $X_{\partial\sigma}$ denote $\operatorname{colim}_{\tau \subsetneq \sigma} X_\tau$, and let $\tilde{X}_{\partial\sigma}$ be the pullback of Z to $X_{\partial\sigma}$.

For a left $\mathbb{Z}[\pi]$-module M let M^t denote the right $\mathbb{Z}[\pi]$-module obtained from the w-twisted involution on $\mathbb{Z}[\pi]$ (see [Ran80b, page 196]).

Convention 7.4. From now on we will often use the convention that a cochain complex can be thought of as a chain complex with the opposite grading. For example, this is needed in part (ii) of our next definition and in Lemma 7.8.

Definition 7.5. Let (σ, o) be an oriented cell of K.

(i) Let $\zeta_{\sigma,o}$ be the image of $\xi_{\sigma,o}$ under the map

$$\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} S_*(\widetilde{X}_\sigma) \xrightarrow{1\otimes AW} \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (S_*(\widetilde{X}_\sigma) \otimes S_*(\widetilde{X}_\sigma)) \cong S_*(\widetilde{X}_\sigma)^t \otimes_{\mathbb{Z}[\pi]} S_*(\widetilde{X}_\sigma),$$

where AW is the Alexander-Whitney map.

(ii) Define a homomorphism of graded abelian groups

$$\operatorname{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi]) \to S_*(\widetilde{X}_\sigma)/S_*(\widetilde{X}_{\partial\sigma})$$

(the cap product) by taking $x \in \operatorname{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi])$ to the image of $\zeta_{\sigma,o}$ under the map

$$S_*(\widetilde{X}_\sigma)^t \otimes_{\mathbb{Z}[\pi]} S_*(\widetilde{X}_\sigma) \xrightarrow{1\otimes x} S_*(\widetilde{X}_\sigma)^t \to S_*(\widetilde{X}_\sigma)^t/S_*(\widetilde{X}_{\partial\sigma})^t$$

(note that $S_*(\widetilde{X}_\sigma)^t/S_*(\widetilde{X}_{\partial\sigma})^t$ is the same graded abelian group as $S_*(\widetilde{X}_\sigma)/S_*(\widetilde{X}_{\partial\sigma})$).

In order for the cap product to be a chain map we need a further assumption.

Definition 7.6. F is *closed* if for each (σ, o) the chain $\xi_{\sigma,o}$ is the sum of the images in $S_*(X_\sigma, \mathbb{Z}^{f_\sigma})$ of the chains $\xi_{\sigma',o'}$, where (σ', o') runs through the oriented cells for which the incidence number $[o, o']$ is $(-1)^{\deg F}$ (see Remark 6.3 for the sign).

Remark 7.7. An equivalent definition of closed uses the functor cl defined in Example 3.11. Given a cell σ of K there is a map of graded abelian groups

$$\mathrm{cl}(\sigma) \to S_*(X_\sigma, \mathbb{Z}^{f_\sigma})$$

which takes $\langle \tau, o \rangle$ to the image of $\xi_{\tau,o}$ in $S_*(X_\sigma, \mathbb{Z}^{f_\sigma})$. F is closed if this is a chain map for each σ.

Lemma 7.8. *If F is balanced and closed and X is well-behaved then the cap product*

$$\operatorname{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi]) \to S_*(\widetilde{X}_\sigma)/S_*(\widetilde{X}_{\partial\sigma})$$

is a chain map for each σ. □

Definition 7.9. F is a K-ad if

(a) it is balanced and closed and X is well-behaved, and

(b) for each (σ, o) the cap product induces an isomorphism

$$H^*(\mathrm{Hom}_{\mathbb{Z}[\pi]}(S_*(\widetilde{X}_\sigma), \mathbb{Z}[\pi])) \to H_{\dim \sigma - \deg F - *}(\widetilde{X}_\sigma, \widetilde{X}_{\partial\sigma}).$$

We write $\mathrm{ad}_{\pi,Z,w}(K)$ for the set of K-ads with values in $\mathcal{A}_{\pi,Z,w}$.

Theorem 7.10. $\mathrm{ad}_{\pi,Z,w}$ *is an ad theory.*

For the proof we need a lemma.

Lemma 7.11. *For $i = 1, 2$, suppose given a group π_i, a properly discontinuous left action of π_i on a simply connected space Z_i, and a homomorphism $w_i : \pi \to \{\pm 1\}$. Also suppose given a ball complex K_i and a K_i-ad F_i with values in $\mathcal{A}_{\pi_i,Z_i,w_i}$. Write $F_i(\sigma, o) = ((X_i)_\sigma, (f_i)_\sigma, (\xi_i)_{\sigma,o})$. Define a pre $(K_1 \times K_2)$-ad $F_1 \times F_2$ with values in $\mathcal{A}_{\pi_1\times\pi_2,Z_1\times Z_2,w_1\cdot w_2}$ by*

$$(F_1 \times F_2)(\sigma \times \tau, o_1 \times o_2) = ((X_1)_\sigma \times (X_2)_\tau, (f_1)_\sigma \times (f_2)_\tau, (\xi_1)_{\sigma,o_1} \times (\xi_2)_{\tau,o_2}).$$

Then $F_1 \times F_2$ is a $(K_1 \times K_2)$-ad. □

Proof of 7.10. The only parts of Definition 3.9 which are not obvious are (g) and (h).

Part (g). Let F be a K'-ad with

$$F(\sigma, o) = (X_\sigma, f_\sigma, \xi_{\sigma,o}).$$

We need to define a K-ad E which agrees with F on each residual subcomplex of K. As in the proof of Theorem 6.5, we may assume by induction that K is a ball complex structure for the n disk with one n cell τ, and that K' is a subdivision of K which agrees with K on the boundary. We only need to define E on the top cell τ of K. We define $E(\tau, o)$ to be $(V_\tau, e_\tau, \theta_{\tau,o})$, where

- $V_\tau = \mathrm{colim}_{\sigma\in K'} X_\sigma$,
- $e_\tau : V_\tau \to Z/\pi$ is the obvious map, and
- $\theta_{\tau,o}$ is
$$\sum_{(\sigma,o')} \xi_{\sigma,o'},$$
where (σ, o') runs through the n-dimensional cells of K' with orientation induced by o.

Then E is closed by Proposition A.1(ii) and the cap product is an isomorphism by Proposition 10.4 below.

Part (h). Let e denote the trivial group and let 1 denote the homomorphism from e to $\{\pm 1\}$. Define an I-ad G with values in $\mathcal{A}_{e,*,1}$ as follows. For a cell σ of I, the identity map id of σ is a singular chain of the space σ; define $G(\sigma, o)$ to be $(\sigma, *, \pm\mathrm{id})$, where $*$ denotes the map to a point and the $\pm$ is $+$ iff o is the standard orientation of σ. Now define the cylinder by $J(F) = F \times G$.

□

Remark 7.12. The description of gluing in the proof just given, together with the proof of Lemma 4.4, shows that addition in the bordism groups of $\mathrm{ad}_{\pi,Z,w}$ is induced by disjoint union.

It remains to compare the bordism groups of this ad theory with the groups $\Omega^P_*(Z/\pi, w)$ defined in [Ran80b, page 243]. Our definition differs from Ranicki's in two ways. First of all, a $*$-ad in our sense is a triple $(X, f : X \to Z/\pi, \xi \in Z_n(X, \mathbb{Z}^f)$ but a geometric Poincaré complex over $(Z/\pi, w)$ in Ranicki's sense is a triple $(X, f : X \to Z/\pi, [X] \in H_n(X, \mathbb{Z}^f)$. This does not affect the bordism groups because of the following lemma.

Lemma 7.13. *Let $(X, f : X \to Z/\pi, \xi)$ be a $*$-ad, and let ξ' be a cycle homologous to ξ. Then the $*$-ads $(X, f : X \to Z/\pi, \xi)$ and $(X, f : X \to Z/\pi, \xi')$ are bordant.*

Proof. Since ξ' is homologous to ξ there is a chain θ with

$$d\theta = \xi' - \xi.$$

Define an I-ad H by letting H take the cells 0,1 and I (with their standard orientations) respectively to (X, f, ξ), (X, f, ξ'), and $(X \times I, h, \xi \times \iota + \theta \times \kappa)$, where h is the composite of the projection $X \times I \to X$ with f, ι is the chain given by the identity map of I, and κ is the 0-chain represented by the point 1. □

The second difference between our definition and Ranicki's is that in [Ran80b] the symbol $\widetilde{X}$ denotes a universal cover of X (that is, a cover which is universal on each component). This presumably means that our bordism groups are different from those in [Ran80b]. Our reason for making this change is that the definition we give is somewhat simpler and seems to provide the natural domain for the symmetric signature (see Section 8). One could, if desired, modify our definition so that the bordism groups would be equal to those in [Ran80b].

8. Example: Symmetric Poincaré ad theories

Fix a ring R with involution.

Definition 8.1. Let $\mathcal{C}$ be the category of chain complexes of free left R modules. An object of $\mathcal{C}$ is *finitely generated* if it is finitely generated in each degree and zero in all but finitely many degrees. An object of $\mathcal{C}$ is *homotopy finite* if it is chain homotopy equivalent to a finitely generated object. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ whose objects are the homotopy finite ones.

Given an object C of $\mathcal{C}$, let C^t be the complex of right R modules obtained from C by applying the involution of R. Give $C^t \otimes_R C$ the $\mathbb{Z}/2$ action that switches the factors.

Let W be the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2]$-modules.

Definition 8.2. A *quasi-symmetric complex of dimension* n is a pair (C, φ), where C is an object of $\mathcal{D}$ and φ is a $\mathbb{Z}/2$-equivariant map

$$W \to C^t \otimes_R C$$

of graded abelian groups which raises degrees by n.

Remark 8.3. (i) A symmetric complex in the sense of Ranicki ([Ran92, Definition 1.6(i)]) is a quasi-symmetric complex for which φ is a chain map.

(ii) The concept of symmetric complex can be motivated as follows. A symmetric bilinear form on a vector space V over a field $\mathbb{F}$ is a $\mathbb{Z}/2$ equivariant map $V \otimes V \to \mathbb{F}$. This is the same thing as an element of $\mathrm{Hom}_{\mathbb{Z}/2}(\mathbb{Z}, V^* \otimes_{\mathbb{F}} V^*)$. In order to generalize this concept to chain complexes we replace V^* by C, $\mathbb{F}$ by R, and Hom by Ext; an element of the Ext group is represented by a symmetric complex. Thus a symmetric complex is a homotopy version of a symmetric bilinear form.

Definition 8.4. We define a category $\mathcal{A}^R$ as follows. The objects of $\mathcal{A}^R$ are the quasi-symmetric complexes. If the dimension of (C, φ) is less than that of (C', φ') then a morphism from (C, φ) to (C', φ') is an R-linear chain map $f : C \to C'$. If the dimensions are equal then a morphism is an R-linear chain map with the property that

$$(f^t \otimes f) \circ \varphi = \varphi'.$$

There are no morphisms that lower dimension.

$\mathcal{A}^R$ is a balanced $\mathbb{Z}$-graded category, where i takes (C, φ) to $(C, -\varphi)$ and $\emptyset_n$ is the n-dimensional object for which C is zero in all degrees.

Definition 8.5. Let (π, Z, w) be a triple as in Section 7. Let $R = \mathbb{Z}[\pi]$ with the w-twisted involution. Define a balanced functor

$$\mathrm{Sig} : \mathcal{A}_{\pi,Z,w} \to \mathcal{A}^R,$$

called the *symmetric signature*, by

$$\mathrm{Sig}(X, f, \xi) = (S_*(\tilde{X}), \varphi_{\tilde{X},\xi}),$$

where $\tilde{X}$ is the pullback of Z to X and $\varphi_{\tilde{X},\xi}$ is the composite

$$W \cong W \otimes \mathbb{Z} \xrightarrow{1\otimes\xi} W \otimes (\mathbb{Z}^w \otimes_R S_*(\widetilde{X})) \cong \mathbb{Z}^w \otimes_R (W \otimes S_*(\widetilde{X}))$$
$$\to \mathbb{Z}^w \otimes_R (S_*(\widetilde{X}) \otimes S_*(\widetilde{X})) \cong S_*(\widetilde{X})^t \otimes_R S_*(\widetilde{X});$$

the unlabeled arrow is induced by the extended Alexander-Whitney map (see [MS03, Definition 2.10(a) and Remark 2.11(a)] for an explicit formula).

Next we must say what the K-ads with values in $\mathcal{A}^R$ are. We will build up to this gradually, culminating in Definition 8.11.

For a balanced pre K-ad F we will use the notation

$$F(\sigma, o) = (C_\sigma, \varphi_{\sigma,o}).$$

Definition 8.6. A map of chain complexes over R is a *cofibration* if it is split injective in each dimension.

Definition 8.7. A functor C from $\mathcal{C}ell^\flat(K)$ to chain complexes over R is called *well-behaved* if the following conditions hold:

(a) C takes each morphism to a cofibration.

(b) For each cell σ of K, the map

$$\operatorname*{colim}_{\tau\subsetneq\sigma} C_\tau \longrightarrow C_\sigma$$

is a cofibration.

For a well-behaved functor C we write $C_{\partial\sigma}$ for $\mathrm{colim}_{\tau\subsetneq\sigma} C_\tau$.

For our next definition, recall Example 3.11.

Definition 8.8. F is *closed* if, for each σ, the map

$$\mathrm{cl}(\sigma) \to \mathrm{Hom}(W, C_\sigma^t \otimes_R C_\sigma)$$

which takes $\langle \tau, o \rangle$ to the composite

$$W \xrightarrow{\varphi_{\tau,o}} C_\tau^t \otimes C_\tau \to C_\sigma^t \otimes_R C_\sigma$$

is a chain map.

In particular, if F is balanced and closed and C is well-behaved then for each σ the composite

$$W \to C_\sigma^t \otimes_R C_\sigma \to (C_\sigma/C_{\partial\sigma})^t \otimes_R C_\sigma$$

is a chain map.

Definition 8.9. Let F be a balanced and closed pre K-ad for which C is well-behaved and let (σ, o) be an oriented cell of K. Choose a chain map $\mathbb{Z} \to W$ which is right inverse to the augmentation and define a chain map

$$\Upsilon_\sigma : \mathrm{Hom}_R(C_\sigma, R) \to C_\sigma/C_{\partial\sigma}$$

to be the composite

$$\mathbb{Z} \otimes \mathrm{Hom}_R(C_\sigma, R) \to W \otimes \mathrm{Hom}_R(C_\sigma, R) \to \big((C_\sigma/C_{\partial\sigma})^t \otimes_R C_\sigma\big) \otimes \mathrm{Hom}_R(C_\sigma, R)$$
$$\to (C_\sigma/C_{\partial\sigma})^t \otimes_R R \cong (C_\sigma/C_{\partial\sigma})^t$$

(note that $(C_\sigma/C_{\partial\sigma})^t$ is the same chain complex as $C_\sigma/C_{\partial\sigma}$).

Remark 8.10. The chain homotopy class of Υ_σ is independent of the choice of the map $\mathbb{Z} \to W$.

Definition 8.11. F is a *K-ad* if
(a) it is balanced and closed and C is well-behaved, and
(b) for each σ the map Υ_σ induces an isomorphism

$$H^*(\mathrm{Hom}_R(C_\sigma, R)) \to H_{\dim\sigma - \deg F - *}(C_\sigma/C_{\partial\sigma}).$$

We write $\mathrm{ad}^R(K)$ for the set of K ads with values in $\mathcal{A}^R$.

Remark 8.12. When K is a simplicial complex, a K-ad is almost the same thing as a symmetric complex ([Ran92, Definition 3.4]) in $\Lambda^*(K)$ ([Ran92, Definition 4.1 and Proposition 5.1]). The only difference is that in [Ran92] the splitting maps $C_\sigma \to C_{\partial\sigma}$ of the underlying graded R-modules are part of the structure.

Theorem 8.13. (i) ad^R *is an ad theory.*

(ii) Sig *induces a morphism of ad theories from* $\mathrm{ad}_{\pi,Z,w}$ *to* $\mathrm{ad}_{\mathbb{Z}[\pi]^w}$ *(where* $\mathbb{Z}[\pi]^w$ *denotes* $\mathbb{Z}[\pi]$ *with the w-twisted involution).*

For the proof we need a product operation on ads. Recall the chain map

$$\Delta : W \to W \otimes W$$

from [Ran80a, page 175].

Lemma 8.14. *For $i = 1, 2$, suppose given a ring with involution R_i, a ball complex K_i and a K_i-ad F_i with values in $\mathcal{A}^{R_i}$. Write $F_i(\sigma, o) = ((C_i)_\sigma, (\varphi_i)_{\sigma,o})$. Define a pre $(K_1 \times K_2)$-ad $F_1 \otimes F_2$ with values in $\mathcal{A}^{R_1 \otimes R_2}$ by*

$$(F_1 \otimes F_2)((\sigma, o), (\tau, o')) = ((C_1)_\sigma \otimes (C_2)_\tau, \varphi_{(\sigma,o),(\tau,o')}),$$

where $\varphi_{(\sigma,o),(\tau,o')}$ is the composite

$$W \xrightarrow{\Delta} W \otimes W \xrightarrow{(\varphi_1)_{\sigma,o} \otimes (\varphi_2)_{\tau,o'}} \big((C_1)_\sigma^t \otimes_{R_1} (C_1)_\sigma\big) \otimes \big((C_2)_\tau^t \otimes_{R_2} (C_2)_\tau\big)$$
$$\cong \big((C_1)_\sigma \otimes (C_2)_\tau\big)^t \otimes_{R_1 \otimes R_2} \big((C_1)_\sigma \otimes (C_2)_\tau\big).$$

Then $F_1 \otimes F_2$ is a $(K_1 \times K_2)$-ad. □

Proof of 8.13. Part (ii) is clear from the definitions.

For part (i), we only need to verify parts (g) and (h) of Definition 3.9.

The proof of part (g) is similar to the corresponding proof in Section 7. Let F be a K'-ad with

$$F(\sigma, o) = (C_\sigma, \varphi_{\sigma,o}).$$

We need to define a K-ad E which agrees with F on each residual subcomplex of K. We may assume that K is a ball complex structure for the n disk with one n cell τ, and that K' is a subdivision of K which agrees with K on the boundary. We only need to define E on the top cell τ of K. We define $E(\tau, o)$ to be $(D_\tau, \kappa_{\tau,o})$, where

- $D_\tau = \operatorname{colim}_{\sigma \in K'} C_\sigma$, and
- $\kappa_{\tau,o}$ is the sum of the composites
$$W \xrightarrow{\varphi_{\sigma,o'}} C^t_\sigma \otimes_R C_\sigma \to D^t_\tau \otimes_R D_\tau,$$
where (σ, o') runs through the n-dimensional cells of K' with orientations induced by o.

Then E is closed by Proposition A.1(ii) and the cap product is an isomorphism by Proposition 10.4 below.

For part (h), we define the cylinder by $J(F) = F \otimes G$, where G is an I-ad with values in $\mathcal{A}^{\mathbb{Z}}$ defined as follows. Let $0, 1, I$ denote the three cells of I, with their standard orientations. Define $G(0)$ to be $(\mathbb{Z}, \epsilon)$ where ϵ is the augmentation, and similarly for $G(1)$. Define $G(I)$ to be $(C_*(\Delta^1), \varphi)$, where C_* denotes simplicial chains and φ is the composite

$$W \cong W \otimes \mathbb{Z} \overset{1\otimes\iota}{\to} W \otimes C_1(\Delta^1) \to C_*(\Delta^1) \otimes C_*(\Delta^1);$$

here ι is the element of $C_1(\Delta^1)$ represented by the identity map and unlabeled arrow is the extended Alexander-Whitney map. □

9. Example: Quadratic Poincaré ad theories

We use the notation of the previous section.

Definition 9.1. A *quasi-quadratic complex of dimension n* is a pair (C, ψ) where C is an object of $\mathcal{D}$ and ψ is an element of $(W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C))_n$.

Definition 9.2. We define a category $\mathcal{A}_R$ as follows. The objects of $\mathcal{A}_R$ are the quasi-quadratic complexes. If the dimension of (C, ψ) is less than that of (C', ψ') then a morphism from (C, ψ) to (C', ψ') is an R-linear chain map $f : C \to C'$. If the dimensions are equal then a morphism is an R-linear chain map with the property that

$$(1 \otimes (f^t \otimes f))\psi = \psi'.$$

There are no morphisms that lower dimension.

$\mathcal{A}_R$ is a balanced $\mathbb{Z}$-graded category, where i takes (C, ψ) to $(C, -\psi)$ and $\emptyset_n$ is the n-dimensional object for which C is zero in all degrees.

A balanced pre K-ad F has the form

$$F(\sigma, o) = (C_\sigma, \psi_{\sigma,o}).$$

Definition 9.3. F is *closed* if, for each σ, the map

$$\mathrm{cl}(\sigma) \to W \otimes_{\mathbb{Z}/2} (C^t_\sigma \otimes_R C_\sigma)$$

which takes $\langle \tau, o \rangle$ to the image of $\psi_{\tau,o}$ is a chain map.

Next we define a nonpositively graded complex of $\mathbb{Z}/2$-modules

$$V_0 \to V_{-1} \to \cdots$$

by letting

$$V_{-n} = \mathrm{Hom}_{\mathbb{Z}/2}(W_n, \mathbb{Z}[\mathbb{Z}/2]).$$

There is an isomorphism

$$W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C) \cong \mathrm{Hom}_{\mathbb{Z}/2}(V, C^t \otimes_R C).$$

The composite

$$N : W \to \mathbb{Z} \to V$$

induces a homomorphism

$$N^* : W \otimes_{\mathbb{Z}/2} (C^t \otimes_R C) \to \mathrm{Hom}_{\mathbb{Z}/2}(W, C^t \otimes_R C)$$

called the *norm map*. We write $\mathcal{N}$ for the functor

$$\mathcal{A}_R \to \mathcal{A}^R$$

which takes (C, ψ) to $(C, N^*(\psi))$.

Definition 9.4. $F \in \mathrm{pre}_R(K)$ is a K*-ad* if
(a) it is balanced and closed and C is well-behaved, and
(b) $\mathcal{N} \circ F$ is a K-ad.

Theorem 9.5. ad_R *is an ad theory.*

For the proof we need a product operation. Ranicki ([Ran80a, pages 174–175]) defines a chain map

$$\Delta : V \to W \otimes V.$$

Lemma 9.6. *For $i = 1, 2$, suppose given a ring with involution R_i and a ball complex K_i. Let $F_1 \in \mathrm{ad}^{R_1}(K_1)$ and write $F_1(\sigma, o) = ((C_1)_\sigma, \varphi_{\sigma,o})$. Let $F_2 \in \mathrm{ad}_{R_2}(K_2)$ and write $F_2(\sigma, o) = ((C_2)_\sigma, \psi_{\sigma,o})$. Define a pre $(K_1 \times K_2)$-ad $F_1 \otimes F_2$ with values in $\mathcal{A}_{R_1 \otimes R_2}$ by*

$$(F_1 \otimes F_2)((\sigma, o), (\tau, o')) = ((C_1)_\sigma \otimes (C_2)_\tau, \omega_{(\sigma,o),(\tau,o')}),$$

where $\omega_{(\sigma,o),(\tau,o')}$ is the element of

$$W \otimes_{\mathbb{Z}/2} (((C_1)_\sigma \otimes (C_2)_\tau)^t \otimes_{R_1 \otimes R_2} ((C_1)_\sigma \otimes (C_2)_\tau))$$

corresponding to the composite

$$\begin{aligned} V \xrightarrow{\Delta} W \otimes V \xrightarrow{\varphi_{\sigma,o} \otimes \psi_{\tau,o'}} & \big((C_1)^t_\sigma \otimes_{R_1} (C_1)_\sigma\big) \otimes \big((C_2)^t_\tau \otimes_{R_2} (C_2)_\tau\big) \\ & \cong \big((C_1)_\sigma \otimes (C_2)_\tau\big)^t \otimes_{R_1 \otimes R_2} \big((C_1)_\sigma \otimes (C_2)_\tau\big). \end{aligned}$$

Then $F_1 \otimes F_2$ is a $(K_1 \times K_2)$-ad.

Proof of Lemma 9.6. First observe that the set of homotopy classes of chain maps from W to a chain complex A is the same as $H_0(A)$. It follows that the diagram

$$\begin{array}{ccc} W & \xrightarrow{N} & V \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ W \otimes W & \xrightarrow{1\otimes N} & W \otimes V \end{array}$$

homotopy commutes. The result follows from this and Lemma 8.14. □

Proof of Theorem 9.5. We only need to verify parts (g) and (h) of Definition 3.9.

Part (g) is proved in the same way as for Theorem 8.13.

For part (h), we define the cylinder by $J(F) = G \otimes F$, where G is the I-ad defined in the proof of Theorem 8.13. □

10. Gluing

Our goal in this section is to prove a result (Proposition 10.4) which completes the proofs of Theorems 7.10, 8.13, and 9.5. First we need some terminology.

Let R be a ring with involution.

Recall Definition 8.1. Let $\mathcal{A}$ be the $\mathbb{Z}$-graded category whose objects of dimension n are pairs (C, ζ), where C is an object of $\mathcal{D}$ and ζ is an n-dimensional element of $C^t \otimes_R C$. If the dimension of (C, ζ) is less than that of (C', ζ') then a morphism from (C, ζ) to (C', ζ') is an R-linear chain map $f : C \to C'$. There are no morphisms which preserve or lower the dimension. The involution i takes (C, ζ) to $(C, -\zeta)$ and $\emptyset_n$ is the n-dimensional object for which C is 0 in all degrees.

A balanced pre K-ad F with values in $\mathcal{A}$ is *closed* if for each σ the elements $\zeta_{\tau,o}$ determine a chain map $\mathrm{cl}(\sigma) \to C_\sigma$. F is a *K-ad* if it is balanced and closed, C is well-behaved, and the cap product

$$H^*(\mathrm{Hom}_R(C_\sigma, R)) \to H_{\dim\sigma - \deg F - *}(C_\sigma / C_{\partial\sigma})$$

(as defined in Definition 7.5(ii)) is an isomorphism for every σ.

Definition 10.1. A *Poincaré pair* is a morphism $(C, \zeta) \to (D, \omega)$ in $\mathcal{A}$ with the property that the pre I-ad G defined by $G(1) = (C, \zeta)$, $G(I) = (D, \omega)$ and $G(0) = \emptyset$ is an ad.

Definition 10.2. Let K be a ball complex and $C : \mathcal{C}ell^\flat(K) \to \mathcal{D}$ a well-behaved functor. Define $C_K \in \mathcal{D}$ to be $\mathrm{colim}_{\sigma \in K}\, C_\sigma$.

Now let (L, L_0) be a ball complex pair such that $|L|$ is an orientable homology manifold with boundary $|L_0|$, and fix an orientation for $|L|$. (For the proofs of Theorems 7.10, 8.13, and 9.5 we only need the special case where $|L|$ is a PL ball).

Definition 10.3. Let $C : \mathcal{C}ell^\flat(L) \to \mathcal{D}$ be a well behaved functor and let

$$\nu : \mathrm{cl} \to C$$

be a natural transformation. Denote the value of ν on $\langle \sigma, o \rangle$ by $\nu_{\sigma,o}$. Define $\nu_L \in C_L$ (resp., $\nu_{L_0} \in C_{L_0}$) to be

$$\sum_{(\sigma,o)} \nu_{\sigma,o},$$

where (σ, o) runs through the top-dimensional cells of L (resp., L_0) oriented compatibly with $|L|$.

Proposition 10.4. *Let F be an L-ad and write $F(\sigma, o) = (C_\sigma, \zeta_{\sigma,o})$. Then*

$$(C_{L_0}, \zeta_{L_0}) \to (C_L, \zeta_L)$$

is a Poincaré pair.

Remark 10.5. The corresponding statements for the ad theories $\mathrm{ad}_{\pi,Z,w}$, ad^R and ad_R are consequences of this.

The rest of this section is devoted to the proof of Proposition 10.4. What we need to show is that the cap product

$$H^*(\mathrm{Hom}_R(C_L, R)) \to H_{\dim|L|-\deg F-*}(C_L/C_{L_0})$$

is an isomorphism.

The first step in the proof is to give an alternate description of $H_{\dim|L|-*}(C_L/C_{L_0})$.

Let

$$B : \mathcal{C}ell^\flat(L) \to \mathcal{D}$$

be a well-behaved functor and consider the chain complex

$$\mathrm{Nat}(\mathrm{cl}, B)$$

of natural transformations of graded abelian groups; the differential is given by

$$\partial(\nu) = \partial \circ \nu - (-1)^{|\nu|}\nu \circ \partial.$$

Define

$$\Phi : \mathrm{Nat}(\mathrm{cl}, B) \to B_L/B_{L_0}$$

by

$$\Phi(\nu) = \nu_L.$$

Then Φ is a chain map by Proposition A.1(ii); note that Φ increases degrees by $\dim|L|$.

Lemma 10.6. (cf. [WW89, Digression 3.11]) Φ *induces an isomorphism*

$$H_*(\mathrm{Nat}(\mathrm{cl}, B)) \to H_{*+\dim|L|}(B_L/B_{L_0})$$

for every well-behaved $B : \mathcal{C}ell^\flat(L) \to \mathcal{D}$.

The proof is deferred to the end of this section.

Continuing with the proof of Proposition 10.4, we observe that

$$\mathrm{Hom}_R(C_L, R) = \mathrm{Nat}_R(C, \underline{R}), \tag{10.1}$$

where Nat_R denotes the chain complex of natural transformations of graded R-modules and $\underline{R}$ denotes the constant functor with value R. There is a cap product

$$\Upsilon : \mathrm{Nat}_R(C, \underline{R}) \to \mathrm{Nat}(\mathrm{cl}, C)$$

which takes ν to the composite

$$\mathrm{cl} \xrightarrow{\zeta} C^t \otimes_R C \xrightarrow{1\otimes\nu} C^t \otimes_R \underline{R} = C^t$$

(note that C^t and C are the same as functors to graded abelian groups). The diagram

$$\begin{array}{ccc} H_{-*}(\mathrm{Nat}_R(C, \underline{R})) & \xrightarrow{H_*\Upsilon} & H_{-*-\deg F}(\mathrm{Nat}(\mathrm{cl}, C)) \\ = \downarrow & & \cong \downarrow \text{Lemma 10.6} \\ H^*(\mathrm{Hom}_R(C_L, R)) & \longrightarrow & H_{\dim|L|-\deg F-*}(C_L/C_{L_0}) \end{array}$$

commutes, so to prove Proposition 10.4 it suffices to show that Υ is a homology isomorphism.

Next observe that

$$\mathrm{Nat}_R(C,\underline{R}) = \lim_{\sigma\in L} \mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R})$$

and that

$$\mathrm{Nat}(\mathrm{cl},C) = \lim_{\sigma\in L} \mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)}).$$

Moreover, the natural maps

$$\lim_{\sigma\in L} \mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R}) \to \mathrm{ho}\lim_{\sigma\in L} \mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R})$$

and

$$\lim_{\sigma\in L} \mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)}) \to \mathrm{ho}\lim_{\sigma\in L} \mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)})$$

are homology isomorphisms by [Hir03, Theorem 19.9.1(2)] (using the fact that C and cl are well-behaved). Thus there are spectral sequences

$$\lim{}^p H_q(\mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R})) \Rightarrow H_{q-p}(\mathrm{Nat}_R(C,\underline{R}))$$

and

$$\lim{}^p H_q(\mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)})) \Rightarrow H_{q-p}(\mathrm{Nat}(\mathrm{cl},C))$$

(see [BK72, Section XI.7] for the construction; note that in the category of chain complexes H_* plays the role of π_*). By [BK72, Proposition XI.6.2] we have $\lim^p = 0$ for $p > \dim |L|$, so these spectral sequences converge strongly.

The cap products

$$\Upsilon|_\sigma : \mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R}) \to \mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)})$$

give a map of inverse systems and hence a map of spectral sequences. By equation (10.1) and Lemma 10.6 the maps

$$H_*(\Upsilon|_\sigma) : H_*(\mathrm{Nat}_R(C|_{\mathcal{C}ell^\flat(\sigma)},\underline{R})) \to H_{*-\deg F}(\mathrm{Nat}(\mathrm{cl},C|_{\mathcal{C}ell^\flat(\sigma)}))$$

agree up to isomorphism with the cap products

$$H^*(\mathrm{Hom}_R(C_\sigma,R)) \to H_{\dim\sigma-*}(C_\sigma/C_{\partial\sigma})$$

which are isomorphisms because F is an ad. Thus the $\Upsilon|_\sigma$ give an isomorphism at E^2. Since $\lim \Upsilon|_\sigma$ is Υ we see that $H_*(\Upsilon)$ is an isomorphism, which completes the proof of Proposition 10.4.

Proof of Lemma 10.6. The proof is similar to the proof of [WW89, Digression 3.11].

First of all, the proof of [WW89, Lemma 3.4] adapts to our situation to show that B is weakly equivalent to a well-behaved functor which is finitely generated (that is, one which takes each σ to a finitely generated complex). The source and target of Φ both preserve weak equivalences (see the argument at the top of page 71 in [WW89]) and so we may assume that B is finitely generated.

Because B is well-behaved, the source and target of Φ both have the property that they take short exact sequences of well-behaved functors to short exact sequences. We give B a decreasing filtration by letting the i-th filtration $B[i]$ take σ to the sum of the images of $B_{\sigma'} \to B_\sigma$ with $\sigma' \subset \sigma$ and $\dim \sigma' \geq i$. Then the sequence

$$0 \to B[i+1] \to B[i] \to B[i]/B[i+1] \to 0$$

is a short exact sequence of well-behaved functors, so it suffices (by induction on i) to show that the lemma is true for the quotients $B[i]/B[i+1]$ when $0 \leq i \leq n$. Now each quotient $B[i]/B[i+1]$ is a direct sum

$$\oplus_{\dim \rho = i} B[\rho]$$

where $B[\rho]_\sigma$ is the image of $B_\rho \to B_\sigma$ if $\rho \subset \sigma$ and 0 otherwise. Next we give $B[\rho]$ an increasing filtration by letting the j-th filtration $B[\rho, j]$ be the functor which takes σ to the part of $B[\rho]_\sigma$ in dimensions $\leq j$. The sequence

$$0 \to B[\rho, j] \to B[\rho, j+1] \to B[\rho, j+1]/B[\rho, j] \to 0$$

is a short exact sequence of well-behaved functors for each j. Since B is finitely generated, it suffices to show that the lemma is true for the quotients $B[\rho, j+1]/B[\rho, j]$.

Fix ρ and j. To lighten the notation let us denote $B[\rho, j+1]/B[\rho, j]$ by A.

The functor A takes ρ to a chain complex which consists of an abelian group (call it $\mathfrak{A}$) in dimension j and 0 in all other dimensions. It takes every cell containing ρ to this same chain complex. Let M be the subcomplex of L consisting of all cells which contain ρ and their faces. Let N be the subcomplex of M consisting of all cells which do not contain ρ. Then the chain complex $\mathrm{Nat}(\mathrm{cl}, A)$ is isomorphic to the cellular cochain complex $C^{j-*}(M, N; \mathfrak{A})$. Next we use results from the Appendix. Proposition A.2 gives a ball complex structure on the pair $(|\mathrm{st}(\hat{\rho})|, |\mathrm{lk}(\hat{\rho})|)$. There is a bijection between the cells of the cells of M which are not in N and the cells of $|\mathrm{st}(\hat{\rho})|$ which are not in $|\mathrm{lk}(\hat{\rho})|$); this bijection preserves incidence numbers and therefore induces an isomorphism

$$C^*(M, N; \mathfrak{A}) \cong C^*(|\mathrm{st}(\hat{\rho})|, |\mathrm{lk}(\hat{\rho})|; \mathfrak{A}).$$

Now

$$H^*(|\mathrm{st}(\hat{\rho})|, |\mathrm{lk}(\hat{\rho})|; \mathfrak{A}) = H^*(|L|, |L| - \hat{\rho}; \mathfrak{A}).$$

Thus $H^*(|\mathrm{st}(\hat{\rho})|, |\mathrm{lk}(\hat{\rho})|; \mathfrak{A})$ is 0 if $* \neq \dim |L|$ or if ρ is in L_0. In the remaining case, we note that $|\mathrm{st}(\hat{\rho})|$ is a homology manifold with boundary $|\mathrm{lk}(\hat{\rho})|$, and thus (by Proposition A.1(i)) the map

$$H^*(|\mathrm{st}(\hat{\rho})|, |\mathrm{lk}(\hat{\rho})|; \mathfrak{A}) \to \mathfrak{A}$$

which takes a cocycle to its value on the sum of the top-dimensional cells of $|\mathrm{st}(\hat{\rho})|$ is an isomorphism.

To sum up, we have shown that if $* \neq j - \dim |L|$ or if ρ is in L_0 then $H_*(\mathrm{Nat}(\mathrm{cl}, A))$ is 0, and that the map

$$H_{j - \dim |L|}(\mathrm{Nat}(\mathrm{cl}, A)) \to \mathfrak{A}$$

which takes ν to ν_L is an isomorphism.

Now if ρ is not in L_0 then A_{L_0} is 0, so $H_*(A_L/A_{L_0})$ is $\mathfrak{A}$ when $* = j$ and 0 otherwise. This proves the lemma in this case.

If ρ is in L_0 then $A_L = A_{L_0}$, so the domain and target of the map in the lemma are both 0. □

11. Functoriality

We begin by considering symmetric Poincaré ad theories.

Let $p : R \to S$ be a homomorphism of rings with involution. We can define a morphism of $\mathbb{Z}$-graded categories $p_* : \mathcal{A}^R \to \mathcal{A}_S$ by taking (C, W, φ) to $(S \otimes_R C, W, \psi)$, where ψ is the composite

$$W \to C^t \otimes_R C \to (S \otimes_R C)^t \otimes_S (S \otimes_R C).$$

This induces a morphism of ad theories $p_* : \mathrm{ad}^R \to \mathrm{ad}^S$. However, ad^R is not a functor of R: if $q : S \to T$ is another homomorphism of rings with involution, then $(qp)_* \neq q_* p_*$, for the simple reason that $T \otimes_S (S \otimes_R C)$ is not equal to $T \otimes_R C$ but only isomorphic to it.

The same problem arises in algebraic K-theory, and Blumberg and Mandell have given a solution in that setting (see the proof of Theorem 8.1 in [BM]). We use their idea in this section.

First, for definiteness, let us define the free R-module generated by a set A to be the set of functions from A to R which are nonzero for only finitely many elements of A. We denote this by $R\langle A\rangle$.

Next we define a category $\mathcal{F}_R$ by letting its objects be sets and letting the morphisms from A to B be the R-module morphisms from $R\langle A\rangle$ to $R\langle B\rangle$. It is easy to check that $\mathcal{F}_R$ is equivalent to the category of free R-modules.

$\mathcal{F}_R$ is a functor of R: given $f : R \to S$ we define $f_* : \mathcal{F}_R \to \mathcal{F}_S$ as follows. f_* is the identity on objects. A morphism in $\mathcal{F}_R$ is given by a (possibly infinite) matrix with values in R; the corresponding morphism in $\mathcal{F}_S$ is obtained by applying f to the entries of this matrix.

If we replace the category of free R-modules by $\mathcal{F}_R$ in Section 8 then $\mathcal{A}_R$ and ad^R become functors of R. The bordism groups are unchanged by Remark 4.8.

Let $\mathcal{R}$ be the category of rings with involution. We write $\mathrm{ad}_{\mathrm{sym}}$ (resp., $\mathrm{ad}_{\mathrm{quad}}$) for the functor from $\mathcal{R}$ to the category of ad-theories that takes R to ad^R (resp., ad^R).

Quadratic Poincaré ad theories can be dealt with in a similar way.

Next we consider geometric Poincaré ad theories. In order for Sig to be compatible with the definition of ad^R just given we need to redefine $\mathcal{A}_{\pi,Z,w}$. Given a map $f : X \to Z/\pi$, let us write f^*Z for the pullback of Z to X (this was denoted $\tilde{X}$ in Section 7). By a *lifting function* for f we mean a function Φ that assigns to each map from a simplex to X a lift to f^*Z. We redefine $\mathcal{A}_{\pi,Z,w}$ to be the category whose objects are quadruples

$$(X,\ f : X \to Z/\pi,\ \xi \in S_*(X, \mathbb{Z}^f), \Phi),$$

where Φ is a lifting function for f; the morphisms are defined exactly as in Definition 7.2 (i.e. Φ plays no role in the definition of morphism). The bordism groups $\Omega_*^{\pi,Z,w}$ are unchanged by Remark 4.8.

Now we redefine Sig. Let (X, f, ξ, Φ) be an object of $\mathcal{A}_{\pi,Z,w}$. For each $p \geq 0$, let C_p be the set of maps $\Delta^p \to X$. The lifting Φ gives an inclusion $C_p \to S_p(f^*Z)$ for each p and this in turn gives an isomorphism

$$\mathbb{Z}[\pi]\langle C_p\rangle \cong S_p(f^*Z).$$

We define a morphism

$$\partial_p : C_p \to C_{p-1}$$

in $\mathcal{F}_{\mathbb{Z}[\pi]}$ to be the composite

$$\mathbb{Z}[\pi]\langle C_p\rangle \cong S_p(f^*Z) \xrightarrow{\partial_p} S_{p-1}(f^*Z) \cong \mathbb{Z}[\pi]\langle C_{p-1}\rangle.$$

These definitions give a chain complex C in $\mathcal{F}_{\mathbb{Z}[\pi]}$. We define $\mathrm{Sig}(X, f, \xi, \Phi)$ to be the triple (C, W, ψ), where W is the standard resolution and ψ is determined by the map φ in Definition 8.5.

Next we consider functoriality of $\mathrm{ad}_{\pi,Z,w}$.

Definition 11.1. Let $\mathcal{T}$ be the category whose objects are the triples (π, Z, w); the morphisms from (π, Z, w) to (π', Z', w') are pairs (h, g), where $h : \pi \to \pi'$ is a homomorphism with $w = w' \circ h$ and g is a π-equivariant map $Z \to Z'$.

A morphism in $\mathcal{T}$ induces a functor $\mathcal{A}_{\pi,Z,w} \to \mathcal{A}_{\pi',Z',w'}$ by taking (X, f, ξ, Φ) to $(X, \bar{g}f, \eta, \Psi)$, where $\bar{g}$ is the map $Z/\pi \to Z'/\pi'$ induced by g, η corresponds to ξ under the isomorphism

$$S_*(X, \mathbb{Z}^f) \cong S_*(X, \mathbb{Z}^{\bar{g}f})$$

(see Definition 7.1), and Ψ is determined by Φ together with the canonical map $f^*Z \to f^*\bar{g}^*Z'$.

With these definitions $\mathrm{ad}_{\pi,Z,w}$ is a functor of (π, Z, w). We write $\mathrm{ad}_{\mathrm{geom}}$ for the functor from $\mathcal{T}$ to the category of ad-theories that takes (π, Z, w) to $\mathrm{ad}_{\pi,Z,w}$.

Finally, let $\rho : \mathcal{T} \to \mathcal{R}$ be the functor which takes (π, Z, w) to $\mathbb{Z}[\pi]$ with the w-twisted involution. Then Sig is a natural transformation from $\mathrm{ad}_{\mathrm{geom}}$ to $\mathrm{ad}_{\mathrm{sym}} \circ \rho$.

12. The cohomology theory associated to an ad theory

Fix an ad theory.

For a ball complex K with a subcomplex L, we will say that two elements F, G of $\mathrm{ad}^k(K, L)$ are *bordant* if there is a $(K \times I, L \times I)$-ad which restricts to F on $K \times 0$ and G on $K \times 1$.

Definition 12.1. Let $T^k(K, L)$ be the set of bordism classes in $\mathrm{ad}^k(K, L)$.

Remark 12.2. (i) $T^k(*)$ is the same as Ω_{-k}.
(ii) For the ad theory in Example 3.11, $T^k(K, L)$ is $H^k(K, L; C)$.

Our goal in this section is to show that T^* is a cohomology theory.

We will define addition in $T^k(K, L)$ using the method of Section 4. First we need a generalization of the functor κ.

Definition 12.3. Let

$$\kappa : \mathcal{C}ell(I \times K, (\{0, 1\} \times K) \cup (I \times L)) \to \mathcal{C}ell(K, L)$$

be the isomorphism of categories which takes $I \times (\sigma, o)$ (where I is given its standard orientation) to (σ, o).

Remark 12.4. κ is incidence-compatible (Definition 3.7(i)) so it induces a bijection

$$\kappa^* : \mathrm{ad}^k(K, L) \to \mathrm{ad}^{k+1}(I \times K, (\{0, 1\} \times K) \cup (I \times L))$$

by part (f) of Definition 3.9.

Now let M and M' be the ball complexes defined in Section 4. Lemma 4.4 generalizes to show that, given $F, G \in \mathrm{ad}^k(K, L)$, there is an $H \in \mathrm{ad}^{k+1}(M \times$

$K, M\times L)$ such that $(\lambda_1\times \mathrm{id})^*H=\kappa^*F$, $(\lambda_2\times id)^*H=\kappa^*G$, and $(\lambda_3\times\mathrm{id})^*H$ and $(\lambda_4\times\mathrm{id})^*H$ are trivial. Then we define $[F]+[G]$ to be

$$[(\kappa^{-1})^*(\lambda_5\times\mathrm{id})^*H].$$

The proof that this is well-defined and that $T^k(K,L)$ is an abelian group is the same as in Section 4.

Next we show that T^k is a homotopy functor.

Using the notation of [BRS76, page 5], let Bi be the category whose objects are pairs of ball complexes and whose morphisms are composites of inclusions of subcomplexes and isomorphisms. Let Bh be the category with the same objects whose morphisms are homotopy classes of continuous maps of pairs.

Proposition 12.5. *For each k, the functor $T^k : Bi \to \mathrm{Ab}$ factors uniquely through Bh.*

The functor $Bh\to\mathrm{Ab}$ given by the lemma will also be denoted by T^k.

For the proof of Proposition 12.5 we need a preliminary fact.

Definition 12.6. (cf. [BRS76, page 5]) An inclusion of pairs $(K_1,L_1)\to(K,L)$ is an *elementary expansion* if

(a) $L_1=L\cap K_1$,

(b) K has exactly two cells (say σ and σ') that are not in K_1, with $\dim\sigma'=\dim\sigma-1$ and $\sigma'\subset\partial\sigma$, and

(c) σ and σ' are either both in L or both not in L.

Lemma 12.7. *If $(K_1,L_1)\to(K,L)$ is an elementary expansion then the restriction*

$$\mathrm{ad}^k(K,L)\to\mathrm{ad}^k(K_1,L_1)$$

is onto.

The proof is deferred to the end of this section.

Proof of Proposition 12.5. The functor ad^k satisfies axioms E and G on page 15 of [BRS76]; axiom E is Lemma 12.7 and axiom G is part (e) of Definition 3.9. Now Proposition I.6.1 and Theorem I.5.1 of [BRS76] show that

$$T^k : Bi\to\mathrm{Set}$$

factors uniquely to give a functor $T^k : Bh\to\mathrm{Set}$. Specifically (with the notation of Definition 2.1) if $f : (|K'|,|L'|)\to(|K|,|L|)$ is a map of pairs then $T^k(f)$ is defined to be $T^k(g)^{-1}T^k(h)$, where g and h are certain morphisms in Bi. But then $T^k(f)$ is a homomorphism, so we obtain a functor $T^k : Bh\to\mathrm{Ab}$. □

Next we observe that excision is an immediate consequence of part (f) of Definition 3.9.

The first step in constructing the connecting homomorphism is to construct a suitable suspension isomorphism.

Lemma 12.8. *κ^* induces an isomorphism*

$$T^kL\to T^{k+1}(I\times L,\{0,1\}\times L).$$

Proof. κ^* is a bijection by Remark 12.4. To see that it is a homomorphism, let $F,G\in\mathrm{ad}^k(L)$ and let $H\in\mathrm{ad}^{k+1}(M\times K, M\times L)$ be as in the definition of addition. Let

$$\theta : \mathcal{C}ell(M\times I\times K,(M\times\{0,1\}\times K)\cup(M\times I\times L))\to\mathcal{C}ell(M\times K, M\times L)$$

be the evident isomorphism. Then $\theta^*(H)$ is an $(M \times I \times K)$-ad with the property that $(\lambda_1 \times \mathrm{id})^*\theta^*(H) = \kappa^*\kappa^*F$, $(\lambda_2 \times \mathrm{id})^*\theta^*(H) = \kappa^*\kappa^*G$, and $(\lambda_3 \times \mathrm{id})^*\theta^*(H)$ and $(\lambda_4 \times \mathrm{id})^*\theta^*(H)$ are trivial. Thus $[\kappa^*F] + [\kappa^*G]$ is $[(\kappa^{-1})^*(\lambda_5 \times \mathrm{id})^*\theta^*(H)]$, which simplifies to $[(\lambda_5 \times \mathrm{id})^*H]$, and this is $\kappa^*[F + G]$. □

Remark 12.9. The statement of the lemma might look strange in view of the fact that, for a space X, $(I \times X)/(\{0, 1\} \times X)$ is homotopic to $\Sigma X \vee S^1$ rather than ΣX. But if E is a cohomology theory then $E^k(X) \cong \tilde{E}^k(X \vee S^0)$, so the lemma agrees with the expected behavior of the suspension map for unreduced cohomology theories.

Now observe that excision gives an isomorphism

$$T^k(I \times L, \{0, 1\} \times L) \xrightarrow{\cong} T^k((1 \times K) \cup (I \times L), (1 \times K) \cup (0 \times L))$$

(where $(1 \times K) \cup (I \times L)$ is thought of as a subcomplex of $I \times K$) and that the map

$$(|(1 \times K) \cup (I \times L)|, |(1 \times K) \cup (0 \times L)|) \to (|I \times K|, |(1 \times K) \cup (0 \times L)|)$$

is a homotopy equivalence of pairs. It follows that the restriction map

$$T^k(I \times K, (1 \times K) \cup (0 \times L)) \to T^k(I \times L, \{0, 1\} \times L)$$

is an isomorphism.

Definition 12.10. The connecting homomorphism

$$T^k(L) \to T^{k+1}(K, L)$$

is the negative of the composite

$$T^k(L) \xrightarrow{\kappa^*} T^{k+1}(I \times L, \{0, 1\} \times L) \xleftarrow{\cong} T^{k+1}(I \times K, (1 \times K) \cup (0 \times L)) \to T^{k+1}(K, L)$$

where the last map is induced by the inclusion

$$(0 \times K, 0 \times L) \to (I \times K, (1 \times K) \cup (0 \times L)).$$

For an explanation of the sign see the proof of Proposition 14.4(ii).

Theorem 12.11. *T^* is a cohomology theory.*

Proof. It only remains to verify that the sequence

$$T^{k-1}K \to T^{k-1}L \to T^k(K, L) \to T^kK \to T^kL$$

is exact for every pair (K, L).

Exactness at $T^k(K)$. We prove more generally that the sequence

$$T^k(K, L) \to T^k(K, M) \to T^k(L, M)$$

is exact for every triple $M \subset L \subset K$.

Clearly the composite $T^k(K, L) \to T^k(K, M) \to T^k(L, M)$ is trivial. On the other hand, if $[F] \in T^k(K, M)$ maps to 0 in $T^k(L, M)$, then there is a bordism $H \in \mathrm{ad}^k(L \times I, M \times I)$ from $F|_L$ to $\emptyset$. We obtain an ad

$$H' \in \mathrm{ad}^k((K \times 1) \cup (L \times I), M \times I)$$

by letting H' be F on $K \times 1$ and H on $L \times I$. The inclusion

$$(K \times 1) \cup (L \times I) \to K \times I$$

is a composite of elementary expansions, so by Lemma 12.7 there is an $H'' \in \mathrm{ad}^k(K \times I, M \times I)$ which restricts to H'. But now $H''|_{K\times 0}$ is in $\mathrm{ad}^k(K, L)$ and is bordant to F, so $[F]$ is in the image of $T^k(K, L)$.

Exactness at $T^k(K, L)$. The composite

$$T^k(I \times K, (1 \times K) \cup (0 \times L)) \to T^k(K, L) \to T^k(K)$$

takes F to $F|_{0\times K}$, but this is bordant to $F|_{1\times K}$ which is 0. It follows that the composite

$$T^{k-1}L \to T^k(K, L) \to T^k(K)$$

is trivial. On the other hand, if $F \in \mathrm{ad}^k(K, L)$ becomes 0 in $T^k(K)$ then there is an $H \in \mathrm{ad}^k(I \times K, (1 \times K) \cup (0 \times L))$ with $H|_{0\times K} = F$. Thus F is in the image of

$$T^k(I \times K, (1 \times K) \cup (0 \times L)) \to T^k(K, L)$$

and hence in the image of the connecting homomorphism.

Exactness at $T^{k-1}L$. The composite

$$T^{k-1}K \to T^{k-1}L \to T^k(K, L)$$

is equal to the composite

$$T^{k-1}K \xrightarrow{\kappa^*} T^k(I \times K, \{0, 1\} \times K) \to T^k(I \times K, (1 \times K) \cup (0 \times L)) \to T^k(K, L)$$

and the composite of the last two maps is clearly trivial. On the other hand, suppose that $x \in T^{k-1}(L)$ maps trivially to $T^k(K, L)$. By definition of the connecting homomorphism, there is a $y \in T^k(I \times K, (1 \times K) \cup (0 \times L))$ such that y restricts to κ^*x in $T^k(I \times L, \{0, 1\} \times L)$ and to 0 in $T^k(0 \times K, 0 \times L)$. Since the restriction map

$$T^k(\{0, 1\} \times K, (1 \times K) \cup (0 \times L)) \to T^k(0 \times K, 0 \times L)$$

is an isomorphism by excision, we see that y restricts trivially to $T^k(\{0, 1\} \times K, (1 \times K) \cup (0 \times L))$. Now the exact sequence of the triple

$$(1 \times K) \cup (0 \times L) \subset \{0, 1\} \times K \subset I \times K$$

implies that there is a $z \in T^k(I \times K, \{0, 1\} \times K)$ that restricts to y. Then z restricts to κ^*x in $T^k(I \times L, \{0, 1\} \times L)$ and therefore $(\kappa^*)^{-1}z \in T^k(K)$ restricts to x. □

Proof of 12.7. Let $F \in \mathrm{ad}^k(K_1, L_1)$. Let σ' and σ be as in the definition of elementary expansion. If σ' and σ are in L then we can extend F to $\mathcal{C}ell(K, L)$ by letting it take σ' and σ to $\emptyset$. So assume that σ' and σ are not in L. Let A be the sub-ball-complex of K which is the union of the cells of $\partial\sigma$ other than σ'. It suffices to show that the restriction of F to $\mathcal{C}ell(A)$ extends to $\mathcal{C}ell(\sigma)$. By Theorem 3.34 of [RS82], the pair (σ, σ') is PL isomorphic to the pair (D^n, S^{n-1}_-) (where n is the dimension of σ, D^n is a standard n-ball and S^{n-1}_- is the lower hemisphere of its boundary). Under this isomorphism A corresponds to a subdivision of the upper hemisphere S^{n-1}_+. Moreover, the pair (D^n, S^{n-1}_+) is PL isomorphic to $(S^{n-1}_+ \times I, S^{n-1}_+ \times 0)$. Thus the pair $(A \times I, A \times 0)$ is PL isomorphic to a subdivision of the pair (σ, A). Part (h) of Definition 3.9 extends F to $\mathcal{C}ell(A \times I)$, and now part (g) of Definition 3.9 gives a corresponding extension of F to $\mathcal{C}ell(\sigma)$. □

13. The spectrum associated to an ad theory

Definition 13.1. Let Δ_{inj} denote the category whose objects are the sets $\{0, \ldots, n\}$ and whose morphisms are the monotonically increasing injections. By a *semisimplicial set* we mean a contravariant functor from Δ_{inj} to Set.

Thus a semisimplicial set is a simplicial set without degeneracies. In the literature these are often called Δ-sets, but this seems awkward because Δ is the category that governs simplicial sets.

The geometric realization of a semisimplicial set is defined by

$$|A| = \big(\coprod \Delta^n \times A_n\big)/\!\sim,$$

where $\sim$ identifies $(d^i u, x)$ with $(u, d_i x)$.

Definition 13.2. Let $*$ denote the semisimplicial set with a single element (also denoted $*$) in each degree. A *basepoint* for a semisimplicial set is a semisimplicial map from $*$.

Remark 13.3. Geometric realization of semisimplicial sets is a left adjoint (for example by [RS71, Proposition 2.1]), but it does not preserve quotients because it does not take the terminal object $*$ to a point.

Now fix an ad theory. First we construct the spaces of the spectrum.

Definition 13.4. (i) For $k \geq 0$, let P_k be the semisimplicial set with n-simplices

$$(P_k)_n = \text{ad}^k(\Delta^n)$$

and the obvious face maps. Give P_k the basepoint determined by the elements $\emptyset$.

(ii) Let Q_k be $|P_k|$.

Next we define the structure maps of the spectrum. For this we will use the semisimplicial analog of the Kan suspension.

Definition 13.5. Given a based semisimplicial set A, define ΣA to be the based semisimplicial set for which the only 0-simplex is $*$ and the (based) set of n simplices is A_{n-1}. The face operators $d_i : (\Sigma A)_n \to (\Sigma A)_{n-1}$ agree with those of A for $i < n$ and d_n takes all simplices to $*$.

Remark 13.6. The motivation for this construction is that the cone on a simplex is a simplex of one dimension higher.

Lemma 13.7. *There is a natural homeomorphism* $\Sigma|A| \cong |\Sigma A|$.

Proof. If $t \in [0, 1]$ and $u \in \Delta^{n-1}$ let us write $\langle t, u\rangle$ for the point $((1 - t)u, t)$ of Δ^n. The homeomorphism takes $[t, [u, x]]$ (where [ ] denotes equivalence class) to $[\langle t, u\rangle, x]$. □

Next observe that for each n there is an isomorphism of $\mathbb{Z}$-graded categories

$$\theta : \mathcal{C}ell(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n + 1\}) \to \mathcal{C}ell(\Delta^n)$$

which lowers degrees by 1, defined as follows: a simplex σ of Δ^{n+1} which is not in $\partial_{n+1}\Delta^{n+1} \cup \{n + 1\}$ contains the vertex $n + 1$. Let θ take σ (with its canonical orientation) to the simplex of Δ^n spanned by the vertices of σ other than $n + 1$ (with $(-1)^{\dim \sigma - 1}$ times its canonical orientation). θ is incidence-compatible (this is the reason for the sign in its definition) so by part (f) of Definition 3.9 it induces a bijection

$$\theta^* : \mathrm{ad}^k(\Delta^n) \to \mathrm{ad}^{k+1}(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}).$$

The composites

$$\mathrm{ad}^k(\Delta^n) \xrightarrow{\theta^*} \mathrm{ad}^{k+1}(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}) \to \mathrm{ad}^{k+1}(\Delta^{n+1})$$

give a semisimplicial map

$$\Sigma P_k \to P_{k+1}.$$

Definition 13.8. Let $\mathbf{Q}$ be the spectrum consisting of the spaces Q_k with the structure maps

$$\Sigma Q_k = \Sigma|P_k| \cong |\Sigma P_k| \to |P_{k+1}| = Q_{k+1}.$$

In the rest of this section we show:

Proposition 13.9. $\mathbf{Q}$ *is an* Ω *spectrum.*

First we observe that the semisimplicial Kan suspension Σ has a right adjoint:

Definition 13.10. For a based semisimplicial set A define a semisimplicial set ΩA by letting the n-simplices of ΩA be the $(n+1)$-simplices x of A which satisfy the conditions

$$d_{n+1}x = * \quad \text{and} \quad (d_0)^{n+1}x = *.$$

The face maps are induced by those of A.

It's easy to check that the adjoint of the map $\Sigma P_k \to P_{k+1}$ is an isomorphism

$$P_k \cong \Omega P_{k+1};$$

it therefore suffices to relate the semisimplicial Ω to the usual one.

Recall ([RS71, page 329]) that a semisimplicial set A is a *Kan complex* if every map $\Lambda_{n,i} \to A$ (where $\Lambda_{n,i}$ is defined on page 323 of [RS71]) extends to a map $\Delta^n \to A$. Proposition 13.9 follows from the next two facts.

Lemma 13.11. *If* A *is a Kan complex then the adjoint of the composite*

$$\Sigma|\Omega A| \cong |\Sigma\Omega A| \to |A|$$

is a weak equivalence.

Lemma 13.12. *For each* k*,* P_k *is a Kan complex.*

Proof of Lemma 13.11. Let S^n denote the based semisimplicial set with one nontrivial simplex in degree n. For a based Kan complex B, Remark 6.5 of [RS71] gives a bijection

$$\pi_n(|B|) \cong [S^n, B]$$

where [,] denotes based homotopy classes of based semisimplicial maps (the homotopy relation is defined at the beginning of [RS71, Section 6]). It is easy to check that ΩA is a Kan complex if A is. It therefore suffices to show that the adjunction induces a map

$$[S^n, \Omega A] \to [\Sigma S^n, A]$$

and that this map is a bijection.

For this, we first observe that for a based semisimplicial set B the set of based semisimplicial maps $S^n \to B$ can be identified with the set (which will be denoted by $\rho_n(B)$) of n-simplices of B with all faces at the basepoint. Moreover, if B is Kan then (by lines -10 to -7 of page 333 of [RS71]) the set $[S^n, B]$ is the quotient of $\rho_n(B)$ by the relation which identifies y and y' if there is a z with $d_0 z = y$, $d_1 z = y'$, and $d_i z = *$ for $i > 1$. The desired bijection is immediate from this and the fact that ΣS^n is S^{n+1}. □

For the proof of Lemma 13.12 we need to introduce a useful class of semisimplicial sets.

Definition 13.13. A semisimplicial set is *strict* if two simplices are equal whenever they have the same set of vertices.

Note that a strict semisimplicial set is the same thing as an ordered simplicial complex.

The geometric realization of a strict semisimplicial set A has a canonical ball complex structure (which will also be denoted by A) and the cells have canonical orientations.

Remark 13.14. We will make important use of the following observation ([Ran92, page 140]): for a pair (A, B) of strict semisimplicial sets, there is a canonical bijection between the set of semisimplicial maps $(A, B) \to (P_k, *)$ and $\mathrm{ad}^k(A, B)$.

Proof of Lemma 13.12. By Remark 13.14, it suffices to show that every element of $\mathrm{ad}^k(\Lambda_{n,i})$ extends to an element of $\mathrm{ad}^k(\Delta^n)$, and this is true by Lemma 12.7. □

14. $\mathbf{Q}$ REPRESENTS T^*

In this section we prove:

Theorem 14.1. *The cohomology theory represented by* $\mathbf{Q}$ *is naturally isomorphic to* T^*.

Remark 14.2. Theorem 14.1 includes as a special case the statement that the semisimplicial sets $\mathbb{L}_n(\Lambda^*(K))$ and $\mathbb{H}^n(K; \mathbb{L}_\bullet(\Lambda))$ in Proposition 13.7 of [Ran92] are weakly equivalent; the statement given in [Ran92] that they are actually isomorphic is not correct (because the sets in the 8th and 9th line of the proof are not isomorphic).

Let $\mathcal{S}$ denote the category of pairs of finite strict semisimplicial sets (see Definition 13.13) and semisimplicial maps. Let $\mathcal{H}$ be the homotopy category of finite CW pairs and let $R : \mathcal{S} \to \mathcal{H}$ be geometric realization. A map (f, g) in $\mathcal{S}$ is a *weak equivalence* if (Rf, Rg) is a weak equivalence in $\mathcal{H}$. Let $w^{-1}\mathcal{S}$ be the category obtained from $\mathcal{S}$ by inverting the weak equivalences.

Lemma 14.3. *R induces an equivalence of categories*

$$w^{-1}\mathcal{S} \to \mathcal{H}$$

Proof. Let $\mathcal{S}'$ be the category of pairs of finite semisimplicial sets and semisimplicial maps, with weak equivalences defined by geometric realization, and let $w^{-1}\mathcal{S}'$ be the category obtained by inverting the weak equivalences. Geometric realization induces an equivalence

$$w^{-1}\mathcal{S}' \to \mathcal{H}$$

by [BRS76, Theorem I.4.3 and Remark I.4.4]. Moreover, the map $w^{-1}\mathcal{S} \to w^{-1}\mathcal{S}'$ is an equivalence because every object of $\mathcal{S}'$ is weakly equivalent to an object of $\mathcal{S}$ (see [BRS76, Proof of Theorem I.4.1]; note that the second derived subdivision of a semisimplicial set is a strict semisimplicial set). □

Theorem 14.1 follows from the lemma and

Proposition 14.4. *There is a natural transformation*

$$\Xi : \mathbf{Q}^*(|A|, |B|) \to T^*(A, B)$$

of functors on $\mathcal{S}$ with the following properties:

(i) Ξ is a bijection when $A = *$ and B is empty.

(ii) The diagram

$$\begin{array}{ccc} Q^k(|B|) & \xrightarrow{\Xi} & T^k(B) \\ \downarrow & & \downarrow \\ Q^{k+1}(|A|, |B|) & \xrightarrow{\Xi} & T^{k+1}(A, B) \end{array}$$

commutes, where the vertical arrows are the connecting homomorphisms.

(iii) Ξ is a homomorphism.

The remainder of the section is devoted to the proof of Proposition 14.4. We begin with the construction of Ξ.

Recall that $T^*(A, B)$ is $\text{ad}^k(A, B)$ modulo the equivalence relation $\sim$ defined by $F \sim G$ if and only if there is an $H \in \text{ad}^k(A \times I, B \times I)$ which restricts to F and G on $A \times 0$ and $A \times 1$. (We remind the reader that we are using the same symbol for a strict semisimplicial set and the ball complex it determines; thus a symbol such as $A \times I$ denotes a product of ball complexes).

There is a similar description of $\mathbf{Q}^*(|A|, |B|)$. By Proposition 13.12 and [RS71, Remark 6.5], $\mathbf{Q}^k(|A|, |B|)$ is the set $[(A, B), (P_k, *)]$ of homotopy classes of semisimplicial maps. The homotopy relation for semisimplicial maps is defined at the beginning of Section 6 of [RS71]; it uses the "geometric product" $\otimes$ defined in [RS71, Section 3]. Using Remark 13.14 above, we see that $\mathbf{Q}^k(|A|, |B|)$ is $\text{ad}^k(A, B)$ modulo the equivalence relation $\sim'$ defined by: $F \sim' G$ if and only if there is an $H \in \text{ad}^k(A \otimes I, B \otimes I)$ which restricts to F on $A \otimes 0$ and to G on $A \otimes 1$.

We can now define Ξ: given an element $x \in \mathbf{Q}^k(|A|, |B|)$, choose an element F of $\text{ad}^k(A, B)$ which represents it and let $\Xi(x)$ be the class of F. To see that this is well-defined, note that $A \otimes I$ is a subdivision of $A \times I$, so by the gluing property of ad theories we see that $F \sim' G$ implies $F \sim G$.

Remark 14.5. The definition of Ξ was suggested by the argument on page 140 of [Ran92].

The definition of $\otimes$ shows that Ξ is the identity map when $A = *$ and B is empty.

Next we check that Ξ is natural. It is obviously natural for inclusions of pairs. If $(f, g) : (A, B) \to (A', B')$ is any semisimplicial map, let M_f and M_g be the mapping cylinders as defined on page 327 of [RS71]; these are strict semisimplicial sets and have the property that there is an inclusion

$$(i, j) : (A, B) \to (M_f, M_g),$$

an inclusion

$$(i', j') : (A', B') \to (M_f, M_g)$$

which is a weak equivalence, and a homotopy $|(i', j')| \circ |(f, g)| \simeq |(i, j)|$. Then $|(f, g)|^* : Q^*(|A'|, |B'|) \to Q^*(|A|, |B|)$ is equal to $(|(i', j')|^*)^{-1}|(i, j)|^*$, and similarly for $(f, g)^* : T^*(A', B') \to T^*(A, B)$. Hence $(f, g)^* \circ \Xi = \Xi \circ |(f, g)|^*$.

For the proof of part (ii) of Proposition 14.4 we need the Kan cone construction (because the Kan suspension of a strict semisimplicial set is not strict in general).

Definition 14.6. Let A be a semisimplicial set. Define a semisimplicial set CA as follows. The 0-simplices of CA are the 0-simplices of A together with a 0-simplex c. For $n \geq 1$ the n simplices of CA are $A_n \coprod A_{n-1}$. If the inclusions of A_n and A_{n-1} in $(CA)_n$ are denoted by f and g then the face maps $d_i : (CA)_n \to (CA)_{n-1}$ are defined by

$$d_i f(x) = f(d_i x)$$

for all i and

$$d_i g(x) = \begin{cases} c & \text{if } n = 1 \text{ and } i = 0 \\ g(d_i x) & \text{if } n > 1 \text{ and } i < n \\ f(x) & \text{if } i = n. \end{cases}$$

We leave it to the reader to check that $|CA| \cong C|A|$, where $C|A|$ denotes $I \wedge (|A|_+)$ (we choose the basepoint of I to be 1). Note that there is an inclusion $A \to CA$ and that the quotient $CA/(A \cup c)$ is $\Sigma(A_+)$ (where A_+ denotes A with a disjoint basepoint).

If A is strict then CA is also.

Proof of Proposition 14.4(ii). The unreduced suspension isomorphism

$$Q^k(|B|) \to Q^{k+1}(C|B|, |B| \cup |c|)$$

is defined as follows: given $f : |B| \to Q_k$ the composite

$$C|B| \xrightarrow{Cf} CQ_k \to \Sigma Q_k \to Q_{k+1}$$

takes $|B| \cup |c|$ to the basepoint, and therefore represents an element of $Q^{k+1}(C|B|, |B| \cup |c|)$.

There is an isomorphism of categories

$$\mu : \mathcal{C}ell(CB, B \cup c) \to \mathcal{C}ell(I \times B, \{0,1\} \times B)$$

defined as follows: a simplex σ of CB which is not in $B \cup c$ corresponds to a simplex σ' of B; let μ take σ (with its canonical orientation) to $I \times \sigma'$ (with $(-1)^{\dim \sigma'}$ times its canonical orientation).

There is a similar isomorphism

$$\nu : \mathcal{C}ell(CA, B \cup c) \to \mathcal{C}ell(I \times A, (1 \times A) \cup (0 \times B)).$$

Both μ and ν are incidence-compatible (Definition 3.7(i)) so part (f) of Definition 3.9 applies.

It is easy to check that the diagram

$$\begin{array}{ccccc}
Q^k(|B|) & & \xrightarrow{\quad\Xi\quad} & & T^k(B) \\
\downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\kappa^*} \\
Q^{k+1}(|CB|,|B\cup c|) & \xrightarrow{\Xi} & T^{k+1}(CB, B\cup c) & \xrightarrow{\mu^*} & T^{k+1}(I\times B, \{0,1\}\times B) \\
\uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\
Q^{k+1}(|CA|,|B\cup c|) & \xrightarrow{\Xi} & T^{k+1}(CA, B\cup c) & \xrightarrow{\nu^*} & T^{k+1}(I\times A, (1\times A)\cup(0\times B)) \\
\downarrow & & & & \downarrow \\
Q^{k+1}(|A|,|B|) & & \xrightarrow{\quad\Xi\quad} & & T^{k+1}(A,B)
\end{array}$$

commutes. The vertical composite on the right is by definition the negative of the connecting homomorphism, so it suffices to show the same for the vertical composite on the left.

The connecting homomorphism

$$Q^k(|B|) \to Q^{k+1}(|A|,|B|)$$

is defined to be the composite

$$\begin{aligned} Q^k(|B|) \xrightarrow{\cong} Q^{k+1}(C|B|, |B|\cup|c|) \cong \tilde{Q}^{k+1}(C|B|/(|B|\cup|c|)) \\ \to \tilde{Q}^{k+1}(|A|\cup C|B|) \xleftarrow{\cong} \tilde{Q}^{k+1}(|A|/|B|) \end{aligned}$$

where the third and fourth maps are induced by the evident quotient maps.

It now suffices to note that the diagram

$$\begin{array}{ccc}
C|B|/(|B|\cup c) & & \\
\uparrow & \searrow^{i} & \\
|A|\cup C|B| & & C|A|/(|B|\cup c) \\
\downarrow & \nearrow_{j} & \\
|A|/|B| & &
\end{array}$$

homotopy commutes, where i takes $t\wedge b$ to the class of $(1-t)\wedge b$ and j takes the class of a to the class of $0\wedge a$. (The homotopy is given by $h(a,s)=s\wedge a$ for $a\in|A|$ and $h(t\wedge b)=s(1-t)\wedge b$.) The negative sign mentioned above comes from the $1-t$ in the definition of i. □

It remains to prove part (iii) of Proposition 14.4.

First recall that for any cohomology theory E the addition in $E^k(|A|,|B|)$ is the composite

$$\begin{aligned} E^k(|A|,|B|)\times E^k(|A|,|B|) = \tilde{E}^k(|A|/|B|)\times\tilde{E}^k(|A|/|B|) \\ \cong \tilde{E}^{k+1}(\Sigma(|A|/|B|))\times\tilde{E}^{k+1}(\Sigma(|A|/|B|)) \xleftarrow{\cong} \tilde{E}^{k+1}(\Sigma(|A|/|B|)\vee\Sigma(|A|/|B|)) \\ \xrightarrow{p^*} \tilde{E}^{k+1}(\Sigma(|A|/|B|)) \cong \tilde{E}^k(|A|/|B|) = E^k(|A|,|B|) \end{aligned}$$

where p is the pinch map.

It therefore suffices to observe that, by part (ii) and naturality, the diagram

$$\begin{array}{ccc} \tilde{Q}^k(|A|/|B|) & \xrightarrow{\Xi} & \tilde{T}^k(|A|/|B|) \\ \cong\downarrow & & \downarrow\cong \\ \tilde{Q}^{k+1}(\Sigma(|A|/|B|)) & \xrightarrow{\Xi} & \tilde{T}^{k+1}(\Sigma(|A|/|B|)) \end{array}$$

commutes, where the vertical arrows are the suspension isomorphisms of the reduced cohomology theories $\tilde{Q}^*$ and $\tilde{T}^*$.

15. The symmetric spectrum associated to an ad theory

Symmetric spectra were originally defined simplicially ([HSS00, Definition 1.2.1]). The topological definition is the obvious analog ([MMSS01, Example 4.2]):

Definition 15.1. A symmetric spectrum $\mathbf{X}$ consists of
(i) a sequence $X_0, X_1, \ldots$ of pointed topological spaces,
(ii) a pointed map $s : S^1 \wedge X_k \to X_{k+1}$ for each $k \geq 0$, and
(iii) a based left Σ_k-action on X_k,
such that the composition

$$S^p \wedge X_k \xrightarrow{S^{p-1}\wedge s} S^{p-1} \wedge X_{k-1} \to \cdots \to X_{k+p}$$

is $\Sigma_p \times \Sigma_k$-equivariant for each $p \geq 1$ and $k \geq 0$.

Our first goal in this section is to define a symmetric spectrum associated to an ad theory. In order to have a suitable Σ_k action we will construct the k-th space of the spectrum as the geometric realization of a k-fold multisemisimplicial set; the Σ_k action will come from permutation of the semisimplicial directions.

By a k-fold multisemisimplicial set we mean a functor from Δ^k_{inj} to sets (see Definition 13.1). Given a multiindex $\mathbf{n} = (n_1, \ldots, n_k)$, let $\Delta^{\mathbf{n}}$ denote the product

$$\Delta^{n_1} \times \cdots \times \Delta^{n_k}.$$

The geometric realization of a k-fold multisemisimplicial set A is

$$|A| = \big(\coprod \Delta^{\mathbf{n}} \times A_{\mathbf{n}}\big)/\sim,$$

where $\sim$ denotes the evident equivalence relation.

Now fix an ad theory.

Definition 15.2. For each $k \geq 1$, define a k-fold multisemisimplicial set R_k by

$$(R_k)_{\mathbf{n}} = \mathrm{ad}^k(\Delta^{\mathbf{n}}).$$

Let M_k be the geometric realization of R_k. For $k = 0$, let R_0 be the set of $*$-ads of degree 0 and let M_0 be R_0 with the discrete topology.

Our next definition gives the left action of Σ_k on M_k. An element of M_k has the form $[u, F]$, where $u = (u_1, \ldots, u_k) \in \Delta^{\mathbf{n}}$, $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$, and [ ] denotes equivalence class. Given $\eta \in \Sigma_k$ let $\epsilon(\eta)$ denote 0 if η is even and 1 if η is odd.

Definition 15.3. Define

$$\eta([u, F]) = [(u_{\eta^{-1}(1)}, \ldots, u_{\eta^{-1}(k)}), i^{\epsilon(\eta)} \circ F \circ \eta_{\#}].$$

Here i is the involution in the target category of the ad theory and $\eta_{\#}$ is the map

$$\mathcal{C}ell(\Delta^{n_{\eta^{-1}(1)}} \times \cdots \times \Delta^{n_{\eta^{-1}(k)}}) \to \mathcal{C}ell(\Delta^{n_1} \times \cdots \times \Delta^{n_k})$$

which takes

$$(\sigma_{\eta^{-1}(1)} \times \cdots \times \sigma_{\eta^{-1}(k)}, o_{\eta^{-1}(1)} \times \cdots \times o_{\eta^{-1}(k)})$$

to

$$(\sigma_1 \times \cdots \times \sigma_k, o_1 \times \cdots \times o_k).$$

It remains to define the suspension maps.

Definition 15.4. (i) For each ball complex K let

$$\lambda : \mathcal{C}ell(\Delta^1 \times K, \partial\Delta^1 \times K) \to \mathcal{C}ell(K)$$

be the incidence-compatible isomorphism of categories which takes $\Delta^1 \times (\sigma, o)$ (where Δ^1 is given its standard orientation) to (σ, o).

(ii) Given $t \in [0,1]$ let $\bar{t}$ denote the point $(1-t, t)$ of Δ^1.

(iii) Given $k \geq 1$ let

$$s : S^1 \wedge M_k \to M_{k+1}$$

be the map which takes $[t, [u, F]]$ to $[(\bar{t}, u), \lambda^*(F)]$.

Proposition 15.5. *The sequence $M_0, M_1, \ldots$, with the Σ_k-actions given by Definition 15.3 and the suspension maps given by Definition 15.4(iii), is a symmetric spectrum.* □

We will denote this symmetric spectrum by $\mathbf{M}$.

Example 15.6. Let us write $\mathbf{M}_{\pi,Z,w}$ (resp., $\mathbf{M}^R$) for the symmetric spectrum associated to $\mathrm{ad}_{\pi,Z,w}$ (resp., ad^R). The morphism of ad theories

$$\mathrm{Sig} : \mathrm{ad}_{\pi,Z,w} \to \mathrm{ad}_{\mathbb{Z}[\pi]^w}$$

(see Theorem 8.13) induces a map

$$\mathbf{M}_{\pi,Z,w} \to \mathbf{M}^{\mathbb{Z}[\pi]^w}.$$

In the remainder of this section we show that $\mathbf{M}$ is weakly equivalent (in an appropriate sense) to the spectrum $\mathbf{Q}$ defined in Section 13.

For $k \geq 1$, let Q'_k be the realization of the semisimplicial set with n-simplexes

$$(R_k)_{(0,\ldots,0,n)}$$

Then Q'_k is homeomorphic to Q_k, and there is an obvious map $Q'_k \to M_k$, so we get a map

$$Q_k \to M_k$$

for $k \geq 1$.

Proposition 15.7. *The map $Q_k \to M_k$ is a weak equivalence.*

Proposition 15.8. *The diagram*

$$\begin{array}{ccc} \Sigma Q_k & \longrightarrow & \Sigma M_k \\ \downarrow & & \downarrow \\ Q_{k+1} & \longrightarrow & M_{k+1} \end{array}$$

commutes up to homotopy.

Before proving these we deduce some consequences. As in [MMSS01], let the forgetful functor from symmetric spectra to ordinary spectra (which are called prespectra in [MMSS01]) be denoted by $\mathbb{U}$. It is shown in [MMSS01] that the right derived functor $R\mathbb{U}$ is an equivalence of homotopy categories.

Corollary 15.9. (i) $\mathbf{M}$ *is a positive* Ω *spectrum (that is, the map* $M_k \to \Omega M_{k+1}$ *is a weak equivalence for* $k \geq 1$*).*

(ii) $R\mathbb{U}$ *takes* $\mathbf{M}$ *to* $\mathbf{Q}$.

(iii) *The homotopy groups of* $\mathbf{M}$ *are the bordism groups of the ad theory.*

Proof. Part (i) is immediate from the proposition.

For part (ii), first recall that $(R\mathbb{U})\mathbf{M}$ is defined to be $\mathbb{U}$ of a fibrant replacement of $\mathbf{M}$. But by [Sch08, Example 4.2] $\mathbf{M}$ is semistable, which means that the map from $\mathbf{M}$ to its fibrant replacement is a π_*-isomorphism. It follows that $(R\mathbb{U})\mathbf{M}$ is (up to weak equivalence) $\mathbb{U}\mathbf{M}$, and it therefore suffices to show that $\mathbf{Q}$ is weakly equivalent to $\mathbb{U}\mathbf{M}$. Define a spectrum $\mathbf{X}$ as follows: X_0 is $*$, X_1 is Q_1, and and for $k \geq 2$ X_k is the iterated mapping cylinder of the sequence of maps

$$\Sigma^{k-1}Q_1 \xrightarrow{\Sigma^{k-2}s} \Sigma^{k-2}Q_2 \xrightarrow{\Sigma^{k-3}s} \cdots \Sigma Q_{k-1} \xrightarrow{s} Q_k$$

The maps $\Sigma X_k \to X_{k+1}$ are defined to be the obvious inclusion maps. Then there are evident weak equivalences $\mathbf{X} \to \mathbf{Q}$ and (using Propositions 15.7 and 15.8) $\mathbf{X} \to \mathbf{M}$, which proves part (ii).

Part (iii) is immediate from part (ii). $\square$

For the proof of Proposition 15.7 we will use an idea adumbrated on page 695 of [WW00].

First we interpolate between Q_k and M_k. For $1 \leq m \leq k$ let R_k^m be the m-fold multisemisimplicial set defined by

$$(R_k^m)_{\mathbf{n}} = (R_k)_{0,\ldots,0,\mathbf{n}}.$$

We have $|R_k^1| = Q_k$ and $|R_k^k| = M_k$, so it suffices to show that the inclusion $|R_k^{m-1}| \to |R_k^m|$ is a weak equivalence for each $m \geq 2$. We will prove this for each k by induction on m, so we assume

(*) $$|R_k^{m'-1}| \to |R_k^{m'}| \text{ is a weak equivalence if } m' < m.$$

Next we observe that the realization $|R_k^m|$ can be obtained by first realizing in the last $m-1$ semisimplicial directions and then realizing in the remaining direction. Namely, for each $p \geq 0$ let $R_k^m[p]$ be the $(m-1)$-fold semisimplicial set with

$$(R_k^m[p])_{\mathbf{n}} = (R_k)_{0,\ldots,0,p,\mathbf{n}}.$$

As p varies we obtain a semisimplicial space $|R_k^m[\bullet]|$ whose realization is $|R_k^m|$. Now R_k^{m-1} is $R_k^m[0]$, and the inclusion $|R_k^{m-1}| \to |R_k^m|$ is the inclusion of the space of 0-simplices $|R_k^m[0]|$ in $|R_k^m|$. It therefore suffices to show that the latter map is a weak equivalence, and this is part (v) of:

Lemma 15.10. (i) *In the semisimplicial space* $|R_k^m[\bullet]|$*, all face maps are homotopy equivalences.*

(ii) *For each* p*, all of the face maps from* $|R_k^m[p]|$ *to* $|R_k^m[p-1]|$ *are homotopic.*

(iii) *The map* $|R_k^m[0]| \to |R_k^m|$ *is a homology isomorphism.*

(iv) *The map* $|R_k^m[0]| \to |R_k^m|$ *is (up to weak equivalence) an* H*-map between grouplike* H*-spaces.*

(v) *The map* $|R_k^m[0]| \to |R_k^m|$ *is a weak equivalence.*

For the proof of the lemma we need an auxiliary construction. Let ad denote the ad theory we have fixed and let $\mathcal{A}$ be its target category. Given a ball complex L we can define a new $\mathbb{Z}$-graded category $\mathcal{A}[L]$ by letting the set of objects in dimension

n be $\mathrm{pre}^{-n}(L)$. Now we define an ad theory $\mathrm{ad}[L]$ with values in $\mathcal{A}[L]$ by letting $\mathrm{ad}[L]^j(K)$ consist of the pre-K-ads which correspond to $(K \times L)$-ads under the bijection

$$\mathrm{pre}[L]^j(K) \cong \mathrm{pre}^j(K \times L).$$

Let us write $\mathbf{Q}[L]$ and $R[L]^m_k$ for the spectrum and the multisemisimplicial sets constructed from the theory $\mathrm{ad}[L]$.

Proof of Lemma 15.10. Part (i). First note that $R^m_k[p]$ is the same thing as $R[\Delta^p]^{m-1}_k$. By the inductive hypothesis (*) we know that $|R[\Delta^p]^{m-1}_k|$ is weakly equivalent to $Q[\Delta^p]_k$. The homotopy groups of $Q[\Delta^p]_k$ are (up to a shift in dimension) the bordism groups of the ad theory $\mathrm{ad}[\Delta^p]$, and inspection of the definitions shows that these are the groups $T^{-*}(\Delta^p)$. This implies that all face maps in $R^m_k[\bullet]$ are weak equivalences, and hence homotopy equivalences since all spaces are CW complexes.

Part (ii) follows from part (i) and the semisimplicial identities.

Part (iii) follows from part (ii) and the homology spectral sequence of a semisimplicial space (cf. [May72, Theorem 11.4]), but note that our situation is simpler because there are no degeneracy maps).

Part (iv): Let $Q[\Delta^\bullet]_k$ denote the semisimplicial space which is $Q[\Delta^p]_k$ in degree p. By the inductive hypothesis (*), it suffices to show that the map

$$Q_k = Q[\Delta^0]_k \to |Q[\Delta^\bullet]_k|$$

is (up to weak equivalence) an H-map between grouplike H-spaces, and this is a consequence of the following commutative diagram (where $\hat{Q}_{k+1}$ denotes the basepoint component of Q_{k+1}; note that $\Omega\hat{Q}_{k+1}$ is the same thing as ΩQ_{k+1}):

$$\begin{array}{ccc}
Q_k & \longrightarrow & |Q[\Delta^\bullet]_k| \\
\simeq \downarrow & & \simeq \downarrow \alpha \\
\Omega\hat{Q}_{k+1} & \longrightarrow & |\Omega\hat{Q}[\Delta^\bullet]_{k+1}| \\
& \searrow & \simeq \downarrow \beta \\
& & \Omega|\hat{Q}[\Delta^\bullet]_{k+1}|
\end{array}$$

Here α is a weak equivalence by [May74, Theorem A.4(ii)], and β is a weak equivalence by [May72, Theorem 12.3] (this is where we need to use basepoint-components).

Part (v) now follows from parts (iii) and (iv) and [Whi78, Corollary IV.3.6 and Corollary IV.7.9]. $\square$

We now turn to the proof of Proposition 15.8. For simplicity we will do the case $k = 2$; the general case is exactly the same but the notation is a little more complicated.

First let us give an explicit description of the maps in the diagram.

Given an element $F \in \mathrm{ad}^2(\Delta^n)$, let us write F' for the corresponding element of $\mathrm{ad}^2(\Delta^0 \times \Delta^n)$. Then the map $Q_2 \to M_2$ takes $[u, F]$ to $[(1, u), F']$ (where 1 denotes the unique element of Δ^0).

Hence the clockwise composite in the diagram of 15.8 takes an element $[t, [u, F]]$ of ΣQ_2 to $[(\bar{t}, 1, u), \lambda^*(F')]$ (see Definition 15.4).

To describe the counterclockwise composite we need some notation. Recall that the homeomorphism in Lemma 13.7 takes $[t,[u,x]]$ to $[\langle t,u\rangle,x]$, where $\langle t,u\rangle = ((1-t)u,t)$. Also, recall the isomorphism

$$\theta : \mathcal{C}ell(\Delta^{n+1}, \partial_{n+1}\Delta^{n+1} \cup \{n+1\}) \to \mathcal{C}ell(\Delta^n)$$

defined after Lemma 13.7.

The map $\Sigma Q_2 \to Q_3$ takes $[t,[u,F]]$ to $[\langle t,u\rangle, \theta^* F]$, and thus the counterclockwise composite in the diagram of 15.8 takes $[t,[u,F]]$ to $[(1,1,\langle t,u\rangle,((\theta^* F)')']$.

Now we need a lemma:

Lemma 15.11. *For every $n \geq 0$ there is an incidence-compatible isomorphism*

$$\mu_n : \mathcal{C}ell(\Delta^1 \times \Delta^0 \times \Delta^{n+1}, (\{1\} \times \Delta^0 \times \Delta^{n+1}) \cup (\Delta^1 \times \Delta^0 \times \{n+1\}) \cup (\{0\} \times \Delta^0 \times \partial_{n+1}\Delta^{n+1})) \to \mathcal{C}ell(\Delta^n \times I)$$

(which lowers degrees by 1) such that

(a) *μ_n takes the cell $\Delta^1 \times \Delta^0 \times \Delta^{n+1}$ (with its standard orientation) to the cell $\Delta^n \times I$ (with its standard orientation).*

(b) *μ_n restricts to a morphism*

$$\mathcal{C}ell(\{0\} \times \Delta^0 \times \Delta^{n+1}, \{0\} \times \Delta^0 \times (\partial_{n+1}\Delta^{n+1} \cup \{n+1\})) \to \mathcal{C}ell(\Delta^n \times \{0\})$$

which agrees with θ.

(c) *μ_n restricts to a morphism*

$$\mathcal{C}ell(\Delta^1 \times \Delta^0 \times \partial_{n+1}\Delta^{n+1}, \partial\Delta^1 \times \Delta^0 \times \partial_{n+1}\Delta^{n+1}) \to \mathcal{C}ell(\Delta^n \times \{1\})$$

which agrees with λ.

(d) *for $0 \leq i \leq n$, μ_n restricts to a morphism*

$$\mathcal{C}ell(\Delta^1 \times \Delta^0 \times \partial_i\Delta^{n+1}, (\{1\} \times \Delta^0 \times \partial_i\Delta^{n+1}) \cup (\Delta^1 \times \Delta^0 \times \{n+1\}) \cup (\{0\} \times \Delta^0 \times \partial_i\partial_{n+1}\Delta^{n+1}))) \to \mathcal{C}ell(\partial_i\Delta^n \times I)$$

which agrees with $i \circ \mu_{n-1}$.

The proof is an easy induction.

Now we can write down the homotopy

$$H : (\Sigma Q_2) \times I \to M_3$$

needed for the case $k=2$ of 15.8:

$$H([t,[u,F]],s) = \begin{cases} [(\bar{t},1,\langle 2ts,u\rangle), \mu_n^*(J(F))] & \text{if } 0 \leq s \leq 1/2, \\ [\overline{(2-2s)t},1,\langle t,u\rangle), \mu_n^*(J(F))] & \text{if } 1/2 \leq s \leq 1, \end{cases}$$

where J is the cylinder (see Definition 3.9(h)). It's easy to check that this is well-defined and that it is equal to the clockwise composite in the diagram of 15.8 when $s=0$ and to the counterclockwise composite when $s=1$.

16. Multiplicative ad theories

Definition 16.1. Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category. A *strict monoidal structure* on $\mathcal{A}$ is a strict monoidal structure $(\boxtimes, \varepsilon)$ (see [ML98, Section VII.1]) on the underlying category such that

(a) the monoidal product $\boxtimes$ adds dimensions and the dimension of the unit element ε is 0,

(b) $i(x \boxtimes y) = (ix) \boxtimes y = x \boxtimes (iy)$ for all objects x and y, and similarly for morphisms,

(c) $x \boxtimes \emptyset_n = \emptyset_n \boxtimes x = \emptyset_{n+\dim x}$ for all n and all objects x, and if $f : x \to y$ is any morphism then $f \boxtimes \emptyset_n$ and $\emptyset_n \boxtimes f$ are each equal to the canonical map $\emptyset_{n+\dim x} \to \emptyset_{n+\dim y}$.

An example is the category $\mathcal{A}_C$ of Example 3.11 when C is a DGA. Another example is the category $\mathcal{A}_{\mathrm{STop}}$, if we redefine the phrase "topological manifold" to mean a topological manifold which is a subset of a Euclidean space.

Assumption 16.2. From now on we will assume that Cartesian products in Section 7 and tensor products in Section 8 are strictly associative (that is, we assume that the monoidal categories Set and Ab have been replaced in those sections by equivalent strict monoidal categories; see [Kas95, Section XI.5]).

With this assumption, the category $\mathcal{A}_{e,*,1}$ defined in Section 7 (where e denotes the trivial group) and, when R is commutative, the category $\mathcal{A}^R$ defined in Section 8 are strict monoidal $\mathbb{Z}$-graded categories.

Remark 16.3. If $\mathcal{A}$ is a $\mathbb{Z}$-graded category with a strict monoidal structure, there is a natural map

$$\boxtimes : \mathrm{pre}^k(K) \times \mathrm{pre}^l(L) \to \mathrm{pre}^{k+l}(K \times L)$$

defined by

$$(F \boxtimes G)(\sigma \times \tau, o_1 \times o_2) = i^{l \dim(\sigma)} F(\sigma, o_1) \boxtimes G(\tau, o_2);$$

this is well-defined, because

$$F(\sigma, -o_1) \boxtimes G(\tau, -o_2) = iF(\sigma, o_1) \boxtimes iG(\tau, o_2) = F(\sigma, o_1) \boxtimes G(\tau, o_2).$$

Definition 16.4. A *multiplicative ad theory* is an ad theory together with a strict monoidal structure on the target category $\mathcal{A}$, such that

(a) the pre $*$-ad with value ε is an ad, and

(b) the map in Remark 16.3 restricts to a map

$$\boxtimes : \mathrm{ad}^k(K) \times \mathrm{ad}^l(L) \to \mathrm{ad}^{k+l}(K \times L).$$

Examples are ad_C when C is a DGA, $\mathrm{ad}_{\mathrm{STop}}$, $\mathrm{ad}_{e,*,1}$, and ad^R when R is commutative; we will put the last two examples in a more general context in the next section.

Theorem 16.5. *The symmetric spectrum* $\mathbf{M}$ *determined by a multiplicative ad theory is a symmetric ring spectrum.*

Remark 16.6. Note that a symmetric ring spectrum satisfies *strict* associativity, not just associativity up to homotopy.

For the proof of Theorem 16.5 we need a lemma. Recall Definitions 15.3 and 15.4.

Lemma 16.7. *Let* $\mathbf{n} = (n_1, \ldots, n_k)$ *and let* $m \geq 0$. *Let* $F \in \mathrm{ad}^k(\Delta^{\mathbf{n}})$ *and let* E *be the* $*$*-ad with value* ε. *Then*

(i) $((\lambda^*)^m E) \boxtimes F = (\lambda^*)^m F$, *and*

(ii) $F \boxtimes ((\lambda^*)^m E) = i^{km} \circ (((\lambda^*)^m E) \boxtimes F) \circ \eta_\#$, *where* $\eta \in \Sigma_{k+m}$ *is the permutation that moves the first* k *elements to the end.*

Proof. Let (σ, o) be the 1-cell of Δ^1 with its standard orientation and let (τ, o') be an oriented cell of $\mathbf{\Delta^n}$ of dimension l. For part (i), we have

$$\begin{aligned}(((\lambda^*)^m E) \boxtimes F)((\sigma, o)^{\times m} \times (\tau, o')) &= i^{km}(((\lambda^*)^m E)((\sigma, o)^{\times m}) \boxtimes F(\tau, o')) \\ &= i^{km}(\varepsilon \boxtimes F(\tau, o')) \\ &= i^{km} F(\tau, o')\end{aligned}$$

and (using Definition 3.7(ii))

$$\begin{aligned}((\lambda^*)^m F)((\sigma, o)^{\times m} \times (\tau, o')) &= (i^{km} \circ F \circ \lambda^m)((\sigma, o)^{\times m} \times (\tau, o')) \\ &= i^{km} F(\tau, o').\end{aligned}$$

For part (ii) we have

$$\begin{aligned}(F \boxtimes ((\lambda^*)^m E))((\tau, o') \times (\sigma, o)^{\times m}) &= i^{lm}(F(\tau, o') \boxtimes ((\lambda^*)^m E)((\sigma, o)^{\times m})) \\ &= i^{lm}(F(\tau, o') \boxtimes \varepsilon) \\ &= i^{lm} F(\tau, o')\end{aligned}$$

and

$$\begin{aligned}&(i^{km} \circ (((\lambda^*)^m E) \boxtimes F) \circ \eta_{\#})((\tau, o') \times (\sigma, o)^{\times m}) \\ &\qquad = i^{km+lm}((((\lambda^*)^m E) \boxtimes F)((\sigma, o)^{\times m} \times (\tau, o'))) \\ &\qquad = i^{lm}(((\lambda^*)^m E)((\sigma, o)^{\times m}) \boxtimes F(\tau, o')) \\ &\qquad = i^{lm}(\varepsilon \boxtimes F(\tau, o')) \\ &\qquad = i^{lm} F(\tau, o').\end{aligned}$$

□

Proof of Theorem 16.5. Recall ([HSS00, Definition 2.2.3]) that the smash product $\mathbf{M} \wedge \mathbf{M}$ is defined to be the coequalizer of

$$\mathbf{M} \otimes \mathbf{S} \otimes \mathbf{M} \underset{r \otimes 1}{\overset{1 \otimes s}{\rightrightarrows}} \mathbf{M} \otimes \mathbf{M}.$$

Here $\otimes$ is the tensor product of the underlying symmetric sequences ([HSS00, Definition 2.1.3]), $\mathbf{S}$ is the symmetric sphere spectrum ([HSS00, Example 1.2.4]), $s : \mathbf{S} \otimes \mathbf{M} \to \mathbf{M}$ is induced by the symmetric spectrum structure of $\mathbf{M}$ ([HSS00, proof of Proposition 2.2.1]), and r is the composite

$$\mathbf{M} \otimes \mathbf{S} \xrightarrow{t} \mathbf{S} \otimes \mathbf{M} \xrightarrow{s} \mathbf{M},$$

where t is the twist isomorphism ([HSS00, page 160]).

The $\boxtimes$ operation of Definition 16.4(ii) gives an associative multiplication

$$m : \mathbf{M} \otimes \mathbf{M} \to \mathbf{M}$$

and we need to show that this induces a map $\mathbf{M} \wedge \mathbf{M} \to \mathbf{M}$. Let

$$\iota : \mathbf{S} \to \mathbf{M}$$

be the map of symmetric spectra that takes the nontrivial element of S^0 to the $*$-ad with value ε. Lemma 16.7 shows that the diagrams

(16.1)
$$\begin{array}{ccc} \mathbf{S}\otimes\mathbf{M} & \xrightarrow{\iota\otimes 1} & \mathbf{M}\otimes\mathbf{M} \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle m} & \\ & \mathbf{M} & \end{array}$$

and

(16.2)
$$\begin{array}{ccccc} \mathbf{M}\otimes\mathbf{S} & \xrightarrow{1\otimes\iota} & \mathbf{M}\otimes\mathbf{M} & & \\ & & & \searrow{\scriptstyle m} & \\ {\scriptstyle t}\downarrow & & & & \mathbf{M} \\ & & & \nearrow{\scriptstyle m} & \\ \mathbf{S}\otimes\mathbf{M} & \xrightarrow{\iota\otimes 1} & \mathbf{M}\otimes\mathbf{M} & & \end{array}$$

commute. These in turn imply that the diagram

$$\begin{array}{ccccc} \mathbf{M}\otimes\mathbf{S}\otimes\mathbf{M} & \xrightarrow{1\otimes s} & \mathbf{M}\otimes\mathbf{M} & & \\ & & & \searrow{\scriptstyle m} & \\ {\scriptstyle t\otimes 1}\downarrow & & & & \mathbf{M} \\ & & & \nearrow{\scriptstyle m} & \\ \mathbf{S}\otimes\mathbf{M}\otimes\mathbf{M} & \xrightarrow{s\otimes 1} & \mathbf{M}\otimes\mathbf{M} & & \end{array}$$

commutes, and hence m induces an associative multiplication

$$\mathbf{M}\wedge\mathbf{M}\to\mathbf{M}.$$

Moreover, diagrams (16.1) and (16.2) imply that the unit diagrams

$$\begin{array}{ccc} \mathbf{S}\wedge\mathbf{M} & \xrightarrow{\iota\wedge 1} & \mathbf{M}\wedge\mathbf{M} \\ & {\scriptstyle \cong}\searrow \quad \swarrow{\scriptstyle m} & \\ & \mathbf{M} & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{M}\wedge\mathbf{S} & \xrightarrow{1\wedge\iota} & \mathbf{M}\wedge\mathbf{M} \\ & {\scriptstyle \cong}\searrow \quad \swarrow{\scriptstyle m} & \\ & \mathbf{M} & \end{array}$$

commute. Thus $\mathbf{M}$ is a symmetric ring spectrum. □

17. Geometric and symmetric Poincaré bordism are monoidal functors

With the notation of Example 15.6, there are product maps

$$\mathbf{M}_{\pi,Z,w}\wedge\mathbf{M}_{\pi',Z',w'}\to\mathbf{M}_{\pi\times\pi',Z\times Z',w\times w'}$$

and

$$\mathbf{M}^R\wedge\mathbf{M}^S\to\mathbf{M}^{R\otimes S}$$

induced by the operations $\times$ and $\otimes$ of Lemmas 7.11 and 8.14. There is also a unit map

$$\mathbf{S} \to \mathbf{M}_{e,*,1}$$

defined as follows. The one-point space gives a $*$-ad of degree 0, and this induces a map of spaces from S^0 to the 0-th space of $\mathbf{M}_{e,*,1}$; the unique extension of this to a map of symmetric spectra is the desired unit map. Similarly there is a unit map

$$\mathbf{S} \to \mathbf{M}^{\mathbb{Z}}$$

determined by the $*$-ad $(\mathbb{Z}, \mathbb{Z}, \varphi)$, where φ is the identity map.

Assumption 16.2 implies that the categories $\mathcal{T}$ and $\mathcal{R}$ introduced in Section 11 are strict monoidal categories.

Definition 17.1. Let $\mathbf{M}_{\text{geom}}$ be the functor from $\mathcal{T}$ to the category of symmetric spectra which takes (π, Z, w) to $\mathbf{M}_{\pi,Z,w}$. Let $\mathbf{M}_{\text{sym}}$ be the functor from $\mathcal{R}$ to the category of symmetric spectra which takes R to $\mathbf{M}^R$.

Theorem 17.2. *$\mathbf{M}_{\text{geom}}$ and $\mathbf{M}_{\text{sym}}$ are monoidal functors. That is, the following diagrams involving $\mathbf{M}_{\text{sym}}$ strictly commute, and similarly for $\mathbf{M}_{\text{geom}}$.*

$$\begin{array}{ccccc}
(\mathbf{M}^R \wedge \mathbf{M}^S) \wedge \mathbf{M}^T & & \xrightarrow{\cong} & & \mathbf{M}^R \wedge (\mathbf{M}^S \wedge \mathbf{M}^T) \\
\downarrow \otimes & & & & \downarrow \otimes \\
\mathbf{M}^{R\otimes S} \wedge \mathbf{M}^T & & & & \mathbf{M}^R \wedge \mathbf{M}^{S\otimes T} \\
& \searrow \otimes & & \swarrow \otimes & \\
& & \mathbf{M}^{R\otimes S\otimes T} & &
\end{array}$$

$$\begin{array}{ccc}
\mathbf{S} \wedge \mathbf{M}^R & \xrightarrow{\cong} & \mathbf{M}^R \\
\downarrow & & \uparrow = \\
\mathbf{M}^{\mathbb{Z}} \wedge \mathbf{M}^R & \xrightarrow{\otimes} & \mathbf{M}^{\mathbb{Z}\otimes R}
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{M}^R \wedge \mathbf{S} & \xrightarrow{\cong} & \mathbf{M}^R \\
\downarrow & & \uparrow = \\
\mathbf{M}^R \wedge \mathbf{M}^{\mathbb{Z}} & \xrightarrow{\otimes} & \mathbf{M}^{R\otimes \mathbb{Z}}
\end{array}$$

The proof is similar to that of Theorem 16.5.

Remark 17.3. One can define a "module functor" over a monoidal functor in the evident way. Then the functor $\mathbf{M}_{\text{quad}}$ from $\mathcal{R}$ to the category of symmetric spectra which takes R to $\mathbf{M}_R$ is a module functor over $\mathbf{M}_{\text{sym}}$.

Remark 17.4. The map

$$\text{Sig} : \mathbf{M}_{\pi,Z,w} \to \mathbf{M}^{\mathbb{Z}[\pi]^w}$$

in Example 15.6 is not a monoidal natural transformation, because the functor which takes a space to its singular chain complex does not take Cartesian products to tensor products. We will return to this point in the sequel.

Appendix A. Ball complex structures on PL manifolds and homology manifolds

We begin with an elementary fact.

Proposition A.1. *Let X be a compact oriented homology manifold of dimension n with a regular CW complex structure such that ∂X is a subcomplex.*

(i) *Let S be the set of n-dimensional cells, with their induced orientations. Then the cellular chain $\sum_{\sigma \in S} \sigma$ represents the fundamental class $[X] \in H_n(X, \partial X)$.*

(ii) *Let T be the set of $(n-1)$-dimensional cells of ∂X, with their induced orientations. Then*

$$\partial \left(\sum_{\sigma \in S} \sigma \right) = \sum_{\tau \in T} \tau$$

Proof. Part (i) follows from the fact that if $\sigma \in S$ and x is in the interior of σ then the image of $[X]$ in $H_n(X, X - \{x\})$ is represented by σ. Applying part (i) to ∂X gives part (ii). □

Next we recall from [McC75] that the concept of barycentric subdivision generalizes from simplicial complexes to ball complexes.

Let K be a ball complex. For each cell σ of K, choose a point $\hat{\sigma}$ in the interior of σ and a PL isomorphism from σ to the cone $C(\partial\sigma)$ which takes $\hat{\sigma}$ to the cone point. With this data, K is a "structured cone complex" ([McC75, page 274]). By [McC75, Proposition 2.1], K has a subdivision $\hat{K}$ which is a simplicial complex with vertices $\hat{\sigma}$. A set of vertices in $\hat{K}$ spans a simplex in $\hat{K}$ if and only if it has the form

$$\{\hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$$

with $\sigma_1 \subset \cdots \subset \sigma_k$.

Recall that if S is a simplicial complex and v is a vertex of S then the *closed star* $\mathrm{st}(v)$ is the subcomplex consisting of all simplices which contain v together with all of their faces. The *link* $\mathrm{lk}(v)$ is the subcomplex of $\mathrm{st}(v)$ consisting of simplices that do not contain v. The realization $|\mathrm{st}(v)|$ is the cone $C(|\mathrm{lk}(v)|)$.

Proposition A.2. *Let K be a ball complex and let σ be a cell of K.*

(i) *For each cell τ with $\tau \supsetneq \sigma$ the subspace $|\mathrm{lk}(\hat{\sigma})| \cap \tau$ is a PL ball, and these subspaces, together with the cells of K contained in $\partial\sigma$, are a ball complex structure on $|\mathrm{lk}(\hat{\sigma})|$.*

(ii) *For each cell τ with $\tau \supset \sigma$ the subspace $|\mathrm{st}(\hat{\sigma})| \cap \tau$ is a PL ball, and these subspaces, together with the cells of $|\mathrm{lk}(\hat{\sigma})|$, are a ball complex structure on $|\mathrm{st}(\hat{\sigma})|$.*

Proof. Let $\tau \supsetneq \sigma$. $|\tau|$ inherits a ball complex structure from K, and $|\mathrm{lk}(\hat{\sigma})| \cap \tau$ (resp., $|\mathrm{st}(\hat{\sigma})| \cap \tau$) is the realization of the link (resp., star) of $\hat{\sigma}$ with respect to this structure. It is a PL ball because $|\tau|$ is a PL manifold and $\hat{\sigma}$ is a point of its boundary. □

Next we recall from [McC75] that the concept of dual cell generalizes from simplicial complexes to ball complexes. For each cell σ of K, let $D(\sigma)$ (resp., $\dot{D}(\sigma)$) be the subcomplex of $\hat{K}$ consisting of simplices $\{\hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$ with $\sigma \subset \sigma_i$ (resp., $\sigma \subsetneq \sigma_i$) for all i.

Two simplices s, s' of a simplicial complex S are *joinable* if their vertex sets are disjoint and the union of their vertices spans a simplex of S; this simplex is called the *join*, denoted $s * s'$. Two subcomplexes A and B of S are joinable if each pair

$s \in A$, $s' \in B$ is joinable, and the join $A * B$ is the subcomplex consisting of the simplices $s * s'$ and all of their faces.

Lemma A.3. *If K is a ball complex and σ is a cell of K then*

$$\mathrm{lk}(\hat{\sigma}, \hat{K}) = \partial\sigma * \dot{D}(\sigma).$$

The proof is immediate from the definitions.

Proposition A.4. *Let (L, L_0) be a ball complex pair such that $|L|$ is a PL manifold of dimension n with boundary $|L_0|$. Let σ be a cell of L of dimension m.*

(i) *If σ is not a cell of L_0 then $|D(\sigma)|$ is a PL $(n-m)$-ball with boundary $|\dot{D}(\sigma)|$ and with $\hat{\sigma}$ in its interior.*

(ii) *If σ is a cell of L_0 then $|\dot{D}(\sigma)|$ is a PL $(n-m-1)$-ball and $|D(\sigma)|$ is a PL $(n-m)$-ball with $|\dot{D}(\sigma)|$ and $\hat{\sigma}$ on its boundary.*

Proof. For part (i), $|\mathrm{lk}(\hat{\sigma})|$ is a PL $(n-1)$-sphere. Lemma A.3 and Theorem 1 of [Mor70] imply that $|\dot{D}(\sigma)|$ is a PL $(n-m-1)$-sphere, and hence $|D(\sigma)|$ is a PL $(n-m)$ ball.

Part (ii) is similar. □

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