Towers of MU-algebras and the generalized Hopkins-Miller theorem

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Abstract

Our results are of three types. First we describe a general procedure of adjoining polynomial variables to A_∞-ring spectra whose coefficient rings satisfy certain restrictions. A host of examples of such spectra is provided by killing a regular ideal in the coefficient ring of MU, the complex cobordism spectrum. Second, we show that the algebraic procedure of adjoining roots of unity carries over in the topological context for such spectra. Third, we use the developed technology to compute the homotopy types of spaces of strictly multiplicative maps between suitable $K(n)$-localizations of such spectra. This generalizes the famous Hopkins-Miller theorem and gives strengthened versions of various splitting theorems.

Key words: S-algebras, topological derivations, Morava K-theories,Witt vectors.

1 Introduction

Our goals in this paper are three-fold. First, we generalize and extend the methods of [6] of constructing A_∞ ring spectra by adjoining "polynomial variables". Namely, suppose that R is a commutative S-algebra in the sense of [4] such that its coefficient ring R_* has no elements of odd degree and the graded ideal I_* in R_* is generated by a regular sequence $u_1, u_2, \ldots$. Then it is known from work of Strickland, [13] that the R-module R/I obtained by killing the ideal I has a structure of an R-ring spectrum. We assume that R/I is actually an R-algebra. Then it turns out that, informally speaking, all R-modules "between" R/I and R also possess structures of R-algebras.

More precisely, there are R-algebras $R[u_k]/u_k^l$ whose coefficient rings are obtained from R_* by killing the sequence $(u_1, u_2, \dots, u_{k-1}, u_k^l, u_{k+1}, \dots$. There are also natural "reduction maps" between these R-algebras.

The basic example is provided by taking $R = MU$, the complex cobordism S-algebra and MU/I be the Eilenberg-MacLane S-algebra H/p. It follows by induction that all MU-algebras obtained by killing any sequence of polynomial generators and/or a prime p have structures of MU-modules.

Second, we consider the question of adjoning roots of unity to an S-algebra. This problem was also treated in the recent preprint [12] by Schwanzl, Vogt and Waldhausen. Their definition of a topological extension has better formal properties than ours but it applies in a far less general situation. We show that one can adjoin roots of unity to such spectra as Morava K-theories $K(n)$, Johnson-Wilson theories $E(n)$ and many other algebras over the complex cobordism S-algebra MU.

Third, we address the problem of computing S-algebra maps between a certain completion $\hat{E}(n)$ of $E(n)$ and an MU-algebra E which is assumed to be "strongly $K(n)$-complete" in some precise sense explained later on in the paper. Examples of such MU-algebras include $\hat{E}(n)$, $K(n)$ and the Artinian completion of the v_n-localization of the Brown-Peterson spectum BP. It turns out that the space of S-algebra maps between $\hat{E}(n)$ and a strongly $K(n)$-complete MU-algebra is homotopically discrete with the set of connected components being equal to the set of multiplicative cohomology operations $\hat{E}(n) \to E$. We call the results of this type the generalized Hopkins-Miller theorem because the original Hopkins-Miller theorem (cf.[10]) asserts that the space of S-algebra self-maps of the spectrum E_n is homotopically equivalent to the (discrete) space of multiplicative operations from E_n to itself. Here E_n is a 2-periodic version of the completed Johnson-Wilson theory $\hat{E}(n)$ which came to be popularly known as the Morava E-theory.

We chose to work with the $2(p^n - 1)$-periodic theory $E(n)$ rather than with E_n. The relation of $E(n)$ to E_n is the same as the relation of the p-local Adams summand of the complex K-theory spectrum KU to KU itself. The advantage of $E(n)$ is that it is smaller than E_n, however it does not admit the action of the full Morava stabilzer group.

One consequence of our generalized Hopkins-Miller theorem is that $\hat{E}(n)$ admits a unique S-algebra structure, the result previously obtained in [1].

Another consequence is that $\hat{E}(n)$ splits off E as an S-algebra for a certain class of strongly $K(n)$-complete MU-algebras E. Such splittings were previously known to be multiplicative only up to homotopy.

Throughout the paper the symbol F_p will denote the prime field with p elements, L/F_p an arbitrary (but fixed) finite separable extension of F_p.

Further $W(L)$ and $W_n(L)$ denote the ring of Witt vectors and the ring of Witt vectors of length n respectively.

2 Adjoining polynomial variables to S-algebras

In this section we assume that R is a fixed commutative S-algebra such that $R_* = \pi_* R$ is a graded commutative ring concentrated in even degrees. We will also assume without loss of generality that R is q-cofibrant in the sense of [4]. All objects under consideration will be R-modules or R-algebras and smash products and homotopy classes of maps will be understood to be taken in the category of R-modules.

We begin by reminding the reader the notions of topological derivations and topological singular extensions of R-algebras. A detailed account can be found in [6]. Let A be an R-algebra and M an A-bimodule. Then the R-module $A \vee M$ has the obvious structure of an R-algebra ("square-zero extension" of A). Consider the set $[A, A \vee M]_{R-alg/A}$ of homotopy classes of R-algebra maps from A to $A \vee M$ which commute with the projection onto A. Then there exists an A-bimodule Ω_A and a natural in M isomorphism

$$[A, A \vee M]_{R-alg/A} \cong [\Omega_A, M]_{A-bimod}$$

where the right hand side denotes the homotopy classes of maps in the category of A-bimodules.

Definition 2.1 *The topological derivations R-module of A with values in M is the function R-module $F_{A\wedge A^{op}}(\Omega_A, M)$. We denote it by $\mathbf{Der}(A, M)$ and its ith homotopy group by $Der^{-i}(A, M)$.*

The A-bimodule Ω_A is constructed as the homotopy fibre of the multiplication map $A \wedge A \to A$. There exists the following homotopy fibre sequence of R-modules:

$$\mathbf{THH}(A, M) \to M \to \mathbf{Der}(A, M) \tag{1}$$

Here $\mathbf{THH}(A, M)$ is the topological Hochschild cohomology spectrum of A with values in M, $\mathbf{THH}(A, M) := F_{A\wedge A^{op}}(A, A)$.

We will frequently use the notion of a primitive operation from A to M. Denote by $m : A \wedge A \to A$ the multiplication map and by $m_l : A \wedge M \to M$ and $m_r : M \wedge A \to M$ the left and right actions of A in M respectively. Then a map $p : A \to M$ is called primitive if p is a "derivation up to homotopy", i.e. the following diagram is homotopy commutative:

$$\begin{array}{ccc} A\wedge A & \xrightarrow{m} & A \\ 1\wedge p\vee p\wedge 1\downarrow & & \downarrow p \\ A\wedge M\vee M\wedge A & \xrightarrow{m_l\vee m_r} & M \end{array}$$

There is a forgetful map $l : Der^*(A, M) \to [A, M]^*$ defined as follows. For any topological derivation $d : A \to A \vee M$ let $l(d)$ be the composite map $A \to A \vee M \to M$ where the last map is just the projection onto the wedge summand. Then it is easy to see that the image of l is contained in the set of primitive operations from A to M.

Suppose we are given a topological derivation $d : A \to A \vee M$. Consider the following homotopy pullback diagram

$$\begin{array}{ccc} X & \to & A \\ \downarrow & & \downarrow \\ A & \stackrel{d}{\to} & A\vee M \end{array}$$

Here the rightmost downward arrow is the canonical inclusion of a retract. Then we have the following homotopy fibre sequence of R-modules:

$$\Sigma^{-1}M \to X \to A \qquad (2)$$

Definition 2.2 *The homotopy fibre sequence (2) is called the topological singular extension associated with the derivation* $d : A \to A \vee M$*.*

For an element $x \in R$ we will denote by R/x the cofibre of the map $R \stackrel{x}{\to} R$ Let I_* be a graded ideal generated by (possibly infinite) regular sequence of elements $(u_1, u_2, \ldots) \in R_*$ of degrees $r_1, r_2, \ldots$. Then we can form the R-module R/I as the infinite smash product of R/x_i. By [13], Proposition 4.8 there is a structure of an R-ring spectrum on R/I. Clearly the coefficient ring of R/I is isomorphic to R_*/I_* where R_*/I_* is understood to be the direct limit of $R_*/(u_1, u_2, \ldots, u_k)$.

Our standing assumption is that R/I has a structure of an R-algebra (i.e. strictly associative). This may seem a rather strong condition but, as we see shortly such a situation is rather typical. In fact P.Goerss proved in [5] that any spectrum obtained by killing a regular sequence in MU, the complex cobordism S-algebra, has a structure of an MU-algebra.

The construction we are about to describe allows one to construct new R-algebras by "adjoining" the indeterminates u_k to the R-algebra R/I. The basic idea is the same as in [6] where Morava K-theories at an odd prime were shown to possess MU-algebra structures. However the arguments we use here are considerably more general, in particular we make no assumption that the prime 2 is invertible. Let us introduce the notation $R/I[u_k]/u_{k*}^l$ for the R_*-algebra

$$lim_{n\to\infty} R_*/(u_1, u_2, \ldots, u_{k-1}, u_k^l, u_{k+1}, \ldots, u_n).$$

This notation emphasizes the analogy with the truncated polynomial algebra even though $R/I[u_k]/u_{k*}^l$ is not always one. (Take, for instance R_* to be the ring of integers and $I_* = (u_1) = p$, a prime number. Then $R/I[u_k]/u_{1*}^l = Z/p^l Z$.) For each l reduction modulo u_k^{l+1} determines a map of R_*-algebras

$$R/I[u_k]/{u_k^{l+1}}_* \to R/I[u_k]/u_{k*}^l.$$

Now we can formulate our main theorem in this section.

Theorem 2.3 *For each k and l there exist R-algebras $R/I[u_k]/u_k^l$ with coefficient rings $R/I[u_k]/u_{k*}^l$ and R-algebra maps*

$$R_{i,k} : R/I[u_k]/u_k^{l+1} \to R/I[u_k]/u_k^l$$

which give the reductions mod u_k^{l+1} on the level of coefficient rings.

Proof. Suppose by induction that the R-algebras $R/I[u_k]/u_k^l$ with the required properties were constructed for $l \leq i$. We will show that there exists an appropriate Bokstein operation from $R/I[u_k]/u_k^i$ to $\Sigma^{|u_k|i+1} R/I$ which allow us to build the next stage.

Consider the cofibre sequence

$$\Sigma^{|u_k|l} R \stackrel{u_k^l}{\to} R \stackrel{\rho_{l,k}}{\to} R/u_k^l \stackrel{\beta_{l,k}}{\to} \Sigma^{|u_k|l+1} R.$$

According to [13] the R-module R/u_k^l admits an associative product $\phi : R/u_k^l \wedge R/u_k^l \to R/u_k^l$ and for any other product ϕ' there exists a unique element $u \in \pi_{2|u_k|l+2} R/u_k^l$ for which $\phi' = \phi + u \circ (\beta_{l,k} \wedge \beta_{l,k})$.

Lemma 2.4 *There exists a map of R-ring spectra $r_{i,k} : R/u_k^{i+1} \to R/u_k^i$ realizing the reduction map mod u_k^{i+1} on coefficient rings. Moreover in the cofibre sequence*

$$R/u_k^{i+1} \stackrel{r_{i,k}}{\to} R/u_k^i \stackrel{\bar{\beta}_{i,k}}{\to} \Sigma^{|u_k|i+1} R/u_i \tag{3}$$

the second map $\bar{\beta}_{k,i} : R/u_k^i \to \Sigma^{|u_k|(i+1)} R/u_k$ is a primitive operation.

Proof. Use induction on i (in fact we only need the inductive assumption in order that R/u_k be a (bi)module spectrum over R/u_k^i.) Consider the following diagram of R-modules:

$$\begin{array}{ccccc}
\Sigma^{|u_k|(i+1)}R & \stackrel{u_k}{\to} & \Sigma^{|u_k|i}R & \to & \Sigma^{|u_k|i}R/u_k \\
u_k^{i+1}\downarrow & & \downarrow u_k^i & & \downarrow \\
R & = & R & \to & pt \\
\downarrow & & \downarrow & & \downarrow \\
R/u_k^{i+1} & \stackrel{r_{i,k}}{\to} & R/u_k^i & \to & \Sigma^{u_k|i+1}R/u_k \\
 & & \downarrow & & || \\
 & & \Sigma^{|u_k|i+1}R & \to & \Sigma^{|u_k|i+1}R/u_k
\end{array}$$

Here the rows and columns are homotopy cofibre sequences of R-modules. The map $r_{i,k} : R/u_k^{i+1} \to R/u_k^i$ is determined uniquely by the requirement that the diagram commute in the homotopy category of R-modules.

We now show that it is possible to choose a product on R/u_k^{i+1} so that $r_{i,k}$ becomes an R-ring spectrum map. Take first any associative product $\phi : R/u_k^{i+1} \wedge R/u_k^{i+1} \to R/u_k^{i+1}$ which exists by [13], Proposition 3.1. As in [13], Proposition 3.15 there is an obstruction $d(\phi) \in \pi_{2|u_k|(i+1)+2}(R/u_k^i)$ for the map $r_{i,k}$ to be homotopy multiplicative. If the product ϕ is changed the obstruction changes according to the formula

$$d(\phi + u \circ (\beta_{i,k} \wedge \beta_{i,k})) = d(\phi) + r_{i,k*}u.$$

Since the map $r_{i,k*} : R_*/u_k^{i+1} \to R_*/u_k^i$ is just the reduction modulo u_k^{i+1} it is surjective and we conclude that there exists a product on R/u_k^{i+1} for which the obstruction vanishes.

Further the map $\bar{\beta}_{i,k} : R/u_k^i \to \Sigma^{|u_k|i+1}R/u_k$ coincides with the composition

$$R/u_k^i \stackrel{\beta_{i,k}}{\to} \Sigma^{|u_k|i+1}R \stackrel{\rho_{i,k}}{\to} \Sigma^{|u_k|i+1}R/u_k^i \to \Sigma^{|u_k|i+1}R/u_k.$$

Here the last map is (a suspension of) the canonical R-ring map $R/u_k^i \to R/u_k$ which exists by the inductive assumption. The composition $\rho_i \circ \beta_i$ is the Bokstein operation which is primitive by [13], Proposition 3.14. It follows that $\beta_{k,i}$ is also primitive and our lemma is proved.

Now taking the smash product of (3) with $R/u_1 \wedge R/u_2 \wedge \ldots \wedge R/u_{k-1} \wedge$ $\wedge R/u_{k+1} \wedge \ldots$ we get the following homotopy cofibre sequence of R-modules:

$$R/I[u_k]/u_k^{i+1} \xrightarrow{R_{i,k}} R/I[u_k]/u_k^i \xrightarrow{Q_{i,k}} \Sigma^{|u_k|i+1}R/I.$$

It follows that $Q_{i,k}$ is a primitive operation and $R/I[u_k]/u_k^{i+1}$ has an R-ring spectrum structure such that $R_{i,k}$ is an R-ring map. (Notice our abuse of notations here in using the symbol $R_{i,k}$ even though $R/I[u_k]/u_k^{i+1}$ is not yet proved to be an R-algebra).

Next we will describe the set of all primitive operations $R/I[u_k]/u_k^i \to \Sigma^* R/I$. Consider the cofibre sequence

$$\Sigma^{|u_l|}R \stackrel{u_l}{\to} R \stackrel{\rho_l}{\to} R/u_l \stackrel{\beta_l}{\to} \Sigma^{|u_l|+1}R.$$

For $l \neq k$ introduce the operation

$$Q_l : R/I[u_k]/u_k^i = R/u_1 \wedge R/u_2 \wedge \ldots \wedge R/u_{k-1} \wedge R/u_k^i \wedge R/u_{k+1} \wedge \ldots \to \Sigma^{|u_l|+1}R/I$$

obtained by smashing $\rho_l \circ \beta_l : R/u_l \to \Sigma^{|u_l|+1}R/u_l$ with the identity map on the remaining smash factors. Clearly the operations Q_l are primitive. The next result shows that Q_l and $Q_{i,k}$ are essentially all primitive operations from $R/I[u_k]/u_k^i$ into (suspensions of) R/I.

Lemma 2.5 *Any primitive operation $R/I[u_k]/u_k^i \to \Sigma^* R/I$ can be written uniquely as an infinite sum $a_k Q_{i,k} + \Sigma_{i \neq k} a_i Q_i$ for $a_i \in R/I_*$.*

Proof. We will only give a sketch since the arguments of Strickland ([13], Proposition 4.17) carry over almost verbatim. Let $\eta : R \to R/I[u_k]/u_k^i$ be the unit map and $J_* = (u_1, u_2 \ldots, u_{k-1}, u_k^i, u_{k+1,\ldots})$ be the kernel of the map $\eta_* : R_* \to R/I[u_k]/u_{k*}^i$. Given a primitive operation $Q : R/I[u_k]/u_k^i \to \Sigma^s R/I$ define the function $d(Q) : J_* \to R/I[u_k]/u_{k*}^i$ as follows. For any $x \in J_*$ we have a cofire sequence

$$\Sigma^{|x|}R \stackrel{x}{\to} R \stackrel{\rho_x}{\to} R/x \stackrel{\beta_x}{\to} \Sigma^{|x|+1}R.$$

Then there is a unique map $f_x : R/x \to R/I[u_k]/u_k^i$ such that $f_x \circ \rho_x = \eta$. Further there is a unique map $y : \Sigma^{|x|+1}R \to \Sigma^s R/I$ such that $Q \circ f_x = y \circ \beta_x$. We define $d(Q)(x) := y \in \pi_{|x|+1-s} R/I_*$.

It follows that the function d actually embeds the set of primitive operations from $R/I[u_k]/u_k^i$ into R/I into $Hom_{R_*}(J/J^2, R/I_*)$. Further it is straightforward to check that

$$d(Q_s)(u_t) = \delta_{st} \ for \ s,t \neq k, \ d(Q_s)(u_k^i) = 0;$$

$$d(Q_{i,k})(u_j) = 0 \ for \ j \neq k, \ d(Q_{i,k})(u_k^i) = 1.$$

Therefore the elements $Q_{i,k}, Q_s, s \neq k$ form a basis in $Hom(J_*/J_*^2, R/I_*)$ dual to the basis $(u_1, u_2, \ldots, u_{k-1}, u_k^i, u_{k+1}, \ldots)$ in J_*/J_*^2 and our lemma is proved.

Now we come to the crucial part of the proof. We are going to show that the Bokstein operation $Q_{i,k}$ can be improved to a topological derivation from which it would follow that $R/I[u_k]/u_k^{i+1}$ is an R-algebra and $R_{i,k} : R/I[u_k]/u_k^{i+1} \to R/I[u_k]/u_k^i$ lifts to an R-algebra map.

The following lemma (interesting in its own right) is preparatory for computing the topological Hochschild cohomology of $R/I[u_k]/u_k^l$ with coefficients in R/I.

Lemma 2.6 *For any regular ideal $J_* = (x_1, x_2, \ldots)$ in R_* and any product on R/J there is a multiplicative isomorphism*

$$\pi_*(R/I \wedge_R R/J^{op}) \cong \Lambda_{R_*/J_*}(\tau_1, \tau_2, \ldots)$$

where $|\tau_i| = |x_i| + 1$.

Proof. The standard spectral sequence arguments show that $\pi_*(R/J \wedge_R R/J^{op})$ is a filtered R_*/J_*-algebra whose associated graded is isomorphic to $\Lambda_{R_*/J_*}(\tau_1, \tau_2, \ldots)$ so the problem is to show that the possible multiplicative extensions are trivial. More precisely one needs to show that all skew-commutators of the elements τ_i in $\pi_*(R/J \wedge_R R/J^{op})$ as well as their squares are zero. Consider the map of R-ring spectra $f : R/J \wedge_R R/J^{op} \to F_R(R/J, R/J)$ which is induced by the structure of a left $R/J \wedge_R R/J^{op}$-module spectrum on R/J. Then f induces a map of R_*/J_*-algebras $f_* : \pi_* R/J \wedge_R R/J^{op} \to \pi_* F_R(R/J, R/J)$. By [13], Proposition 4.15 the ring $\pi_* F_R(R/J, R/J)$ is a completed exterior algebra over R_*/J_*, in particular it is (graded) commutative. Therefore all possible nontrivial relations between the elements τ_i map to zero in the R_*/J_*-algebra $\pi_* F_R(R/J, R/J)$. Since these relations belong to R_*/I_* and f_* is R_*/I_*-linear we conclude that they are actually zero and $\pi_*(R/J \wedge_R R/J^{op})$ is in fact a (graded) commutative R_*/J_*-algebra. This finishes the proof of lemma 2.6.

Remark 2.7 *The R_*/J_*-algebra $\pi_*(R/J \wedge_R R/J)$ need not be commutative in general. For example take $R/J = MU/2$, the reduction of MU modulo 2. Then it can be shown using the methods of [8] that $MU/2 \wedge_{MU} MU/2$ is an $MU/2_*$-algebra on one generator in degree 1 whose square is equal to $x_1 \in MU/2_*$. We will discuss this and related phenomena elsewhere.*

Lemma 2.8 *There is the following isomorphism of graded R_*-modules:*

$$grTHH^*(R/I[u_k]/u_k^i, R/I) = R/I_*[\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k, \ldots];$$

$$grDer^{*-1}(R/I[u_k]/u_k^i, R/I) = R/I_*[\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k, \ldots]/(R/I_*).$$

Here $gr(?)$ denotes the associated graded module. Moreover the image of the forgetful map

$$Der^*(R/I[u_k]/u_k^i, R/I) \to [R/I[u_k]/u_k^i, R/I]^*$$

is the set of all primitive operations from $R/I[u_k]/u_k^i$ to $\Sigma^* R/I$.

Proof. Denote by J_* the ideal in R generated by $(u_1, u_2, \ldots, u_{k-1}, u_k^i, u_{k+1}, \ldots)$. Then we have the following spectral sequence

$$Ext^{**}_{\pi_* R/I[u_k]/u_k^i \wedge R/I[u_k]/u_k^i{}^{op}}(R/I[u_k]/u_{k*}^i, R/I_*)$$

$$= Ext^{**}_{\pi_* R/J \wedge R/J^{op}}(R/J_*, R/I_*) \Rightarrow THH^*(R/I[u_k]/u_k^i, R/I).$$

By Lemma 2.6 $\pi_* R/J \wedge R/J^{op} = \Lambda_{R/J_*}(\tau_1, \tau_2 \ldots)$. Therefore

$$Ext^{**}_{\pi_* R/J \wedge R/J^{op}}(R/J_*, R/I) = R/I_*[\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_i, \ldots]$$

where $|\tilde{u}_l| = -|u_l| + 2$ for $l \neq k$ and $|\tilde{u}_k| = -|u_k| i + 2$. Our spectral sequence collapses and we obtain the desired isomorphism

$$grTHH^*(R/I[u_k]/u_k^i, R/I) = R/I_*[\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_i, \ldots].$$

The isomorphism involving $Der^*(R/I[u_k]/u_k^i, R/I)$ is obtained similarly.

Now consider the spectral sequence

$$Ext^{**}_{R*}(R/I[u_k]/u_{k*}^i, R/I_*) = Hom(\Lambda(z_1, z_2, \ldots), R/I_*) \Rightarrow [R/I[u_k]/u_k^i, R/I]^*.$$

Here $|z_l| = |u_l| + 1$ for $l \neq k$ and $|z_k| = |u_k| i + 1$. This spectral sequence is easily seen to collapse and denoting by $\tilde{z}_i$ the elements in the dual basis in $\Lambda(z_1, z_2, \ldots)$ we identify its $E_2 = E_\infty$-term with $\Lambda_{R/I_*}(\tilde{z}_1, \tilde{z}_2, \ldots)$. (Of course this is an identification only as R/I_*-modules, not as rings.) It is clear that the elements z_l correspond to the primitive operations $Q_l : R/I[u_k]/u_k^i \to \Sigma^{|u_l|+1} R/I$ for $l \neq k$ whilst z_k corresponds to $Q_{i,k} : R/I[u_k]/u_k^i \to \Sigma^{|u_l|i+1} R/I$.

Furthermore the forgetful map l (operating on the level of E_2-terms) sends $\tilde{u}_l \in grDer^*(R/I[u_k]/u_k^i, R/I)$ to $\tilde{z}_l$. In other words all primitive operations $R/I[u_k]/u_k^i \to \Sigma^* R/I$ are covered by l up to higher filtration terms. Since the image of $l : Der^*(R/I[u_k]/u_k^i, R/I) \to [R/I[u_k]/u_k^i, R/I]^*$ is contained in the subspace of the primitive operations no higher filtration terms are present and we conclude that all Bokstein operations Q_l and $Q_{i,k}$ are in the image of l. With this Lemma 2.8 is proved.

So we proved that there exists a topological derivation $\tilde{Q}_{i,k} : R/I[u_k]/u_k^i \to R/I[u_k]/u_k^i \vee \Sigma^{|u_k|i+1} R/I$ such that its composition with the projection onto the wedge summand $R/I[u_k]/u_k^i \vee \Sigma^{|u_k|i+1}R/I \to \Sigma^{|u_k|i+1}R/I$ is the Bokstein operation $Q_{i,k}$. Associated with $\tilde{Q}_{i,k}$ is a topological singular extension

$$\Sigma^{|u_k|i}R/I \to X \to R/I[u_k]/u_k^i$$

such that X is weakly equivalent to $R/I[u_k]/u_k^{i+1}$. In other words the R-module $R/I[u_k]/u_k^{i+1}$ admits a structure of an R-algebra so that the reduction map $R/I[u_k]/u_k^{i+1} \to R/I[u_k]/u_k^i$ is an R-algebra map. The inductive step is completed and Theorem 2.3 is proved.

The singular extension of R-algebras

$$R/I[u_k]/u_k^{i+1} \xrightarrow{R_{i,k}} R/I[u_k]/u_k^i \xrightarrow{Q_{i,k}} \Sigma^{|u_k|i+1}R/I$$

is of fundamental importance to us. We will call the R-algebra $R/I[u_k]/u_k^{i+1}$ an elementary extension of $R/I[u_k]/u_k^i$. Sometimes we will use the notation $R/I[u_k]$ for $holim_{i\to\infty}R/I[u_k]/u_k^i$. Of course the coefficient ring of $R/I[u_k]$ is not always a polynomial algebra on R/I_*.

Now let $R= MU$, the complex cobordism spectrum. It is well-known that MU is a commutative S-algebra and $MU_* = Z[x_1, x_2, \ldots]$, the polynomial algebra on infinitely many generators in even degrees.

Corollary 2.9 *Let J_* be a regular ideal in $MU_* = Z[x_1, x_2, \ldots]$ generated by any subsequence of the regular sequence $p^{\nu_0}, x_1^{\nu_1}, x_2^{\nu_2}, \ldots$ where $\nu_l \geq 1$ for all l. Then MU/J admits a structure of an MU-algebra.*

Proof. First assume that the element p_0^ν does not belong to J_*. Denote by I_* be the ideal in MU_* generated by all polynomial generators $(x_1, x_2, \ldots)$. Then MU/I is the integral Eilenberg-MacLane spectrum HZ which possesses a canonical structure of an MU-algebra (even a commutative MU-algebra).

Taking successive elementary extensions corresponding to elements $x_{i_k}^{\nu_{i_k}}$ with $\nu_{i_k} > 1$ we construct an MU-algebra

$$\widetilde{MU}/J = ((HZ[x_{i_1}]/x_{i_1}^{\nu_{i_1}})[x_{i_2}]/x^{\nu_{i_k}})\ldots.$$

Similarly taking elementary extensions corresponding to those polynomial generators $\{u_1, u_2, \ldots\}$ whose powers are not in J_* and passing to the (inverse) limit we construct the MU-algebra $\widetilde{MU}/J[u_1, u_2, \ldots]$ whose underlying MU-module is MU/J.

If the element p^{ν_0} does belong to J_* the only difference is that we start our induction with the Eilenberg-MacLane MU-algebra HZ/p^{ν_0} instead of HZ. With this Corollary 2.9 is proved.

Remark 2.10 *Notice that our method also shows that the MU-algebra structures on MU/J for different J are compatible in the sense that various reduction maps given by killing elements $x_i^{\nu_i}$ are actually MU-algebra maps.*

Recall that the Brown-Peterson spectrum BP is obtained from MU by killing all polynomial generators in MU_* except for $v_i = x_{2(p^i-1)}$ and localizing at the prime p. Therefore $BP_* = Z_{(p)}[v_1, v_2, \ldots]$. It follows that BP possesses an MU-algebra structure. We define

$$BP < n >= BP/(v_i, i > n) = HZ_{(p)}[v_1, v_2, \ldots, v_n]$$

$$P(n) = BP/(v_i, i < n) = HF_p[v_n, v_{n+1}, \ldots]$$

$$B(n) = v_n^{-1}P(n) = v_n^{-1}HF_p[v_n, v_{n+1}, \ldots]$$

$$k(n) = BP/(v_i, i \neq n) = HF_p[v_n]$$

$$K(n) = v_n^{-1}BP/(v_i, i \neq n) = HF_p[v_n, v_n^{-1}]$$

$$E(n) = v_n^{-1}BP/(v_i, i > n) = HZ_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}]$$

Since inverting an element in the coefficient ring is an instance of Bousfield localization and Bousfield localization preserves algebra structures we conclude that all spectra listed above admit structures of MU-algebras.

We conclude this section with a few remarks about the commutativity of the products on the MU-algebras considered. First in the case of the odd prime p BP and other spectra derived from it have coefficient rings concentrated in degrees congruent to 0 mod 4 and therefore all products are unique up to homotopy and automatically commutative. If p=2 then Strickland shows that BP still admits a structure of a commutative MU-ring spectrum, but this does not follow directly from our construction. We conjecture that in the context of Corollary 2.9 one can choose an MU-algebra structure on MU/J compatible with any given structure of an MU-ring spectrum on MU/J.

The situation with $E(n)$ is different. The spectrum $E(n)$ is Landweber exact regardless of the prime p. Therefore it possesses a unique homotopy associative and commutative multiplication compatible with its structure (in the classical sense) of an MU-algebra spectrum. Our results show that this multiplication can be improved to an MU-algebra structure which in turn gives an A_∞-structure. On the other hand Strickland shows in [13] that at the prime 2 there is no homotopy commutative multiplication on $E(n)$ in the category of MU-modules (to get a homotopy commutative product one has to choose a different set of generators). Therefore, quite curiously, $E(n)$ is a homotopy commutative A_∞-ring spectrum and simaltaneously a MU-algebra that is not commutative even up to homotopy.

3 Separable extensions of R-algebras

We keep our convention that R is a commutative S-algebra with coefficient ring R_* concentrated in even degrees. Let A be an R-algebra with coefficient ring A_* and $A_* \subset \overline{A}_*$ a ring extension in the usual algebraic sense. In this section we consider the problem of finding an R-algebra $\overline{A}$ together with an R-algebra map $A \to \overline{A}$ wich realizes the given extension $A_* \subset \overline{A}_*$ in homotopy. If such an algebra $\overline{A}$ exists we say that the algebraic extension $A_* \subset \overline{A}_*$ admits a topological lifting.

We now describe a general framework in which it is possible to prove that topological liftings exist. Suppose that $R_* \to \overline{R}_*$ is a separable extension of R. That means that R_* is a subring of a graded commutative ring $\overline{R}_*$ and $\overline{R}_*$ is a separable algebra over R_*, i.e. $\overline{R}_*$ is a projective module over $\overline{R}_* \otimes_{R_*} \overline{R}_*$. In addition we assume that $\overline{R}_*$ is a projective R_*-module. Then for any ideal I_* in R_* the R_*-algebra $\overline{R_*/I}_* := R_*/I_* \otimes_{R_*} \overline{R}_*$ is a separable R_*/I_*-algebra so $R_*/I_* \to \overline{R_*/I}_*$ is also a separable extension.

Further assume that the ideal I_* is generated by a regular sequence $(u_1, u_2, \ldots)$ (possibly infinite), the R-module R/I is supplied with a structure of an R-algebra and the algebraic extension $R_*/I_* \to \overline{R_*/I}_*$ admits a topological lifting. In other words there exists an R-algebra $\overline{R/I}$ and an R-algebra map $R/I \to \overline{R/I}$ realizing the given separable extension on the level of coefficient rings.

Recall that in the previous section we constructed an R-algebra structure on the R-module

$$R/I[u_k]/u_k^l := R/u_1 \wedge R/u_2 \ldots \wedge R/u_{k-1} \wedge R/u_k^l \wedge R/u_{k+1} \wedge \ldots.$$

Then we have the following

Theorem 3.1 *There exist R-algebras $\overline{R/I[u_k]/u_k^l}$ realizing in homotopy the R_*-module $\overline{R/I[u_k]/u^l_{k*}} := R/I[u_k]/u^l_{k*} \otimes_{R_*} \overline{R}$ and reduction maps*

$$\overline{R_{i,k}} : \overline{R/I[u_k]/u_k^{l+1}} \to \overline{R/I[u_k]/u_k^l}.$$

Moreover the algebraic extension of rings $R/I[u_k]/u^l_{k*} \to \overline{R/I[u_k]/u^l_{k*}}$ admits a topological lifting so that the following diagram of R-algebras is homotopy commutative:

$$\begin{array}{ccc} R/I[u_k]/u_k^{l+1} & \xrightarrow{R_{i,k}} & R/I[u_k]/u_k^l \\ \downarrow & & \downarrow \\ \overline{R/I[u_k]/u_k^{l+1}} & \xrightarrow{\overline{R_{i,k}}} & \overline{R/I[u_k]/u_k^l} \end{array}$$

Proof. Proceeding by induction suppose that the R-algebras $\overline{R/I[u_k]/u_k^l}$ with required properties were constructed for $l \leq i$ Consider the spectral sequence

$$Tor_{**}^{R_*}(\overline{R/I[u_k]/u^i_{k*}}, \overline{R/I[u_k]/u^i_{k*}}^{op})$$
$$= \overline{R}_* \otimes_{R_*} \overline{R}_* \otimes_{R_*} Tor_{**}^{R_*}(R/I[u_k]/u^i_{k*}, R/I[u_k]/u^{i\ op}_{k*})$$
$$= \overline{R}_* \otimes_{R_*} \overline{R}_* \otimes_{R_*} \Lambda_{R/I[u_k]/u^i_{k*}}(\tau_1, \tau_2, \ldots)$$
$$= \overline{R/I[u_k]/u^i_{k*}} \otimes_{R/I[u_k]/u^i_{k*}} \overline{R/I[u_k]/u^i_{k*}} \otimes \Lambda(\tau_1, \tau_2, \ldots)$$
$$\Rightarrow \pi_* \overline{R/I[u_k]/u^i_k} \wedge_R \overline{R/I[u_k]/u^i_k}^{op}.$$

Since by Lemma 2.6

$$\pi_* R/I[u_k]/u^i_k \wedge_R R/I[u_k]/u^{i\ op}_k = \Lambda_{R/I[u_k]/u^i_{k*}}(\tau_1, \tau_2, \ldots)$$

we see that the exterior generators τ_i are permanent cycles and it follows that our spectral sequence collapses multiplicatively so the ring $\pi_* \overline{R/I[u_k]/u^i_k} \wedge_R \overline{R/I[u_k]/u^i_k}^{op}$ is isomorphic to the exterior algebra on $\tau_1, \tau_2, \ldots$ with coefficients in $\overline{R/I[u_k]/u^i_{k*}} \otimes_{R/I[u_k]/u^i_{k*}} \overline{R/I[u_k]/u^i_{k*}}$.

Further since $\overline{R/I[u_k]/u^i_{k*}}$ is a separable $R/I[u_k]/u^i_{k*}$-algebra we see that

$$grDer^{*-1}(\overline{R/I[u_k]/u^i_k}, \overline{R/I})$$
$$= \overline{R/I[u_k]/u^i_{k*}} \otimes_{R/I[u_k]/u^i_{k*}} \otimes \overline{R/I}_*[\tilde{u_1}, \tilde{u_2}, \ldots]/(\overline{R/I}_*).$$

Similarly

$$grDer^{*-1}(R/I[u_k]/u^i_k, \overline{R/I})$$
$$= \overline{R/I[u_k]/u^i_{k*}} \otimes_{R/I[u_k]/u^i_{k*}} \otimes \overline{R/I}_*[\tilde{u_1}, \tilde{u_2}, \ldots]/(\overline{R/I}_*).$$

Therefore the map $\mathbf{Der}(\overline{R/I[u_k]/u^i_k}, \overline{R/I}) \to \mathbf{Der}(R/I[u_k]/u^i_k, \overline{R/I})$ induced by the R-algebra map $R/I[u_k]/u^i_k \to \overline{R/I[u_k]/u^i_k}$ is an isomorphism in homotopy.

Consider the following homotopy commutative diagram of R-algebras:

$$\begin{array}{ccc} \overline{R/I[u_k]/u^i_k} & \xrightarrow{d} & \overline{R/I} \vee \Sigma^{|u_k|i+1} \overline{R/I} \\ \uparrow & & \uparrow \\ R/I[u_k]/u^i_k & \longrightarrow & R/I \vee \Sigma^{|u_k|i+1} \overline{R/I} \\ || & & \uparrow \\ R/I[u_k]/u^i_k & \xrightarrow{Q_{i,k}} & R/I \vee \Sigma^{|u_k|i+1} R/I \end{array} \qquad (4)$$

Here the horizontal arrows are topological derivation. The lower arrow is the Bokstein operation $Q_{i,k}$ constructed in the previous section,the middle arrow is the only one that makes the lower square commute. Since derivations of $\overline{R/I[u_k]/u_k^i}$ with values in $\overline{R/I}$ are in one-to-one correspondence with derivations of $R/I[u_k]/u_k^i$ with values in $\overline{R/I}$ the upper horizontal arrow d exists making the upper square commute.

Now the derivation d gives rise to a topological singular extension

$$\Sigma^{|u_k|i}\overline{R/I} \to ? \to \overline{R/I[u_k]/u_k^i}$$

which on the level of coefficient rings reduces to the algebraic singular extension

$$\Sigma^{|u_k|i}\overline{R/I}_* \to \overline{R/I[u_k]/u_k^{i+1}}_* \to \overline{R/I[u_k]/u_k^i}_*$$

Therefore we can denote ? by $\overline{R/I[u_k]/u_k^{i+1}}$ and because of the diagram (4) we have a map of topological singular extensions

$$\begin{array}{ccccc} \Sigma^{|u_k|i}R/I & \to & R/I[u_k]/u_k^{i+1} & \to & R/I[u_k]/u_k^i \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{|u_k|i}\overline{R/I} & \to & \overline{R/I[u_k]/u_k^{i+1}} & \to & \overline{R/I[u_k]/u_k^i} \end{array}$$

With this the inductive step is completed and Theorem 3.1 is proved.

We now show that Theorem 3.1 gives a way of adjoining roots of unity to various MU-algebras. Let $F_p \hookrightarrow L$ be a finite separable extension of the field F_p (typically obtained by adjoining roots of an irreducible factor of the cyclotomic polynomial $x^{p^n-1}-1$ for some n). Let $R = MU_p$, the p-completion of the complex cobordism spectrum MU and $\overline{R}_* = W(L) \otimes MU_*$ where $W(L)$ is the ring of Witt vectors of L. Since L is a separable extension of F_p the algebra $W(L)$ is separable over the p-adic integers $\hat{Z}_p$ and it follows that $W(L) \otimes MU_*$ is a separable algebra over $MU_{p*} = \hat{Z}_p \otimes MU_*$.

Corollary 3.2 *Let J_* be the ideal in $MU_* = Z[x_1, x_2, \ldots]$ generated by $(p, x_{i_1}, x_{i_2}, \ldots)$ where x_{i_k} are polynomial generators in MU_*. Then for any finite separable extension $F_p \hookrightarrow L$ there exist MU_p-algebras MU/J, MU/J_L, $MU/J_{W(F_p)}$ and $MU/J_{W(L)}$ together with the commutative diagram in the homotopy category of MU_p-algebras*

$$\begin{array}{ccc} MU/J_{W(F_p)} & \to & MU/J_{W(L)} \\ \downarrow & & \downarrow \\ MU/J & \to & MU/J_L \end{array} \tag{5}$$

which reduces in homotopy to the following diagram of MU_*-algebras:

$$\begin{array}{ccc} \hat{Z}_p \otimes MU_*/(x_{i_1}, x_{i_2} \dots) & \to & W(L) \otimes MU_*/(x_{i_1}, x_{i_2} \dots) \\ modp \downarrow & & \downarrow modp \\ MU_*/J_* & \to & L \otimes MU_*/J_* \end{array}$$

Proof. Denote by $u_1, u_2, \dots$ the collection of polynomial generators of MU_* which are not in J_*. There is an isomorphism of rings $MU_*/J_* \cong F_p[u_1, u_2, \dots]$. Let I be the maximal ideal $(p, x_1, x_2, \dots)$ in $MU_* = Z[x_1, x_2, \dots]$. Then MU/I is the Eilenberg-MacLane spectrum HF_p and the canonical map $MU_p \to MU/I$ is a commutative S-algebra map. Therefore MU/I is an MU_p-algebra (even a commutative MU_p-algebra). Further the extension $F_p \hookrightarrow L$ determines a map of commutative S-algebras $HF_p \to HL$ which is also a map of commutative MU_p-algebras. Using Theorem 3.1 we can adjoin the variable u_i to the topological extension $HF_p \to HL$ and obtain the tower of topological extensions

$$\{HF_p[u_i]/[u_i^k] \to HL[u_i]/[u_i^k]\}.$$

Passing to the limit we get the topological extension $HF_p[u_i] \to HL[u_i]$.

This procedure could be repeated so we can adjoin another polynomial generator to the extension $HF_p[u_i] \to HL[u_i]$ (or any number of polynomial generators). In this way we construct the extension

$$HF_p[u_1, u_2, \dots] \to HL[u_1, u_2, \dots]$$

which is the same as the extension $MU/J \to MU/J_L$ for the ideal $J_* = (p, x_{i_1}, x_{i_2}, \dots)$ in MU_*. At this point we adjoin the indeterminate corresponding to the prime p. We then get the following diagram of MU-algebras:

$$\begin{array}{ccccccccc} MU/J & \leftarrow & MU/J_{W_2(F_p)} & \leftarrow & \dots & \leftarrow & MU/J_{W_n(F_p)} & \leftarrow \\ \downarrow & & \downarrow & & & & \downarrow & \\ MU/J_L & \leftarrow & MU/J_{W_2(L)} & \leftarrow & \dots & \leftarrow & MU/J_{W_n(L)} & \leftarrow \end{array}$$

where the MU_p-algebras $MU/J_{W_n(F_p)}$ and $MU/J_{W_n(L)}$ realize the MU_{p*}-algebras $W_n(F_p) \otimes MU_*/(x_{i_1}, x_{i_2}, \dots)$ and $W_n(L) \otimes MU_*/(x_{i_1}, x_{i_2}, \dots)$ respectively.

Finally taking the (homotopy) inverse limit we get an MU_p-algebra map $MU/J_{W(F_p)} \to MU/J_{W(L)}$ which realizes in homotopy the extension of rings $\hat{Z}_p \otimes MU/J_* \to W(L) \otimes MU/J_*$. Corollary 3.2 is then proved.

Remark 3.3 *It is not hard to prove that the diagram (5) is equivariant with respect to $Gal(L/F_p)$ in a suitable sense but we save this observation for future work.*

4 Localized towers of MU-algebras

In this section we investigate the behaviour of the towers of MU-algebras constructed in the previous section under Bousfield localization. We start with an almost trivial

Theorem 4.1 *Let*

$$I \to A \to B \tag{6}$$

be a singular extension of R-algebras for a commutative S-algebra R. Then for any R-module M the cofibre sequence

$$I_M \to A_M \to B_M \tag{7}$$

is also a singular extension of R-algebras (here $?_M$ denotes Bousfield localization with respect to M)

Proof. The singular extension (6) is associated with a certain topological derivation $d : B \to B \vee \Sigma M$ so that there is a homotopy pullback square of R-algebras

$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ B & \stackrel{d}{\to} & B \vee \Sigma I \end{array}$$

where the vertical map $B \to B \vee \Sigma M$ is the canonical inclusion of the wedge summand. Localizing this diagram with respect to M we get the homotopy pullback diagram

$$\begin{array}{ccc} A_M & \to & B_M \\ \downarrow & & \downarrow \\ B_M & \stackrel{d_M}{\to} & B_M \vee \Sigma I_M \end{array}$$

Therefore (7) is a singular extension associated with the topological derivation $B_M \stackrel{d_M}{\to} B_M \vee \Sigma I_M$ and the theorem is proved.

We now specialize to the case $R = MU$. Let I_* be a regular ideal $(u_{i_1}, u_{i_1}, \ldots)$ in MU_* such that the elements u_{i_k} have positive degrees and MU/I is an MU-algebra. Consider the MU-algebra $MU/I[u_{i_1}, u_{i_1}, \ldots, u_{i_k}]$ obtained by adjoining the indeterminates $u_{i_1}, u_{i_1}, \ldots, u_{i_k}$ to MU/I as explained in Section 2. Set

$$MU/I[[u_{i_2}, u_{i_3}, \ldots, u_{i_k}]][u_{i_1}, u_{i_1}^{-1}] = MU/I[u_{i_2}, u_{i_3}, \ldots, u_{i_k}]_{MU/I[u_{i_1}, u_{i_1}^{-1}]},$$

the Bousfield localization of the MU-algebra $MU/I[u_{i_2}, u_{i_3}, \ldots, u_{i_k}]$ with respect to $MU/I[u_{i_1}, u_{i_1}^{-1}] := u_{i_1}^{-1} MU/I[u_{i_1}]$. The reason for this notation is the following

Proposition 4.2 *The coefficient ring of* $MU/I[[u_{i_2}, u_{i_3}, \ldots, u_{i_k}]][u_{i_1}, u_{i_1}^{-1}]$ *as an* MU_**-algebra is* $MU/I_*[[u_{i_2}, u_{i_3}, \ldots, u_{i_k}]][u_{i_1}, u_{i_1}^{-1}]$.

Proof. We have the following tower of MU-algebras:

$$MU/I[u_{i_1}] \leftarrow MU/I[u_{i_1}][u_{i_2}]/u_{i_2}^2 \leftarrow \ldots \leftarrow MU/I[u_{i_1}][u_{i_2}]/u_{i_2}^l \leftarrow \ldots$$

Bousfield-localizing it with respect to $MU/I[u_{i_1}, u_{i_1}^{-1}]$ we get the tower

$$MU/I[u_{i_1}, u_{i_1}^{-1}] \leftarrow MU/I[u_{i_1}, u_{i_1}^{-1}][u_{i_2}]/u_{i_2}^2 \leftarrow \ldots$$

Notice that the first term of the localized tower as well as its successive subquotients (isomorphic to suspensions of $MU/I[u_{i_1}, u_{i_1}^{-1}]$) are $MU/I[u_{i_1}, u_{i_1}^{-1}]$-local and therefore so is its homotopy inverse limit. Denote this limit by $MU/I[u_{i_1}, u_{i_1}^{-1}][[u_{i_2}]]$, clearly

$$MU/I[u_{i_1}, u_{i_1}^{-1}][[u_{i_2}]]_* = MU/I_*[u_{i_1}, u_{i_1}^{-1}][[u_{i_2}]]$$

Further the canonical map

$$holim(MU/I[u_{i_1}][u_{i_2}]/u_{i_2}^l) \simeq MU/I[u_{i_1}, u_{i_2}] \to MU/I[u_{i_1}, u_{i_1}^{-1}][[u_{i_2}]]$$

is $MU/I[u_{i_1}, u_{i_1}^{-1}]$-equivalence and we conclude that

$$MU/I[u_{i_1}, u_{i_1}^{-1}][[u_{i_2}]] \cong MU/I[u_{i_1}, u_{i_2}]_{MU/I[u_{i_1}, u_{i_1}^{-1}]} \tag{8}$$

We then proceed by adjoining u_{i_3}. There is a tower of MU-algebras

$$MU/I[u_{i_1}, u_{i_2}] \leftarrow MU/I[u_{i_1}, u_{i_2}][u_{i_3}]/u_{i_3}^2 \leftarrow \ldots$$

where the successive stages

$$\Sigma^{|u_{i_3}|s} MU/I[u_{i_1}, u_{i_2}][u_{i_3}] \to MU/I[u_{i_1}, u_{i_2}][u_{i_3}]/u_{i_3}^{s+1} \to MU/I[u_{i_1}, u_{i_2}][u_{i_3}]/u_{i_3}^s$$

are singular extensions of MU-algebras. Localizing this tower with respect to $MU/I[u_{i_1}, u_{i_1}^{-1}]$ and using (8) and Theorem 4.1 we get the tower

$$MU/I[[u_{i_2}]][u_{i_1}, u_{i_1}^{-1}] \leftarrow MU/I[[u_{i_2}]][u_{i_1}, u_{i_1}^{-1}][u_{i_3}]/u_{i_3}^2 \leftarrow \ldots.$$

whose homotopy limit

$$holim(MU/I[[u_{i_2}]][u_{i_1}, u_{i_1}^{-1}][u_{i_3}]/u_{i_3}^k) = MU/I[[u_{i_2}, u_{i_3}]][u_{i_1}, u_{i_1}^{-1}]$$

is clearly weakly equivalent to the localization of $MU/I[u_{i_1}, u_{i_2}, u_{i_3}]$ with respect to $MU/I[u_{i_1}, u_{i_1}^{-1}$. Moreover

$$MU/I[[u_{i_2}, u_{i_3}]][u_{i_1}, u_{i_1}^{-1}]_* = MU/I_*[[u_{i_2}, u_{i_3}]][u_{i_1}, u_{i_1}^{-1}]$$

Repeating this process for $u_{i_4}, u_{i_4}, \dots, u_{i_k}$ we obtain the desired isomorphism

$$MU/I[[u_{i_2}, u_{i_3}, \dots, u_{i_k}]][u_{i_1}, u_{i_1}^{-1}]_* = MU/I_*[[u_{i_2}, u_{i_3} \dots, u_{i_k}]][u_{i_1}, u_{i_1}^{-1}].$$

Our main application of the developed localization techniques is to Johnson-Wilson theories $E(n)$.

Recall that the MU-algebra spectrum $E(n)$ has the coefficient ring $E(n)_* = Z_{(p)}[v_1, v_2, \dots, v_{n-1}][v_n, v_n^{-1}]$. The MU_*-algebra structure on $E(n)_*$ is defined by the correspondence $x_i \to 0$ for $i \neq 2(p^n - 1)$ and $x_{2(p^n-1)} \to v_n$ where x_i are polynomial generators of MU_*. We have the following

Corollary 4.3 *There exist MU-algebras $\hat{E}(n)$, $\hat{E}_{W(L)}(n)$, $K(n)$, $K(n)_L$, and the homotopy commutative diagram of MU-algebras*

$$\begin{array}{ccc} \hat{E}(n) & \to & \hat{E}_{W(L)}(n) \\ \downarrow & & \downarrow \\ K(n) & \to & K(n)_L \end{array} \tag{9}$$

which realizes in homotopy the diagram of MU_*-algebras

$$\begin{array}{ccc} \hat{Z}_p[[v_1, v_2, \dots, v_{n-1}]][v_n, v_n^{-1}] & \to & W(L)[v_1, v_2, \dots, v_{n-1}][[v_n, v_n^{-1}]] \\ \downarrow & & \downarrow \\ Z/pZ[v_n, v_n^{-1}] & \to & L[v_n, v_n^{-1}] \end{array}$$

Proof. We first construct the map of MU-algebras $k(n) \to k(n)_L$ by adjoining v_n to the extension $HF_p \to HL$. Here $k(n)$ and $k(n)_L$ are the connective Morava K-theories with coefficient rings $k(n)_* = F_p[v_n]$ and $k(n)_* = L[v_n]$. Adjoining indeterminates $p, v_1, v_2, \dots, v_{n-1}$ we construct the homotopy commutative diagram of MU-algebras

$$\begin{array}{ccc} \widehat{BP} < n > & \to & \widehat{BP} < n >_{W(L)} \\ \downarrow & & \downarrow \\ k(n) & \to & k(n)_L \end{array} \tag{10}$$

which realizes in homotopy the diagram

$$\begin{array}{ccc} \hat{Z}_p[v_1, v_2, \dots, v_{n-1}][v_n] & \to & W(L)[v_1, v_2, \dots, v_{n-1}][v_n] \\ \downarrow & & \downarrow \\ Z/pZ[v_n] & \to & L[v_n] \end{array}$$

According to Proposition 4.2 Bousfield localization of $\widehat{BP} < n >$ with respect to $K(n)$ on the level of coefficient rings amounts to inverting v_n and completing at the ideal $(v_1, v_2, \dots, v_{n-1})$. Further notice that since $\pi_* \widehat{BP} < n >_{W(L)}$ is a free $\pi_* \widehat{BP} < n >$-module of finite rank (which is equal to the degree of the field extension L/F_p) the MU-algebra $\widehat{BP} < n >_{W(L)}$ is a finite cell $\widehat{BP} < n >$-module and therefore its $K(n)$-localization is equivalent to

$$\widehat{BP} < n >_{K(n)} \wedge_{\widehat{BP}<n>} \widehat{BP} < n >_{W(L)}$$

and its coefficient ring is

$$W(L) \otimes \pi_* \widehat{BP} < n >_{K(n)} = W(L)[[v_1, v_2, \dots, v_{n-1}]][v_n, v_n^{-1}].$$

Therefore we obtain the diagram (9) by localizing (10) with respect to $K(n)$ in the category of MU-algebras. Corollary 4.3 is proved.

5 Computing spaces of S-algebra maps

In this section we investigate spaces of strictly multiplicative maps from $\hat{E}(n)_{W(L)}$ to certain MU-algebras which we call called "strongly $K(n)_L$-complete.". As a consequence we obtain versions of the Hopkins-Miller theorem as well as splitting theorems for such spectra. Such theorems (in a weaker, up to homotopy form) were previously obtained by methods of formal group theory, cf. [2].

Let A, B be S-algebras, which we will assume without loss of generality to be q-cofibrant in the sense of [4]. That means in particular, that the topological space of S-algebra maps $B \to A$ has the "correct" homotopy type (i.e. the one that depends only on the homotopy type of A and B as S-algebras). Denote this topological space by $F_{S-alg}(B, A)$. Also denote the set of multiplicative up to homotopy maps from A to B by $Mult(A, B)$.

We recall one result from [6] which will be needed later on.

Theorem 5.1 *Let $\Sigma^{-1}M \to X \to A$ be a singular extension of S-algebras associated with a derivation $d : A \to A \vee M$ and $f : B \to A$ a map of R-algebras. Then f lifts to an R-algebra map $B \to X$ iff a certain element in $Der^0(B, M)$ is zero. Assuming that a lifting exists the map*

$$F_{R-alg}(B, X) \to F_{R-alg}(B, A)$$

has homotopy fibre over the point $f \in F_{R-alg}(B, A)$ weakly equivalent to $\Omega^\infty \mathbf{Der}(B, \Sigma^{-1}M)$ (the 0th space of the spectrum $\mathbf{Der}(X, \Sigma^{-1}M)$).

Recall from the previous section that the spectrum $\hat{E}(n)_{W(L)}$ with coefficient rings $W(L)[[v_1, v_2, \ldots, v_{n-1}]][v_n, v_n^{-1}]$ has a structure of an S-algebra.

We say that the generalized Hopkins-Miller theorem holds for an S-algebra E if $\pi_0 F_{S-alg}(\hat{E}(n)_{W(L)}, E) = Mult(\hat{E}(n)_{W(L)}, E)$ whilst $\pi_i F_{S-alg}(\hat{E}(n)_{W(L)}, E) = 0$ for $i > 0$.

Proposition 5.2 *The generalized Hopkins-Miller theorem holds for the S-algebra $K(n)_L$.*

Proof. Consider the Bousfield-Kan spectral sequence (cf.[3]) for the mapping space $F_{S-alg}(A, B)$. The identification of the E_2-term is standard and we refer the reader to [10] for necessary details. Here we only mention that the key ingredient in this identification is the existence of the Kunneth formula

$$K(n)_{L*}(\hat{E}(n)_{W(L)} \wedge \hat{E}(n)_{W(L)})$$

$$= K(n)_{L*}(\hat{E}(n)_{W(L)}) \otimes_{K(n)_{L*}} K(n)_{L*}(\hat{E}(n)_{W(L)}).$$

This is the result we need:

$$E_2^{st} = Der^{st}_{K(n)_{L*}}(\hat{E}(n)_{W(L)*}K(n)_L, K(n)_{L*}) \ for \ s, t \neq 0;$$

$$E_2^{00} = Mult(\hat{E}(n)_{W(L)}, K(n)_L).$$

Here $Der_k^{**}(?, ?)$ is defined for a graded k-algebra R_* and a graded R_*-bimodule M_* as the shifted Hochschild cohomology:

$$Der_k^{**}(A_*, M_*) := HH_k^{*+1*}(A_*, M_*)$$

(the second grading reflects the fact that A_* and M_* are graded objects).

Further

$$\hat{E}(n)_{W(L)*}K(n)_L = \hat{E}(n)_* K(n) \otimes W(L) \otimes L = E(n)_* K(n) \otimes L \otimes L$$

and computations in [9], Chapter VI show that

$$E(n)_* K(n) = \Sigma_* = F_p[v_n, v_n^{-1}][t_k | k > 0]/(t_k^{p^n} - v_n^{p^k - 1} t_k)$$

where the degree of t_k is $2(p^k - 1)$. We have:

$$HH^{**}_{K(n)_{L*}}(\Sigma_* \otimes L \otimes L, K(n)_{L*}) = HH^{**}_{K(n)_*}(\Sigma_* \otimes L, K(n)_* \otimes L)$$

$$= HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) \otimes HH^*_{F_p}(L, L) = HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) \otimes L.$$

An easy cohomological calculation (due to A.Robinson, cf. [11]) shows that $HH^{**}_{K(n)_*}(\Sigma_*, K(n)_*) = K(n)_*$ and therefore

$$HH^{**}_{K(n)_{L*}}(\hat{E}(n)_{W(L)_*}K(n)_L) = K(n) \otimes L = K(n)_{L*} \tag{11}$$

Thus the Bousfield-Kan spectral sequence reduces to its corner term and Proposition 5.2 is proved.

Remark 5.3 *Notice that even though in Proposition 5.2 and we used the results in the previous section that spectra $K(n)_L$, and $\hat{E}(n)_{W(L)}$ admit S-algebra structures we did not specify which structures are used. In other words Proposition 5.2 is valid with any choice of S-algebra structures on the spectra in question.*

We can now regard $K(n)_L$ as a bimodule over $\hat{E}(n)_{W(L)}$ by choosing an S-algebra map $\hat{E}(n)_{W(L)} \to K(n)_L$ (which is supplied by Propositions 5.2). Therefore it makes sense to consider spectra of topological Hochschild cohomology and topological derivations of $\hat{E}(n)_{W(L)}$ with values in $K(n)_L$. It turns out that these spectra do not depend up to weak equivalence which bimodule structure we choose.

Proposition 5.4 *The canonical map of S-modules*

$$\mathbf{THH}(\hat{E}(n)_{W(L)}, K(n)_L) \to K(n)_L;$$

is a weak equivalences. Furthermore the S-module $\mathbf{Der}(\hat{E}(n)_{W(L)}, K(n)_L)$ is contractible.

Proof. Since there is a cofibre sequences of S-modules

$$\Sigma^{-1}\mathbf{Der}(\hat{E}(n)_{W(L)}, K(n)_L) \to \mathbf{THH}(\hat{E}(n)_{W(L)}, K(n)_L) \to K(n)_L$$

it suffices to prove the statement about **THH**. We have the following spectral sequence

$$Ext^{**}_{\hat{E}(n)_{W(L)_*}\hat{E}(n)_{W(L)}}(\hat{E}(n)_{W(L)_*}, K(n)_{L*}) \Rightarrow THH^*(\hat{E}(n)_{W(L)}, K(n)_L) \tag{12}$$

Since $\hat{E}(n)_{W(L)}$ is a Landweber exact theory the $\pi_*\hat{E}(n)_{W(L)}$ - module $\hat{E}(n)_{W(L)_*}\hat{E}(n)_{W(L)}$ is flat and by flat base change we obtain

$$Ext^{**}_{\hat{E}(n)_{W(L)_*}\hat{E}(n)_{W(L)}}(\hat{E}(n)_{W(L)_*}, K(n)_{L*})$$

$$= Ext^{**}_{\hat{E}(n)_{W(L)_*}K(n)_L}(K(n)_{L*}, K(n)_{L*})$$

$$= HH^*_{K(n)_{L*}}(\hat{E}(n)_{W(L)*}K(n)_L, K(n)_{L*}).$$

The isomorphism (11) shows that the spectral sequence (12) collapses giving the weak equivalence

$$\mathbf{THH}(\hat{E}(n)_{W(L)}, K(n)_L) \simeq K(n)_L$$

and our theorem is proved.

We now describe a particular class of MU-algebras E which will be called strongly $K(n)_L$ - complete and for which the set of multiplicative maps $\hat{E}(n)_{W(L)} \to E$ has an especially simple form.

Let $\nu = \{\nu_0, \nu_1, \ldots, \hat{\nu}_{2(p^n-1)}, \ldots\}$ (i.e.the $2(p^n-1)$th spot is missed) be a sequence of positive integers. Associated to ν is the regular sequence $p^{\nu_0}, x_1^{\nu_1}, x_2^{\nu_2}, \ldots$ in MU_*. Consider the MU-module

$$MU(\nu) := x^{-1}_{2(p^n-1)} MU_{W(L)}/(p^{\nu_0}, x_1^{\nu_1}, x_2^{\nu_2}, \ldots)$$

$$= v_n^{-1} MU_{W(L)}/(p^{\nu_0}, x_1^{\nu_1}, x_2^{\nu_2}, \ldots).$$

Since $MU(\nu)$ can be constructed from $K(n)_L$ by a sequence (possibly infinite) of elementary extensions we see that there is a structure of an MU-algebra on $MU(\nu)$ compatible with the MU_*-algebra structure on $v_n^{-1}MU(\nu)_* = W(L) \otimes v_n^{-1}MU_*/(p^{\nu_0}, x_1^{\nu_1}, x_2^{\nu_2}, \ldots)$. Notice that there is a reduction MU-algebra map $MU(\nu) \to K(n)_L$. Furthermore the filtration on $MU(\nu)$ by powers of the maximal ideal has subquotients isomorphic to products of copies of $K(n)_{L*} = L[v_n, v_n^{-1}]$.

Let E and F be two MU-algebras supplied with an MU-algebra map $E \to K(n)_L$ and $F \to K(n)_L$. We say that an MU-algebra map $E \to F$ is *compatible* if it is a map over $K(n)_L$.

Definition 5.5 *The class $SC(K(n)_L)$ of strongly $K(n)_L$-complete MU-algebras is the smallest class of MU-algebras over $K(n)_L$ that satisfies the following two axioms:*
(i) $SC(K(n)_L)$ contains $MU(\nu)$ for any $\nu = \{\nu_1, \nu_2, \ldots, \hat{\nu}_{2(p^n-1)}, \ldots\}$ as above;
(ii) For any sequential inverse system of compatible maps in $SC(K(n)_L)$ its homotopy inverse limit is also in $SC(K(n)_L)$.

In other words an MU-algebra is $K(n)_L$-complete if it can be built from $K(n)_L$ by taking elementary extensions and (homotopy) inverse limits. If

$E_1 \leftarrow E_2 \leftarrow \ldots$ is an inverse system in $SC(K(n)_L)$ then $\pi_* holim(E_n) = lim_{n\to\infty}\pi_* E_n$ since coefficients rings of E_n are all in even degrees. Therefore we never have to worry about lim^1-problems.

Examples of strongly $K(n)_L$ - complete MU-algebras include $K(n)_L$, $\hat{E}(n)_{W(L)}$ and also $v_n^{-1}\widehat{BP}_{W(L)}$ and $v_n^{-1}\widehat{P}(n)_{W(L)}$, the Artinian completions of $v_n^{-1}BP_{W(L)}$ and $v_n^{-1}P(n)_{W(L)}$

Let E be a strongly $K(n)_L$-complete MU-algebra. The canonical S-algebra map (which is also an MU-algebra map) $E \to K(n)_L$ determines via composition the morphism of topological spaces (unbased)

$$i : F_{S-alg}(E(n)_{W(L)}, E) \to F_{S-alg}(E(n)_{W(L)}, K(n)_L) \qquad (13)$$

Proposition 5.6 *The map (13) is a weak equivalence. In particular* $F_{S-alg}(E(n)_{W(L)}, E)$ *is a homotopically discrete space. Moreover the* MU*-module* $\mathbf{Der}(E(n)_{W(L)}, E)$ *is contractible.*

Proof. First assume that $E = MU(\nu)$ where all ν_l except for ν_i are equal to one, i.e. that $E = K(n)_L[x_k]/x_k^{\nu_i}$. The we have the tower

$$K(n)_L \leftarrow K(n)_L[x_k]/x_k^2 \leftarrow \ldots \leftarrow K(n)_L[x_k]/x_k^{\nu_i} = E.$$

Applying to this tower the functor $F_{S-alg}(E_{W(L)}, ?)$ we get the tower of topological spaces

$$F_{S-alg}(E_{W(L)}, K(n)_L) \leftarrow F_{S-alg}(E_{W(L)}, K(n)_L[x_k]/x_k^2) \leftarrow \ldots$$
$$\leftarrow F_{S-alg}(E_{W(L)}, K(n)_L[x_k]/x_k^{\nu_i}) = F_{S-alg}(E_{W(L)}, E) \qquad (14)$$

Using Theorem 5.1 and Proposition 5.4 we see that the homotopy fibre of each map

$$F_{S-alg}(E_{W(L)}, K(n)_L[x_k]/x_k^s) \leftarrow F_{S-alg}(E_{W(L)}, K(n)_L[x_k]/x_k^{s+1})$$

is weakly equivalent to the space $\Omega^\infty \mathbf{Der}(E_{W(L)}, \Sigma^{|x_k|s} K(n)_L)$ which is contractible. Hence all maps in (14) are weak equivalences of spaces and in particular $F_{S-alg}(E_{W(L)}, K(n)_L[x_k]/x_k^{\nu_i})$ is weakly equivalent to $F_{S-alg}(E_{W(L)}, K(n)_L$. Similarly $\mathbf{Der}(E_{W(L)}, K(n)_L[x_k]/x_k^{\nu_i})$ is weakly equivalent to $\mathbf{Der}(E_{W(L)}, K(n)_L$ and is contractible.

Further obvious induction shows that the statement of Proposition 5.6 holds for an MU-algebra of the form $MU(\nu)$ where all ν_l except for for a finite number are equal to one. Passing to the limit we obtain the result for an arbitrary MU-algebra $MU(\nu)$.

Now using transfinite induction suppose that the conclusion of Proposition 5.6 holds for an inverse system $\{E_l\}$ of strongly $K(n)_L$-complete MU-algebras consisting of compatible maps and let $E = holim\ E_l$. Applying the functor $Map_{S-alg}(E_{W(L)}, ?)$ to $\{E_l\}$ we get an inverse system of topological spaces $\{F_{S-alg}(E_{W(L)}, E_l)\}$. By inductive assumption all spaces $F_{S-alg}(E_{W(L)}, E_l)$ are weakly equivalent to $F_{S-alg}(E_{W(L)}, K(n)_L)$ and the maps in the inverse system $\{F_{S-alg}(E_{W(L)}, E_l)\}$ respect this equivalence. We conclude that the space $F_{S-alg}(E_{W(L)}, E)$ is also weakly equivalent to $F_{S-alg}(E_{W(L)}, K(n)_L)$. Similar arguments show that $\mathbf{Der}(E_{W(L)}, K(n)_L)$ is contractible and Proposition 5.6 is proved.

Remark 5.7 *We see that for any strongly $K(n)_L$-complete MU-algebra E the canonical S-algebra map $\hat{E}(n)_{W(L)} \to K(n)_L$ lifts uniquely to a map $\hat{E}(n)_{W(L)} \to E$. Therefore E is naturally a (bi)module over $\hat{E}(n)_{W(L)}$.*

The following lemma descibes the structure of cohomology operations from $\hat{E}(n)_{W(L)}$ to a strongly $K(n)_L$-complete MU-algebra E.

Lemma 5.8 *Let E be a strongly $K(n)_L$-complete MU-algebra. Then the evaluation map*

$$E^*\hat{E}(n)_{W(L)} \to Hom_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)_*}\hat{E}(n)_{W(L)}, E_*)$$

is an isomorphism.

Proof. We have the following natural isomorphism of S-modules

$$F_S(\hat{E}(n)_{W(L)}, E) \cong F_{\hat{E}(n)_{W(L)}}(\hat{E}(n)_{W(L)} \wedge \hat{E}(n)_{W(L)}, E)$$

Consider further the spectral sequence

$$\begin{aligned} &Ext^{**}_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, E_*) \\ \Rightarrow &[\hat{E}(n)_{W(L)} \wedge \hat{E}(n)_{W(L)}, E]^*_{(\hat{E}(n)_{W(L)})} = [\hat{E}(n)_{W(L)}, E]^* \end{aligned} \tag{15}$$

We claim that all higher Ext groups vanish so that our spectral sequence reduces to $Ext^{0*}_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)_*}\hat{E}(n)_{W(L)}, E_*)$. (This would clearly give us the statement of the lemma). To see this first assume that E is of the form $MU(\nu)$ for some sequence ν.

Then since for the maximal ideal $\mathcal{M}$ of E_* the E_*-module $\mathcal{M}^n/\mathcal{M}^{n+1}$ is isomorphic to direct product of copies of $K(n)_{L*}$ it is enough to prove that

$$Ext^i_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_L) = 0$$

for $i > 0$ (here we suppressed the second, internal grading on Ext from the notations). Since the $\hat{E}(n)_{W(L)*}$-module $\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}$ is flat we have by flat base change:

$$Ext^i_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_{L*})$$
$$= Ext^i_{K(n)_{L*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)} \otimes_{\hat{E}(n)_{W(L)*}} K(n)_{L*}, K(n)_{L*})$$

and the last Ext-group is zero for $i > 0$ since $K(n)_{L*}$ is injective as a graded module over itself.

Therefore our claim about the vanishing of higher Ext-groups in (15) is proved for $E = MU(\nu)$. To get the general case suppose that E is the homotopy inverse limit of strongly $K(n)_L$-complete MU-algebras E_l for which higher Ext's do vanish. Then

$$Ext^{i*}_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, E_*)$$
$$= Ext^{i*}_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, lim_{l\to\infty} E_{l*})$$
$$lim_{l\to\infty} Ext^{i*}_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, E_{l*}) = 0$$

for $i > 0$. With this Lemma 5.8 is proved.

We can now formulate our main

Theorem 5.9 *The generalized Hopkins-Miller theorem holds for any strongly $K(n)_L$-complete MU-algebra E.*

Proof. By Proposition 5.6 we know that the space $Map_{S-alg}(\hat{E}(n)_{W(L)}, E)$ is homotopically discrete with

$$\pi_0(Map_{S-alg}(\hat{E}(n)_{W(L)}, E)) = Hom_{K(n)_{L*}-alg}(\Sigma_* \otimes L, K(n)_{L*}).$$

So it remains to show that an arbitrary multiplicative operation $\hat{E}(n)_{W(L)} \to E$ lifts to an S-algebra map. Since any such operation determines an $\hat{E}(n)_{W(L)*}$-algebra map $\hat{E}(n)_{W(L)*}E(n)_{W(L)} \to E_*$ we conclude by Lemma 5.8 that there is an injective map

$$Mult(\hat{E}(n)_{W(L)}, E) \to Hom_{\hat{E}(n)_{W(L)*}-alg}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, E_*) \quad (16)$$

We claim that any $\hat{E}(n)_{W(L)*}$-algebra map $\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)} \to K(n)_{L*}$ lifts uniquely to a map $\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)} \to E_*$ so that there is an isomorphism

$$Hom_{\hat{E}(n)_{W(L)*}-alg}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, E_*$$
$$\cong Hom_{\hat{E}(n)_{W(L)*}-alg}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_{L*}) \quad (17)$$

To see this assume first that E is of the form $MU(\nu)$. Induction up the filtration of the $\hat{E}(n)_{W(L)*}$-module E_* by the powers of the maximal ideal shows that our claim is equivalent to the vanishing of certain Hochschild cohomology classes in $HH^*_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_{L*})$. Flat base change gives an isomorphism

$$HH^*_{\hat{E}(n)_{W(L)*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_{L*})$$

$$= HH^*_{K(n)_{L*}}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)} \otimes_{\hat{E}(n)_{W(L)*}} K(n)_{L*}, K(n)_{L*})$$

$$= HH^*_{K(n)_{L*}}(K(n)_{L*}\hat{E}(n)_{W(L)}, K(n)_{L*}) = K(n)_{L*}.$$

We see that all higher Hochschild cohomology groups are zero and our claim is proved (in the special case $E = MU(\nu)$). Then the standard by now arguments (which we omit) show that (17) holds for any E.

Further we have an obvious change of rings isomorphism

$$Hom_{\hat{E}(n)_{W(L)*}-alg}(\hat{E}(n)_{W(L)*}\hat{E}(n)_{W(L)}, K(n)_{L*})$$

$$= Hom_{K(n)_{L*}-alg}(K(n)_{L*}\hat{E}(n)_{W(L)}, K(n)_{L*})$$

The last term corresponds bijectively by Proposition 5.6 to homotopy classes of S-algebra maps from $\hat{E}(n)_{W(L)}$ to E. This shows that any multiplicative operation $\hat{E}(n)_{W(L)} \to E$ lifts to an S-algebra map and Theorem 5.9 is proved.

Remark 5.10 *The class of strongly $K(n)_L$-complete MU-algebras is probably not the most general one for which the conclusion of Theorem 5.9 holds. It is plausible that one has the generalized Hopkins-Miller theorem for any $K(n)_L$-local MU-algebra but such a result seems out of reach at present.*

Tracing the proofs of Proposition 5.2 and Theorem 5.9 we see that they don't depend on the choice of the S-algebra structure on $\hat{E}(n)_{W(L)}$ as long as the ring spectrum structure is fixed. (But they do depend on the choice of the S-algebra structure on E.) This gives the following

Corollary 5.11 *The spectrum $\hat{E}(n)_{W(L)}$ has a unique (up to a noncanonical isomorphism) structure of an S-algebra compatible with its structure of a ring spectrum.*

Proof. Let $\hat{E}'(n)_{W(L)}$ denote the S-algebra whose underlying ring spectrum is equivalent to $\hat{E}(n)_{W(L)}$ but the S-algebra structure is possibly different. Then the given multiplicative up to homotopy weak equivalence

$\hat{E}'(n)_{W(L)} \to \hat{E}(n)_{W(L)}$ can be lifted to an S-algebra map which shows that the would-be exotic S-algebra structure on $\hat{E}'(n)_{W(L)}$ is actually isomorphic to the standard one.

Remark 5.12 *A similar statement about the uniqueness of the A_∞ structure on $\hat{E}(n)$ was proved by A.Baker in [1]. However the proof in the cited reference was based on the obstruction theory of A.Robinson and it is unclear at this time to what extent this theory carries over in the present context.*

Theorem 5.9 allows one to obtain splittings of various strongly $K(n)_L$-complete MU-algebras. For example consider $v_n^{-1}\widehat{BP}_{W(L)}$, the Artinian completion of the v_n-localization of the Brown-Peterson spectrum $BP_{W(L)}$ (cf.[2] concerning Artinian completions). Then $v_n^{-1}\widehat{BP}_{W(L)}$ is a strongly $K(n)_L$-complete MU-algebra obtained from $K(n)_{W(L)}$ by adjoining the collection of indeterminates $\{p, v_1, v_2, \dots\}$:

$$v_n^{-1}\widehat{BP}_{W(L)} = K(n)_{W(L)}[[v_1, v_2, \dots]].$$

There is a canonical map $v_n^{-1}\widehat{BP}_{W(L)} \to \hat{E}(n)_{W(L)}$ which on the level of coefficient ring reduces to killing the ideal $(v_{n+1}, v_{n+2}, \dots)$ in $v_n^{-1}\widehat{BP}_{W(L)_*}$. From Theorem 5.9 we conclude that any S-algebra map $\hat{E}(n)_{W(L)} \to \hat{E}(n)_{W(L)}$ lifts uniquely to an S-algebra map $\hat{E}(n)_{W(L)} \to v_n^{-1}\widehat{BP}_{W(L)}$. In particular the identity map $id : \hat{E}(n)_{W(L)} \to \hat{E}(n)_{W(L)}$ can be so lifted and we obtain

Theorem 5.13 *The S-algebra map $v_n^{-1}\widehat{BP}_{W(L)} \to \hat{E}(n)_{W(L)}$ admits a unique S-algebra splitting $\hat{E}(n)_{W(L)} \to v_n^{-1}\widehat{BP}_{W(L)}$*

An analogous theorem was proved (in a weaker, up to homotopy form) in [2].

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